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Microlocal Analysis of Infrared Singularities

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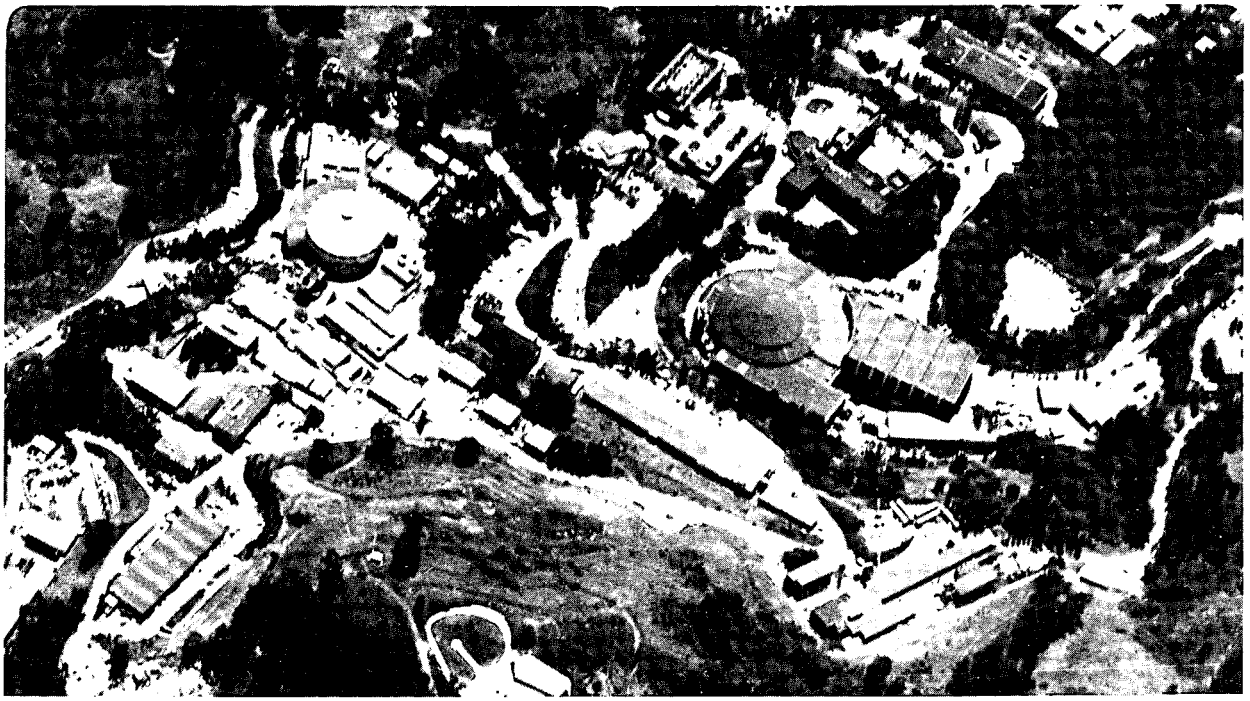

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Microlocal Analysis of Infrared Singularities *

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§0. Introduction.

Probably the most secure feature of relativistic quantun theory is the physical-region analytic structure: the singularities of any scattering function are confined to certain surfaces, called Landau-Nakanishi surfaces[1], or Landau surfaces for short, and in a neighborhood of any regular point of such a surface the scattering function is, normally, the boundary value of a function holomorphic on a certain specified kind of domain. Moreover, if all the relevant particles have strictly positive mass then the singularity on the Landau surface specified by $\varphi = 0$, at a regular point with grad $\varphi \neq 0$, normally has the form $A\varphi^{\lambda}(\log \varphi)^{\nu}$ where A is non-singular, $\lambda = \frac{3}{2}N - 2n + \frac{3}{2}$, $\nu = 1$ or 0 according as λ is a non-negative integer on not, and N and n are the numbers of internal lines and vertices in the "Landau diagram" associated with this Landau surface[2]. These analytic properties seem quite secure because they follow from general properties that do not depend on the short-range (i.e., high-energy) contributions that can lead to ultraviolet divergences. Moreover, this physical-region singularity structure is the momentum-space form of an asymptotic spacetime structure that corresponds to classical physics in just the way demanded by the correspondence principle: the physical-region momentum-space singularity structure corresponds to the classical property that a stable particle, left undisturbed, moves with constant velocity, and neither disappears nor generates

clones of itself, in the course of an undisturbed motion.[4]

This correspondence-principle connection evidently requires that this same form $\varphi^{\lambda}(\log \varphi)^{\nu}$ be maintained modulo weaker singularities in the presence of radiative corrections coming from massless particles. For, an electron appears neither to disappear nor generate clones of itself during its asymptotic motion, as it would be effectively forced to do if λ or ν were altered by the effects of the massless photons.

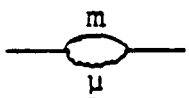
Several difficulties block the direct extension of the earlier calculations of the values of λ and ν to cases in which (radiative) corrections due to exchange of photons, which are massless, are included. The first is the infrared divergence: straightforward perturbative calculations lead to divergent integrals, when they are confined to mass-shell manifolds. The essential ideas needed to circumvent these divergences are well-known[5]: one must separate out for special treatment the classical part of the problem, and use the fact that photons of sufficiently small energy are not observed. These ideas allow the calculations to be reorganized according to the principle that the classical part must be treated as a whole.

These ideas have recently been applied to the problem of the elucidation of the effects of radiative corrections on the singularity structure of scattering functions, and on the asymptotic behavior in spacetime.[6,7] In ref.6 the photon interaction was separated into a "classical" part and a "quan-

tum" part. The contributions from "classical photons", which by definition have "classical" couplings at each end, can then be treated exactly, without using any limiting procedure involving a fictitious nonzero photon mass, and these classical contributions sum to a unitary operator. The expected asymptotic behavior in spacetime demanded by the correspondence principle was then deduced, under the condition that the radiative corrections that arise from (massless) photons coupled via the residual "quantum" couplings lead to no infrared divergences, and to no disruption of the dominant contributions to the physical-region singularity structure described above.

The dominant term arising from the quantum coupling has in place of the usual photon coupling factor γ_μ rather the factor $\gamma_\mu \not{k}$. Here $\not{k} = k^\nu \gamma_\nu$, the quantities k^ν are components of the photon momentum-energy vector, and the γ_μ are Dirac matrices. The extra factor $\not{k}$, which vanishes at $k = 0$, is expected to remove all infrared divergences. But if one wishes to examine the asymptotic behavior in spacetime, or the corresponding momentum-space analytic structure, then difficulties still remain. These arise from the singular character of the photon propagator $(k^2+i0)^{-1}$ at $k = 0$.

In the present context the usual procedure for avoiding this problem by introducing a fictitious photon mass μ, which is set to zero at the end of the computation is not efficient. For the character of the singularity surface $k^2 - \mu^2 = 0$ changes abruptly when the photon mass μ is set to zero, and

this change induces an abrupt change in the character of the singularity on or near the Landau surface $\varphi = 0$, when the photon mass is set to zero. For example, if we consider the (renormalized) Feynman function for the self-energy diagram 

, the exponent λ at the threshold $p^2 = (m+\mu)^2$ is equal to $1/2$ for $\mu \neq 0$ and equal to 1 for $\mu = 0$. Thus one is left with the highly nontrivial problem of proving that the $\mu \to 0$ limiting procedure yields precisely the answer that would be obtained if one used the true propagator $(k^2+i0)^{-1}$.

Our approach is to deal directly with the photon propagator $(k^2+i0)^{-1}$, without introducing any limiting procedure. This makes the mathematical situation different in principle from the familiar one associated with the propagators of massive particles. The main content of this paper consists, then, basically of a rigorous treatment of the problems associated with the singular character of the photon propagator at $k = 0$, in the context of the simplest nontrivial example.

The initial problem in setting up such a formulation is to give well-defined meaning to photon propagator $(k^2+i0)^{-1}$. This is done in §1, where a rigorous meaning is given to the formal expression

$$(0.1) \qquad (k^2+i0)^{-1} = -2\pi i\ \delta(k_0-\sqrt{\vec{k}^2})(2k_0)^{-1} + ((k_0-i0)^2 - \vec{k}^2)^{-1}$$

and its $k \mapsto -k$ transform

$$(0.2) \qquad (k^2+i0)^{-1} = -2\pi i\ \delta(k_0+\sqrt{\vec{k}^2})(2k_0)^{-1} + ((k_0+i0)^2 - \vec{k}^2)^{-1}.$$

Here, and in what follows, $\vec{k}$ denotes (k_1, k_2, k_3), and $\vec{k}^2 = \sum_{j=1}^{3} k_j^{\,2}$. The apparent ambiguity of $\delta(k_0 \pm \sqrt{\vec{k}^2})(2k_0)^{-1}$ at $k = 0$ will be removed in §1, and $(k^2+i0)^{-1}$ will then be expressed (in two different ways) as a sum of two well-defined hyperfunctions. (Cf. (1.10)). Each of the two terms in either of those expressions has important properties which we will exploit. But in none of these terms is there a separation of the singularities into two disjoint parts, one confined to the region of positive photon energy and the other confined to the region of negative photon energy. This separation holds for the massive particle case, and it plays an important role in the usual derivations of singularity properties. The failure of these properties forces us to devise new methods, in order to deal with the $k = 0$ contributions.

A specific aim of our analysis is to show that in our simple case the discontinuity around the singularity surface under consideration is given by Cutkosky-type formulas.[8] This property does not hold for the original Feynman function, with photon couplings γ_μ, because of infrared divergences. In particular, for our case in which the external particles

are all neutral the original Feynman function is well-defined (i.e., it is infrared finite), and so is the discontinuity. But the expression for the discontinuity obtained from the $\mu \to 0$ limiting procedure is a sum of four Cutkosky-type functions each of which is infrared divergent. In our case, with couplings $\gamma_\mu \not{k}$, we find, in the end, that the discontinuity is given by the same Cutkosky-type formula that was formally obtained from the $\mu \to 0$ limiting procedure. Now, however, the four discontinuities are, as expected, infrared finite, due to the extra power of k in each quantum coupling. The formula allows one to exhibit explicitly the character of the singularity at $\mathcal{S} = 0$, and confirms that the character of the dominant singularity is not altered by the radiative corrections, as was demanded in ref. 6.

The scattering functions, considered as functions of real momentum vectors, are hyperfunctions: in a sufficiently small real open neighborhood Ω of any point p, the scattering function $f(p')$ can be represented as a sum of terms, each of which is a boundary value $f_j(p')$ of a function $\tilde{f}_j(p'+iq')$ that is holomorphic on an open region $p' \in \Omega$, $q' \in \Gamma_j$, where Γ_j is a domain that tends to an open cone with vertex at $q' = 0$ as q' tends to zero. We use the framework of microlocal analysis, which is described in refs 9, 10. Very briefly, the formalism rests on the concept of the "singularity spectrum" of a hyperfunction. For each point $x = (x_1, \cdots, x_n)$ in a real manifold M where the hyperfunc-

tion is <u>not</u> analytic there is a set of pairs $(x; \sqrt{-1}\, w)$, where the "cotangential" components $w(\neq 0)$, which are defined up to a strictly positive scalar multiple, are determined as "duals" of the set of allowed directions along which the real point x is to be approached if the hyperfunction is to be represented as a sum of boundary values of holomorphic functions. The totality of such pairs $(x; \sqrt{-1}w)$ is called the singularity spectrum of f, and it is denoted as S.S.f. (The singularity spectrum is, by its definition, a subset of the (pure imaginary) spherical cotangent bundle, which is usually denoted $\sqrt{-1}\, S^*M$.)

Two important properties of hyperfunctions used repeatedly in this work are:

(A) The product $f = \prod_{\ell \in L} f_\ell$ of finitely many hyperfunctions f_ℓ is a well-defined hyperfunction at x if there is no solution to $\sum_{\ell \in L(x)} \alpha_\ell w_\ell = w = 0$, $\alpha_\ell \geq 0$, $\sum_{\ell \in L(x)} \alpha_\ell \neq 0$, $\alpha_\ell = 0$ $(\ell \notin L(x))$, where $L(x) = \{\ell \in L;\ f_\ell$ is not analytic at $x\}$ and each w_ℓ $(\ell \in L(x))$ belongs to a pair $(x; \sqrt{-1}\, w_\ell)$ associated with f_ℓ (i.e., $(x; \sqrt{-1}\, w_\ell)$ belongs to the singularity spectrum of f_ℓ). If there is no such solution $w = 0$ then the singularity spectrum of f over the base point x is contained in the set of pairs $(x; \sqrt{-1}\, w)$ such that w is a solution of the above set of equations. (Cf. Ref. <u>10</u>, p.109.)

(B) If $f(x,y)$ is a hyperfunction then the integral $F(x) = \int dy f(x,y)$ is a well-defined hyperfunction provided the support of $f(x,y)$ in y, as x ranges over a compact set, is compact. Further, the singularity spectrum of $F(x)$ is

confined to the set of pairs $(x; \sqrt{-1}\, u)$ such that $((x,y); (\sqrt{-1}\, u, \sqrt{-1}\cdot 0))$ is in the singularity spectrum of $f(x,y)$. In particular, $F(x)$ is analytic if there is no point of the form $(x,y; (\sqrt{-1}\, u, \sqrt{-1}\cdot 0))$ $(u \neq 0)$ in the singularity spectrum of $f(x,y)$, provided that the condition on the support of $f(x,y)$ is satisfied.

In §1, we define $(k^2+i0)^{-1}$ as a hyperfunction in accordance with (0.1) and (0.2), and give conditions on the singularity spectra of the terms appearing on the right-hand side of these equations. In §2, we define our problem, which is to determine the effects of a radiative correction to a charged-particle triangular closed loop. In §3 it is noticed that the relevant scattering function is not infrared divergent, and has a singularity on the Landau surface $\mathcal{Y} = 0$ associated with the triangle diagram. In §4, §5, §6 and §7 we derive the discontinuity around this surface. The method is the same as was employed in refs 3 and 11: the scattering function f^+ is transformed into $\Delta + f^-$, where Δ is zero in $\mathcal{Y} < 0$, and f^+ and f^- have plus $i0$ and minus $i0$ continuations around $\mathcal{Y} = 0$. Then Δ is the discontinuity. The character of the singularity at $\mathcal{Y} = 0$ is then discussed in §8.

§1. The Photon Propagator.

Our first task is to make the Feynman photon propagator $(k^2+i0)^{-1}$ well-defined within our framework. Here k is the energy-momentum four-vector of the photon, $(k_0,k_1,k_2,k_3) = (k_0,\vec{k})$ and k^2 means $k_0^2 - \vec{k}^2$. At $k=0$ the meaning of the symbol $+i0$ is not clear, because $\mathrm{grad}_k\, k^2$ vanishes there. This is in contrast to the case of massive particle, for which the propagator is $(p^2-m^2+i0)^{-1}$ $(m>0)$. Then $\mathrm{grad}_p(p^2-m^2) \neq 0$ if $p^2-m^2=0$, and the symbol $+i0$ acquires unambiguous meaning. (Cf. Ref. 10, p.89.)

One way, natural both from mathematical and physical viewpoints, is to start from the retarded propagator $R(k)$ and the advanced propagator $A(k)$; they are, by definition, given respectively by

$$(1.1) \qquad R(k) = ((k_0+i0)^2 - \vec{k}^2)^{-1}$$

and

$$(1.2) \qquad A(k) = ((k_0-i0)^2 - \vec{k}^2)^{-1}.$$

It is known that each of these two functions, $R(k)$ and $A(k)$, can be realized as the boundary value of a holomorphic function. (Ref. 10, p.90.)

More specifically, their singularity spectra are given as follows:

(1.3) $\quad$ S.S. $R(k) \subset \{(k;\sqrt{-1}w) \in \sqrt{-1}S^*\mathbb{R}^4;$

$k^2=0,\ k\neq 0,\ w = c\,\mathrm{sgn}(k_0)\mathrm{grad}_k k^2 = c\,\mathrm{sgn}(k_0)k \quad (c>0)\}$

$\cup \{(k;\sqrt{-1}w);\ k=0,\ w^2 \underset{\mathrm{def}}{=} w_0^2 - \sum_{j=0}^{3} w_j^2 \geq 0 \ \text{ and } \ w_0>0\},$

(1.4) $\quad$ S.S. $A(k) \subset \{(k;\sqrt{-1}w) \in \sqrt{-1}S^*\mathbb{R}^4;$

$k^2=0,\ k\neq 0,\ w = -c\,\mathrm{sgn}(k_0)k \quad (c>0)\} \cup \{(k;\sqrt{-1}w);\ k=0,$

$w^2 \geq 0,\ w_0<0\}.$

Here, and in what follows, we identity $\mathrm{grad}_k k^2$ with k, using the Minkowsky metric. (Ref. 10, p.90.)

Furthermore, thanks to the four-dimensionality, we can verify that the d'Alembertian $\Box = \partial^2/\partial k_0^2 - \sum_{j=1}^{3} \partial^2/\partial k_j^2$ annihilates both $R(k)$ and $A(k)$. (Ref. 10, p.150.) Hence, Sato's lemma on microlocal ellipticity (Ref. 10, p.140) entails

(1.5) $\quad$ S.S. $R(k) \subset \{(k;\sqrt{-1}w);\ w^2=0\}$

and

(1.6) $\quad$ S.S. $A(k) \subset \{(k;\sqrt{-1}w);\ w^2=0\}.$

That is, the cotangential component of the singularity spectra of $R(k)$ and $A(k)$ are confined to the light cone, provided we restrict our considerations to the four-dimensional world. These extra relations (1.5) and (1.6) will be used effectively

in later sections.

Now, by the result (A) stated in §0, we find that $\theta(k_0)R(k)$, $\theta(k_0)A(k)$ and $\theta(k_0)(R(k)-A(k))/(-2\pi i)$ are all well-defined. Here, and in what follows, $\theta(k_0)$ denotes the Heaviside function. Since $\theta(k_0)(R(k)-A(k))/(-2\pi i)$ coincides with $\delta(k^2)$ for $k_0>0$, and vanishes identically for $k_0<0$, we denote it by $\delta^+(k^2)$. Then, the massless propagator $(k^2+i0)^{-1}$ is, by definition,

$$(1.7) \qquad (-2\pi i)\delta^+(k^2) + A(k).$$

Note that it can be expressed also in the following form:

$$(1.8) \qquad (+2\pi i)\delta^-(k^2) + R(k).$$

Here $\delta^-(k^2)$ is, by definition,

$$(1.9) \qquad \frac{1}{2\pi i}\,\theta(-k_0)(A(k) - R(k)).$$

Summing up, we have

$$(1.10) \qquad \frac{1}{k^2+i0} = (-2\pi i)\delta^+(k^2) + A(k)$$

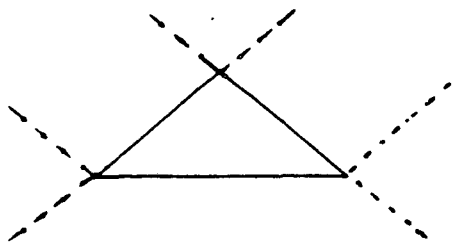
$$= (+2\pi i)\delta^-(k^2) + R(k).$$

This relation will be used frequently in the ensuing discussion.

§2. Definition of the Problem.

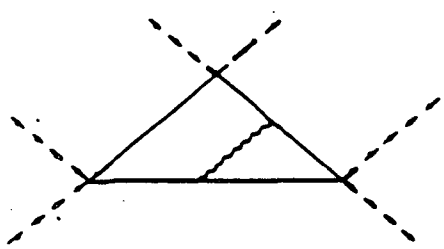
Throughout this article, we denote by D the following triangle graph:

D:



Each solid line is associated with a charged particle of mass $m>0$. Each dashed line is a neutral particle. Let D' denote the following graph, which corresponds to an electromagnetic correction:

D':



Here the wiggly line is associated with the photon propagator $g^{\mu\nu}/(k^2+i0)$, where $g^{\mu\nu}$ is the Minkowsky metric tensor. We call each end point of the wiggly line a photon vertex.

The modified Feynman function, which is the quantum coupled function associated with the graph D', is given by the following formula (2.1). There, and in the sequel, we denote by e a constant (electric charge). We also use the symbol $\not{p}$ to denote $p^\mu\gamma_\mu = g^{\mu\nu}\gamma_\mu p_\nu$, where γ_μ $(\mu = 0,1,2,3)$ denotes the Dirac gamma matrix.

$$(2.1)\qquad F_{QQ}(q) = \int \frac{d^4k}{(2\pi)^4}\frac{g^{\mu\nu}}{k^2+i0}\int\frac{d^4p_1}{(2\pi)^4}$$

$$\mathrm{Tr}\left[F_{Q\mu}(p_1,k)V_1F'_{Q\nu}(-p_1+q_1,k)V_2\,\frac{(-\not p_1+\not q_1+\not q_2+m)}{(-p_1+q_1+q_2)^2-m^2+i0}V_3\right]$$

$$\times\,(2\pi)^4\delta^4(q_1+q_2+q_3),$$

where V_j $(j = 1,2,3)$ is some Dirac matrix, $F_{Q\mu}(p_1,k)$ is, by definition, given by

$$(2.2)\qquad G_\mu(p_1,k) - \int_0^1 d\lambda G_\mu(p_1 + \lambda k,\ 0)$$

with

$$G_\mu(p_1,k) = \frac{(\not p_1 + m)(-ie\gamma_\mu)(\not p_1 + \not k + m)}{(p_1^2 - m^2 + i0)((p_1+k)^2 - m^2 + i0)}\ ,$$

and $F'_{Q\nu}$ is the similar function for the other photon vertex.

As our concern in this article is the infra-red problem, i.e., the problems arising from points near $k=0$, we neglect the ultra-violet problem, i.e., the contribution to the integral from large k.

By performing the λ-integration in (2.2) explicitly, we find that $F_{Q\mu}(p_1,k)$ has the form

$$(2.3)\qquad \Phi_\mu(p_1,k) + \rho_\mu(p_1,k),$$

where $\Phi_\mu(p_1,k)$ has a pole on both $\{p_1^2=m^2\}$ and $\{(p_1+k)^2=m^2\}$, and it does not vanish when $k^2=0$, whereas $\rho_\mu(p_1,k)$ is the sum of terms that do not have these properties. The explicit form of $\Phi_\mu(p,k)$ is

$$(2.4) \qquad \frac{(\not{p}_1 + m)\gamma_\mu \not{k}}{(p_1^2 - m^2 + i0)((p_1 + k)^2 - m^2 + i0)} ,$$

and the actual computation shows that $\rho_\mu(p_1,k)$ contributes singularities to $F_{QQ}(q)$ along the leading Landau-Nakanishi surface $L_0^+(D)$ that are weaker than those coming from $\Phi_\mu(p_1,k)$. (See ref. 12.)

Hence the most singular part of $F_{QQ}(q)$ is given by the combination of $\Phi_\mu(p_1,k)$ and the similar function $\Phi_\nu(p_1+q_1,-k)$ determined by $F'_{Q\nu}(p_1+q_1,-k)$. This fact confirms a formula given in ref. 6 obtained by studying the asymptotic behavior of the corresponding function (i.e., the inverse Fourier transform) in position space.

By virtue of these results we may focus our attention on the integral $\Phi(q)$ defined by replacing in (2.1) $F_{Q\mu}$ by Φ_μ and $F_{Q\nu}$ by Φ_ν.

§3. Properties of $\Phi(q)$.

It is an easy consequence of the results (A) and (B) in §0 that $\Phi(q)$ is a well-defined hyperfunction, provided we ignore any possible problem associated with the contributions at large k. Moreover, near $L_0^+(D)$ its singularities are confined to $L_0^+(D)$. It is also a $+i0$ boundary value along $L_0^+(D)$. (See ref. 12, Appendix B.) Hence we can consider its discontinuity $\Delta(q)$ around $L_0^+(D)$. This latter function coincides with $\Phi(q)$ in the $+i0$-direction (i.e., its difference with $\Phi(q)$ has no singularity associated with $+i0$-direction), and it vanishes below the threshold $L_0^+(D)$).

Our specific task here is to show that the discontinuity is represented as a sum of functions that are determined diagramatically, are easily calculable, and, in particular, are of the kind proposed by Cutkosky for the massive-particle case. From the formula we may obtain the singularity structure of $\Phi(q)$ near $L_0^+(D)$. In deriving the discontinuity formula, the factor $\not{k}$ in the numerator Φ_μ and Φ_ν play a decisively important role; this factor, which the ordinary Feynman function lacks, makes the integral convergent near $k=0$.

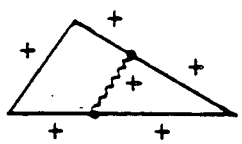
§4. The Discontinuity around $L_0^+(D)$.

We now turn to the derivation of the discontinuity formula for $\Phi(q)$ around the surface $L_0^+(D)$. Our method is based on a graphical analysis combined with microlocal analysis.

In what follows we identify a graph with an associated function. The symbols $\overset{+}{\text{———}}$, $\overset{-}{\text{———}}$ and $\text{—}\!\!\!

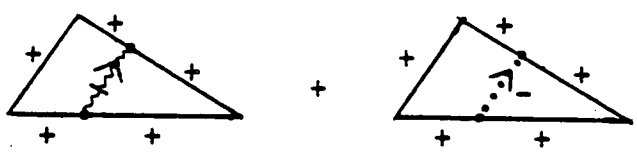
For any two graphs D_1 and D_2 the symbol $D_1 \cong D_2$ means that the function associated with D_1 minus the function associated with D_2 is a $(-i0)$-boundary value at a generic point of $L_0^+(D)$. For the sake of the simplicity of notations, we will omit the external lines of each graph in the sequel.

The first step of our graphical analysis is to decompose



into

(4.2)



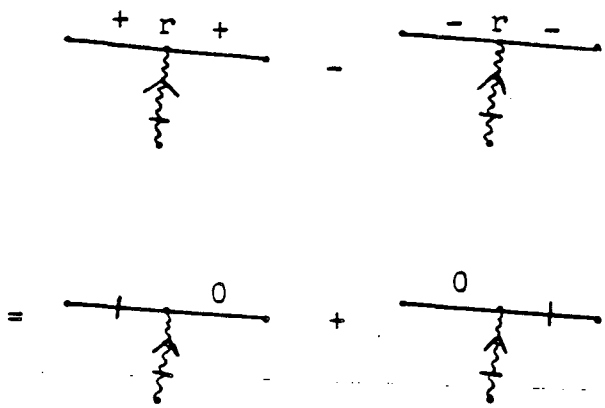
We call the first (resp., second) graph D_0 (resp., D_1). To manipulate D_0 and D_1, we will in the next two sections first prepare some auxiliary results. Then the required decompositions of D_0 and D_1 will be given in §7.

§5. Polar Coordinates.

As noted earlier, our reasoning relies heavily on the extra factor $\not{k}$ at each photon vertex. To make full use of this factor, we sometimes use the polar coordinate system (r, Ω) in k-space; $k = r\Omega$ $(r \geq 0)$ with $\Omega\tilde{\Omega} = 1$, where

$\tilde{\Omega} = (\Omega_0, -\Omega_1, -\Omega_2, -\Omega_3)$. Then we put a symbol r on a photon vertex if the vertex is associated with a factor $\not{k}$. Using this symbol we obtain the following relation:

(5.1)



The line segments with a symbol zero represents the factor $r((p')^2-m^2)^{-1}$, where p' is the energy-momentum vector associated with the line, subjected to the constraints $\Omega^2 = 0$ and $(p'') = m^2$ that arise from the slashes on the other two lines. Each of these factors associated with the zero line is non-singular in (r, Ω)-variables, since $p^2 = m^2 > 0$ and $\Omega^2 = 0$ entail $p\Omega \neq 0$.

The result (5.1) is an immediate consequence of

(5.2)
$$\frac{r}{p^2-m^2\pm i0} \cdot \frac{1}{p^2-m^2+2rp\Omega\pm i0}$$

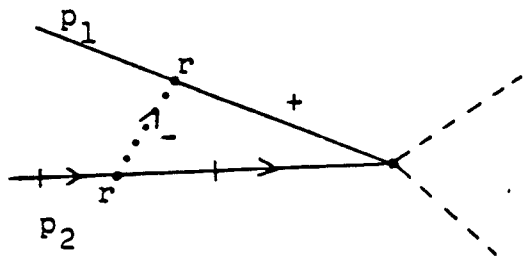
$$= \frac{1}{p^2-m^2\pm i0} \cdot \frac{1}{2p\Omega} - \frac{1}{2p\Omega} \cdot \frac{1}{p^2-m^2+2rp\Omega\pm i0}$$

(the signs $\pm i0$ should be uniformly used).

§6. Some Auxiliary Results.

Another needed auxiliary result pertains to the analyticity of the function associated with the following graph:

(6.1)



The associated function $I(p_1, p_2)$ takes the form (6.2) below in the (r, Ω)-variable. There $\delta << m$ is a strictly positive constant that is introduced in order to restrict our considerations to contributions coming from a neighborhood of $k = 0$, and $P^{\mu\nu}$ is a polynomial in p and $r\Omega$:

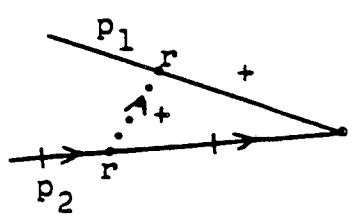
$$(6.2) \qquad \int_0^\delta r^2 dr \int d^4\Omega\,\delta(\Omega\tilde{\Omega}-1)P^{\mu\nu}\,\frac{\Omega_\mu\Omega_\nu\delta(-2p_2\Omega+r\Omega^2)}{((\Omega_0-i0)^2-\vec{\Omega}^2)((p_1+r\Omega)^2-m^2+i0)}\,.$$

Using the results (A) and (B) in §0, we can easily verify that this integral defines a well-defined hyperfunction. However, we cannot see immediately that there is no net contribution to the singularity from the endpoint $r = 0$. To obtain that result we use the following trick: Let us consider the coordinate transformation $(r, \Omega) \longmapsto (-r, -\Omega)$. Then, $I(p_1, p_2)$ can be expressed in the form

$$(6.3) \qquad \frac{1}{2}\int_{-\delta}^{\delta} r^2 dr \int d^4\Omega\,\delta(\Omega\tilde{\Omega}-1)P^{\mu\nu}\,\frac{\Omega_\mu\Omega_\nu\delta(-2p_2\Omega+r\Omega^2)}{((\Omega_0-i0)^2-\vec{\Omega}^2)((p_1+r\Omega)^2-m^2+i0)}\,.$$

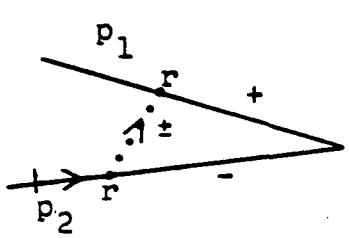
This form has no endpoint contribution from $r = 0$. The integral given in (6.3) is analytic except at points where contributions from the endpoints $r = \pm\delta$ are relevant. To convert $I(p_1, p_2)$ to the form of (6.3), it suffices to note that the $(-i0)$ in the first denominator should be understood as $-i0/\mathrm{sgn}\, r$ if $r \neq 0$. This fact, combined with the trivial fact that $r\Omega$ is invariant under the coordinate transformation in question, guarantees that the integrand of the integral I is invariant under the coordinate transformation $(r, \Omega) \longmapsto (-r, -\Omega)$. The required result then follows immediately. The same reasoning gives also this same analyticity property for the function specified by the following graph:

(6.4)



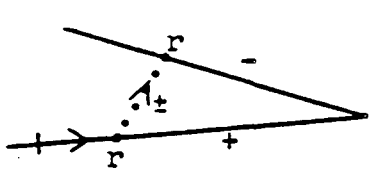
The relations (1.3) ∿ (1.6) guarantee the analyticity of the functions specified by either

(6.5)



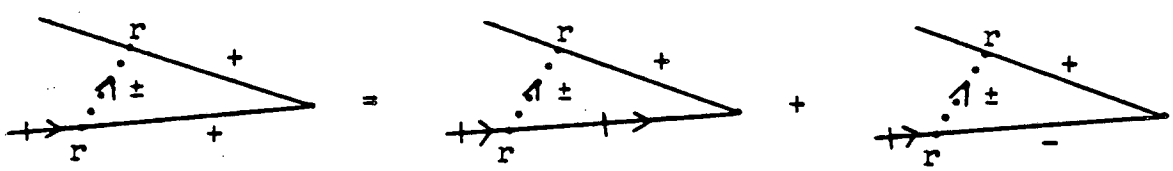
or

(6.6)



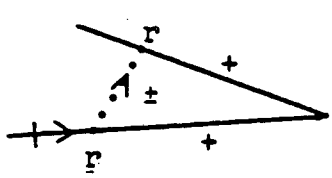
In fact, using the results (A) and (B) in §0, we deduce from (1.3) ∿ (1.6) that the analyticity (at a generic point of $L_0^+(D)$) of these functions fails only when $\alpha_1 p_1 + \alpha_2 p_2 = w$ holds for a non-zero light cone vector w with $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 \neq 0$ and $\alpha_1(p_1^2 - m^2) = \alpha_2(p_2^2 - m^2) = 0$. Therefore $\alpha_1\alpha_2 \neq 0$, and hence $p_1^2 = p_2^2 = m^2$. But this cannot happen as $p_{1,0}$ and $p_{2,0}$ are of the same sign at the relevant points.

It follows from the definitions that



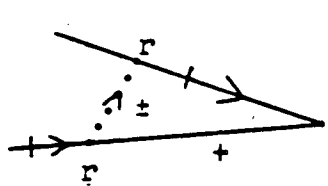
Thus we have the same analyticity property also for the graphs

(6.7)

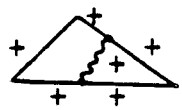


Combining the graph (6.6) and (6.7) we also find this same analyticity for the graphs

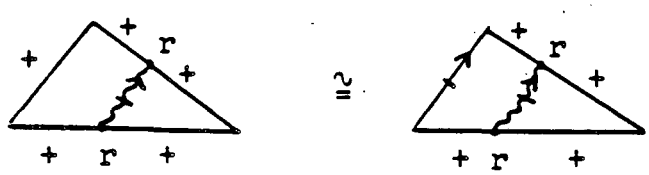
(6.8)



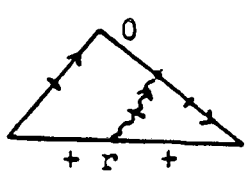
§7. Derivation of a Discontinuity Formula.

Let us now transform  into a sum of functions supported above the threshold $L_0^+(D)$, modulo some $(-i0)$-boundary value along generic points of $L_0^+(D)$. The transformation given below relies on the results (A) and (B) in §0 supplemented by the results obtained in the previous two sections.

Let us first consider D_0. Then, using the result in §5, we find the following;

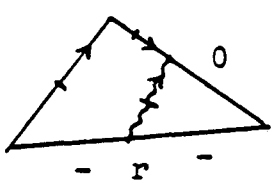
(7.1) $D_0 =$ 

$=$

$+$ 

$\cong$

$=$



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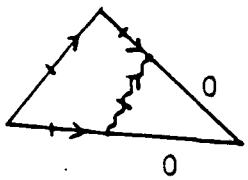

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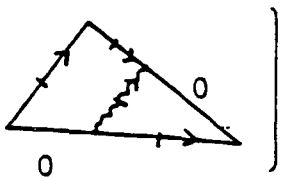

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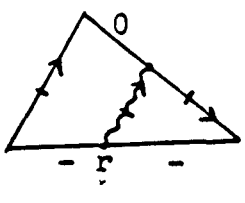

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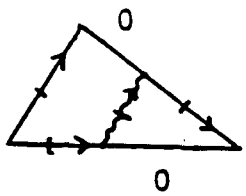

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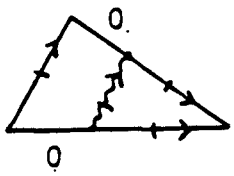
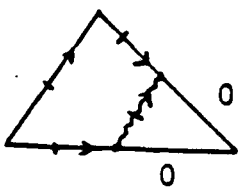


$\cong$



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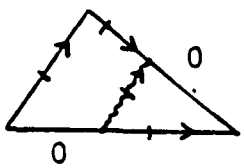

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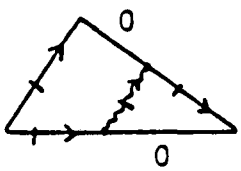

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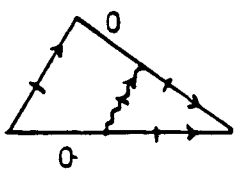
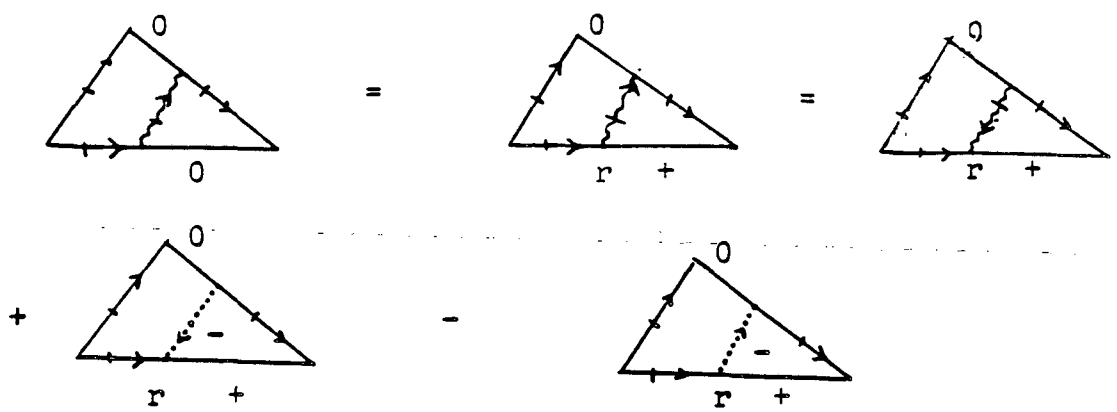


.

Let D_j $(j=2,3,4,5)$ denote the j-th graph in the last expression. Observing the direction of the energy flows, we immediately see that D_2, D_3 and D_5 vanish below the threshold $L_0^+(D)$. Hence they are a part of the discontinuity function as they stand. To manipulate D_4 further we again apply (4.1):

(7.2)



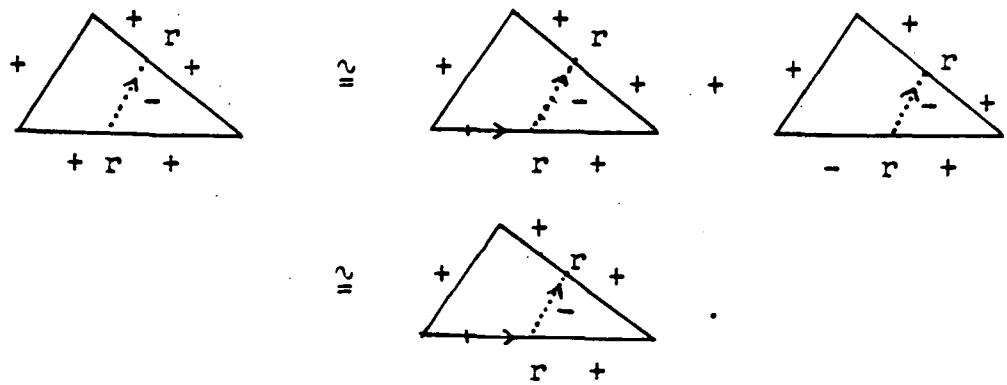
Then, in view of the analyticity established in §6 (which lets the small triangle involving the dotted minus line to be contracted to a point), we obtain

(7.3)

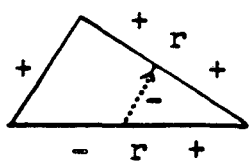
The right-hand side of (7.3), which we call D_d, vanishes below the threshold $L_0^+(D)$, and hence it is a part of the discontinuity function.

Combining (7.1) and (7.3), we find that the discontinuity function for D_0 is a sum of the terms D_2, D_3, D_5 and D_d.

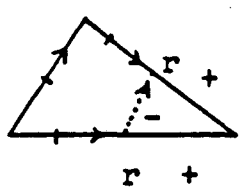
Next let us consider D_1. For D_1, we find the following:

(7.4) 

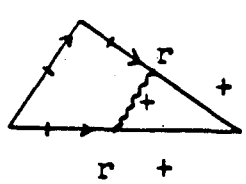
Here we have used (1.4) and (1.6) to obtain

(7.5)  $\cong 0$.

In view of the analyticity for the graph in (6.7), we find the discontinuity function for the last term in (7.4) is given by

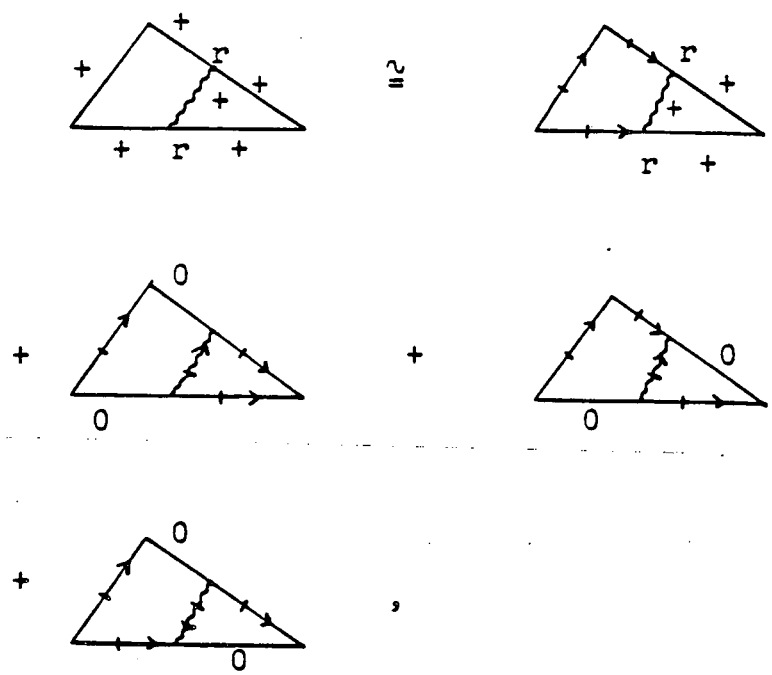
(7.6)  .

Combining this contribution with the one corresponding to D_2 we obtain, using (4.1),

(7.7) $D_a \underset{\text{def}}{=}$  .

Summing up the results obtained so far, we find the following formula:

(7.8)

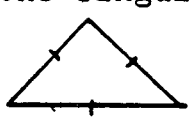


where each term on the right-hand side is supported in the region lying above $L_0^+(D)$.

In what follows, we denote by D_b, D_c and D_d respectively the last three terms in the right-hand side of (7.8).

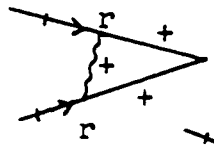
§8. Explicit Form of D_a, D_b, D_c and D_d.

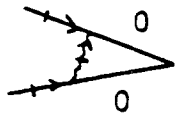
Let us now show that

(I) The character of the singularity of D_a along $L_0^+(D)$ is the same as that of 

and that

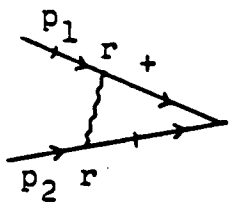
(II) The singularities of D_b, D_c and D_d along generic points of $L_0^+(D)$ are strictly weaker than that of D_b.

The assertion (I) is an immediate consequence of the analyticity of 

, which follows from (6.7) and the analyticity of 

.

The analysis of D_c and that of D_d are similar. Let us consider D_c. The function associated with



is, by definition, of the following form:

(8.1) $I_c(p_1,p_2)$

$$= \int d^4\Omega A(p_1,p_2,\Omega)\delta^+(\Omega^2)\delta(\Omega\tilde{\Omega}-1)$$

$$\int_0^\delta \delta^+(p_2^2-m^2-2rp_2\Omega)r^2dr,$$

where $A(p_1,p_2,\Omega)$ is non-singular. As the end-point contribution from $r=\delta$ does not give any singularities at the generic point of $L_0^+(D)$, we neglect it. Then we find

$$(8.2) \qquad I_c = \int d^4\Omega A(p_1,p_2,\Omega)\frac{(p_2^2-m^2)^2\theta(p_2^2-m^2)}{(2p_2\Omega)^3}\,\delta^+(\Omega^2)\,\delta(\Omega\tilde{\Omega}-1).$$

Since $p_2\Omega \neq 0$ at relevant points, the integral I_c has the form

$$(8.3) \qquad B(p_1,p_2)(p_2^2-m^2)^2\theta(p_2^2-m^2),$$

where B is non-singular.

Thus the function corresponding to D_c has the form

$$\int C(p,q_1,q_2)\,\delta^+(p^2-m^2)\,\delta^+((p+q_1)^2-m^2)\ \times$$

$$\times\ ((q_2-p)^2-m^2)\,\theta((q_2-p)^2-m^2)\,d^4p,$$

where C is non-singular. Hence it follows from a result in Ref. 10, p.422 that the above integral has the form

$$(8.4) \qquad A_c(q_1,q_2)\varphi(q_1,q_2)^2\theta(\varphi(q_1,q_2)),$$

where A_c is analytic and φ is a defining function of $L_0^+(D)$ (with its sign suitably chosen). This is much weaker than D_a, which has the form $A_a(q_1,q_2)\theta(\varphi(q_1,q_2))$.

Finally, let us consider D_b. Taking into account the

extra r^2 in the numerator, we find that it assumes the following form:

$$(8.5)\qquad \int d^4\Omega\,\delta^+(\Omega^2)\,\delta(\Omega\tilde{\Omega}-1)\int_0^\delta r\,dr\int d^4p\,A(p,q_1,q_2,\Omega,r)\ \times$$

$$\times\ \delta^+(p^2-m^2)\,\delta^+((q_1+p+r\Omega)^2-m^2)\,\delta^+((q_2-p-r\Omega)^2-m^2),$$

where A is non-singular. By performing the p-integration, we find the p-integral has the form

$$(8.6)\qquad B(q_1,q_2,\Omega,r)\,\theta(\varphi(q_1+r\Omega,q_2-r\Omega)),$$

where B is again non-singular. Let $\psi(q_1,q_2,\Omega,r)$ denote $\varphi(q_1+r\Omega,q_2-r\Omega)$. Then we find

$$(8.7)\qquad \frac{\partial\psi}{\partial r}\Big|_{r=0} = \Omega\cdot\Big(\frac{\partial\varphi}{\partial q_1}\Big|_{r=0} - \frac{\partial\varphi}{\partial q_2}\Big|_{r=0}\Big).$$

On the other hand, the Landau-Nakanishi equations tell us

$$(8.8)\qquad \frac{\partial\varphi}{\partial q_1}\Big|_{r=0} - \frac{\partial\varphi}{\partial q_2}\Big|_{r=0} = -\alpha p$$

with $\alpha > 0$, $p^2=m^2$, if (q_1,q_2) lies in $L_0^+(D)$. Hence if $\Omega^2=0$ and $\Omega_0 > 0$, the right-hand side of (8.7) is strictly negative. Since $\psi\big|_{r=0} = \varphi(q_1,q_2)$ holds, the r-integration should be done from φ/c to δ with some non-zero analytic function c. Hence, again neglecting the contribution from

the end-point $r=\delta$, we find that D_b has the form

$$(8.9)\quad \int d^4\Omega\delta^+(\Omega^2)\delta(\Omega\tilde{\Omega}-1)C(q_1,q_2,\Omega)\varphi(q_1,q_2)^2\theta(\varphi(q_1,q_2)),$$

where C is non-singular. Hence the Ω-integration can be trivially done to give the form

$$(8.10)\quad A_b(q_1,q_2)\varphi(q_1,q_2)^2\theta(\varphi(q_1,q_2)).$$

This is again much weaker than D_a. Thus we have verified both (I) and (II).

References

1. L. D. Landau, in Proc. of Kiev Conference on High-Energy Physics (1959); Nuclear Physics, 13, 181 (1959). N. Nakanishi, Prog. Theor. Phys. 21, 135 (1959).
2. R. J. Eden, P. V. Landshoff, D. I. Olive and J. C. Polkinghorne, The Analytic S-matrix, Cambridge U.P., p.57. See also T. Kawai and H. P. Stapp, Publ. RIMS, Kyoto Univ., 12 Suppl., 155 (1977).
3. H. P. Stapp, J. Math. Phys., 9, 1548 (1968). J. Coster and H. P. Stapp, J. Math. Phys., 10, 371 (1969); 11, 1441 (1970); 11, 2743 (1970); 16, 1288 (1975). H. P. Stapp, in Structural Analysis of Collision Amplitudes, ed. R. Balian and D. Iagolnitzer, North Holland (1976). D. Iagolnitzer, The S-matrix, North-Holland (1978).

4. C. Chandler and H. P. Stapp, J. Math. Phys., 10, 826 (1969). D. Iagolnitzer and H. P. Stapp, Commun. Math. Phys. 14, 15 (1969). H. P. Stapp, in Structural Analysis of Collision Amplitudes (ibid.). D. Iagolnitzer, The S-matrix (ibid.).

5. F. Bloch and A. Nordsieck, Phys. Rev., 52, 54 (1937). D. Yennie, S. Frautchi and H. Suura, Ann. Phys. (N.Y.) 13, 379 (1961).

6. H. P. Stapp, Phys. Rev., 28, 1386 (1983).

7. E. D'Emilio and M. Mintchev, Phys. Rev., 27, 1840 (1983). M. V. Bergere and L. Szymanowski, Phys. Rev., 26, 3550 (1982). H. D. Dahmen, T. Manual and D. Manakos, Phys. Rev. (1987), to be published.

8. R. E. Cutkosky, J. Math. Phys., 1, 429 (1960).

9. M. Sato, T. Kawai and M. Kashiwara, Lect. Notes in Math., No.287 (Springer), 265 (1973).

10. M. Kashiwara, T. Kawai and T. Kimura, Foundations of Algebraic Analysis, Princeton U. P. (1986).

11. T. Kawai and H. P. Stapp, Publ. RIMS, Kyoto Univ., 12 Suppl., 155 (1977).

12. T. Kawai and H. P. Stapp, Infrared finiteness and analytic structure, in preparation.

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