Submitted to Nuclear Physics B

LBL-2427
Preprint c.2

RECEIVED
LAWRENCE
RADIATION LABORATORY

MAY 1974

LIBRARY AND
DOCUMENTS SECTION

OPTICAL THEOREMS FOR THREE-TO-THREE PROCESSES

Joseph Coster and Henry P. Stapp

January 21, 1974

LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY

Prepared for the U. S. Atomic Energy Commission
under Contract W-7405-ENG-48

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

OPTICAL THEOREMS FOR THREE-TO-THREE PROCESSES

Joseph Coster*

Physics Department
Western Illinois University
Macomb, Illinois 61455

and

Henry P. Stapp†

Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

January 21, 1974

ABSTRACT

The three-to-three inclusive optical theorem expresses the inclusive cross section for the process $a+b \to c+\text{anything}$ as a multiple of a certain discontinuity of the scattering function for the process $a+b+c \to a+b+c$. This theorem is derived here from S-matrix principles as a special case of a general formula that expresses in terms of physical scattering functions all of the single, double, and higher-order multiple discontinuities formed from the members of a large set of different boundary values of an arbitrary three-to-three scattering function. These different boundary values correspond to taking the limit $\text{Im}\, k \to 0$ from different combinations of sides of the normal-threshold cuts in the various channel invariants. The formula is exact and global: all singularities are taken into account and the result holds at all energies, not just near thresholds.

The formula expresses in terms of physical scattering functions certain discontinuities that have in appropriate limits, according to the tenets of Regge theory, known expressions in terms of Regge parameters. It thus opens the way to an enlargement of the scope of Regge-Mueller analyses.

I. INTRODUCTION

The inclusive optical theorem expresses the inclusive cross section for a process $a+b \to c+\cdots+f+\text{anything}$ as a multiple of a certain discontinuity of the scattering function for the process $a+b+c+\cdots+f \to a+b+c+\cdots+f$. The recent theoretical interest in this theorem stems largely from the work of Mueller and Tan. Mueller [1] showed that important properties of inclusive cross sections could be derived from the assumption that certain matrix elements of products of currents enjoy Regge behavior. Tan [2] then observed that Mueller's special assumption about matrix elements of currents can be replaced, with the aid of the inclusive optical theorem, by the general S-matrix assumption that discontinuities of scattering functions across leading normal threshold cuts enjoy Regge behavior.

The inclusive optical theorem itself was first proved in ref. [3], on the basis of field theory. One aim of the present work is to give an S-matrix proof for the special case in which $c+\cdots+f = c$. A heuristic S-matrix derivation of this result has been given by Tan [2], who used crossing arguments. However, many cuts stand in the way of on-mass-shell continuations to the crossed channels, and the effects of these cuts were not fully analyzed by Tan. The present proof does not depend on crossing arguments, and thus avoids the difficulties inherent in that approach.

A second and more important aim of this paper is to derive a certain general discontinuity formula. This formula contains, as a particular instance, the inclusive optical theorem for the case $c + \cdots + f = c$. Its importance lies in the fact that each of the discontinuities given by the formula is expected to enjoy Regge behavior. Moreover, each of these discontinuities is expressed in terms of physical scattering amplitudes. Hence the formula, together with the Regge hypothesis, imposes numerous direct constraints on physical scattering functions, opening the way to an enlargement of the scope of Regge-Mueller analyses.

This general discontinuity formula will be described presently. First the inclusive optical theorem and certain related formulas are reviewed.

The inclusive optical theorem is essentially a formula for the discontinuity of the scattering function across a certain basic cut, evaluated on specified sides of each of the other basic cuts. These basic cuts are cuts in the channel energies that start at the lowest normal thresholds and extend to plus infinity. This formulation is based on field theory, and the use of channel energies as basic variables, rather than channel invariants, stems from the fact that the scattering function has simple cut-plane analyticity in the complex energy variables if the three-momenta are all held fixed and real [3]. In particular, the singularities are confined to the union of the surfaces $\mathrm{Im}\, E_g = 0$, where E_g is the channel energy associated with channel g.

For the three-to-three case the complex energy space is five dimensional, since one energy variable is fixed by energy conservation.

There are sixteen channels g associated with basic cuts. These are the one direct or total-energy channel, the three initial subenergy channels, the three final subenergy channels, and the nine cross-energy channels that are defined by sets consisting of one initial particle and two final particles, or by the complementary sets consisting of two initial particles and one final particle.

The sixteen planes $\mathrm{Im}\, E_g = 0$ divide the five dimensional space of imaginary energies into 2282 regions called zones. Each zone is a cone-shaped region with apex at the origin. The scattering function is analytic in each zone, and has, for each zone, a corresponding boundary value, which is defined by letting the complex point $p' = (p'_1, \cdots, p'_6)$ approach the real boundary point $p = (p_1, \cdots, p_6)$ from within that zone. The inclusive optical theorem, for the case $c + \cdots + f = c$, is essentially a formula for the difference between two of these 2282 zone boundary values.

In the S-matrix framework the mass-shell constraints are retained. Hence the energies are real if the three-momenta are real. Consequently, the zones defined above do not intersect the (mass-shell) domain of definition of the scattering function. It is therefore necessary to define the boundary values by a different procedure. This procedure uses the sixteen channel invariants s_g, instead of the sixteen energy energies E_g.

The sixteen channel invariants s_g are not independent variables: they are functions of the eight independent scalar invariants. Suppose for a moment, however, that these sixteen variables could be treated as independent variables, and that the scattering function had only normal-threshold singularities, and

hence enjoyed cut-plane analyticity in $(s_1,\cdots,s_{16})$ space, with the singularities confined to the union of the planes $\mathrm{Im}\, s_g = 0$. In this case there would be 2^{16} boundary values, one corresponding to each combination of sides of the 16 cuts $\mathrm{Im}\, s_g = 0$. Stated differently, for every one of the 2^{16} subsets G of the set E of sixteen indices g there would be a boundary value $M^G(s_1,\cdots,s_{16})$ obtained by approaching the real boundary point $(s_1,\cdots,s_{16})$ from the lower-half s_g plane for every g in G, and from the upper half s_g plane for every g in the complement $\bar{G} \equiv E - G$ of G.

Actually the sixteen variables s_g are not independent, and the singularities of the scattering function are not confined to the surfaces $\mathrm{Im}\, s_g = 0$. Nevertheless, there is a set of 2^{16} functions $M^G(p)$ that is analogous to the set of 2^{16} functions $M^G(s_1,\cdots,s_{16})$ defined above. A large number of these functions $M^G(p)$ satisfy the primary properties to be described next, and the rest satisfy a weakened version of these properties.

The primary properties of the functions $M^G(p)$ are as follows:

(1) $M^G(p)$ is analytic at all real p, except on certain Landau singularity surfaces.

(2) $M^G(p)$ is a single analytic function: it continues into itself around each Landau singularity surface by some infinitesimal detour.

(3) $M^G(p)$ continues into itself around each normal-threshold singularity surface $s_g = (\sum m_j)^2$ by passing into the lower- or upper-half s_g plane according to whether g is contained in G or $\bar{G}$. Here $\sum m_j$ is some sum of physical particle masses.

(4) $M^G(p) - M^{Gg}(p)$ vanishes if $s_g(p)$ is less than its value on the leading g-channel normal threshold. Here $Gg \equiv G \cup g$.

(5) $M(p) = M^{\emptyset}(p)$ is the physical scattering function, which is the connected part of the S-matrix, divided by $(2\pi)^4\, \delta^4(\sum p_i - \sum p_f)$.

(6) $M^E(p) = -M^\dagger(p)$. This property is hermitian analyticity: the scattering function evaluated below all the cuts is minus the hermitian conjugate scattering function.

It might seem that these properties would be easy to satisfy: one might try to define the functions M^G by simply continuing the scattering function according to the prescribed rule around every normal-threshold singularity surface, and continuing it according to any arbitrary rule around every other singularity surface. This procedure does not work. For the way in which the function is continued around a normal-threshold singularity surface determines the way in which it must be continued around any non-normal-threshold singularity surfaces that emerge from it. And this determination fixes, in turn, the way in which the function must be continued around any normal-threshold singularity surfaces into which this non-normal-threshold singularity surface merges. Consequently, it is not evident that a set of 2^{16} functions $M^G(p)$ satisfying the principal defining properties exists. And given that such a set exists it is not clear that these functions can be expressed in terms of <u>physical</u> scattering functions alone, as contrasted to unphysical boundary values of the anaytic continuations of scattering functions.

A set of functions $M^G(p)$ that satisfy the primary defining properties is constructed in later sections by first constructing two sets of partial solutions, the T^G and the $\bar{T}^G$.

The 2^{16} functions T^G satisfy properties (1), (4), (5), and (6), with T^G replacing M^G, and half of (3); the function T^G continues into itself around each normal-threshold singularity $s_g = (\sum m_j)^2$ for $g \in G$ by passing into the lower-half s_g plane. Similarly, the 2^{16} functions $\overline{T}^G$ satisfy properties (1), (4), (5), and (6), with $\overline{T}^{E-G}$ replacing M^G, and the other half of (3); the function $\overline{T}^G$ continues into itself around each normal threshold singularity $s_g = (\sum m_j)^2$ for $g \in \overline{G}$ by passing into the upper-half s_g plane. For 26,018 of the $2^{16} = 65,536$ possible sets G the identity $T^G = \overline{T}^{E-G}$ holds. For these values of G the functions M^G are defined by

$$M^G = T^G = \overline{T}^{E-G} .$$

Then the final condition, property (2), is proved. This is nontrivial, because each $M^G(p)$ is constructed as a sum of functions $M_H(p)$ only one of which, $M_\emptyset(p) = M^\emptyset(p) = M(p)$, is a single analytic function.

Each of the component functions M_H except $M_\emptyset = M$ is a unitarity-type sum of products two or more physical scattering functions, or their complex conjugates. In a field theory framework off-mass-shell extensions of the scattering functions are introduced. Thus in that framework the functions M_H, and hence also the functions M^G, have off-mass-shell extensions. Near the off-mass-shell point $p_1 = \cdots = p_6 = 0$ all of the differences $M^G(p) - M^{Gg}(p)$, vanish, and hence all of the functions $M^G(p)$ are equal. Thus in this off-mass-shell framework the 26,018 functions M^G are all different boundary values of the analytically continued scattering function. They are, accordingly, called boundary values of the analytically continued scattering function, even though no <u>mass-shell</u> path of

continuation that links each function M^G to the physical function M is constructed in the present work.

By virtue of property (3) the boundary value $M^G(p)$ is a boundary value from below the normal-threshold cuts corresponding to channels $g \in G$ and from above the normal-threshold cuts corresponding to channels $g \in \overline{G}$. For brevity this boundary value $M^G(p)$ is called the scattering function evaluated below the cuts $g \in G$ and above the cuts $g \in \overline{G}$. Similarly the difference $M^G - M^{Gg} \equiv M_g{}^G$ is called the discontinuity across the cut g evaluated below the cuts $g' \in G$ and above the cuts $g' \in E - Gg$. The higher-order multiple discontinuities are defined similarly.

For the remaining $2^{16} - 26,018$ values of G no functions $M^G(p)$ that strictly satisfy all of the primary properties have been found, and we believe that none exist. However, it is convenient, for reasons to be discussed later, to enlarge the set of 26,018 boundary values M^G into a full set of 2^{16} functions M^G by means of generalized Steinmann relations. This is discussed next.

The Steinmann relations can be formulated in terms of the notion of overlapping channels. Two channels are said to <u>overlap</u> if and only if neither of the two complementary subsets of particles that defines one channel is contained in either of the two complementary subsets that define the other. The Steinmann relations are equivalent to the following <u>Steinmann discontinuity property</u>: if two channels g and h overlap then the discontinuity across the cut g does not depend on whether it is evaluated above or below the cut h. The ordinary Steinmann relations assert that this property holds for all discontinuities formed from the 2282 zone boundary values, with

the cuts g and h identified as two of the channel-energy cuts that separate these zones. The generalized Steinmann relations assert that the Steinmann discontinuity property holds for all discontinuities $M^G - M^{Gg} \equiv M_g{}^G$ formed from the 2^{16} functions M^G.

It is not obvious that there is a set of 2^{16} functions M^G that includes the 26,018 boundary values M^G defined above and that also satisfies the generalized Steinmann relations. However, there is such a set, and it is unique. In this set the remaining functions M^G are given by $M^G = T^G$ or $M^G = \bar{T}^{E-G}$ according to whether $\bar{G}$ or G contains the direct channel label $g = t$. This set of 2^{16} functions M^G satisfies the properties (1), (4), (5), and (6), and, for each G, half of property (3). The remaining half of (3) is disrupted by certain singularities that are associated with closed loop diagrams. Thus property (3) holds for the $2^{16} - 26{,}018$ functions $M^G(p)$ in a tree-diagram approximation.

Property (2) does not hold in general for the $2^{16} - 26{,}018$ functions M^G. Thus these functions cannot, in general, be identified as boundary values of the analytically continued scattering function. However, the singularities that block the continuation are also associated with closed loop diagrams. The situation is therefore this: for 26,018 values of G there are boundary values $M^G(p)$ of the analytically continued scattering function that satisfy all of the primary properties. This set of 26,018 boundary values is uniquely imbedded in a set of 2^{16} functions $M^G(p)$ that satisfy (1), (4), (5), and (6), together with the generalized Steinmann relations. These extra $2^{16} - 26{,}018$ functions satisfy the two remaining properties, (2) and (3), in a tree-diagram approximation.

The 2^{16} functions $M^G(p)$ are connected in a natural way to the 2282 zone boundary values. Each of the 2282 zones lies below some set G of basic (channel energy) cuts, and above the rest. Thus each of the 2282 zone boundary values corresponds to one of the 2^{16} functions M^G, and in fact to one of the 26,018 boundary values M^G. It is shown in ref. [4] that each of the zone boundary values is equal to the corresponding function M^G. Thus the full set of functions M^G is an extension of the set of 2282 zone boundary values to a set of 2^{16} functions that satisfies the generalized Steinmann relations. This extension is unique [4]. Moreover, each of the functions M^G can be expressed as a linear combination of the 2282 zone boundary values [4]. These results imply the uniqueness of the set of 2^{16} functions $M^G(p)$ insofar as one demands both their agreement with the zone boundary values and the validity of the generalized Steinmann relations.

A formula is given in Section II that compactly expresses in terms of physical scattering functions each of the 2^{16} functions M^G, and each of the single, double, and multiple discontinuities formed from the set of 2^{16} functions $M^G(p)$. The relevance of this formula to Regge theory is now discussed.

Regge behavior in its simplest form is simply a fall-off property of the scattering amplitude itself in certain limits. However, under the impetus of Mueller's work the hypothesis of Regge behavior was extended to cover also discontinuities across the basic normal threshold cuts. To get Mueller's results it is sufficient to assume merely that the particular discontinuity that occurs in the inclusive optical theorem enjoys Regge behavior. However, it then

becomes natural to assume that the discontinuities across the other basic cuts also enjoy Regge behavior.

This expanded concept of Regge behavior was explored in detail and it was soon recognized that the Steinmann relations impose important conditions on the structure of the Regge vertices [5]. Ultimately, on the basis of many works, Weis [6] obtained a general formula for discontinuities that consolidates the tenets of Regge theory with the conditions imposed by the Steinmann relations.

A fundamental aspect of the Regge hypothesis for scattering functions is that the stipulated behavior holds for the actual scattering function itself, not merely for some part of the amplitude, or for some approximation to the amplitude. Similarly, the stipulated behavior of the discontinuities should hold for the actual discontinuities themselves, not merely for parts of the discontinuities, or for approximations to the discontinuities. Thus the question arises: What are the discontinuities to which the Weis formula applies. The problem is that this formula refers to discontinuities associated with the various normal threshold cuts, but it is not specified exactly how the functions on the various sides of the normal threshold cuts are to be defined. To the extent that the formulas are to be restricted to the discontinuities formed from the 2282 zone boundary values defined by the channel-energy cuts, the answer is clear. However, the Regge considerations are formulated in an S-matrix framework, and there is no indication there that the formulas should be limited in this way.

Within the S-matrix framework it seems natural to define the discontinuities in question as the discontinuities formed from the 26,018 boundary values $M^G(p)$. These boundary values correspond to

continuations around the normal threshold singularities in the prescribed fashions. Moreover, they are part of the unique extension of the set of 2282 zone boundary values to a set of 2^{16} functions $M^G(p)$ satisfying the generalized Steinmann relations. In the derivation of the Weis formulas there is no restriction to the 2282 zone boundary values, and hence the Steinmann relations used there are actually the generalized Steinmann relations. Thus the boundary values to which these formulas apply must evidently satisfy these generalized relations. This requirement uniquely determines the functions $M^G(p)$.

By the same argument the remaining $2^{16} - 26{,}018$ functions from which the Weis discontinuities are formed must be the remaining functions $M^G(p)$. These remaining functions are not boundary values of the analytically continued scattering function. However, they must, as linear combinations of the 2282 zone boundary values, fall off in the Regge manner, if the zone boundary values do. Hence there seems to be no reason to restrict the Weis formulas to include only those discontinuities formed from the 26,018 boundary values. If the remaining discontinuities are to be defined at all as differences of well-defined functions, then these functions must be the $M^G(p)$.

It seems therefore reasonable to propose, within the general framework of contemporary Regge theory, that the discontinuities specified by the Weis formula are the discontinuities formed from the 2^{16} functions M^G. This hypothesis, together with the general discontinuity formula discussed earlier, adds substantial new conditions to Regge theory, for it allows the detailed Weis expressions in terms of Regge parameters to be identified with corresponding expressions in terms of physical scattering functions.

The first application of this hypothesis, which will be described in a later work, is the derivation of a unitarity type relation for particle-reggeon scattering amplitudes. This relation is identical in form to the unitarity equation for a two-particle scattering amplitude, except that one initial particle is replaced by a reggeon and one final particle is replaced by a reggeon.

The plan of the paper is as follows. In section II the notation is introduced and the general formula that defines the 2^{16} functions M^G, and the discontinuities formed from them, is described. The rule that identifies the 26,018 boundary values M^G is given at the end of that section, in eq. (2.25). In section III a preliminary discussion of the functions T^G and $\overline{T}^G$ is given. This discussion is heuristic, because it expresses each of the functions T^G and $\overline{T}^G$ as a formal infinite sum of bubble-diagram functions. Three properties of these formal expressions are identified in section IV as the defining properties of the functions T^G and $\overline{T}^G$, and a fundamental analytic property of these functions is derived from their defining properties alone. In section V the functions T^G and $\overline{T}^G$ are expressed in a well-defined way as sums that, like unitarity sums, reduce to finite sums of bubble-diagram functions on any finite region in p space. Then it is shown that $T^G = \overline{T}^{E-G}$ for the 26,018 sets G specified by eq. (2.25).

In section VI the functions $M^G = T^G = \overline{T}^{E-G}$ are shown to satisfy the six primary properties. The proof is divided into two parts. First a proof is given in a slightly enlarged theoretical framework in which it is assumed that the physical scattering functions, and hence the functions M^G, can be extended infinitesimally

off the mass shell. Then the analytic continuations that connect the functions M^G on different sides of the Landau surfaces can pass through slightly off-mass-shell regions. In this off-mass-shell framework the rule for continuing around any singularity surface can be expressed as a function of the Landau diagram D that corresponds to the surface, without specifying the particular point p on the surface near which the continuation takes place. In the second part of the proof the mass-shell constraints are rigorously observed. In this mass-shell framework the continuation depends both on the diagram D and on the point p near which the continuation takes place.

Certain general properties of Landau surfaces are derived in appendix A. In particular, it is shown that each Landau surface is confined to an algebraic variety of dimension at least one less than the dimension of the mass shell, and that the union of the Landau singularity surfaces associated with each of the functions M^G also has this property locally.

II. NOTATION AND RESULTS

A <u>channel</u> g is defined by a separation of the complete set of incoming and outgoing particles of a reaction into two complementary disjoint subsets, J_g and $\overline{J}_g$, each of which has at least two particles. For a $3 \to 3$ process there are twenty-five channels. These are the direct or total-energy channel t, the three initial subenergy channels i, the three final subenergy channels f, the nine cross-energy channels (if), and nine other cross-energy channels, which will remain unnamed. These channels are defined in fig. 2.1. For definiteness the sets J_g and $\overline{J}_g$ are defined so that J_g contains at least two final particles.

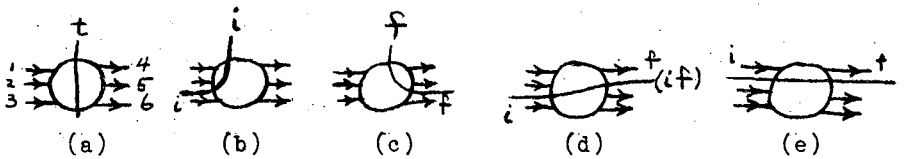


Fig. 2.1. Definition of the channels.

Throughout this paper the index i stands for 1,2, or 3. A line i is a line corresponding to one of the three initial particles; the channel i is the corresponding initial subenergy channel specified in fig. 2.1. The index f always stands for 4,5, or 6, and is used to label final lines, and also the corresponding final subenergy channels, as indicated in fig. 2.1c.

Stability conditions preclude the existence of normal-threshold singularities in the nine channels of type (e) of fig. 2.1, and these channels will henceforth be ignored. Thus the complete set E of channel labels g is the set of sixteen elements
$E \equiv \{t,1,\cdots,6,(14),\cdots,(36)\}$.

An arbitrary line will generally be represented by the letter j. The channel invariant s_g is the square of the sum of the momentum-energy vectors k_j of either one of the two sets J_g or $\bar{J}_g$:

$$s_g \equiv \left(\sum_{j\in J_g} \varepsilon_j p_j\right)^2 = \left(\sum_{j\in \bar{J}_g} \varepsilon_j p_j\right)^2 .$$

The symbol ε_j is a sign that is plus or minus according to whether j is an f or an i. The real physical momentum-energy of particle j is denoted by p_j.

The set E has 2^{16} different subsets G. For each of these G there is a function M^G. In this section a formula is given that expresses each of these 2^{16} functions M^G, and every single, double, and higher-order multiple discontinuity formed from these M^G, in terms of physical scattering functions for other processes. Certain properties of these functions M^G are derived in later sections.

The function $M^G = M^G(p)$ is a function of the set of six real on-mass-shell energy-momentum conserving four-vectors $p \equiv (p_1,\cdots,p_6)$. It is convenient to call these functions M^G by the names that would be appropriate if the sixteen channel invariants s_g were independent variables. Thus M^G is called the function evaluated below all the cuts g in G and above all the cuts g in the complement $\bar{G} \equiv E - G$ of G. Similarly the difference

$$M_g \equiv M - M^g \tag{2.1}$$

is called the discontinuity across the cut g. And for any h in $\bar{G}$ the difference

$$M_h{}^G \equiv M^G - M^{Gh} \tag{2.2}$$

is called the discontinuity across the cut h evaluated below all the cuts g in G and above all the cuts g not in $Gh \equiv G \cup h$. The cut g means the cut lying in $\mathrm{Im}\, s_g = 0$.

There are, in addition to single discontinuities, also double discontinuities, and higher-order multiple discontinuities. The double discontinuity across a pair of different cuts h and k is the discontinuity across the cut h of the discontinuity across the cut k:

$$M_{hk} \equiv (M - M^k) - (M^h - M^{hk})$$
$$= M - M^k - M^h + M^{hk} \quad . \tag{2.3}$$

Similarly the multiple discontinuity across the set of cuts $H = (h_1, h_2, \cdots, h_m)$, with the h_i all different, is the discontinuity across h_1 of the discontinuity across h_2 $\cdots$ of the discontinuity across h_m. It is equal to

$$M_H \equiv \sum_{H' \subset H} (-1)^{n(H')} M^{H'} \tag{2.4a}$$

where $M_\emptyset = M^\emptyset = M$ and the sum runs over all different subsets H' of H, including the empty set $\emptyset$, and $n(H')$ is the number of elements of H'. Similarly, for $G \cap H = \emptyset$,

$$M_H{}^G \equiv \sum_{H' \subset H} (-1)^{n(H')} M^{GH'} \tag{2.4b}$$

is the multiple discontinuity across the set of cuts H evaluated below all the cuts g in G and above all the cuts g not in $GH \equiv G \cup H$.

Note that the multiple discontinuity $M_H{}^G$ is independent of the order of the elements in H. It is also independent of the order of the elements of G. These sets are regarded as unordered sets.

The formulas (2.4a) and (2.4b) can be inverted to give

$$M^G = \sum_{G' \subset G} (-1)^{n(G')} M_{G'} \tag{2.5a}$$

and, for $G \cap H = \emptyset$,

$$M_H{}^G = \sum_{G' \subset G} (-1)^{n(G')} M_{HG'} \quad . \tag{2.5b}$$

Equation (2.5a) is just a special case of (2.5b). These formulas, which are derived in appendix B, express all 2^{16} functions M^G, and also all the single, double, and higher-order multiple discontinuities formed from them, evaluated on all possible sides of all the remaining cuts, in terms of the various multiple discontinuities M_H.

Our general discontinuity formula is equation (2.5), together with explicit formulas for all of the functions M_H occurring on its right-hand side. Most of these functions M_H vanish by virtue of the generalized Steinmann relations.

The generalized Steinmann relations assert that the discontinuity $M_h{}^G$ across any cut h is independent of whether it is evaluated above or below any "crossed cut" g. Two cuts g and h are said to be crossed if and only if the corresponding channels overlap: i.e., if and only if each of the four sets $J_g \cap J_h$, $\bar{J}_g \cap J_h$, $J_g \cap \bar{J}_h$, and $\bar{J}_g \cap \bar{J}_h$ is nonempty. This requirement is equivalent to the condition that none of the four sets J_g, J_h, $\bar{J}_g$, and $\bar{J}_h$ be a subset of any of the others.

Two different cuts $h \in E$ and $k \in E$ are crossed if and only if one of the following five conditions holds: (2.6)

(a) both are initial subenergy cuts i,

(b) both are final subenergy cuts f,

(c) both are cross-energy cuts (if),

(d) one is a cross-energy cut (if), and one is the total energy cut t.

(e) one is a cross energy cut (if) and the other is a subenergy cut that is neither i nor f.

It is convenient to speak of g as either a label, a channel, or a cut. Hence a set of g's can be called a set of labels, a set of channels, or a set of cuts.

A necessary and sufficient condition for the generalized Steinmann relation to hold is that

$$M_H = 0 \quad \text{if } H \text{ contains any pair of crossed cuts.} \tag{2.7}$$

It is immediately evident from (2.5) that this condition is sufficient: (2.7) and (2.5b) ensure that M_h^G is independent of the presence in G of any cut g such that g and h are crossed. Conversely, if the M^G satisfy the generalized Steinmann relations then (2.7) follows from (2.4a). For if g and h are two crossed cuts in H then for every term $(-1)^{n(H')} M^{H'}$ in (2.4a) such that neither g nor h is in H' there are three other terms with H' replaced by gH', hH', and ghH', respectively, and the sum of these four terms will vanish, by virtue of the independence of M_h^G upon whether g is in G.

The conditions (2.6) entail that every set H of more than three cuts contains a pair of crossed cuts. In fact, the generalized Steinmann relations are equivalent to the following set of conditions:

$$M_H = 0 \quad \text{for } n(H) > 3 , \tag{2.8a}$$

$$M_{ii'} = M_{ii'h} = 0 , \tag{2.8b}$$

$$M_{ff'} = M_{ff'h} = 0 , \tag{2.8c}$$

$$M_{(if)(i'f)} = M_{(if)(if')} = M_{(if)(i'f')} =$$

$$= M_{(if)(i'f)h} = M_{(if)(if')h} = M_{(if)(i'f')h} = 0 , \tag{2.8d}$$

$$M_{(if)t} = M_{(if)th} = 0 , \tag{2.8e}$$

and

$$M_{i'(if)} = M_{(if)f'} = M_{i'(if)h} = M_{(if)f'h} = 0 , \tag{2.8f}$$

where i and i' are different elements of the set {1,2,3}, f and f' are different elements of the set {4,5,6}, h is an arbitrary element of E, and $Hh \equiv H \cup h$.

The conditions (2.8) reduce the number of nonzero M_H to 68. Moreover, all these are obtained from twelve basic forms by inserting particular integers for i and f. It is therefore feasible to exhibit explicitly all the nonzero M_H.

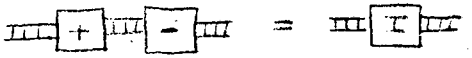
Because topological connections are of central importance it is convenient to represent M_H in diagrammatic notation [7]. The S matrix is represented by a plus box:

S = . (2.9a)

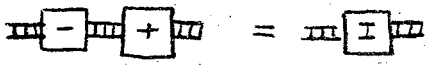
The inverse of the S matrix (or $S^\dagger$) is represented by a minus box:

$S^{-1} = S^\dagger =$. (2.9b)

Thus unitarity says that

 (2.9c)

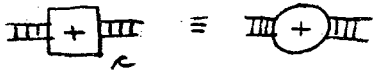
and



(2.9d)

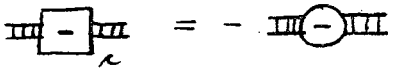
where the I-box represents the identity operator. The shaded strips represent arbitrary sets of lines, and there is an implied unitarity-type sum over all (mass-shell) values of all possible sets of intermediate-particle variables [8].

The connected part of S is represented by a plus bubble:



(2.9e)

and the connected part of $S^{\dagger} = S^{-1}$ is represented by <u>minus</u> the minus bubble:*



(2.9f)

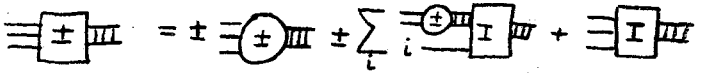
The cluster decomposition of the S-matrix reads, in various special cases that are needed below,

(2.9g)

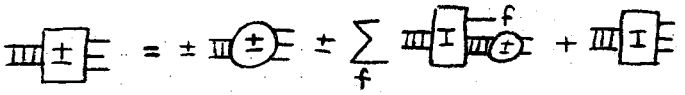
---

* The notation differs from that of ref. [7] by the extra minus sign in eq. (2.9f), which is introduced to make the minus bubble represent the continuation of the scattering function to below all the cuts. Also, in the present work the diagram are to be read from left to right, with initial lines coming first. Sums of diagrams represent the corresponding sums of functions.

(2.9h)

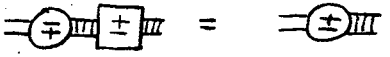


(2.9i)



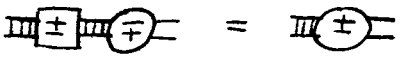
(2.9j)

where ± is plus or minus throughout each equation. Two frequently used identities are



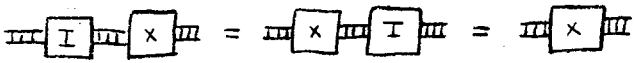
(2.9k)

and



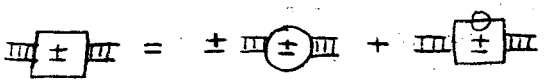
(2.9ℓ)

They follow immediately from unitarity and (2.9g) and (2.9h), respectively, together with the property of the I-box [9],



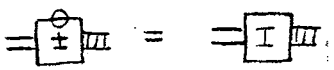
(2.9m)

where the X box represents any combination of boxes and bubbles. Another frequently used symbol is defined by



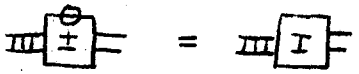
(2.9n)

This equation combined with (2.9g) and (2.9h) gives



(2.9o)

and



(2.9p)

These two equations will be used later.

The function $M_\emptyset \equiv M^\emptyset \equiv M$ is the connected part of the physical scattering amplitude:

$M_\emptyset$ =

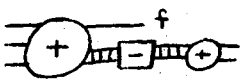
(2.10)

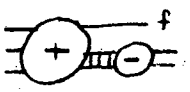
The sixteen single discontinuities M_g are

M_i =

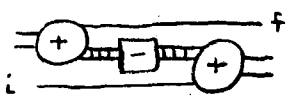
=

(2.11a)

M_f = 

= 

(2.11b)

$M_{(if)}$ = 

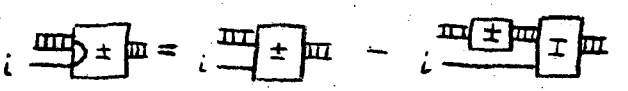
(2.11c)

and

M_t =

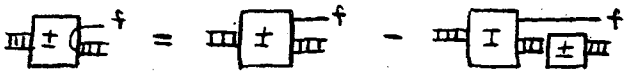
(2.11d)

It is convenient to introduce special symbols to represent the sum of terms of S (or of $S^\dagger$) that have special connectedness properties. The symbol defined by



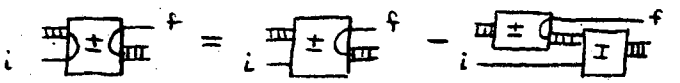
(2.12a)

can be shown [10] to represent the sum of the terms of S(or $S^\dagger$) in which the initial line i is connected to some nontrivial bubble*; i.e., it represents the sum of terms in which the line i does not go straight through. Similarly the symbol

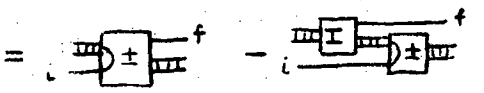


(2.12b)

represents the sum of terms of S (or $S^\dagger$) in which the final line f does not go straight through. Finally, the symbol



(2.12c)

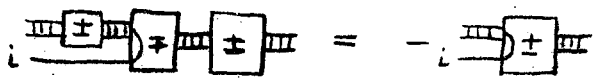


(2.12d)

represents the sum of terms of S (or $S^\dagger$) in which neither i nor f go straight through. Two frequently used identities, which follow from (2.12a,b) and (2.9c,d,m), are

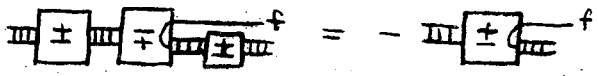
---

* A trivial bubble is a bubble that is connected to only two lines. It is usually represented by a dot.



(2.12e)

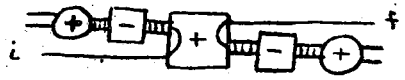
and



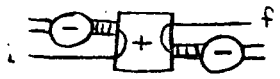
(2.12f)

In terms of these symbols the nonvanishing M_{gh} are given by

M_{if} =

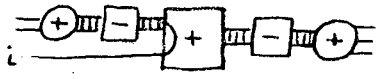


=

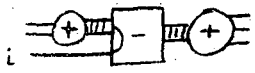


(2.13a)

M_{it} =

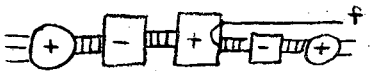


=



(2.13b)

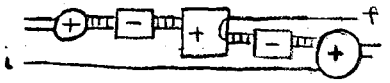
M_{tf} =



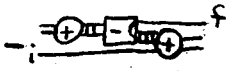
=

(2.13e)

$M_{i(if)}$ =

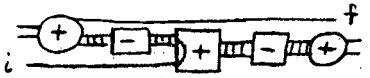


=

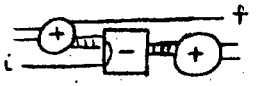


(2.13d)

$M_{(if)f}$ =



=

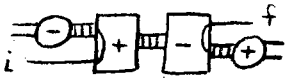


(2.13e)

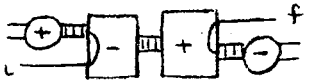
The nonvanishing functions M_{ghk} are

M_{itf} =

=



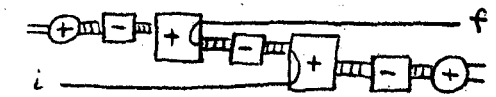
=



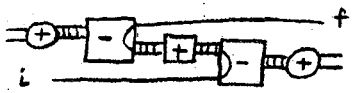
(2.14a)

and

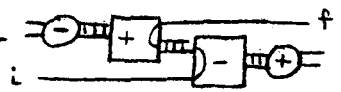
$M_{1(if)f}$ =



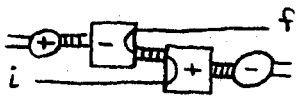
=



= −



= −



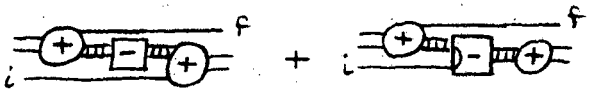
(2.14b)

The first form given for each of these functions M_H, although longer than the succeeding ones, exhibits a systematic rule: there is a minus box for each h in H, and these minus boxes occur between the parts of plus boxes that contain nontrivial bubbles on which the appropriate external lines terminate.

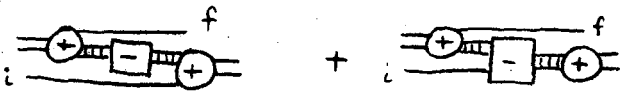
To show how these formulas work we calculate $M^f_{(if)}$, which is the discontinuity across the cut (if) evaluated below the cut f, but above all the other cuts. Using in order equations (2.5), (2.11c), (2.12a), (2.13e), (2.12a), (2.9m), and (2.9ℓ), one obtains

$M^f_{(if)} = M_{(if)} - M_{(if)f}$

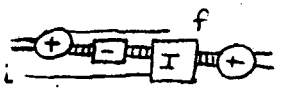
=



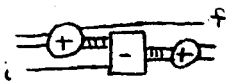
=



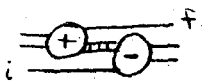
−



=



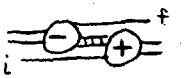
=



(2.15a)

In a similar way one obtains

$M^i_{(if)}$ =



(2.15b)

These formulas (2.15) contain the inclusive optical theorem for the case c+···+f = c.

The formula (2.5a) expresses the 2^{16} functions M^G as the function $M_\emptyset = M^\emptyset = M$, which is the function evaluated above all the cuts, plus the sum of discontinuities that shift the point of evaluation, to below the set of cuts G. There is an equivalent formula that expresses M^G as the function $\overline{M} \equiv -M^\dagger$, which is the function evaluated below all the cuts, plus the sum of discontinuities that moves the point of evaluation to above the complementary set of cuts $\overline{G} \equiv E - G$.

To exhibit these other formulas we introduce for any function F formed from boxes and bubbles the notation

$$\overline{F} = -F^\dagger \quad , \tag{2.16a}$$

where dagger represents hermitian conjugate. In particular, for any function F represented by a single diagram one has

$$\overline{F} = -(-1)^{N_b} F(+ \leftrightarrow -) \quad , \tag{2.16b}$$

where N_b is the number of explicitly appearing bubbles in the diagram, and $F(+ \leftrightarrow -)$ is the function represented by the diagram obtained from the one representing F by reversing the sign inside each bubble, box, and modified box [such as occurs in (2.12) and (2.9n)].

The functions M^G all satisfy the important property [see (5.58)]

$$M^G = \overline{M}^{\overline{G}} \quad . \tag{2.17}$$

But application of $F \to \overline{F}$ and $G \to \overline{G}$ to (2.5a) gives

$$M^G = \overline{M}^{\overline{G}} = \sum_{G' \subset \overline{G}} (-1)^{n(G')} \overline{M}_{G'} \quad . \tag{2.18a}$$

Since the sums in (2.5a) and (2.18a) are over complementary sets, one of these formulas for M^G may have fewer terms than the other.

Similarly, the application of $F \to \overline{F}$ and $G \to \overline{G}$ to (2.5b) gives, for $\overline{G} \cap H = \emptyset$,

$$\overline{M}_H^{\overline{G}} = \sum_{G' \subset \overline{G}} (-1)^{n(G')} \overline{M}_{HG'} \quad . \tag{2.18b}$$

These functions $\overline{M}_H^{\overline{G}}$ satisfy (see (2.4b))

$$\begin{aligned}
\overline{M}_H^{\overline{G}} &= \sum_{H' \subset H} (-1)^{n(H')} \overline{M}^{\overline{G}H'} \\
&= \sum_{H'' \subset H} (-1)^{n(H-H'')} \overline{M}^{\overline{G}(H-H'')} \\
&= (-1)^{n(H)} \sum_{H'' \subset H} (-1)^{n(H'')} \overline{M}^{\overline{(G-H)H''}} \\
&= (-1)^{n(H)} M_H^{G-H} \quad ,
\end{aligned} \tag{2.19}$$

or, equivalently, for $G \cap H = \emptyset$

$$M_H^G = (-1)^{n(H)} \overline{M}_H^{\overline{GH}} \quad . \tag{2.20}$$

The function $\overline{M}_H^{\overline{G}}$ (for $H \cap \overline{G} = \emptyset$) is the multiple discontinuity across the set of cuts H evaluated above the set of cuts $g \varepsilon \overline{G}$ and below the set of cuts $g \varepsilon G-H$. The extra factor $(-1)^{n(H)}$ in (2.19) and (2.20) reflects the fact that the multiple discontinuities $\overline{M}_H^{\overline{G}}$ are calculated by the rule "function below the cut minus function above the cut". The sets of cuts referred to in (2.19) are shown in fig. 2.2

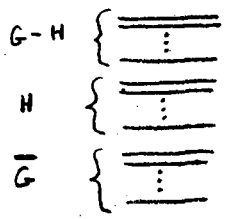
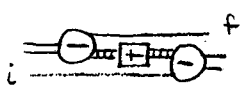


Fig. 2.2. The function $\overline{M}_H^{\overline{G}}$ represents the multiple discontinuity across the set of cuts H (calculated by the rule below minus above) evaluated above the set of cuts $\overline{G}$, and below the set of cuts $G-H$.

As an example of these alternative formulas note that the discontinuity across the (if) cut evaluated below all other cuts is given by (2.20), (2.16), and (2.11c) as

$$M_{(if)}^{\overline{(if)}} = -\overline{M}_{(if)} =$$

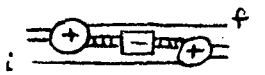


(2.21)

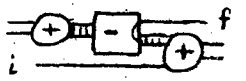
This same discontinuity is given by (2.5), (2.8), and the definitions (2.11c), (2.13d), (2.13e), and (2.14b), as

$$M_{(if)}^{\overline{(if)}} = M_{(if)} - M_{i(if)} - M_{(if)f} + M_{i(if)f}$$

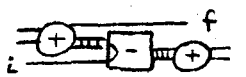
=



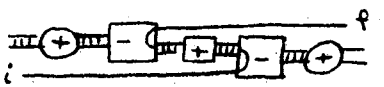
+



+



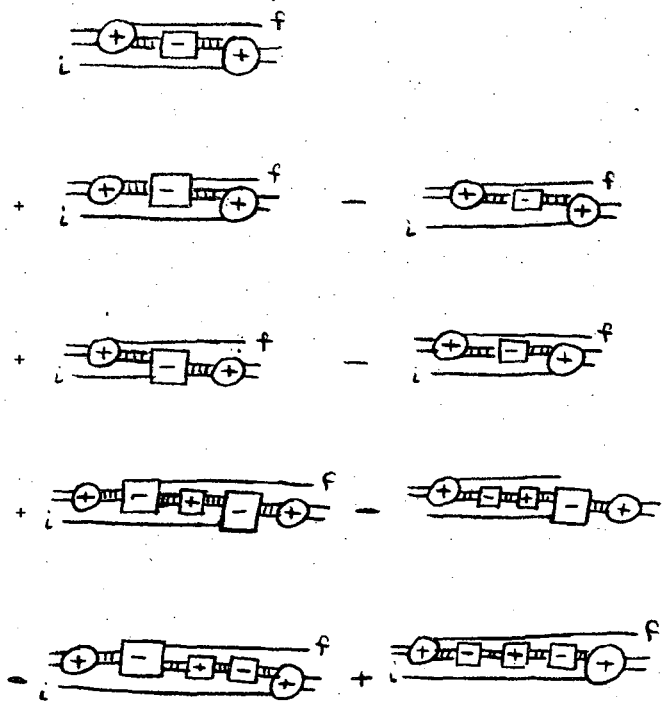
+



(2.22)

Introducing the definitions (2.12), and the identity property (2.9m) of the I-box, one obtains

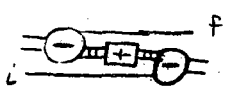
$M^{\overline{(if)}}_{(if)}$ =



(2.23)

which, by virtue of the unitarity equation (2.9d) and the identities (2.9k,ℓ,m), becomes

$M^{\overline{(if)}}_{(if)}$ =



(2.24)

which agrees with (2.21).

These results, together with several other single discontinuities that can be derived in similar ways are summarized in fig. 2.3. Other valid formulas follow from these by a reflection about a vertical axis together with the substitution $i \leftrightarrow f$. Still others follow from the uniform substitution $+ \leftrightarrow -$ for all signs

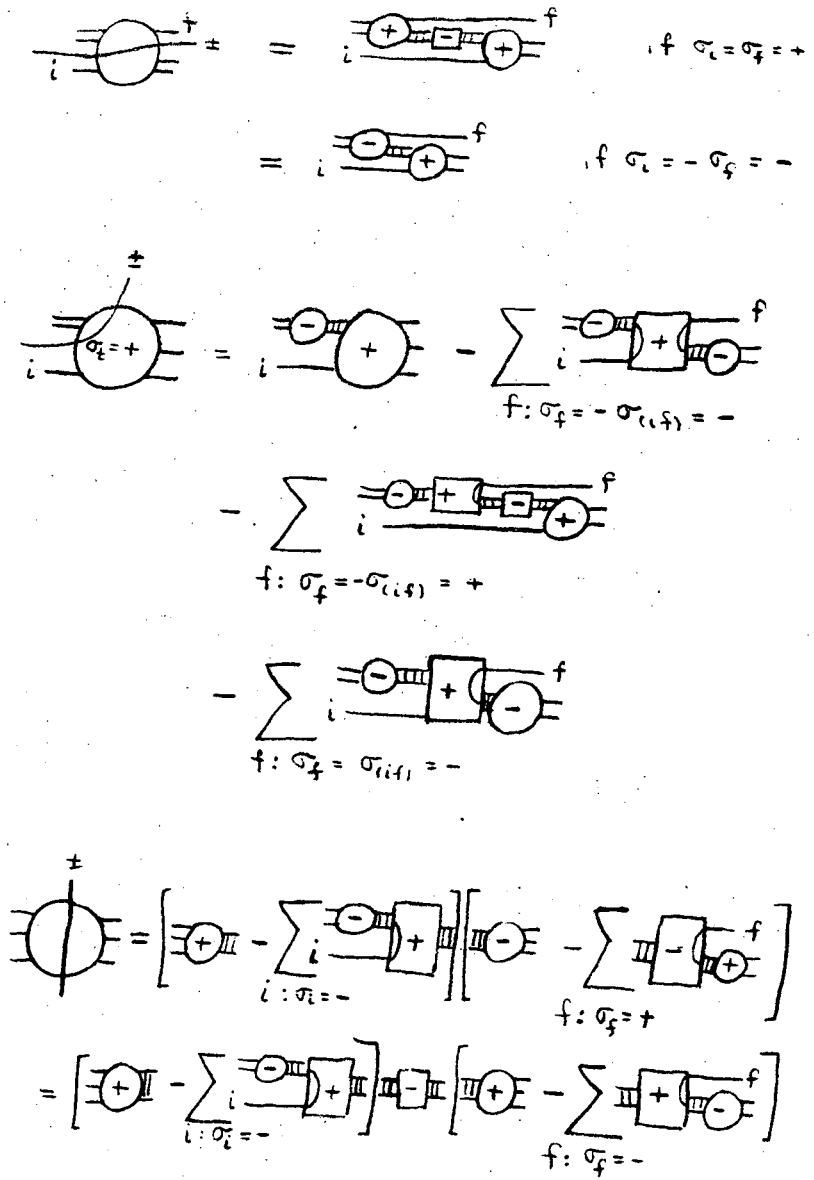


Fig. 2.3. The single discontinuity formulas. –

σ_g and all signs inside bubbles, boxes, and modified boxes. The $\pm$ sign standing outside the bubble on the left-hand side is not to be changed: it signifies that the discontinuity on the left-hand side is defined to be the function above the cut minus the function below the cut. That is, it is the difference $M(\sigma_g = +) - M(\sigma_g = -)$.

The formulas described above define the 2^{16} functions M^G, and all the discontinuities formed from them. In the following sections it will be shown that 26,018 of these functions have nice analyticity properties, and, in particular, continue in a well-defined way around each real p singularity surface. The 26,018 functions M^G that have this property can be identified in the following way: For any set G let the set of signs η_g be defined by the condition that η_g is plus if g lies in G and minus if g lies in $\overline{G}$. Then M^G is one of the 26,018 boundary values if and only if there is no pair (i,f) such that the following conditions are all satisfied:

$$\eta_{(if)} = \eta_t = -\eta_i = -\eta_f \quad . \tag{2.25}$$

III. HEURISTICS

The properties of the functions M^G will be derived in section VI from the properties of a similar set of functions T^G. These functions T^G are heuristically defined in this section as formal infinite sums of bubble diagram functions. Three properties of the functions T^G will be identified in section IV as their defining properties, and finite expressions for them will be obtained in section V.

If one introduces the definition

$$R = S - I \tag{3.1}$$

then unitarity takes the form

$$R = -R^\dagger - R^\dagger R \quad .$$

Iteration gives, formally,

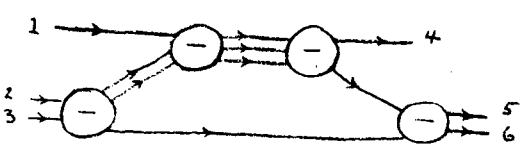
$$R = \sum_{n=1}^{\infty} (-R^\dagger)^n \quad . \tag{3.2}$$

This is an infinite series expansion for R in terms of $R^\dagger$.

The connected part of (3.2) can be expressed in the form [11]

$$R_c = S_c \equiv \overline{M} = \sum_{B^-} F^{B^-} \equiv T \quad . \tag{3.3}$$

Here, as throughout this paper, the symbol B^- represents a bubble diagram every bubble b of which is a (nontrivial) minus bubble. The sum in (3.3) is over all bubble diagrams B^- with the appropriate external lines, and F^{B^-} is the bubble diagram function corresponding to the bubble diagram B^-. A typical bubble diagram B^- is

B^- = 

(3.4)

and the corresponding function F^{B^-} is the product of the four indicated functions $\overline{M} = -S_c^\dagger$, integrated over the physical values of the variables associated with the six sets of intermediate lines. Further details can be found in ref. [11].

By definition a bubble diagram B is required to have its bubbles partially-ordered by the condition that every line that connects two bubbles of B runs from the right-hand side of one bubble to the left-hand side of another bubble that stands completely to the right of the bubble from which the line came. This means that all lines can be drawn as directed lines that point from left to right. A diagram such as the one in fig. 3.1 is not a bubble diagram.

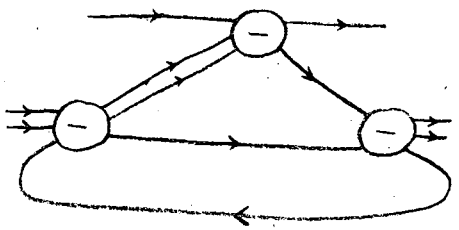


Fig. 3.1. A diagram that is not a bubble diagram.

Consider an arbitrary channel g. It is defined by a separation of the set $(1,\cdots,6)$ into two complementary disjoint sets J_g and $\bar{J}_g$. The set $\mathcal{B}^-$ of all B^- (with initial lines 1,2,3 and final lines 4,5,6) can be separated into two sets $\mathcal{B}_g$ and $\mathcal{B}^g$ by the following rule: B^- belongs to $\mathcal{B}_g$ or $\mathcal{B}^g$ according to whether B^- has or does not have an explicit g-channel cut set.

<u>Definition</u>. An <u>explicit g-channel cut set of B</u> is a set of internal lines of B which if cut separates B into two connected bubble diagrams $B(J_g)$ and $B(\bar{J}_g)$, where $B(J_g)$ contains all the external lines $j\varepsilon J_g$, and $B(\bar{J}_g)$ contains all the external lines $j\varepsilon\bar{J}_g$.

Moreover, each line L_j of the cut set must be directed from $B(\bar{J}_g)$ to $B(J_g)$.

Thus, for example, the diagram B^- of (3.4) belongs to $\mathcal{B}_g$ for $g = 1,4$, and t, and to $\mathcal{B}^g$ for all other g in E.

This decomposition of $\mathcal{B}^-$ induces a corresponding decomposition

$$T = T_g + T^g \tag{3.5a}$$

of the infinite series expansion T of M given in (3.3). Here

$$T_g = \sum_{B^-\varepsilon\,\mathcal{B}_g} F^{B^-} \tag{3.5b}$$

and

$$T^g = \sum_{B^-\varepsilon\,\mathcal{B}^g} F^{B^-} . \tag{3.5c}$$

For each channel g one can make this decomposition $T = T_g + T^g$. Moreover, for any sum F of bubble diagram functions F^B one can use (3.3) to give an expansion of F in terms of bubble diagram functions F^{B^-}, and then use the same procedure to define a separation of F into two parts;

$$F = F_g + F^g , \tag{3.6}$$

where F_g corresponds to a sum of B^- each of which has an explicit g-channel cut set, and F^g corresponds to a sum of B^- none of which has an explicit g-channel cut set. In particular, one can write

$$T_h = T_{hg} + T_h^{\,g} \tag{3.7}$$

and

$$T^h = T^h_g + T^{hg} \quad . \tag{3.8}$$

These definitions entail that $T_{gh} = T_{hg}$: both symbols represent the sum of F^{B^-} over those B^- that have both an explicit g-channel cut set and explicit h-channel cut set. In a similar way one can define T_{ghk}, T_{ghkm}, etc., all of which are independent of the order of their subscripts. From this symmetry property it follows that all of the various functions $T_H^{\ G}$ with both upper and lower indices are independent of the order of their indices. For example, (3.7) gives

$$T_h^{\ g} = T_h - T_{hg} \tag{3.9a}$$

while (3.6) gives

$$T^g_{\ h} = (T - T_g)_h = T_h - T_{gh} \quad , \tag{3.9b}$$

so that $T_h^{\ g} = T^g_{\ h}$.

An expansion formally similar to (2.5) follows directly from (3.6). For example,

$$T^{gh} = (T - T_g)^h = T^h - T_g^{\ h}$$

$$= T - T_h - T_g + T_{gh} \quad . \tag{3.10}$$

More generally, one finds

$$T^G = \sum_{H \subset G} (-1)^{n(H)} T_H \quad . \tag{3.11a}$$

The functions $\overline{T}^G$ and $\overline{T}_H$ defined by $\overline{T}^G = -(T^G)^\dagger$ and $\overline{T}_H = -(T_H)^\dagger$ satisfy

$$\overline{T}^G = \sum_{H \subset G} (-1)^{n(H)} \overline{T}_H \quad . \tag{3.11b}$$

IV. ANALYTIC PROPERTIES OF T^G AND $\overline{T}^G$

A. The Structure Theorem

The proofs of analytic properties will be based on a theorem that specifies the analytic properties of an arbitrary bubble-diagram function F^B. This theorem has been described in detail in ref. [12]. A resumé of its main content is given here.

The main assumption of the theorem is a set of physical-region analyticity properties called the normal analytic structure. This analytic structure is equivalent to the S-matrix macrocausality condition [13]. It is also a formal property of the perturbation theory expression for any scattering function. Roughly it is the property that the singularities of the physical scattering function are confined to positive-α Landau surfaces, and that in some real neighborhood of almost any point p on any one of these surfaces the physical scattering function is the limit of an analytic function from directions lying in a certain cone. The rules specifying these directions are called the plus $i\varepsilon$ rules.

The Landau surfaces are surfaces associated with diagrams called Landau diagrams. A Landau diagram is a topological diagram consisting of a set of directed line segments L_j and a set of point vertices V_r. The topological structure of the diagram is specified

by a set of structure coefficients ε_{jr} defined as follows:

$$\varepsilon_{jr} = \begin{cases} +1 & \text{if } L_j \text{ terminates at } V_r \\ -1 & \text{if } L_j \text{ originates at } V_r \\ 0 & \text{otherwise} \end{cases} \qquad (4.1)$$

The lines of the diagram are classified as incoming, outgoing, or internal according to the rule:

$$L_j \text{ is } \begin{cases} \text{incoming if } \varepsilon_{jr} \geqslant 0 \text{ for all } r \\ \text{outgoing if } \varepsilon_{jr} \leqslant 0 \text{ for all } r \\ \text{internal otherwise} \end{cases}$$

The incoming and outgoing lines are collectively called external lines.

Each internal and external line L_j of a Landau diagram D is associated with a physical particle (of positive mass $m_j > 0$), and with a momentum-energy four-vector p_j. Each internal line L_j is associated also with a scalar parameter α_j. The Landau equations corresponding to D are the mass-shell constraints

$$p_j^2 - m_j^2 = 0 \quad (\text{all } j), \qquad (4.2a)$$

the conservation-law constraints

$$\sum_j p_j \varepsilon_{jr} = 0 \quad (\text{all } r), \qquad (4.2b)$$

the Landau loop equations

$$\sum_{j\varepsilon \mathrm{Int}} \alpha_j p_j \eta_{j\ell} = 0 \quad (\text{all } \ell\varepsilon L), \qquad (4.2c)$$

and the nontriviality condition

$$\sum_{j\varepsilon \mathrm{Int}} \alpha_j^2 - 1 = 0 . \qquad (4.2d)$$

The set Int is the set of indices that label the internal lines of D, L is the set of indices that label the closed loops that can be constructed on the internal lines of D, and $\eta_{j\ell}$ is the number of times loop ℓ passes along line L_j in the positive direction minus the number of times loop ℓ passes along line L_j in the negative direction. (It is sufficient to consider a set of linearly independent loops; then the $\eta_{j\ell}$ can be restricted to ± 1, and zero.)

Let p represent the set of momentum-energy vectors p_j corresponding to the <u>external</u> lines of D. Then the complex Landau surface $L(D)$ is the set of complex points p such that for some choice of the complex p_j and α_j associated with the internal lines L_j of D the Landau equations corresponding to D can all be satisfied.

To describe the real Landau surface corresponding to D let a sign σ_j be introduced for each internal line L_j of D. The set of signs σ_j is denoted by σ. Then the real Landau surface $L(D^\sigma)$ is the set of points p that satisfy the above Landau equations and also the conditions

$$\mathrm{Im}\, p_j = 0 \quad \text{all } j \,, \tag{4.2e}$$

$$\mathrm{Im}\, \alpha_j = 0 \quad \text{all } j\varepsilon \mathrm{Int} \,, \tag{4.2f}$$

$$p_j^{\,0} > 0 \quad \text{all } j \,, \tag{4.2g}$$

and

$$\sigma_j\, \alpha_j > 0 \quad \text{all } j\varepsilon \mathrm{Int} \,. \tag{4.2h}$$

The equations (4.2a) through (4.2h) are called the Landau equations corresponding to D^σ.

The sign σ_j is allowed to be $\pm$, in which case the corresponding equation (4.2h) is eliminated from the equations that define $L(D^\sigma)$.

The symbol D^+ represents a D^σ with all $\sigma_j = +$. The corresponding Landau surface $L(D^+)$ is called a positive-α Landau surface. The union of all $L(D^+)$ is called L^+.

The Landau equations corresponding to D^+ have a simple physical interpretation [14,15]. The significance of the mass-shell and conservation-law constraints is obvious. The significance of the Landau loop equations is this: they ensure that the "displacement" vectors

$$\Delta_i = \alpha_i\, p_i \tag{4.3}$$

fit together to form a geometric diagram in a four-dimensional space. This geometric diagram has the topological structure specified by D^+, and it can be interpreted as a space-time diagram representing a possible classical multiple-scattering process in which point particles scatter at point vertices. The conditions (4.2e) through (4.2h) ensure that positive energy is carried forward in time on each

leg of this multiple-scattering process. The parameter α_j is the proper time associated with L_j, divided by the mass m_j. This geometrical interpretation of the solutions of the positive-α Landau equations makes it clear that $L(D^+)$ can be nonempty only if the vertices of D^+ can be partially ordered by the condition that each internal line L_j of D^+ point from left to right.

These space-time diagrams are called <u>space-time representations of D^σ</u>. For every solution of the Landau equations corresponding to D^σ there is an associated space-time representation of D^σ. The vectors from an arbitrary origin to the vertices V_r of the space-time representation are denoted by ω_r. The line L_i is represented by the four-vector

$$\Delta_i = \sum_r \varepsilon_{ir}\, \omega_r \,. \tag{4.4}$$

One of the conclusions of the structure theorem is an $i\varepsilon$ rule of continuation for bubble diagram functions. This rule is a simple generalization of the plus $i\varepsilon$ rule, and it is most easily described by first describing the plus $i\varepsilon$ rule itself.

Let $\omega = (\omega_1, \cdots, \omega_m)$ represent the set of ω_r corresponding to a space-time representation of some D^σ. Let D^+ be some positive-α diagram, and let p be some point on $L(D^+)$. Let $\Omega(D^+,p)$ be the union of all ω that correspond to space-time representations of D^+ having external lines specified by the set of (external) variables p. Let $\Omega^+(p)$ be the union of the sets $\Omega(D^+,p)$ over all positive-α diagrams.

The plus $i\varepsilon$ rules can be stated in terms of a cone $C^+(p)$ that is closely related to $\Omega^+(p)$. Let $q \equiv (q_1, q_2, \cdots, q_6)$ represent the imaginary part of the complexification of $p \equiv (p_1, p_2, \cdots, p_6)$. Then

$$C^+(p) \equiv \left\{ q : -\sum_{\substack{j\varepsilon \mathrm{Ext} \\ r\varepsilon \mathrm{Ver}}} \varepsilon_{jr}\, q_j \cdot \omega_r > 0 \quad \text{for all } \omega\varepsilon\Omega^+(p) \right\} . \tag{4.5a}$$

Here Ext is the set of indices j that label the components p_j of $p = (p_1, \cdots, p_6)$, and Ver is the set of indices r of the components ω_r of ω. By virtue of the Landau equations the cone $C^+(p)$ can be written in the alternative form [16]

$$C^+(p) = \left\{ q : \sum_{j\varepsilon \mathrm{Int}} q_j \cdot \Delta_j(\omega) > 0 \quad \text{for all } \omega\varepsilon\Omega^+(p) \right\} , \tag{4.5b}$$

where $\Delta_j(\omega)$ is defined by (4.4), Int is the set of indices labelling internal lines of the diagram specified by ω, and the q_j for $j\varepsilon$Int are <u>any</u> set of four-vectors that satisfy for every r the momentum-energy conservation-law constraints $\sum q_j\, \varepsilon_{jr} = 0$, where the external q_j are fixed by q.

The plus $i\varepsilon$ rule says this: Let p be any point of $\mathcal{M}$, which is the real mass shell restricted by momentum-energy conservation. Let $C(p)$ be any cone that is closed apart from its missing apex $q = 0$, and that is contained in $C^+(p')$ for all points p' in some real neighborhood of p. Let $\mathcal{M}_c$ be the complex mass shell, restricted by momentum-energy conservation, and suppose the

intersection of $\mathcal{M}_c$ with $\{q\varepsilon C(p)\}$ has p on its boundary. Then p has a real neighborhood $\mathcal{N}(p) \subset \mathbb{R}^{4n}$ such that the physical scattering function in $\mathcal{N}(p) \cap \mathcal{M}$ is the boundary value (in a distribution sense) of a function that is analytic [15] in the set

$$\mathcal{M}_c \cap \{p'\varepsilon\, \mathcal{N}(p)\} \cap \{q\varepsilon C(p)\} \cap \{q\varepsilon N\}$$

where $N \subset \mathbb{R}^{4n}$ is some neighborhood of $q = 0$.‡ This statement of the $i\varepsilon$ rule will be used presently.

The first main conclusion of the structure theorem [17] is this*:

(1) Let B be any bubble diagram, and let $F^B(p)$ be the corresponding bubble diagram function. Then $F^B(p)$ is analytic at all real (mass-shell) points p not lying on

$$L(B) \equiv \bigcup_{D^\sigma \subset B} L(D^\sigma) , \tag{4.6}$$

where D^σ is <u>contained</u> in B (i.e., $D^\sigma \subset B$) if and only if D^σ can be constructed by replacing each plus bubble b of B by either a point vertex V_b or a connected positive-α Landau diagram D_b^+ such that $L(D_b^+) \neq \emptyset$, and by replacing each minus bubble b of

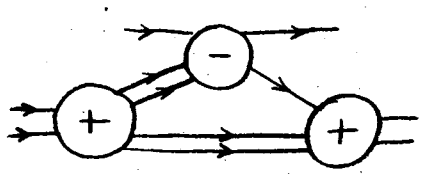
---

* The contraction condition occurring in the statement of the theorem given in reference [17] is here replaced by the rule that condition (4.2h) is relaxed for lines L_j that are explicit lines of B itself. This new version is slightly stronger than the original version, but follows from essentially the same argument.

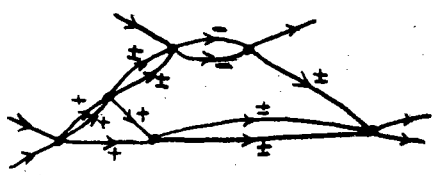
‡ $\mathbb{R}^{4n}$ is the real 4n-dimensional space, and n is the number of particles, which is six in the case under consideration.

B by either a point vertex V_b or a connected negative-α Landau diagram D_b^- such that $L(D_b^-) \neq \emptyset$. The initial and final lines of b are to match the incoming and outgoing lines of D_b. Thus each line L_j of $D^\sigma \subset B$ is either an internal line of a D_b^+ or D_b^-, in which case it carries the sign $\sigma_j = +$ or $\sigma_j = -$, respectively, or it is a line L_j of the original bubble diagram B. In this latter case this line is assigned the sign $\sigma_j = \pm$, which means that the corresponding parameter α_j can be positive, negative, or zero. These lines of B itself are sometimes called <u>explicit</u> lines.

An example of a $D^\sigma \subset B$ is given by

B =  (4.7a)

and

D^σ =  (4.7b)

The internal lines of each D_b^+ and D_b^- are drawn so as to lie inside the corresponding bubble b.

The set $\Omega(D^\sigma,p)$ is defined, in analogy to $\Omega(D^+,p)$, as the set of all ω that correspond to space-time representations of D^σ that correspond to solutions at p of the Landau equations corresponding to D^σ. And $\Omega^B(p)$ is defined in analogy to $\Omega^+(p)$ as

$$\Omega^B(p) \equiv \bigcup_{D^\sigma \subset B} \Omega(D^\sigma,p) \tag{4.8}$$

Finally, $C^B(p)$ is defined, in analogy to $C^+(p)$, as

$$C^B(p) \equiv \left\{ q : -\sum_{\substack{j\varepsilon Ext \\ r\varepsilon Ver}} \varepsilon_{jr}\, q_j\cdot\omega_r > 0 \quad \text{for all } \omega \text{ in } \Omega^B(p) \right\} \tag{4.9a}$$

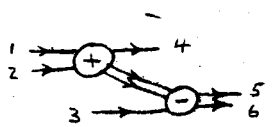
$$= \left\{ q : \sum_{j\varepsilon Int} q_j\cdot\Delta_j(\omega) > 0 \quad \text{for all } \omega \text{ in } \Omega^B(p) \right\} . \tag{4.9b}$$

The second main consequence of the structure theorem is this:

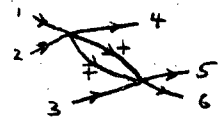
(a) The functions $F^B(p)$ satisfy an $i\varepsilon$ rule that is the same as the plus $i\varepsilon$ rule described above, except that $C^B(p)$ replaces $C^+(p)$.

If $C^B(p)$ is empty then the $i\varepsilon$ rule is devoid of content: no assertion about analyticity properties of F^B at p is made.

An important case where $C^B(p)$ is empty is the case in which p lies on $L(D^+(B))$, where $D^+(B)$ is the positive-α diagram obtained by contracting all the bubbles of B to point vertices. For example, if

B =  (4.10a)

then

$D^+(B) =$ 

(4.10b)

To see that $C^B(p)$ is empty in this case note that if p lies on $L(D^+(B))$ then it also lies on $L(D^-(B))$, because the Landau equations are invariant under the transformation $\alpha_j \to -\alpha_j$, $\sigma_j \to -\sigma_j$. But the change $\alpha_j \to -\alpha_j$ reverses all vectors Δ_j, $j \varepsilon \mathrm{Int}$. But then the $C^B(p)$ defined in (4.9b) is empty, since both $D^+(B) \subset B$ and $D^-(B) \subset B$. Thus no analyticity properties are asserted for F^B at p on $L(D^+(B))$.

It is in fact well-known [18] that F^B is identically zero on one side of the surface $L(D^+(B))$, but not (in general) on the other.

The example just given illustrates a simple way in which the analytic continuation of a sum F of bubble diagram functions can be blocked: for some Landau diagram D^σ the function F has a Landau singularity surface corresponding to both D^σ and $D^{-\sigma}$, where $D^{-\sigma}$ is D^σ with all the signs σ_j reversed. The $i\varepsilon$ rules associated with $D^{-\sigma}$ are opposite to those associated with D^σ, and hence the structure theorem provides no way to continue the sum F past the surface $L(D^\sigma) = L(D^{-\sigma})$.

This situation in which a function can have singularity surfaces associated with both a diagram D^σ and also the associated diagram $D^{-\sigma}$ is the canonical situation in which continuation is blocked. However, for a full proof that continuation is never blocked one must also rule out the possibility that surfaces

corresponding to topologically different diagrams conspire to block the continuation. This will be done in section VI.

The proofs of section VI depend on certain analyticity properties of the functions T^G and $\overline{T}^G$. These properties are derived in subsection D by combining the results described in this subsection with the properties of the functions T^G and $\overline{T}^G$ described in the next subsection.

B. Properties of the T^G and $\overline{T}^G$

In this subsection three properties of the functions T^G are described. These properties are satisfied by the formal expressions for these functions given in section III, and can be considered to be the defining properties of these functions as will be discussed in the next subsection. The analytic properties of the functions T^G and $\overline{T}^G$ that are derived in section D follow from these properties alone, and hence apply, in particular, to the well-defined expressions for these functions given in section V.

The three properties of the T^G are now described.

Property 1

Each T^G can be written in the form

$$T^G = \sum_{H \subset G} (-1)^{n(H)} T_H \tag{4.11a}$$

where the T_H are a set of 2^{16} functions that can be written in the form

$$T_H = \sum_{B \varepsilon \mathcal{B}_H} F^B \tag{4.11b}$$

where every B in $\mathcal{B}_H$ has an explicit h-channel cut set for every h in H. [These cut sets are defined above (3.5).] Property 1 follows, for the infinite series expressions, from (3.11a).

Property 2

Each T^G can be converted solely by means of the unitarity and cluster decomposition properties of S from the form given in (4.11) to the form

$$T^G = \sum_{B\varepsilon \mathcal{B}^G} F^B \quad , \tag{4.12}$$

where no $D^\sigma \subset \mathcal{B}^G$ has a positive-α g-channel cut set for any g in G. Here the following definitions are used:

Definition A positive-α g-channel cut set of D^σ is a set of internal lines of D^σ such that D^σ is separated by the cutting precisely once of every line of this set into two connected diagrams $D^\sigma(J_g)$ and $D^\sigma(\bar{J}_g)$ such that $D^\sigma(J_g)$ contains all the external lines $j\varepsilon J_g$ and $D^\sigma(\bar{J}_g)$ contains all the external lines $j\varepsilon\bar{J}_g$. Moreover, each line L_j of the cut set must be directed from $D^\sigma(\bar{J}_g)$ to $D^\sigma(J_g)$ and have a sign $\sigma_j = +$ or $\pm$.

Definition A $D^\sigma \subset \mathcal{B}$ is a D^σ such that for some B the conditions $D^\sigma \subset B\varepsilon\mathcal{B}$ hold.

Remark For any collection $\mathcal{B}$ of diagrams B^- having only minus bubbles the statement no $D^\sigma \subset \mathcal{B}$ has a positive-α g-channel cut set is equivalent to the statement no $D^\sigma \subset \mathcal{B}$ has an explicit g-channel cut set. This is because only explicit lines of B^- can belong to a positive-α g-channel cut set. Thus property 2 follows from the meaning given in (3.5) and (3.6) of superscripts.

Property 3

$$T_\emptyset = T^\emptyset = T = M \ . \tag{4.13}$$

Remark In equations such as (4.11a) and (4.12) involving sums of functions F^B over sets $B\varepsilon\mathcal{B}$ it is to be understood that the diagrams B of $\mathcal{B}$ can eventually have signs or other numerical coefficients, and that the functions F^B inherit these coefficients.

The functions $\bar{T}^G$ and $\bar{T}_H$ are defined, in accordance with (2.16), by

$$\bar{T}^G = -(T^G)^\dagger$$

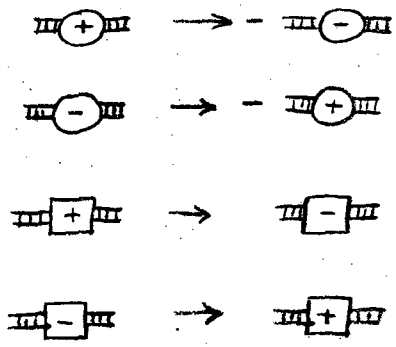
$$= \sum_{B\varepsilon\bar{\mathcal{B}}^G} F^B \tag{4.14a}$$

and

$$\bar{T}_H = -(T_H)^\dagger$$

$$= \sum_{B\varepsilon\bar{\mathcal{B}}_H} F^B \ . \tag{4.14b}$$

where $\bar{\mathcal{B}}^G$ and $\bar{\mathcal{B}}_H$ are obtained from $\mathcal{B}^G$ and $\mathcal{B}_H$ by the mapping



(4.14c)

followed by an overall sign change. The last two mappings in (4.14c) apply also to the special boxes defined by (2.9n), (2.12a), (2.12b), and (2.12c).

Negative-α g-channel cut sets are defined in the same way as positive-α g-channel cut sets [see (4.12)] except that "positive-α" is replaced by "negative-α", and $\sigma_j = +$ or $\pm$ is replaced by $\sigma_j = -$ or $\pm$. It is thus clear that no $D^\sigma \subset \bar{\mathcal{B}}^G$ has a negative-α g-channel cut set for any g in G.

C. Uniqueness of the T^G

In section V a set of 2^{17} well-defined functions T^G and T_H satisfying properties (1), (2), and (3) is constructed. The question of uniqueness arises: Can there be two different sets of 2^{17} functions T^G and T_H satisfying these three properties?

In this section it is shown that the functions T^G and T_H are unique in the following sense: Let T^G and T_H be members of any set of 2^{17} functions that satisfy properties (1), (2), and (3). Let T_-^G and T_H^- be the infinite series expressions obtained by introducing for each plus bubble of the expressions (4.12) and (4.11b) for T^G and T_H, respectively, the expansion (3.3), and then combining together the different terms that are multiples of each of the distinct possible minus-bubble diagram functions F^{B^-}. Then T_-^G and T_H^- are precisely the infinite sums that were represented in section III by the symbols T^G, T_H. This result justifies the use of the same symbols T^G and T_H to represent, on the one hand, the infinite series expressions defined in section III, and, on the other hand, the finite expressions that are obtained in section V.

For a complete proof of uniqueness one should, strictly speaking, show that two different well-defined expressions in terms of bubble-diagram functions that have the same expression in terms of minus bubble diagram functions are in fact equal. This can probably be done. However, it is not necessary for our purposes. All that we need is <u>some</u> set of well-defined functions that satisfy the properties (1), (2), and (3), and these functions will be taken to be the functions defined in section V. We doubt that others exist, but the uniqueness of the M^G is based in any case on the generalized Steinmann relations.

Because uniqueness is not really essential to our argument the proof of it will be simply sketched, rather than presented in full detail.

If the infinite series expansion (3.3) obtained from unitarity and the cluster decomposition is introduced back into the unitarity equation one finds that it is identically satisfied: i.e., if (3.3) is introduced into $SS^{\dagger} - I = 0$ then one obtains the result

$$\sum_{B^{-}\varepsilon \mathcal{B}^{-}_{\emptyset}} F^{B^{-}} = 0 \tag{4.15}$$

where $\mathcal{B}^{-}_{\emptyset}$ is empty. This means that if any function

$$F = \sum_{B\varepsilon \mathcal{B}^{F}} F^{B} \tag{4.16a}$$

is converted by means of (3.3) to an infinite series

$$F = \sum_{B^{-}\varepsilon \mathcal{B}^{-F}} F^{B^{-}} \tag{4.16b}$$

then $\mathcal{B}^{-F}$ is invariant under a change in form of F generated by the application of unitarity: i.e., if F and F' are equal by virtue of unitarity and the cluster decomposition of S then

$$\mathcal{B}^{-F} = \mathcal{B}^{-F'} . \tag{4.16c}$$

This is true because (4.15) implies that any identity among bubble diagram functions that follows from unitarity and cluster properties is identically satisfied when all the components are expanded in terms of $F^{B^{-}}$ functions: i.e., unitarity acts as the identity in the minus-bubble representation.

Let T^{G} and T_{H} be members of a set of 2^{17} functions that satisfy (1), (2), and (3). The formula (4.11a) for T^{G} entails (see Appendix B) that the functions T_{H}^{G} defined for $G\cap H = \emptyset$ by

$$T_{H}^{G} = \sum_{G'\subset G} (-1)^{n(G')} T_{HG'} \tag{4.17a}$$

satisfy

$$T_{H}^{G} = \sum_{H'\subset H} (-1)^{n(H')} T^{GH'} \tag{4.17b}$$

and also

$$T_{H}^{G} = \sum_{\substack{K',K'' \\ K'\cup K''=K \\ K'\cap K''=\emptyset}} T^{GK'}_{HK''} \tag{4.17c}$$

where in (4.17c) the sum is over all nonintersecting sets K' and K'' whose union is any fixed set K that does not intersect $G\cup H$.

A special case of (4.17c) is

$$T^{G} = \sum_{K'\subset \overline{G}} T^{GK'}_{\overline{GK'}} . \tag{4.18}$$

This gives T^{G} as a sum over terms of the form $T_{\overline{K}}^{K}$. Equations (4.17a) and (4.17b) give

$$T_{\overline{K}}^{K} = \sum_{G'\subset K} (-1)^{n(G')} T_{\overline{K}G'} \tag{4.19a}$$

and

$$T_{\overline{K}}^{K} = \sum_{H' \subset \overline{K}} (-1)^{n(H')} T^{KH'} . \tag{4.19b}$$

The equality of the right-hand sides of (4.19a) and (4.19b) follows from (4.11a) alone: it is identically satisfied if (4.11a) is introduced into (4.19b). According to property (2) the equality of the expressions (4.11) and (4.12) for T^G follows just from unitarity and the cluster decomposition of S. Thus, according to (4.16c), equation (4.11a) with T^G and T_H replaced by T_-^G and T_H^-, respectively, is identically satisfied: both sides have the same minus-bubble representation. But then the replacements of the T^G and T_H on the right-hand sides of (4.19b) and (4.19a) by T_-^G and T_H^-, respectively, must yield the same answer: both procedures must give the same formal expression

$$T_{\overline{K}}^{K} = \sum_{B^- \varepsilon \mathcal{B}_{\overline{K}}^{-K}} F^{B^-} , \tag{4.20}$$

where $\mathcal{B}_{\overline{K}}^{-K}$ is some well-defined set of B^-.

Let $\mathcal{B}^{-G}$ be the set of B^- obtained by the expansion of the $B \varepsilon \mathcal{B}^G$ in terms of B^-'s. And let $\mathcal{B}_H^-$ be the set of B^- obtained from the series expansion of $\mathcal{B}_H$ in terms of B^-'s. The characteristic properties of $\mathcal{B}^G$ and $\mathcal{B}_H$ are not destroyed by the series expansion. That is, no $D^\sigma \subset \mathcal{B}^{-G}$ can contain a positive-α g-channel cut set for any g in G, and every $B \varepsilon \mathcal{B}_H^-$

contains an explicit h-channel cut set for every h in H. Moreover, the condition that no $D^\sigma \subset \mathcal{B}^{-G}$ contains a positive-α g-channel cut set is equivalent to the condition no $B \varepsilon \mathcal{B}^{-G}$ contains an explicit g-channel cut set, since all lines L_j of any B^- with $\sigma_j = +$ or $\pm$ are explicit lines of B^-. Thus, by virtue of (4.19), $\mathcal{B}_{\overline{K}}^{-K}$ contains a B^- only if B^- has no explicit g-channel cut set for any g in K, and has an explicit h-channel cut set for every h in $\overline{K}$. But then any given B^- can be contained in one and only of the sets $\mathcal{B}_{\overline{K}}^{-K}$, namely the one such that $\overline{K}$ is the set of all g such that B^- contains an explicit g-channel cut set.

Each B^- must in fact be contained exactly once (with coefficient plus one) in the union of all $\mathcal{B}_{\overline{K}}^{-K}$. This follows from (3.3), (4.13), and (4.18) for the special case $G = \emptyset$. Thus each $\mathcal{B}_{\overline{K}}^{-K}$ is, by virtue of (4.11-13), exactly the set of all B^- such that B^- contains an explicit g-channel cut set for every g in $\overline{K}$, and contains no explicit g-channel cut set for any g in K. That is, each $\mathcal{B}_{\overline{K}}^{-K}$ is uniquely defined by the conditions (4.11-13). Thus, by virtue of (4.18) and (4.20), every T_-^G is also uniquely defined by the defining properties (1), (2), and (3) of T^G. The T_H^- are uniquely defined by (4.20) and

$$T_H = \sum_{K' \subset \overline{H}} T_{HK'}^{\overline{HK'}} . \tag{4.21}$$

D. Analytic Properties of T^G and $\overline{T}^{\overline{G}}$

For any g in G consider the Landau surface $L(D_g^+) = L(D_g^-)$ corresponding to the pair (D_g^+, D_g^-) of g-channel normal-threshold diagrams. [See fig. 4.1.] The diagram D_g^+ has a

positive-α g-channel cut set. Hence property (4.12) implies that no D_g^+ satisfies $D_g^+ \subset \mathcal{B}^G$. If only normal threshold diagrams need to be considered then the generalized $i\varepsilon$ rule stated below (4.9) says that the function T^G continues into itself around $L(D_g^+) = L(D_g^-)$ by the rule associated with the negative-α diagram D_g^-. This rule is the minus $i\varepsilon$ rule, which prescribes a detour into the lower-half s_g plane.

This argument extends immediately to a large class of singularities: property (4.12) excludes from T^G all singularities associated with diagrams D^σ that can be contracted to any positive-α diagram $D_{g'}^+$, for any g' in G.

By a similar argument $\overline{T}^G$ can have no singularities associated with diagrams that can be contracted to any D_g^- for any g in G. If $T^G = \overline{T}^{E-G}$, then this function has no singularities associated with any diagram D^σ having a positive-α g-channel cut set for any g in G, or having a negative-α g-channel cut set for any g in E-G.

This property is the basis of the proof in section VI of the analyticity properties of the 26,018 boundary values M^G. Before giving that proof we shall, in the following section, construct finite representations for the functions T^G and $\overline{T}^G$, and show that $T^G = \overline{T}^{E-G}$ for the values of G defined by (2.25)

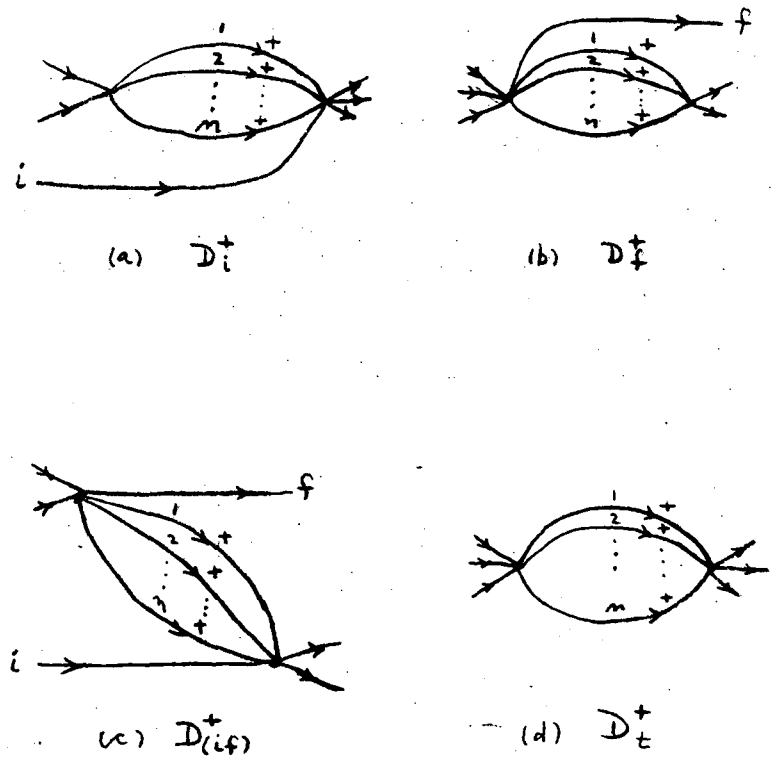


Fig. 4.1. All connected positive-α Landau diagrams with 3 incoming lines, 3 outgoing lines, and 2 vertices are shown. Line i is any one of the initial lines 1, 2, 3, and line f is any one of the final lines 4, 5, 6. The plus signs on the internal lines indicate that the corresponding Landau α's are positive. The number of the internal lines, n, is an arbitrary positive integer. The 6 diagrams of (a) and (b), the 9 diagrams of (c), and the diagram of (d) are called sub-energy diagrams, cross-energy diagrams, and total-energy diagram, respectively. The connected negative-α g-channel diagram D_g^- is obtained from D_g^+ by simply changing all the plus signs σ_j to minus signs.

V. CONSTRUCTION OF THE T^G

In this section a set of functions T^G is exhibited that satisfies the three properties listed in section IV. The procedure is to make an ansatz for the functions T_H, and then to show that these three properties hold. The ansatz is that

$$T_H = M_H \tag{5.1a}$$

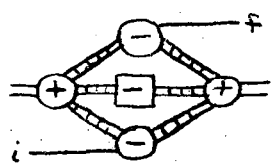
in all cases except those given by the formulas

$$T_{(if)t} = T_{i(if)t} = T_{(if)ft} = T_{i(if)ft}$$

$$= D_{if} \,. \tag{5.1b}$$

The functions M_H are defined by (2.7-14), and the functions D_{if} are defined by

D_{if} =



(5.1c)

Inspection of (5.1) and (2.11-14) shows that (4.11b) is satisfied. Thus if T^G is defined by (4.11a) then property (1) is satisfied. Property (3), i.e., eq. (4.13), follows from (4.11) and (5.1). Thus it only remains to prove property (2), which is that the expression for T^G given in (4.11) can be converted solely by means of unitarity and cluster properties to the form (4.12).

The conditions imposed by (4.12) can be compactly stated with the aid of the following

<u>Definition</u> A function F is said to belong to $\mathcal{R}^G$ if and only if F can be expressed, using only unitarity and cluster properties, in the form

$$F = \sum_{B\varepsilon \mathcal{B}^F} F^B \tag{5.2}$$

where no $D^\sigma \subset \mathcal{B}^F$ has a positive-α g-channel cut set for any g in G.

In terms of this definition (4.12) is the requirement that for every G the function T^G belongs to $\mathcal{R}^G$.

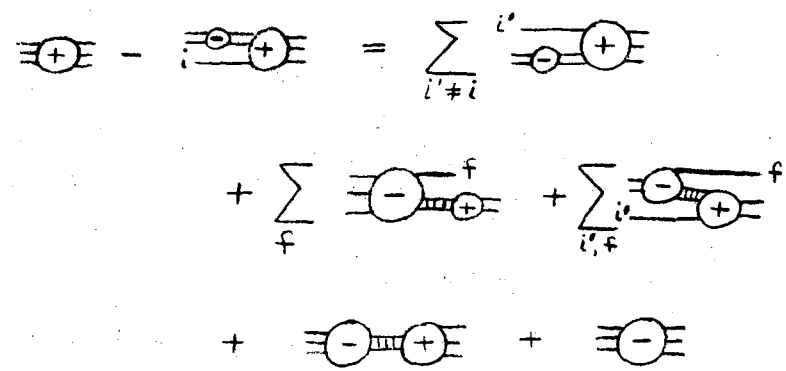
This property (4.12) must be proved for each of the 2^{16} possible sets G. The sixteen special cases in which G consists of a single element $G\varepsilon E$ are covered by

<u>Proposition 5.1</u> $T^g \equiv T - T_g$ belongs to $\mathcal{R}^g$.

<u>Proof</u>

<u>Case 1 g = i</u>

Unitarity (2.9d), and the cluster decomposition formulas (2.9i), (2.9j), and (2.9m) give



(5.3)

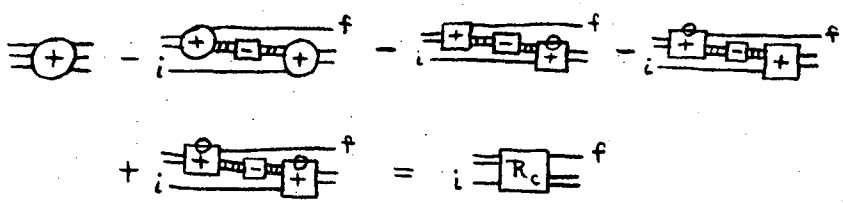
The left-hand side of (5.3) is $T - T_i$. The right-hand side is the required expression for T^i, for it is clear by inspection† that every term in this expression belongs to $\mathcal{R}^i$.

Case 2 $g = f$

The proof is essentially the same as for case 1: one merely uses the alternative form (2.9c) of unitarity, in which the minus box appears on the right-hand side instead of the left-hand side.

Case 3 $g = (if)$

Equation (5.9) of ref. [9], specialized to the present case, reads

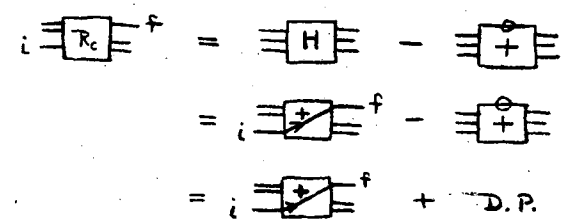


(5.4a)

† No $D^\sigma \subset B$ can have a positive-α g-channel cut set if any minus bubble of B touches external lines from both J_g and $\bar{J}_g$. And no $D^\sigma \subset B$ can have a positive-α g-channel cut set if B is not a connected diagram. These remarks will be used repeatedly in the arguments that follow, and will be indicated always by the dagger symbol that identifies this footnote.

where the function represented by the R_c box appearing on the right-hand side belongs, as explained below, to $\mathcal{R}^{(if)}$. The last three terms on the left-hand side of (5.4a) are disconnected, by virtue of (2.9o) and (2.9p), and hence cancel the disconnected terms on the right-hand side. This gives the required result.

The fact that the function represented by the R_c box in (5.4a) belongs to $\mathcal{R}^{(if)}$ follows from results of ref. [9]. Comparison of eqs. (5.6), (5.8), and (5.9) of [9] shows that



(5.4b)

where D.P. is a sum of disconnected parts and H is a sum of F^B over a set $\mathcal{B}$ of B with the property that for any $D^\sigma \subset \mathcal{B}$ the line i can be connected to line f by a directed path from i to f that consists of a sequence of L_j having the property that every one of these L_j with $\sigma_j = +$ or $\pm$ points in the direction of this path from i to f. This directed path is indicated in the

second line of (5.4b). [The lines L_j with $\sigma_j = -$ can be ignored]. The existence of such a path is implied by (5.7) of ref. [9]. It ensures that no $D^\sigma \subset \mathcal{B}$ has a positive-α (if)-channel cut set.

<u>Case 4 $g = t$</u>

Let A, B, and C denote certain bubble-diagram functions, and let A and C each be decomposed into a sum of two bubble-diagram functions so that

$$A = A' + A'' \tag{5.5a}$$

and

$$C = C' + C'' \ . \tag{5.5b}$$

Then one finds, trivially, that

$$ABC = A'BC + ABC' - A'BC' + A''BC'' \ . \tag{5.6}$$

Here the product form indicates the usual product of bubble-diagram functions so that, for example, in ABC the outgoing lines of the bubble diagram corresponding to A are identical with the ingoing lines of the bubble diagram corresponding to B.

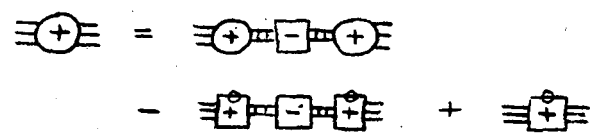
Consider the special case of (5.6) where A and C each represent the plus (minus) box and where B represents the minus (plus) box. Unitarity [Eq. (2.9c,d)] takes the form

$$AB = BC = I \ . \tag{5.7}$$

Substituting (5.7) into (5.6) one obtains

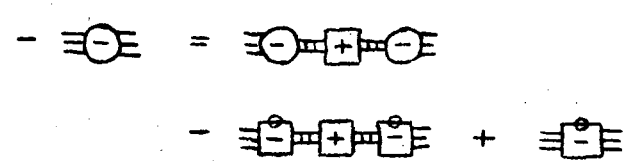
$$A = A' + C' - A'BC' + A''BC'' \ . \tag{5.8}$$

Suppose, specifically, that in (5.8) A and C represent the plus box and that B represents the minus box. Also let A' and C' each denote the circled plus box and let A" and C" each denote the plus bubble [so that eqns. (5.5a) and (5.5b) are a form of (2.9n)]. Then (5.8) takes the form



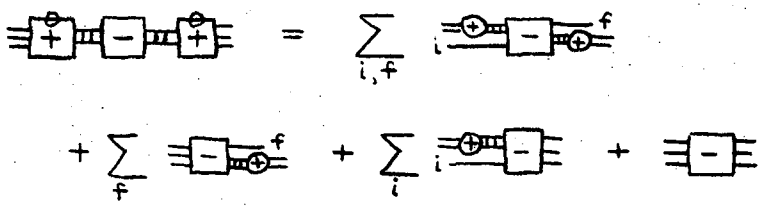
(5.9a)

Similarly, one finds



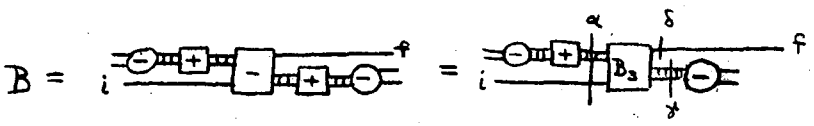
(5.9b)

Equations (2.9n), (2.9i), and (2.9j) show that



(5.10)

It is clear by inspection† that the last three terms of (5.10) belong to $\mathcal{R}^t$. The first term can be written with the aid of (2.9k,ℓ) in the form



(5.11)

where the vertical lines α, γ, and δ cut through the sets of lines that are to be identified with the sets of lines labelled by α, γ, and δ in (6.1) of [9], respectively, and β is identified with the empty set.

Suppose that B is replaced by an equivalent B' as is explained in corollary 6.1 of [9]. Let D^σ be any $D^\sigma \subset B'$ and suppose that D^σ has a positive-α t-channel cut set. Then the right end point of every line of γ must lie in $D^\sigma(J_t)$. This is true because no line of a positive-α cut set of D^σ can be an

outgoing line of the right minus bubble or an internal line of a negative-α Landau diagram D_b^- corresponding to this minus bubble. Similarly, line δ is in $D^\sigma(T_t)$. Hence, all the conditions required by the last sentence of Corollary 6.1 of [9] are met [To convert the notation of [9] to that of Case 4 of the present paper use the correspondence: $\omega = \omega_1 \to \{1,2,3\}$, $\omega' = \omega_2' \to \{4,5,6\}$, connected $\overline{X}(C,\omega') \to D^\sigma(J_t)$, connected $\overline{X}(C,\omega) \to D^\sigma(\overline{J}_t)$, simple positive (ω,ω') cut set with $\overline{X}(C,\omega')$ and $\overline{X}(C,\omega)$ connected → positive-α t-channel cut set.]. Thus, if that part of D^σ that corresponds to B_3' (B_3' is equivalent to B_3) is denoted by D_3^σ, then all points of $D_3^\sigma - \alpha$ lie in $D^\sigma(J_t)$. This means that the part of D^σ that corresponds to the part of B' lying to the right of the set of lines α must lie in $D^\sigma(J_t)$. But then the remaining part of D^σ must be disconnected, and the existence of a positive-α t-channel cut set is precluded.

Proposition 5.1 is a special case of

<u>Proposition 5.2</u> The function

$$T^G = \sum_{H \subset G} (-1)^{n(H)} T_H \tag{5.12}$$

can be converted by means of the unitarity and cluster decomposition properties of S to the form

$$T^G = \sum_{B \in \mathcal{B}^G} F^B , \tag{5.13}$$

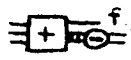
where no $D^\sigma \subset \mathcal{B}^G$ has a positive-α g-channel cut set for any g in G.

That is, the T^G defined by (5.12), with the T_H defined by (5.1), belongs to $\mathcal{R}^G$.

<u>Proof</u> Several quantities that occur often in the proof are defined as follows:

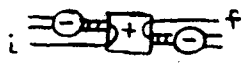
$A =$ (5.14a)

$A_i =$ (5.14b)

$A_f =$  (5.14c)

$A_{if} =$

$=$

$+$  $+$ D.P.

$$= T^i_{(if)} + T^f_{(if)} + \overline{T}_{(if)} + T_{if} + \text{D.P.} \qquad (5.14d)$$

where D.P. stands, in general, for any sum of disconnected parts. The first two quantities appearing in the last line of (5.14d) are defined by

$$T^i_{(if)} \equiv T_{(if)} - T_{i(if)}$$

$$\equiv M_{(if)} - M_{i(if)}$$

$\equiv M^i_{(if)} =$ (5.14e)

where the calculation in (2.15b) is used, and

$$T^f_{(if)} \equiv T_{(if)} - T_{(if)f}$$

$$\equiv M_{(if)} - M_{(if)f}$$

$\equiv M^f_{(if)} =$ (5.14f)

where the calculation in (2.15a) is used. The second and third quantities in the last line of (5.14d) are defined by (5.1), (2.11c), (2.13a), and the definition

$$\overline{F^B} = -(F^B)^\dagger$$

$$= -(-1)^{N_b} F^B(+ \leftrightarrow -) \ , \qquad (5.14g)$$

where N_b is the number of (explicitly appearing) bubbles in B and $F^B(+ \leftrightarrow -)$ is the bubble diagram function for the bubble

diagram $B(+ \leftrightarrow -)$ obtained from B by changing the sign inside each bubble, box, and modified box.

Each of the above equations can be converted to another one by the application of bars to each term. For example, from (5.14d) one obtains

$$\overline{A}_{if} = \overline{T}^{i}_{(if)} + \overline{T}^{f}_{(if)} + T_{(if)} + \overline{T}_{if} + \text{D.P.} \quad . \tag{5.14h}$$

Using (5.14), (4.13), (2.10), and (5.1a) one may write (5.9a) and (5.10) in the form

$$T = T_t + \overline{A} - \sum_i \overline{A}_i - \sum_f \overline{A}_f + \sum_{i,f} \overline{A}_{if} + \text{D.P.} \quad . \tag{5.15a}$$

From (5.9b) one obtains

$$\overline{T} = \overline{T}_t + A - \sum_i A_i - \sum_f A_f + \sum_{i,f} A_{if} + \text{D.P.} \quad . \tag{5.15b}$$

For any G in E this equation can be written as

$$A - \sum_{i\varepsilon G} A_i - \sum_{f\varepsilon G} A_f + \sum_{i\varepsilon G, f\varepsilon G} A_{if} = F_1 \tag{5.16a}$$

where

$$F_1 \equiv \overline{T} - \overline{T}_t + \sum_{i\varepsilon\overline{G}} A_i + \sum_{f\varepsilon\overline{G}} A_f - \sum_{i\varepsilon\overline{G},f} A_{if} - \sum_{f\varepsilon\overline{G},i} A_{if}$$

$$+ \sum_{i\varepsilon\overline{G}, f\varepsilon\overline{G}} A_{if} + \text{D.P.} \quad . \tag{5.16b}$$

The unrestricted sums over i or f are sums over all three values of i or f. As mentioned earlier, the index i is always restricted in this paper to the values 1,2, and 3, and the index f is restricted to 4,5, and 6. A sum over $i\varepsilon G$ is a sum over those indices $i = g\varepsilon G$ that label initial subenergy channels, and the sum over $i\varepsilon\overline{G}$ is the sum over the remaining indices in the set (1,2,3).

From (5.14b), the cluster expansion (2.9j), and (5.14e) one obtains

$$A_i = \sum_f T^{i}_{(if)} + T_i + \text{D.P.} \quad . \tag{5.17a}$$

Under the substitution $F \to \overline{F}$ this equation becomes

$$\overline{A_i} = - \sum_f T^{f}_{(if)} + \overline{T_i} + \text{D.P.} \tag{5.17b}$$

where we have used (5.14e) and (5.14f). Similarly,

$$A_f = \sum_i T^{f}_{(if)} + T_f + \text{D.P.} \tag{5.18a}$$

and

$$\overline{A_f} = - \sum_i T^{i}_{(if)} + \overline{T_f} + \text{D.P.} \quad . \tag{5.18b}$$

Substituting the right-hand sides of (5.17a), (5.18a), and (5.14d) into the second, third, and fourth terms on the left-hand side of (5.16a), respectively, one obtains

$$T - \sum_{i\varepsilon G} T_i - \sum_{f\varepsilon G} T_f + \sum_{i\varepsilon G, f\varepsilon G} T_{if} - \sum_{i\varepsilon G, f\varepsilon \overline{G}} T^i_{(if)}$$

$$- \sum_{i\varepsilon \overline{G}, f\varepsilon G} T^f_{(if)} + \sum_{i\varepsilon G, f\varepsilon G} \overline{T}_{(if)} = F_1 \quad . \tag{5.19}$$

The equation

$$- \overline{T}_{(if)} = T_{(if)} - T_{i(if)} - T_{(if)f} + T_{i(if)f} \tag{5.20}$$

was proved in section II [See (2.21) and (2.22)]. This equation yields trivially

$$- \sum_{i\varepsilon G, f\varepsilon G} \overline{T}_{(if)} = \sum_{i\varepsilon G, f\varepsilon G, (if)\varepsilon G} \Big[T_{(if)} - T_{i(if)} - T_{(if)f}$$

$$+ T_{i(if)f} \Big] - \sum_{i\varepsilon G, f\varepsilon G, (if)\varepsilon \overline{G}} \overline{T}_{(if)} \quad , \tag{5.21}$$

where the condition $(if)\varepsilon G$ or $(if)\varepsilon \overline{G}$ under the summation sign means that there is a sum over the pairs (i,f), and that this sum is to be restricted both by any other appearing conditions on i or f, and also by the condition that (if) be an element of G or $\overline{G}$, respectively. One also finds trivially from (5.14e) and (5.14f) that

$$\sum_{i\varepsilon G, f\varepsilon \overline{G}} T^i_{(if)} = \sum_{i\varepsilon G, f\varepsilon \overline{G}, (if)\varepsilon G} \Big[T_{(if)} - T_{i(if)} \Big]$$

$$+ \sum_{i\varepsilon G, f\varepsilon \overline{G}, (if)\varepsilon \overline{G}} T^i_{(if)} \tag{5.22}$$

$$\sum_{i\varepsilon \overline{G}, f\varepsilon G} T^f_{(if)} = \sum_{i\varepsilon \overline{G}, f\varepsilon G, (if)\varepsilon G} \Big[T_{(if)} - T_{(if)f} \Big]$$

$$+ \sum_{i\varepsilon \overline{G}, f\varepsilon G, (if)\varepsilon \overline{G}} T^f_{(if)} \quad . \tag{5.23}$$

Substituting (5.21), (5.22), and (5.23) into (5.19) one obtains

$$T + F_2 = F_3 \tag{5.24}$$

where

$$F_2 \equiv - \sum_{i\varepsilon G} T_i - \sum_{f\varepsilon G} T_f + \sum_{i\varepsilon G, f\varepsilon G} T_{if} - \sum_{(if)\varepsilon G} T_{(if)}$$

$$+ \sum_{i\varepsilon G, (if)\varepsilon G} T_{i(if)} + \sum_{f\varepsilon G, (if)\varepsilon G} T_{(if)f}$$

$$- \sum_{i\varepsilon G, f\varepsilon G, (if)\varepsilon G} T_{i(if)f} \tag{5.25a}$$

and

$$F_3 \equiv F_1 - \sum_{i\varepsilon\bar{G},f\varepsilon\bar{G},(if)\varepsilon G} T_{(if)} - \sum_{i\varepsilon G,f\varepsilon G,(if)\varepsilon\bar{G}} \bar{T}_{(if)}$$

$$+ \sum_{i\varepsilon G,f\varepsilon\bar{G},(if)\varepsilon\bar{G}} T^{i}_{(if)} + \sum_{i\varepsilon\bar{G},f\varepsilon G,(if)\varepsilon\bar{G}} T^{f}_{(if)} \quad . \qquad (5.25b)$$

At this point it is convenient to consider separately two different cases.

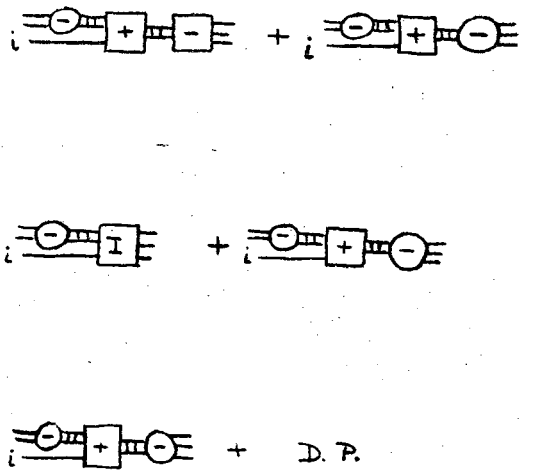
<u>Case 1 ($t\varepsilon\bar{G}$)</u>

Suppose G does not contain t. Then (5.24) becomes

$$\sum_{H\subset G} (-1)^{n(H)} T_H \equiv T^G = F_3 \quad : \qquad (5.26)$$

Indeed, the terms appearing on the left-hand side of (5.24) are identical to those appearing on the left-hand side of (5.26), by virtue of the fact that all the $T_H = M_H$ for $H \subset G$ not explicitly listed in (2.10), (2.11), (2.13), or (2.14) vanish. Thus the proof for case 1 can be completed by showing that F_3 belongs to $\mathcal{R}^G$.

The cluster property (2.9j) and unitarity give

$A_i - \sum_f A_{if} =$ 

(5.27)

Similarly,

$A_f - \sum_i A_{if} =$

(5.28)

After substituting the right-hand sides of (5.27) and (5.28) into (5.16b), one sees† that

$$F_1 - \sum_{i\varepsilon\bar{G},f\varepsilon\bar{G}} A_{if} \qquad (5.29)$$

belongs to $\mathcal{R}^G$.

After replacing the plus bubbles in $T^{i}_{(if)}$ and in $T^{f}_{(if)}$ by the left-hand sides of (2.9k) and (2.9ℓ) one sees† that the last three terms on the right-hand side of (5.25b) belong to $\mathcal{R}^G$.

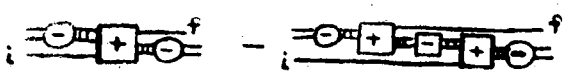
Hence, F_3 belongs to $\mathcal{R}^G$ if

$$\sum_{i\varepsilon\bar{G},f\varepsilon\bar{G}} A_{if} - \sum_{i\varepsilon\bar{G},f\varepsilon\bar{G},(if)\varepsilon G} T_{(if)} = \sum_{i\varepsilon\bar{G},f\varepsilon\bar{G},(if)\varepsilon\bar{G}} A_{if}$$

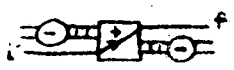
$$+ \sum_{i\varepsilon\bar{G},f\varepsilon\bar{G},(if)\varepsilon G} \left(A_{if} - T_{(if)}\right) \qquad (5.30)$$

belongs to $\mathcal{R}^G$. One sees† from (5.14d) that this is true for the first term on the right-hand side of (5.30).

From (5.6) of [9] one obtains

$A_{if} - T_{(if)} =$ 

$=$

$\equiv$  (5.31)

The H-box is the expression given in (5.7) of [9]. It is a sum over a set $\mathcal{B}$ of bubble diagrams B such that any $D^\sigma \subset B \varepsilon \mathcal{B}$ contains a path $\mathcal{P}$ from i to f that consists of lines L_j with the following property: each line L_j with sign $\sigma_j = +$ or $\pm$

points in the direction of $\mathcal{P}$, i.e., from i to f. This property is indicated by the internal line in the box appearing in the last line of (5.31). This line precludes the existence of a positive-α (if)-channel cut set in the $A_{if} - T_{(if)}$. Thus one sees† from (5.31) that the second term on the right-hand side of (5.30) belongs to $\mathcal{R}^G$. This completes the proof for case 1.

Case 2 ($t\varepsilon G$)

Suppose G contains t. The equation

$$-\overline{T_t} = T_t - \sum_i T_{it} - \sum_f T_{ft} + \sum_{i,f} T_{itf} \qquad (5.32)$$

is now needed.

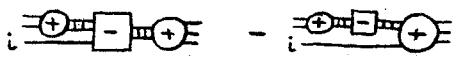
Proof of (5.32)

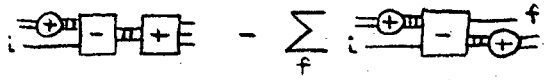
Equations (5.15a) and (5.15b) give

$$-\overline{T_t} = T_t - \sum_i \overline{A_i} - \sum_f \overline{A_f} + \sum_{i,f} \overline{A}_{if} - \sum_i A_i - \sum_f A_f$$

$$+ \sum_{i,f} A_{if} + \text{D.P.} \quad . \qquad (5.33)$$

Using (5.1), (2.13b), (2.12a), (2.9j), (2.9k), (5.14), and (2.11a) one obtains

$- T_{it} =$ 

$=$ 

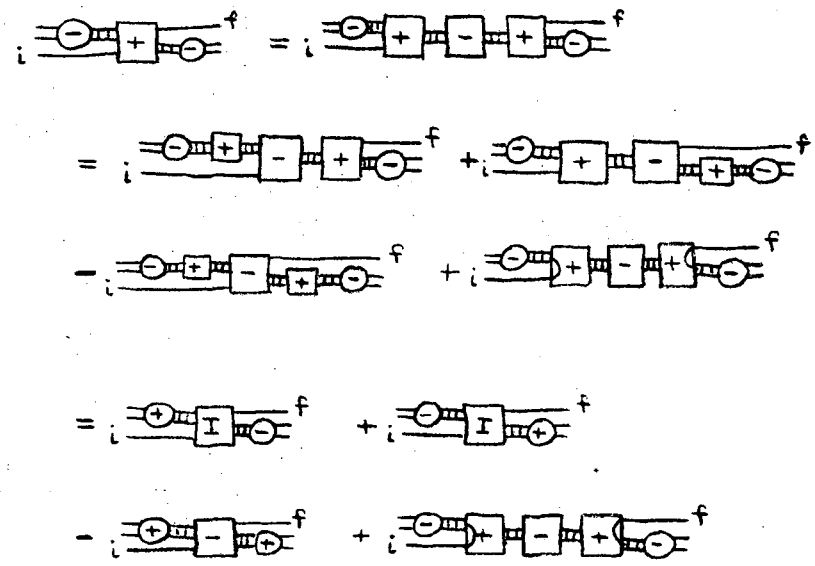
$-$

$$= \sum_f \overline{A}_{if} - \overline{A}_i - T_i + \text{D.P.} \quad . \tag{5.34}$$

Similarly,

$$- T_{ft} = \sum_i \overline{A}_{if} - \overline{A_f} - T_f + \text{D.P.} \quad . \tag{5.35}$$

By virtue of (2.9c), (5.6), (2.12a), (2.12b), (2.9k), and (2.9ℓ) one has



(5.36)

so that, by (5.14), (5.1), and (2.14a),

$$T_{itf} = - T^i_{(if)} - T^f_{(if)} + A_{if} - \overline{A}_{if} + \text{D.P.} \quad . \tag{5.37}$$

Combining (5.33), (5.34), (5.35), (5.37), and using (5.17a) and (5.18a) one obtains (5.32).

By virtue of (5.34), (5.35), and (5.37) one may write (5.32) in the form

$$- \overline{T}_t = - F_4 + \sum_{i\varepsilon\overline{G},f} \overline{A}_{if} - \sum_{i\varepsilon\overline{G}} \overline{A}_i - \sum_{i\varepsilon\overline{G}} T_i + \sum_{i,f\varepsilon\overline{G}} \overline{A}_{if}$$

$$- \sum_{f\varepsilon\overline{G}} \overline{A}_f - \sum_{f\varepsilon\overline{G}} T_f$$

$$+ \left(\sum_{i\varepsilon\overline{G},f} + \sum_{i,f\varepsilon\overline{G}} - \sum_{i\varepsilon\overline{G},f\varepsilon\overline{G}} \right) \left(- T^i_{(if)} - T^f_{(if)} + A_{if} - \overline{A}_{if} \right)$$

$$+ \text{ D.P.} \tag{5.38}$$

where

$$F_4 \equiv - T_t + \sum_{i\varepsilon G} T_{it} + \sum_{f\varepsilon G} T_{ft} - \sum_{i\varepsilon G,f\varepsilon G} T_{itf} \quad . \tag{5.39}$$

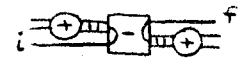
Consider the term $\sum_{i\varepsilon\overline{G},f\varepsilon\overline{G}} \overline{A}_{if}$ appearing in (5.38). Equations (5.14e,f, and h) allow it to be written as

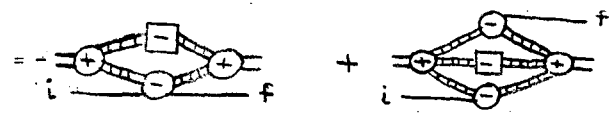
$$\sum_{i\varepsilon\overline{G},f\varepsilon\overline{G}} \overline{A}_{if} = \sum_{i\varepsilon\overline{G},f\varepsilon\overline{G},(if)\varepsilon\overline{G}} \overline{A}_{if}$$

$$- \sum_{i\varepsilon\overline{G},f\varepsilon\overline{G},(if)\varepsilon G} \left(T^i_{(if)} + T^f_{(if)} - T_{(if)} - \overline{T}_{if} \right)$$

$$+ \text{ D.P.} \quad . \tag{5.40}$$

The term $- \overline{T}_{if}$ can be decomposed into two parts:

$- \overline{T}_{if} =$ 

$=$ 

$$= C_{if} + D_{if} \quad . \tag{5.41}$$

The term C_{if} is the part of $- \overline{T}_{if}$ in which the two lines i and f touch a single minus bubble, and D_{if} is the part in which the two lines i and f do not touch a single minus bubble.

Substituting (5.40) into (5.38) gives an expression for $- \overline{T}_t^-$ that can be introduced into the expression (5.16b) for F_1. Substituting this new expression for F_1 into (5.25b) and using (5.17a), (5.17b), (5.18a), (5.18b), and (5.41) one finds that Eq. (5.24) takes the form

$$T + F_2 + F_4 + F_5 = F_6 \tag{5.42}$$

where

$$F_5 \equiv \sum_{i\varepsilon\overline{G},f\varepsilon\overline{G},(if)\varepsilon G} D_{if} \equiv D^G \tag{5.43}$$

and

$$F_6 \equiv \overline{T} - \sum_{i\varepsilon\overline{G}} \overline{T}_i - \sum_{f\varepsilon\overline{G}} \overline{T}_f + \sum_{f\varepsilon\overline{G},(if)\varepsilon\overline{G}} T^i_{(if)}$$

$$+ \sum_{i\varepsilon\overline{G},(if)\varepsilon\overline{G}} T^f_{(if)} - \sum_{i\varepsilon G,f\varepsilon G,(if)\varepsilon\overline{G}} \overline{T}_{(if)}$$

$$+ \sum_{i\varepsilon\overline{G},f\varepsilon\overline{G},(if)\varepsilon\overline{G}} \overline{A}_{if} - \sum_{i\varepsilon\overline{G},f\varepsilon\overline{G},(if)\varepsilon G} C_{if}$$

$$+ \text{D.P.} \quad . \tag{5.44}$$

[The function D^G is defined by (5.43) only for the present case $t\varepsilon G$. See (5.49).]

According to (5.1)

$$T_{(if)t} = T_{i(if)t} = T_{(if)ft} = T_{i(if)ft} = D_{if} \quad . \tag{5.45}$$

A trivial consequence of (5.45) and (5.43) [for the present $t\varepsilon G$] is that

$$D^G = \sum_{(if)\varepsilon G} T_{(if)t} - \sum_{i\varepsilon G,(if)\varepsilon G} T_{i(if)t}$$

$$- \sum_{f\varepsilon G,(if)\varepsilon G} T_{(if)ft} + \sum_{i\varepsilon G,f\varepsilon G,(if)\varepsilon G} T_{i(if)ft} \quad . \tag{5.46}$$

If one uses the representation (5.46) of F_5, then, for G containing t,

$$T + F_2 + F_4 + F_5 = \sum_{H\subset G} (-1)^{n(H)} T_H \quad . \tag{5.47}$$

Thus, the proof of Proposition 5.2 for Case 2 can be completed by showing that F_6 belongs to $\mathcal{R}^G$.

Inspection† of the bubble diagrams B that correspond to the bubble-diagram functions F^B of F_6 [see (5.14), (2.11), and (5.41)] shows that all terms belong to $\mathcal{R}^G$ except possibly

$$\sum_{i\varepsilon\overline{G},f\varepsilon\overline{G},(if)\varepsilon\overline{G}} \overline{A}_{if} \quad . \tag{5.48}$$

The function $\overline{A}_{if}$ is defined in (5.14d,g). Introducing the expression (5.11), and making use of the argument of Case 4 of Proposition 5.1, one sees that this term is also in $\mathcal{R}^G$. This completes the proof of Proposition 5.2.

<u>Proposition 5.3</u>

$$T^G - D^G = \overline{T}^{\overline{G}} - \overline{D}^{\overline{G}} \tag{5.49}$$

where D^G is defined by (5.46) if G contains t, and by $D^G = 0$ if G does not contain t. The quantities $\overline{T}^{\overline{G}}$ and $\overline{D}^{\overline{G}}$ are defined for all sets $\overline{G}$ by

$$\overline{T}^{\overline{G}} \equiv -(T^{\overline{G}})^{\dagger} \tag{5.50a}$$

and

$$\overline{D}^{\overline{G}} \equiv -(D^{\overline{G}})^{\dagger} \quad . \tag{5.50b}$$

<u>Proof</u> It is sufficient to prove (5.49) for the case in which G contains t. For the application of hermitian conjugation and a sign change to (5.49) gives,

$$\overline{T}^{G} - \overline{D}^{G} = T^{\overline{G}} - D^{\overline{G}} \quad ,$$

which is (5.49) with G replaced by $\overline{G}$.

Applying this same transformation to (5.20), and performing a summation, one obtains

$$-\sum_{i\varepsilon\overline{G},f\varepsilon\overline{G},(if)\varepsilon\overline{G}} T_{(if)} =$$

$$= \sum_{i\varepsilon\overline{G},f\varepsilon\overline{G},(if)\varepsilon\overline{G}} \left(\overline{T}_{(if)} - \overline{T}_{i(if)} - \overline{T}_{(if)f} + \overline{T}_{i(if)f}\right) . \tag{5.51}$$

Applying this same transformation to (5.14e) and (5.14f) one obtains

$$-T^{f}_{(if)} = \overline{T}_{(if)} - \overline{T}_{i(if)} \tag{5.52}$$

and

$$-T^{i}_{(if)} = \overline{T}_{(if)} - \overline{T}_{(if)f} \quad . \tag{5.53}$$

Consider the case in which G contains t. If on the right-hand side of (5.44) one substitutes for $\overline{A}_{if}$ the right-hand side of

(5.14h) and then uses (5.51), (5.52), (5.53), and (5.41), one finds that

$$F_6 = \overline{T} - \sum_{i\varepsilon\overline{G}} \overline{T}_i - \sum_{f\varepsilon\overline{G}} \overline{T}_f - \sum_{(if)\varepsilon\overline{G}} \overline{T}_{(if)} + \sum_{i\varepsilon\overline{G},f\varepsilon\overline{G}} \overline{T}_{if}$$

$$+ \sum_{i\varepsilon\overline{G},(if)\varepsilon\overline{G}} \overline{T}_{i(if)} + \sum_{f\varepsilon\overline{G},(if)\varepsilon\overline{G}} \overline{T}_{(if)f}$$

$$- \sum_{i\varepsilon\overline{G},f\varepsilon\overline{G},(if)\varepsilon\overline{G}} \overline{T}_{i(if)f} + \sum_{i\varepsilon\overline{G},f\varepsilon\overline{G},(if)\varepsilon G} D_{if}$$

$$= \overline{T}^{\overline{G}} + D^{G} \quad . \tag{5.54}$$

Inserting (5.54) into (5.42) and using (5.47) one obtains the required (5.49).

<u>The Functions M^G</u>

The function T^G was defined by (5.12) as

$$T^{G} \equiv \sum_{H\subset G} (-1)^{n(H)} \; T_H \quad . \tag{5.55}$$

The function M^G was defined by (2.5a) as

$$M^{G} \equiv \sum_{K\subset G} (-1)^{n(K)} \; M_K \quad . \tag{5.56}$$

According to (5.1) $T_H = M_H$ except when H contains, for some (i,f), one of the four sets $\{(if),t\}$, $\{i,(if),t\}$, $\{(if),t,f\}$, $\{i,(if),f,t\}$. Hence, by (5.55) and (5.56) $T^G = M^G$ if G does not include t. If G does include t, then (5.47) together with the definitions of F_2 [see (5.25a)], F_4 [see (5.39)], and F_5 [see (5.43) and (5.46)] show that

$$T^G = M^G + F_5 \quad .$$

Since $D^G = 0$ if G does not include t and is equal to F_5 if G does include t, the above result, can be summarized in the equation

$$T^G = M^G + D^G \quad . \tag{5.57}$$

Then (5.49) gives, for all G,

$$M^G = \overline{M}^{\overline{G}} = -(M^{\overline{G}})^\dagger \quad , \tag{5.58}$$

which is (2.17).

<u>Remark</u> Let the G in (5.58) be the complete set E. Then (5.58) says that M^E, the function evaluated below all the cuts, is

$$M^E = \overline{M}^\phi \equiv \overline{M} \equiv -M^\dagger \quad . \tag{5.59}$$

This result is called the Hermitian analyticity property of the scattering function. The more general property (5.58) says that the function M^G that corresponds to M plus the discontinuities that take the function to its value below the set of cuts g in G, is equal to the function $\overline{M}$ plus the discontinuities that take the function to above the set of cuts g in $\overline{G}$. The analogous result

with T in place of M does not hold in general, as is shown by (5.49).

The function D^G is defined to be zero if t lies in $\overline{G}$. For t in G it can be expressed in the form (5.43). Thus for the set of 26,018 values of G defined by (2.25) one finds

$$D^G = \overline{D}^{\overline{G}} = 0 \quad . \tag{5.60a}$$

In these cases one has, by virtue of (5.57) and (5.58),

$$M^G = T^G = \overline{T}^{\overline{G}} \quad . \tag{5.60b}$$

The analytic properties of these functions M^G are discussed in section VI. It is already evident that these functions M^G satisfy the properties (4), (5), and (6) described in the introduction.

VI. ANALYTIC PROPERTIES OF THE M^G

A. Geometric Representations of Landau Diagrams

The proofs of analyticity properties given in this section are based on the existence of two different geometric representations of Landau diagrams. These two representations are discussed in this subsection.

Each internal line L_j of a Landau diagram has a well-defined direction: L_j is directed from the vertex V_r with $\varepsilon_{jr} = -1$ to the vertex V_r with $\varepsilon_{jr} = +1$. This direction is the direction of flow of positive energy. An arrow is often placed on L_j to indicate this direction, and L_j is said to <u>point</u> in the direction of this arrow, i.e., from the vertex V_r with $\varepsilon_{jr} = -1$ to the vertex V_r with $\varepsilon_{jr} = +1$. The external lines L_j are also directed: each incoming line is directed toward a vertex V_r with

$\varepsilon_{jr} = +1$; each outgoing line L_j is directed away from a vertex V_r with $\varepsilon_{jr} = -1$. [One can introduce trivial two-line vertices to take care of the trivial cases in which a line goes straight through the diagram without touching any nontrivial vertex.]

The first geometric representation of a Landau diagram D^σ is the space-time representation discussed in section IV. Each space-time representation of D^σ represents a particular solution of the Landau equations corresponding to D^σ, and corresponds to some particular point p on $L(D^\sigma)$. In this representation each internal line L_j of D^σ is represented by a space-time four-vector $\Delta_j = \alpha_j p_j$. The Landau equation $p_j^0 > 0$ entails that Δ_j point in the direction of increasing time if σ_j is plus, and in the direction of decreasing time if σ_j is minus. That is, the vertex V_r with $\varepsilon_{jr} = +1$ lies later than the vertex V_r with $\varepsilon_{jr} = -1$ if $\sigma_j = +$, but lies earlier if $\sigma_j = -$.

These conditions on the directions of the four-vectors Δ_j impose a partial ordering requirement on the vertices of the Landau diagram D^σ. In particular, for any positive-α diagram D^+ with nonempty $L(D^+)$ the vertices must satisfy the partial ordering condition that the diagram can be drawn so that each internal line segment L_j points from left to right. Likewise, for any negative-α diagram D^- with nonempty $L(D^-)$ the vertices must also satisfy the partial ordering condition that the diagram can be drawn so that each internal line segment L_j points from left to right. To see this, one simply orders the vertices of the Landau diagram D^+ from left to right in accordance with the increasing time of the vertices of the space-time representations of D^+, and orders the vertices of D^-

from left to right in accordance with decreasing time of the vertices of the space-time representations of D^-.

Consider now any bubble diagram B. By definition this diagram can be drawn so that every (explicit) internal line runs from the right-hand side of one bubble to the left-hand side of a bubble that stands completely to the right of the first bubble. Thus each line of B can be drawn as a line that points from left to right.

Consider next any $D^\sigma \subset B$. If one orders the bubbles of B in the way just described, so that all explicit lines of B point from left to right, and then draws each D_b^+ and D_b^- of $D^\sigma \subset B$ as a small diagram lying completely inside the corresponding bubble b, with all of its internal lines pointing from left to right, then all the lines of $D^\sigma \subset B$ will point from left to right. [See (4.7b)]

The representation of a $D^\sigma \subset B$ as a diagram in which every line L_j points from left to right is called a flow diagram: positive energy flows always from left to right in a flow diagram. This uniformity of directions of the lines of a flow diagram is to be contrasted with the nonuniformity of directions in the space-time representations of D^σ. In the space-time representations the vectors Δ_j point in the direction of increasing time if σ_j is plus, and in the direction of decreasing time if σ_j is minus. For any $D^\sigma \subset B$ it must be possible to draw both a flow-diagram representation of D^σ and also a space-time representation of D^σ, if $L(D^\sigma)$ is nonempty.

The constraints imposed on D^σ by the existence of the flow diagram representation can be expressed in terms of the concept of a flow line.

Definition 6.1 A flow line is an ordered sequence of internal line segments L_j of a Landau diagram such that the leading end point $(\varepsilon_{jr} = +1)$ of any L_j in the sequence except the last one is the trailing end point $(\varepsilon_{jr} = -1)$ of the next one in the sequence. Thus positive energy flows always in the same direction along a flow line: it flows from the trailing vertex of the first L_j of the sequence to the leading end point of the final L_j. These two vertices are called the initial and final vertices of the flow line. In the flow diagram representation of D^σ the final vertex of any flow line stands to the right of the initial vertex of that flow line.

Definition 6.2 A flow line $V_r \to V_s$ is a flow line with initial vertex V_r and final vertex V_s.

Definition 6.3 A maximal flow line is a flow line that is not a proper subsequence of any other flow line.

B. Space-time Ordering Theorem

The proof of the analytic properties of M^G is based on a theorem proved in this subsection. This theorem depends on two lemmas.

Lemma 6.1 Let V_r and V_s be two distinct vertices of a connected Landau diagram D. Suppose X is a set of lines of D such that the cutting precisely once of each line of X separates D into two disjoint diagrams D_r and D_s, where V_r lies in D_r and V_s lies in D_s. Then there is a subset X^c of X such that the cutting precisely once of each line of X^c separates D into two diagrams D_r^c and D_s^c such that D_r^c is a connected diagram that contains V_r, and D_s^c is a connected diagram that contains V_s, and the two diagrams D_r^c and D_s^c exhaust D.

Proof The cutting of the lines of the set X separates D into a number of connected parts. Let $D_s^{c'}$ be the connected part containing V_s. Let X_s be the set consisting of the lines of X that lie partly in $D_s^{c'}$. Let X^c be the subset of X_s that consists of each line L_j such that the cut in L_j can be reached from V_r by a path in D that is not cut by the set of cuts in the lines of X_s. This set X^c is the desired set: cutting precisely once each line of X^c separates D into the two connected parts D_r^c and D_s^c.

It is clear that cutting the lines of X^c disconnects the part of D that is connected to V_r from the part that is connected to V_s. For any path in D from V_r to V_s would have to enter $D_s^{c'}$ at the cut on some line of X_s. But the first such cut reached by this path must be a cut on a line of X^c. Thus the cutting of the lines of X^c definitely separates D into at least two connected parts D_r^c and D_s^c. What must now be shown is that these two parts exhaust D.

Consider the diagram D cut on the lines of X^c. Suppose there is a point x that is not connected in this cut diagram to either V_r or V_s. This point x is connected in the original connected diagram D to the point V_r by some path P. Since x is, by assumption, not connected to V_r in the cut diagram the path P must pass through the cut in at least one line of X^c. Let C be the first cut in X^c reached on the path P from x to V_r. Then x must be connected in the cut diagram to one side or the other of the cut C. But each cut in the lines of X^c is connected in the cut diagram on one side to V_s, since X^c is a subset of X_s and on the other side to V_r, since each cut of X^c can be connected to

V_r by a path not cut by any cut in X_s. Thus x must be connected in the cut diagram to either V_r or V_s, contrary to the original assumption about x. Thus no such x can exist. This means that the two connected parts D_r^c and D_s^c of the cut diagram exhaust it.

Lemma 6.2 Let $V_r \to V_s$ be a flow line of a flow diagram D^σ. Let X be the set of lines of D^σ that are cut by a plane T that lies perpendicular to the flow axis, that lies between V_r and V_s, and that touches no vertices of D^σ. Then the subset X^c of X of lemma 6.1 is such that each L_j in X^c has its leading end point in D_s^c and its trailing end point in D_r^c.

Proof The plane T cuts D^σ into the parts D_r and D_s, where D_r lies to the left of T and D_s lies to the right of T. The construction in lemma 6.1 ensures that $D_s^{c'}$ is a subset of D_s. Hence $D_s^{c'}$ also lies to the right of T. Moreover, X^c is a subset of the set X_s of lines of X that touch $D_s^{c'}$. Thus every line of X^c touches $D_s^{c'}$, and hence has its leading end point in $D_s^{c'}$, which is contained in D_s^c. Thus the trailing end points of the $L_j \varepsilon X^c$ must lie in D_r^c.

Definition 6.4 Let V_r and V_s be two vertices of a Landau diagram D^σ. Let η be a plus or minus sign. A $V_r \overset{\eta}{\to} V_s$ cut set of D^σ is a set of internal lines of D^σ such that D^σ is separated by the cutting precisely once of every line of this set into two connected diagrams $D^\sigma(V_r)$ and $D^\sigma(V_s)$, where V_r lies in $D^\sigma(V_r)$ and V_s lies in $D^\sigma(V_s)$, and such that each line L_j of the cut set points from $D^\sigma(V_r)$ to $D^\sigma(V_s)$, and has a sign $\sigma_j = \eta$ or $\pm$.

Theorem 6.1 Let B be any connected bubble diagram. Let D^σ be any $D^\sigma \subset B$ that has a flow line $V_r \to V_s$. Let η be either plus or minus. Suppose D^σ has no $V_r \overset{\eta}{\to} V_s$ cut set. Then for every space-time representation of D^σ the vectors ω_r and ω_s to the vertices V_r and V_s satisfy

$$\omega_s - \omega_r \varepsilon V^{-\eta} \quad , \tag{6.1}$$

where V^+ is the open forward light cone and V^- is the open backward light cone.

Proof Suppose η is plus. And suppose that the points V_r and V_s both lie inside some single minus bubble b (i.e., they are vertices of the D_b^- that replaces b in the construction of $D^\sigma \subset B$). In this case the flow line $V_r \to V_s$ must consist wholly of line segments L_j that also lie inside this minus bubble, and hence have signs $\sigma_j = -$. This is because the condition that the bubbles of B be partially ordered precludes the possiblity that a flow line $V_r \to V_s$ begin and end in the same b, but pass outside b. But if the lines L_j of $V_r \to V_s$ all carry minus signs then the Landau equations that define the space-time representations of D^σ entail that each of the corresponding Δ_j point into the backward light-cone. Thus, by virtue of the ordering conditions on the L_j that make up a flow line, condition (6.1) will be satisfied.

Suppose, on the other hand, that V_r and V_s do not lie in the same minus bubble. Then one can construct a $V_r \overset{+}{\to} V_s$ cut set of D^σ. To do this, simply draw the flow diagram D^σ by first making all the minus bubbles b of B extremely tiny, and then replacing each tiny minus bubble b by a tiny D_b^-. The D_b^+ are not made tiny.

Since the minus bubbles are tiny, and V_r and V_s do not lie inside the same minus bubble, one can draw a plane T that lies perpendicular to the flow axis, that lies between V_r and V_s, and that does not touch any vertex of D^σ or any line that lies inside any of the tiny minus bubbles. The X^c of lemma 6.2 is then a $V_r \ddagger V_s$ cut set of D^σ. Thus the assumption of the theorem is not satisfied in this case, and (6.1) need not be proved. This completes the proof for the case $\eta = +$. For the case $\eta = -1$ the proof is completely analogous.

C. Skeleton Diagrams

Each flow diagram D has a unique skeleton diagram D_s, which is constructed as follows. Consider the set of maximal flow lines of D. Regard as equivalent any two of them that touch exactly the same set of external vertices. (External vertices are vertices that touch external lines.) Draw a diagram consisting of the external vertices of D, the external lines of D, and one internal (flow) line ℓ for each equivalence class of maximal flow lines of D. This line ℓ is drawn so that it touches precisely those external vertices that are touched by each member of the corresponding equivalence class. Now delete any line ℓ that touches a set of external vertices that is a proper subset of the set of external vertices touched by any other line ℓ'. The resulting diagram D_s called the skeleton of D.

Each flow diagram D having three incoming lines and three outgoing lines has a unique skeleton diagram D_s that is one of the 76 skeleton diagrams shown in fig. 6.1.

In constructing these diagrams use is made of the stability requirements, which demand that each nontrivial vertex have at least two incoming lines, and at least two outgoing lines. Each maximal

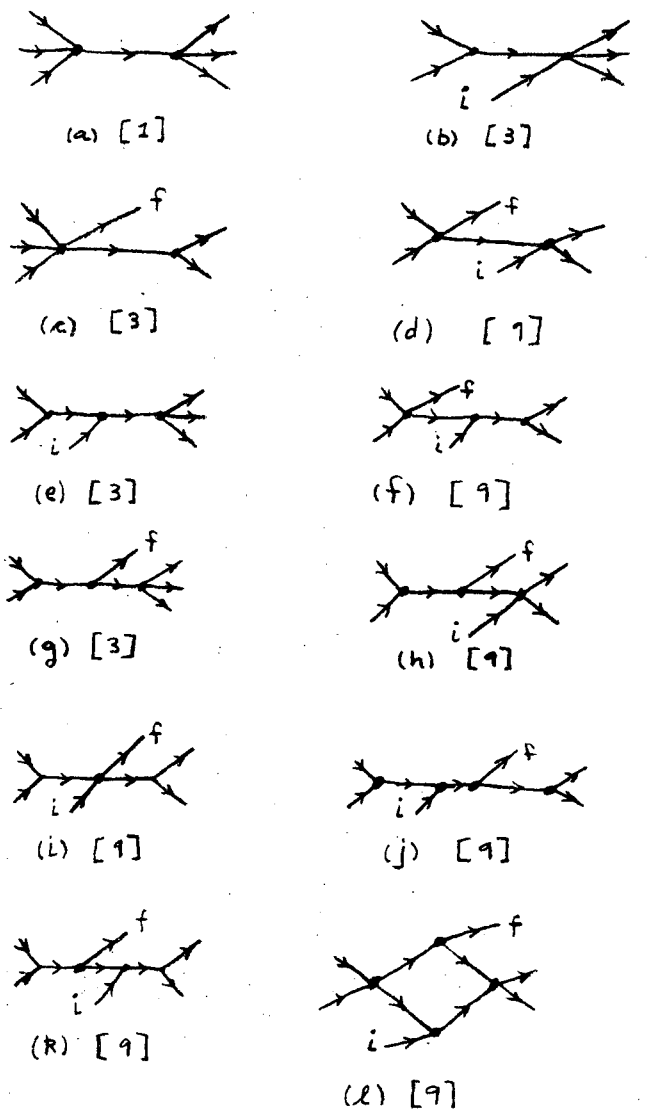


Fig. 6.1. The 76 skeleton diagrams for $3 \to 3$ processes. The indices i and f run over (1,2,3), and (4,5,6) respectively. The number in square brackets below each figure is the number of skeleton diagrams represented by that figure.

flow line must therefore begin at a vertex that has at least two incoming external lines, and must end at a vertex that has at least two outgoing external lines.

D. Path of Continuation (Off-Mass-Shell)

The aim of the present subsection is to construct for each of the 26,018 boundary values M^G and for each possible singularity surface $L(D^\sigma)$ of M^G a path of continuation that continues the function M^G into itself around $L(D_\sigma)$. However, the complications arising from the mass-shell constraints are ignored. When these constraints are ignored the rule of continuation can be formulated so that it depends only on G and on the skeleton of D^σ. And for a given skeleton D_s the rule depends on G only through the question of whether certain critical g's associated with D_s lie in G or $\overline{G}$. These critical g's are those that label the critical channels of D_s, which are now described.

For each skeleton D_s there is a unique set of critical channels g. The critical channels g corresponding to a given D_s are the channels g such that D_s can be separated, by cutting some of its internal lines ℓ, into two connected parts $D_s(J_g)$ and $D_s(\overline{J}_g)$, where $D_s(J_g)$ contains all the external lines $j\varepsilon J_g$, and $D_s(\overline{J}_g)$ contains all the external lines $j\varepsilon\overline{J}_g$. Moreover, all the cut lines ℓ run from $D_g(\overline{J}_g)$ to $D_g(J_g)$. The critical channels corresponding to several skeletons D_s are indicated in fig. 6.2. [For each g the sets J_g and $\overline{J}_g$ must be selected so that J_g contains at least two indices f, and $\overline{J}_g$ contains at least two indices i. Otherwise the conditions given above cannot be satisfied.]

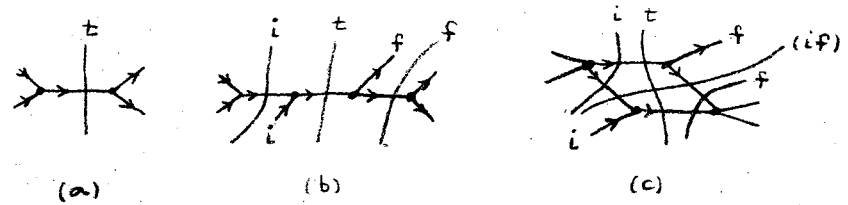


Fig. 6.2. The critical channels g of several skeleton diagrams D_s are indicated by lines g that separate D_s into the two parts $D_s(\overline{J}_g)$ and $D_s(J_g)$.

For any given G a set of signs η_g is defined as follows [see (2.25)]: the sign η_g is plus if g lies in G and is minus if g lies in $\overline{G}$. Symbolically, η_g can be defined by the set of conditions

$$g \,\varepsilon\, G^{\eta_g} \qquad \text{(for every } g \,\varepsilon\, E) \tag{6.2a}$$

where

$$G^+ = G \ , \tag{6.2b}$$

and

$$G^- = \overline{G} \ . \tag{6.2c}$$

The rules of continuation to be constructed here apply only to the 26,018 boundary values M^G defined by (2.25). For these functions one has, according to (5.60),

$$M^G = T^G = \overline{T}^{\overline{G}} \ . \tag{6.3}$$

Thus M^G can, by virtue of proposition 5.2 and eq. (4.14), be written in two alternative forms:

$$M^G = \sum_{B \varepsilon \mathcal{B}^G_+} F^B , \qquad (6.4+)$$

and

$$M^G = \sum_{B \varepsilon \mathcal{B}^G_-} F^B , \qquad (6.4-)$$

where the notation $\mathcal{B}^G_+ \equiv \mathcal{B}^G$ and $\mathcal{B}^G_- \equiv \bar{\mathcal{B}}^{\bar{G}}$ is used. The sets $\mathcal{B}^G_+$ and $\mathcal{B}^G_-$ have the following properties: no $D_\sigma \subset \mathcal{B}^G_+$ has a positive-α g-channel cut set for any g in G; no $D_\sigma \subset \mathcal{B}^G_-$ has a negative-α g-channel cut set for any g in $\bar{G}$.

If one introduces the notation of (6.2) and writes $+\alpha$ for positive-α and $-\alpha$ for negative-α then the properties of $\mathcal{B}^G_+$ and $\mathcal{B}^G_-$ can be combined into the following statement:

For any G and g let $\eta = \eta_g$. Then no $D^\sigma \subset \mathcal{B}^G_\eta$ has an $\eta\alpha$ g-channel cut set.

The rule for continuing M^G past any $L(D^\sigma)$ will be derived by combining this property of $\mathcal{B}^G_\eta$ with theorem 6.1, and then using the structure theorem described in section IV.

Consider any fixed G. Let D_s be the skeleton shown in fig. 6.2(a). The rule for continuing $M^G(p)$ past all $L(D^\sigma)$ that correspond to D^σ having this skeleton D_s will now be derived.

The diagram D_s [i.e., fig. 6.2(a)] has only one critical channel g, namely the channel $g = t$. Let η be the η_t defined in

(6.2). Let M^G be represented by the formula (6.4η). Then according to the statement just given of property (4.12), no $D^\sigma \subset \mathcal{B}^G_\eta$ has an $\eta\alpha$ t-channel cut set.

This result implies that the suppositions of theorem 6.1 are valid for every $B \varepsilon \mathcal{B}^G_\eta$ and every $D^\sigma \subset B$ such that D^σ has the skeleton D_s, provided V_r and V_s are identified as the initial and and final vertices of D_s, respectively. This is because any $V_r \overset{\eta}{\to} V_s$ cut set of D^σ is also an $\eta\alpha$ t-channel cut set of D^σ [See (4.12)] Indeed, since D^σ has only the two external vertices V_r and V_s shown in fig. 6.2(a), any cutting of the internal lines of D^σ that separates D^σ into two parts $D^\sigma(V_r)$ and $D^\sigma(V_s)$ with V_r in $D^\sigma(V_r)$ and V_s in $D^\sigma(V_s)$ must also separate D^σ into two parts $D^\sigma(\bar{J}_g)$ and $D^\sigma(J_g)$ such that $D^\sigma(\bar{J}_g)$ contains all the external lines $j \varepsilon \bar{J}_g$ and $D^\sigma(J_g)$ contains all the external lines $j \varepsilon J_g$. In fact, $D^\sigma(\bar{J}_g)$ would be identical to $D^\sigma(V_r)$, and $D^\sigma(J_g)$ would be identical to $D^\sigma(V_s)$.

Since the suppositions of theorem 6.1 hold, the conclusion holds: eq. (6.1) is true for all space-time representations of all $D^\sigma \subset \mathcal{B}^G_\eta$ that have as skeleton the D_s of fig. 6.2(a), with V_r and V_s identified in the manner described.

According to (4.6) and (6.4η), the singularities of M^G are confined to the union of $L(D^\sigma)$ over $D^\sigma \subset \mathcal{B}^G_\eta$. Suppose p lies on $L(D^\sigma)$ for $D^\sigma \subset \mathcal{B}^G_\eta$ only if D^σ has skeleton D_s. [This supposition will be removed later] Then, by virtue of the conclusion stated in the preceeding paragraph, eq. (6.1) holds for all ω in the set

$$\Omega(p) = \bigcup_{B\varepsilon \mathcal{B}^G_\eta} \Omega^B(p) \quad , \tag{6.5a}$$

where $\Omega^B(p)$ is defined in (4.8). Thus if $C(p)$ is defined by

$$C(p) = \bigcap_{B\varepsilon \mathcal{B}^G_\eta} C^B(p) \quad , \tag{6.5b}$$

where $C^B(p)$ is defined in (4.9), then (6.1) and (4.9) show that $C(p)$ contains all points q that satisfy, with $\varepsilon_f = +1$ and $\varepsilon_i = -1$,

$$\sum_{j\varepsilon J_t} \varepsilon_j q_j = - \sum_{j\varepsilon \bar{J}_t} \varepsilon_j q_j \; \varepsilon \; V^{-\eta} \quad . \tag{6.6}$$

But then, according to consequence (2) of the structure theorem, which is described below (4.9), all of the (finite number of) bubble diagram functions F^B that occur in the expression (6.4η) are boundary values of functions that are analytic at all mass-shell points sufficiently near p for which q satisfies (6.6). [Strictly speaking, the cone of **analyticity** is not $V^{-\eta}$ itself but rather any cone that is contained with its closure, apart from the apex at $q = 0$, in $V^{-\eta}$. This slight diminuation of all cones of analyticity will always be taken as understood in the discussion that follows.]

This means that M^G can be continued about all $L(D^\sigma)$ that correspond to D^σ having the skeleton D_s of fig. 6.2(a) by means of a small detour through the region (6.6).

Next let D_s be fig. 6.2(b). The rule for continuing any fixed M^G past all $L(D^\sigma)$ associated with this D_s is now derived.

Consider first the critical channel i of D_s. Let η be η_i, and use the representation of M^G given by (6.4η). Let the B of Theorem 6.1 be any $B\varepsilon \mathcal{B}^G_\eta$, and let D^σ be any $D^\sigma \subset B\varepsilon \mathcal{B}^G_\eta$ having skeleton D_s. Finally, let the vertices V_r and V_s of Theorem 6.1 be the vertices of fig. 6.2(b) that stand just to the left and just to the right of the i-channel line of fig. 6.2(b), respectively. [i.e., V_r is the left-most vertex of fig. 6.2(b), and V_s is the vertex at which the external line i terminates.]

We know that no $D^\sigma \subset \mathcal{B}^G_\eta$ has an $\eta\alpha$ i-channel cut set. This immediately implies that no $D^\sigma \subset \mathcal{B}^G_\eta$ having skeleton D_s has a $V_r \overset{\eta}{\to} V_s$ cut set in which the last two vertices of fig. 6.2(b) lie in $D^\sigma(V_s)$. And the possibility of any other $V_r \overset{\eta}{\to} V_s$ cut set is ruled out by the existence of the flow lines represented by the two right-most lines ℓ of fig. 6.2(b): these lines would run the wrong way across any cut separating a $D^\sigma(V_r)$ containing V_r from $D^\sigma(V_s)$ containing V_s if either of the remaining two vertices were to lie in $D^\sigma(V_r)$. This is shown in fig. 6.3, which is explained in the text that follows.

The possible $V_r \overset{\eta}{\to} V_s$ cut sets can be examined by placing V_r and V_s on the left- and right-hand sides of a vertical line, and placing the other vertices on the two sides of this line in all possible combinations. There can be a $V_r \overset{\eta}{\to} V_s$ with the parts $D^\sigma(V_r)$ and $D^\sigma(V_s)$ of D^σ containing the external vertices lying, respectively, on the left- and right-hand side of the vertical line only if all the lines of the skeleton diagram run from left to right across the vertical line. Only figure (a) satisfies this condition. However, this way [fig. 6.3(a)] of achieving a $V_r \overset{\eta}{\to} V_s$ cut set of a

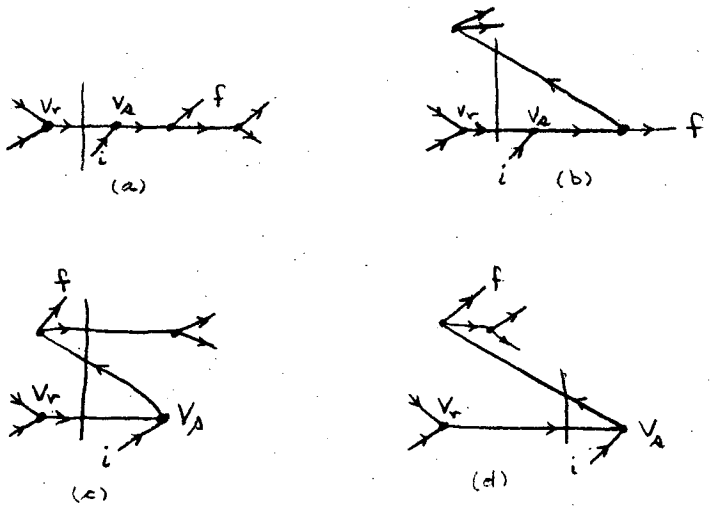


Fig. 6.3. Diagrams for examining possible $V_r \overset{\eta}{\to} V_s$ cut sets.

$D^\sigma \subset \mathcal{B}^G_\eta$ having D_s as skeleton is ruled out by the property of $\mathcal{B}^G_\eta$.

Since the suppositions of Theorem 6.1 hold, the conclusion holds: every space-time representation of every $D^\sigma \subset \mathcal{B}^G_\eta$ that has D_s [fig. 6.2(b)] as its skeleton is such that the vector $\Delta = \omega_s - \omega_r$ lies in $V^{-\eta}$. Here ω_s and ω_r are the space-time vectors to the leading and trailing vertices V_s and V_r of ℓ.

Essentially the same argument can be made for the pair of vertices lying on either side of the t line in fig. 6.2(b), and also for the pair of vertices lying on either side of the f line. The results of the three similar arguments can be summarized in a systematic way.

For each of the three critical channels g associated with fig. 6.2(b) define

$$\Delta^g \equiv \omega_s{}^g - \omega_r{}^g \quad , \tag{6.7}$$

where $\omega_s{}^g$ and $\omega_r{}^g$ are the four-vectors to the vertices $V_s{}^g$ and $V_r{}^g$ that lie just to the right and just to the left of the line g in fig. 6.2(b) corresponding to channel g. (For example, $\omega_s{}^t$ and $\omega_r{}^t$ are the vectors to the vertices lying at end points of the lines f and i, respectively.) Then the property of $\mathcal{B}^G_\eta$ for $\eta = \eta_g$, together with Theorem 6.1, shows that, for each critical channel g of D_s,

$$\Delta^g \in V^{-\eta_g} \tag{6.8}$$

for every space-time representation of every $D^\sigma \subset \mathcal{B}^G_{\eta_g}$ having as skeleton the D_s of fig. 6.2(b).

For any D^σ having as skeleton the D_s of fig. 6.2(b) the function occurring on the right-hand side of (4.9a) can be written in the form

$$-\sum_{\substack{j\in \mathrm{Ext} \\ r\in \mathrm{Ver}}} q_j\, \varepsilon_{jr}\, \omega_r = \sum_g q^g\, \Delta^g \quad , \tag{6.9}$$

where g runs over the critical channels associated with fig. 6.2(b), and q^g is the imaginary part of the momentum-energy that runs along the line ℓ of fig. 6.2(b) that is cut by the line g corresponding to channel g. In particular,

$$q^i = q_4 + q_5 + q_6 - q_i \tag{6.10a}$$

$$q^t = q_4 + q_5 + q_6 = q_1 + q_2 + q_3 \ , \tag{6.10b}$$

and

$$q^f = q_4 + q_5 + q_6 - q_f \ . \tag{6.10c}$$

The various q^g are indicated in fig. 6.4.

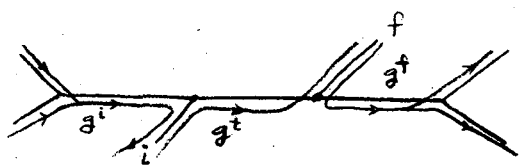


Fig. 6.4. Each q^g can be considered to be the imaginary energy carried along an open path that contains only one internal line.

We shall now temporarily ignore the mass-shell constraints, and suppose that our functions M^G can be extended some small finite distance off-mass-shell, and that the only singularity surfaces encountered in some sufficiently small neighborhood of any real mass-shell point p are the singularity surfaces obtained in the mass-shell theory. If continuations through these off-mass-shell regions are thus allowed, then the rules of continuation can be stated in a simple way, which will be described next. The complications associated with the restriction to the mass shell, and with the possible conspiracies among singularities corresponding to different skeleton graphs will be considered in the following subsection.

The results (6.8) and (6.9) together with the $i\varepsilon$ rule stated below (4.9) show that if g labels any one of the three critical channels of D_s and if p lies on $L(D^\sigma)$ for $D^\sigma \subset \mathcal{B}^G_{\eta_g}$ only if D^σ has skeleton D_s, then M^G is the boundary value of a function that is analytic near p in the q-space region satisfying

$$q^g \ \varepsilon \ V^{-\eta_g} \tag{6.11a}$$

and

$$q^{g'} = 0 \tag{6.11b}$$

where g' runs over the critical channels $g' \neq g$. This equation gives three different regions of analyticity, one for each critical channel of D_s. The boundary values are all the same (distribution) M^G. Thus the generalized edge of the wedge theorem [19] implies that the functions in these three domains are parts of one single analytic function that is analytic near p also in the q-space region restricted only by

$$q^g \ \varepsilon \ V^{-\eta_g} \qquad \text{all critical } g \ . \tag{6.12}$$

Equation (6.12) gives a domain through which M^G can be continued (off-mass-shell) past all singularities surfaces $L(D^\sigma)$ of M^G that correspond to D^σ having as skeleton the diagram D_s shown in fig. 6.2(b). Essentially the same argument applies for any skeleton diagram of fig. 6.1 that does not have closed loops [i.e., the D_s of figs. 6.2(a)-(k)]. For any given one of these <u>tree</u> diagrams D_s the q^g associated with each critical channel g of D_s is a well-defined combination of external vectors, and the domain

This completes the derivation of the off-mass-shell $i\varepsilon$ rules associated with the various individual skeleton diagrams.

E. Path of Continuation (On-mass-shell)

The rule obtained above for continuing M^G past $L(D^\sigma)$ is simple because it prescribes a set of allowed detours that is independent of the particular point p on $L(D^\sigma)$. However, the rule is deficient because in many cases the set of allowed detours contains none that remain on the mass shell. Moreover, possible conflicts between the rules associated with different skeleton diagrams have not been considered: it is conceivable that a set of points lying on surfaces $L(D^\sigma)$ associated with different skeleton diagrams might conspire to block the continuation. These two deficiencies can be remedied by considering paths of continuation that depend on the real point p around which the continuation takes place.

To derive this on-mass-shell continuation we shall need one assumption that goes beyond the usual general S-matrix principles. This assumption is roughly that different Landau surfaces do not coincide "by accident".

To formulate this assumption precisely it is convenient to introduce three definitions.

Two space-time representations r and r' are said to be externally similar if and only if every external trajectory line ℓ_j' of r' can be brought into coincidence with the corresponding line ℓ_j of r by means of a single (positive or negative) scale change and a single overall space-time translation of r' relative to r. The external trajectory line ℓ_j is the space-time line that contains the external trajectory associated with the external variable $p_j \in p$.

Two representations can be externally similar only if they generate the same point $p \equiv (p_1, \cdots, p_n)$, since only then are the corresponding lines ℓ_j and ℓ_j' parallel.

Two space-time representations r and r' are said to be externally equivalent if and only if (1) the two representations are externally similar, and (2) the transformation that brings every pair of lines ℓ_j and ℓ_j' into coincidence also brings the corresponding pairs of external vertices v_j and v_j' in coincidence.

A major part of an analytic subvariety V is a subset of V that lies in no analytic subvariety V' of dimension less than that of V. [An analytic subvariety V is a surface that can be defined locally as the zeros of a finite set of (locally) analytic functions: i.e., for each point p in the imbedding space there is a neighborhood U of p and a finite set of functions $f_1(p), \cdots, f_r(p)$, analytic in U, such that the subvariety V coincides in U with the simultaneous zeros of all of the functions $f_1(p), \cdots, f_r(p)$. If this set of functions is, for every U, the same fixed set of polynomials then V is called an algebraic subvariety.]

Suppose X is a major part of some codimension-one analytic subvariety Z of the mass shell (restricted by momentum-energy conservation) $\mathcal{M}$; i.e., X lies in an $N-1$ dimensional analytic subvariety Z of $\mathcal{M}$, but in no $N-2$ dimensional analytic subvariety of $\mathcal{M}$, where $N = 3n - 4$ is the dimension of $\mathcal{M}$. Let D^σ and $D^{\sigma'}$ be two Landau diagrams. Suppose for each point $p \in X$ there is a representation $r(p)$ of D^σ and a representation $r'(p)$ of $D^{\sigma'}$ such that $r(p)$ and $r'(p)$ are two externally similar representations that generate the same point p.

Situations of this kind occur often. A simple example is one in which D^σ is a normal threshold diagram D_g^+, and $D^{\sigma'}$ is obtained from D_g^+ by joining together at a third vertex some of the internal lines of D_g^+. These two Landau diagrams D^σ and $D^{\sigma'}$ generate the same normal-threshold singularity surface $s_g = \left(\sum m_i\right)^2$. A slightly more complicated case is obtained by adding to $D^\sigma = D_g^+$ another positive-α internal line, together with a negative-α internal line, associated with the same mass, that runs in the opposite direction. This diagram $D^{\sigma'}$ also generates the same surface $s_g = \left(\sum m_j\right)^2$.

There are many other ways in which a major part X of some codimension-one analytic subvariety Z of $\mathcal{M}$ can be generated by two different sets of externally similar representations. But in all known cases the two representations $r(p)$ and $r'(p)$ that generate the same point p are, for almost all points $p \varepsilon X$, not only externally similar, but also externally equivalent. The reason is that different representations that generate the same surface X can be constructed by making changes in the internal structure that do not affect the external structure. But if the external structure is substantially changed then the equations are essentially different, and the surfaces cannot be expected to coincide.

It is probably impossible to prove that a major part of codimension-one analytic subvariety of $\mathcal{M}$ can never be generated by two sets of externally similar representations r and r' that are not externally equivalent. For although the equations that must be satisfied are highly overdetermined, there seems to be no way to rule out the possibility that in some case the extra equations are satisfied purely by accident. However, we do not believe that such an accident ever

actually happens, and shall assume that it does not. Specifically, we shall assume the following: Suppose D^σ and $D^{\sigma'}$ are two (possibly identical) Landau diagrams and that for every point p in some set $Y \subset L(D^\sigma) \cap L(D^{\sigma'})$ there is a pair of representations $r(p)$ and $r'(p)$ such that (1) $r(p)$ is a representation of D^σ and $r'(p)$ is a representation of $D^{\sigma'}$, (2) $r(p)$ and $r'(p)$ both generate p, (3) $r(p)$ and $r'(p)$ are externally similar, and (4) $r(p)$ and $r'(p)$ are not externally equivalent. Then Y contains no major part X of any codimension-one analytic subvariety Z of $\mathcal{M}$; i.e., $Y \cap Z$ is confined to an analytic subvariety of $\mathcal{M}$ of codimension two (dimension $N-2$).

This assumption, called assumption A, is further discussed in appendix A, where it is used to derive the following result: Let $\mathcal{D}$ be the set of all Landau diagrams D^σ such that for some G, η, and $B \subset \mathcal{B}_\eta^G$ the condition $D^\sigma \subset B$ holds. That is,

$$\mathcal{D} \equiv \{D^\sigma : D^\sigma \subset \mathcal{B}_\eta^G \quad \text{for some } G \text{ and } \eta\} . \tag{6.17}$$

Then the set

$$L \equiv \bigcup_{D^\sigma \varepsilon \mathcal{D}} L(D^\sigma) \tag{6.18}$$

lies in a codimension-one analytic subvariety V_L of $\mathcal{M}$. [For each U the defining functions $f_1(p), \cdots, f_r(p)$ of V_L are in fact, polynomials, but this result is not needed.] Furthermore, it is shown that there is a codimension-two analytic subvariety W of $\mathcal{M}$ such that each point p in $L-W$ has the following three properties:

(a) There is a neighborhood U of p such that $L \cap U$ is a real codimension-one analytic submanifold of $\mathcal{M}$. This means, in particular that the surface L near p is a smooth codimension-one surface with a well-defined smoothly varying normal.

(b) If $\mathcal{R}$ is the set of all representations of all $D^\sigma \subset \mathcal{D}$, then all of the $r(p) \in \mathcal{R}$ that generate p are externally equivalent.

(c) If $r(p) \in \mathcal{R}$ generates p then the external trajectory lines ℓ_j of $r(p)$ do not all pass through a common point; i.e., there is no space-time point v that lies on every external trajectory line ℓ_j of $r(p)$.

The union of all of the singularity surfaces of all of the functions M^G is confined to L. Hence the union of all the singularity surfaces of any individual M^G is confined to L. The set W is a codimension-two analytic subvariety of $\mathcal{M}$, and hence it can not block the analytic continuation. Thus it is sufficient to prove that each M^G can be continued past L-W.

Inspection of (4.9) shows that the $i\varepsilon$ rule at any point p is determined by the locations of the external vertices of all of the representations $r(p)$ that generate p. It will be noted later that the condition on the domain of analyticity $C(p)$ that arises from any <u>individual</u> $r(p) \in \mathcal{R}$ that generates any $p \in L$-W defines a full upper-half plane of analyticity in appropriate mass-shell variables. Thus it is sufficient to show that the conditions on $C(p)$ associated with all of the different representations that generate any $p \in L$-W are identical.

Property (b) ensures that, apart from positive or negative scale changes and overall space-time translations, the location of each individual external vertex v_j is the same in every representation $r(p) \in \mathcal{R}$ that generates any fixed $p \in L$-W. The positive scale changes and the overall space-time translations do not effect the $i\varepsilon$ rule. Thus, in view of (6.4), it need only be shown that for each point $p \in L$-W, and each G, there is a sign η such that if $\mathcal{R}^G_\eta$ is the restriction of $\mathcal{R}$ to representations of diagrams $D^\sigma \subset \mathcal{B}^G_\eta$, then no two $r(p) \in \mathcal{R}^G_\eta$ that generate p are related by a negative scale change, where a negative scale change is a scale change that changes the signs of all of the difference vectors $\omega_s - \omega_r$.

Let $\mathcal{R}^G_\eta(D_s)$ be the set of all representations of all diagrams $D^\sigma \subset \mathcal{B}^G_\eta$ that have skeleton D_s. It was shown in the preceding subsection that for each G and D_s there is a pair of external vertices V_r and V_s and a sign η such that for all representations $r(p) \in \mathcal{R}^G_\eta(D_s)$ the following condition holds:

$$\omega_s - \omega_r \in V^{-\eta} \quad . \tag{6.19}$$

This condition is not maintained by a negative scale change. Thus for any G, D_s, and $p \in L$-W no two of the externally equivalent $r(p) \in \mathcal{R}^G_\eta(D^s)$ that generate p can be related by a negative scale change. This result would complete the proof, were it not for the possibility of conspiracies among the singularities associated with different skeletons D_s.

To complete the proof it will now be shown that for each G and $p \in L$-W there is a pair of external vertices V_r and V_s

and a sign η such that (6.19) holds for all representations $r(p) \varepsilon \mathcal{R}^G_\eta$ that generate p. This condition precludes the possibility that any two of the externally equivalent $r(p) \varepsilon \mathcal{R}^G_\eta$ that generate p are related by a negative scale change. Therefore all of the representations $r(p) \varepsilon \mathcal{R}^G_\eta$ that generate p give exactly the same constraint on the region $C(p)$ of analyticity.

To obtain this result let p be any fixed point in L-W. Consider all of the representations $r(p) \varepsilon \mathcal{R}$ that generate p. The six external vertices V_i of any such representation lie on a set of n' space-time points, where $n' = 2, 3$, or 4. [The case $n' = 1$ is ruled out by property (c).]

If $n' = 4$ then each representation $r(p) \varepsilon \mathcal{R}$ that generates the fixed p must be a representation of a $D^\sigma \varepsilon \mathcal{D}$ that has one of three possible skeletons. Apart from trivial modifications these three skeletons are the three indicated in fig. 6.8. The different lines in this figure are identified by their slopes. The vertical lines can be either initial or final.

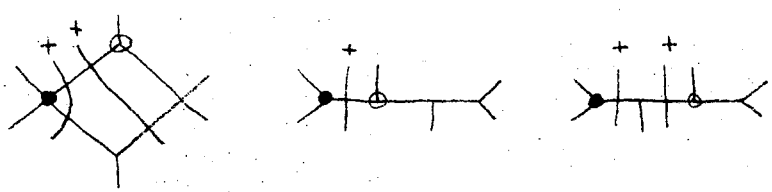


Fig. 6.8. The three skeletons for the case $n' = 4$.

Consider first the box diagram D_s of fig. 6.8. It was shown in section VI.D. that for each G the skeleton diagram D_s has a preferred line $V_r \to V_s$. Let this preferred line be, for example, the upper left-hand internal line of D_s. Then the vertices V_r and V_s are indicated by the heavy dot and little circle, respectively. In the figure the common signs η_g of the two lines g that cross the preferred line are shown as plus. Equation (6.14) gives (6.19), with η equal to these two common signs η_g. For the second skeleton in fig. 6.8 this same condition (6.19) holds as a consequence of (6.8), and the condition that the sign η be equal to these two common signs η_g. For the third skeleton the same condition (6.19) holds for the same reasons, together with the fact that the sum of two vectors in $V^{-\eta}$ also lies in $V^{-\eta}$. Thus all of the $r(p) \varepsilon \mathcal{R}^G_\eta$ that generate this arbitrary $n' = 4$ point p satisfy the same condition (6.19). Hence no two can be related by a negative scale change.

Consider next the case $n' = 2$. The possible skeletons are shown in fig. 6.9.

Let η be fixed to be the sign η_g associated with the channel g defined by the first of these diagrams. In the figure η is shown as plus. The various signs η_g of the lines g in the second (i.e., box) diagram are then fixed by (2.25), together with the requirement that the line joining the two heavy dots cannot be a preferred line, nor can the line joining the two little circles be a preferred line. For with every preferred line there is a light-cone condition (6.14), and such a condition would contradict the $n' = 2$ condition that the two vertices represented by heavy dots lie at the same space-time point, or the analogous condition

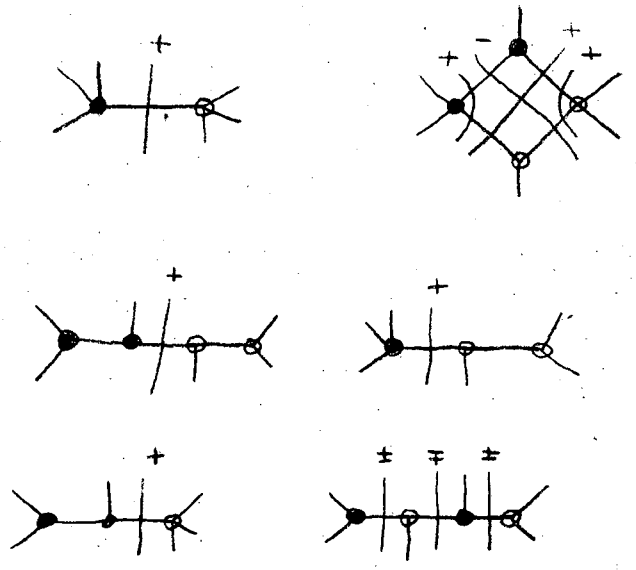


Fig. 6.9. The skeletons for the case $n' = 2$.

on the little circles. The remaining two lines of the box are then the two preferred lines, and the condition (6.14) associated with them gives the same condition (6.19) as was obtained from the first skeleton.

For the next three diagrams of fig. 6.9 the condition (6.8) immediately gives the same condition as was obtained from the first two diagrams. For the final diagram the $n' = 2$ condition that the two vertices represented by heavy dots lie at the same space-time point, and the similar condition for the little circles, together with (6.8), require that the signs associated with the lines g be alternating, as shown. The upper case gives the desired condition

(6.19). The lower case contradicts (2.25). Thus for all $n' = 2$ points there is a common condition (6.19), and a negative scale change is again precluded.

For the $n' = 3$ points similar arguments work. There are essentially two cases. The first is shown in fig. 6.10.

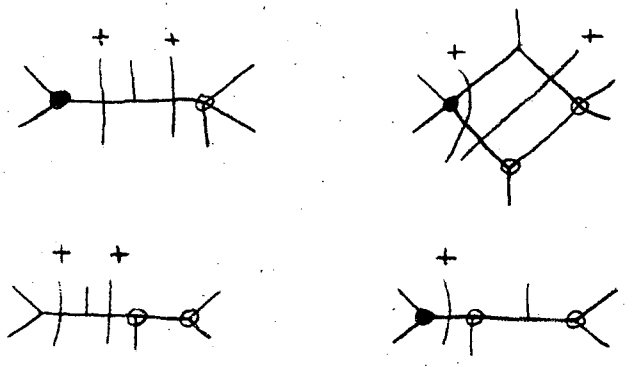


Fig. 6.10. The skeletons and signs for the first case $n' = 3$.

In this first case the two signs in the first diagram are the same, say plus. In this case the V_r and V_s are chosen to be the heavy dot and little circle respectively. Then the remaining three diagrams show, by arguments essentially the same as those given above, that the same condition (6.19) holds in all four cases.

The second $n' = 3$ case is shown in fig. 6.11.

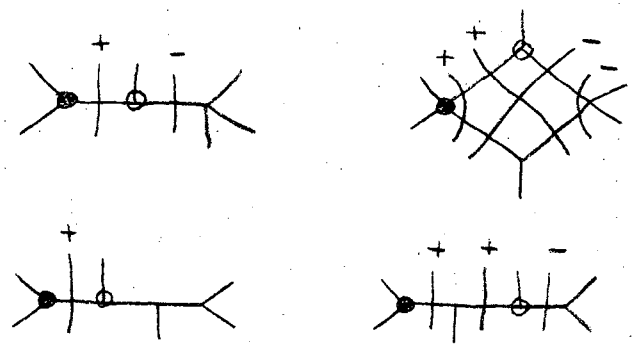


Fig. 6.11. The skeletons and signs for the second case $n' = 3$.

In this case the two signs in the first diagram are opposite, say plus and minus as shown. The signs in the box diagram are then forced to be those shown by eq. (2.25) and the $n' = 3$ condition that the lower right-hand line not be a preferred line. The signs in the last diagram are forced to be those shown by eq. (2.25) and the $n' = 3$ condition that the last two signs be opposite. These diagrams show that the same condition (6.19) holds in all four cases.

The arguments just given show that for any fixed G and fixed $p \in L\text{-}W$ there is a sign η such that all of the $r(p) \in \mathcal{R}^G_\eta$ that generate p give precisely the same condition on the cone of analyticity $C(p)$. It remains to show that the corresponding domain of analyticity contains a path that remains in the mass shell. It was shown in ref. [15] that each individual representation $r(p)$ restricts $C(p)$ to a full half plane in appropriate mass-shell variables, provided the external trajectory lines ℓ_j of $r(p)$ do not all intersect at a common point. This condition is satisfied at each point $p \in L\text{-}W$, by virtue of property (c).

APPENDIX A. PROPERTIES OF LANDAU SURFACES

Several properties of Landau surfaces are derived in this appendix. A principal result is that each Landau surface $L(D^\sigma)$ lies in a codimension-one algebraic subvariety of $\mathcal{M}$, where $\mathcal{M}$ is the $3n - 4 \equiv N$ dimensional mass shell restricted by momentum-energy conservation. Assumption A of section VI.E is used to obtain this result.

An earlier proof that Landau surfaces lie in algebraic varieties has been given by Chandler [20] and Kershaw [21]. However, in those works the Landau surfaces were defined by the α-form of the Landau equations [22]. These α-form equations are not equivalent to the original Landau equations, for they have extra solutions, which arise from points in α space where a certain discriminant $C(\alpha)$ vanishes. These extra solutions are called second-type singularities [23]. There are cases, for example the Landau surface associated with the ice-cream-cone diagram, where these so-called second-type singularities cover the entire mass shell $\mathcal{M}$ (i.e., the resolvent is identically zero [24]). Thus these earlier proofs do not yield the result we need, namely that each Landau surface $L(D^\sigma)$ is confined to a <u>codimension-one</u> algebraic subvariety of $\mathcal{M}$.

The singularities of the functions M^G lie, by virtue of the structure theorem, on a union Landau surfaces $L(D^\sigma)$. Each Landau surface $L(D^\sigma)$ is defined by a corresponding set of equations (4.2 a-h). These equations are essentially the original Landau equations. Hence the functions M^G do not have second-type singularities in their original real domain of definition.

Each real Landau surface $L(D^{\sigma})$ lies in a corresponding complex Landau surface $L(D)$ defined by eqs. (4.2 a-d). It is convenient to eliminate the conservation-law equation (4.2b) by introducing loop momenta. Let q_ℓ be the momentum-energy associated with closed loop ℓ and let k_e be the momentum-energy associated with the open loop e. Then the momentum-energy p_j of line j is

$$p_j = p_j(k,q) \equiv \sum_\ell \eta_{j\ell}\, q_\ell + \sum_e \eta_{je}\, k_e \quad , \qquad (A.1)$$

where the $\eta_{j\ell}$ are as in (4.2c), and the η_{je} are the analogous quantities for the open loops e. The $(n-1)$ independent open loops e enter and leave the diagram D via the n external lines of D. For definiteness the open loop e is assumed to leave D via external line e and to enter D via external line n. Then k_e equals $\pm p_e$ for $e = 1,\cdots,n-1$. (The lower sign is for initial e.)

With the aid of (A.1) the Landau equations (4.2 a-d) can be reduced to the equations

$$\sum_j \alpha_j\, p_j(k,q)\, \eta_{j\ell} = 0 \qquad (\text{all } \ell) \quad , \qquad (A.2)$$

$$p_j^{\,2}(k,q) - m_j^{\,2} = 0 \qquad (\text{all } j) \quad , \qquad (A.3)$$

and

$$\sum \alpha_j^{\,2} - 1 = 0 \quad . \qquad (A.4)$$

The coefficient $\eta_{j\ell}$ is zero if j labels an external line. Thus the α_j needed in (A.2) include only those corresponding to internal lines. However, parameters α_j corresponding to external lines can be introduced, and the external-loop analog of (A.2) considered:

$$\sum \alpha_j\, p_j(k,q)\, \eta_{je} = 0 \qquad (\text{all } e). \qquad (A.5)$$

Equations (A.2-4) define the Landau surface $L(D)$. Equations (A.2-5) define an associated subsurface $L^x(D) \subset L(D)$. Equation (A.5) is the requirement that all of the external trajectory lines pass through some common point. That is, each external line is assigned a parameter α_j, and hence also a space-time interval $\Delta_j = \alpha_j\, p_j$, and all of the external loops are required to be closed loops. This condition is property (c) of section VI.E.

Let $z \equiv (z_1,\cdots,z_s) \equiv (k,q,\alpha)$ denote the set of components of the vectors k_e and q_ℓ together with the α_j for $j\varepsilon\text{Int}$. Then the Landau equations (A.2-4) are a set of algebraic (i.e., polynomial) equations

$$g_j(z) = 0 \quad . \qquad (A.6)$$

The set of points $z\varepsilon\mathbb{C}^s$ that satisfy all of the equations (A.6) is the algebraic subvariety $V(g) \equiv V$ of $\mathbb{C}^s$. The set V^x defined by

$$V^x \equiv \{z\varepsilon V : (A.5) \text{ is soluble}\} \qquad (A.7)$$

is the set of point $z\varepsilon\mathbb{C}^s$ that satisfy both (A.6) and (A.5) for some set of α_j with $j\varepsilon$Ext.

Let k be the set of open loop momenta $(k_1,\cdots,k_{n-1})$, and let $g_j^{ex}(k)$ be the functions

$$g_j^{ex}(k) = p_j^2(k) - m_j^2 \qquad j\varepsilon\text{Ext} \quad . \tag{A.8}$$

By the mass shell $\mathcal{M}$ we mean now the algebraic subvariety of k space

$$\mathcal{M} \equiv V(g_j^{ex}) \equiv \{k : g_j^{ex}(k) = 0 \quad \text{all} \quad j\varepsilon\text{Ext}\} \ . \tag{A.9}$$

The <u>restricted mass shell</u> $\mathcal{M}'$ consists of all points $k\varepsilon\mathcal{M}$ except those rare points for which each external momentum-energy vector $p_j(k)$ is parallel to every other one. The restricted mass shell $\mathcal{M}'$ is an N-dimensional analytic submanifold of k space [15]. Thus for each point $k\varepsilon\mathcal{M}'$ there is a mass-shell neighborhood $U \subset \mathcal{M}'$ of k, and a set $(Z_1,\cdots,Z_N)$ consisting of N of the components of the loop momenta k_e such that the equation

$$\text{rank}\left(\frac{\partial(g_j^{ex},Z_m)}{\partial k_e^{\mu}}(k)\right) = 4(n-1) \tag{A.10}$$

holds for all points k in U. These variables $Z_1,\cdots,Z_N$ are local analytic coordinates of the mass shell in U. That is, U is isomorphic [25] to its image $Z(U) \subset \mathbb{C}^N$. The mapping $Z : \mathbb{C}^{4(n-1)} \to \mathbb{C}^N$ is the projection of k space onto Z space.

Let $\phi: \mathbb{C}^s \to \mathbb{C}^{4(n-1)}$ denote the projection of z space onto k space. Then

$$\phi(V) = L(D) \ , \tag{A.11}$$

where $V \equiv V[D]$ is the subvariety of $\mathbb{C}^s$ defined by the Landau equations (4.2 a-d) associated with the Landau diagram D.

Some properties of Landau surfaces will now be developed in a series of lemmas. The quantities ϕ, V, Z_m, g_j, etc. occurring in these lemmas are the quantities defined above. And in particular U is an open subset at the restricted mass shell $\mathcal{M}'$ that is isomorphic to its image $Z(U)$ in Z space. Equation (A.10) holds for all $k\varepsilon U$, and $\phi^{-1}U$ is the set of point z such that $\phi(z) = k$ lies in U. The function $Z(z) \equiv Z(k,q,\alpha)$ is defined by $Z(k,q,\alpha) = Z(k)$.

<u>Lemma A1</u> For any point $z\varepsilon\left[V \cap \phi^{-1}U\right]$ there is a set of coefficients $\alpha_j'(z)$ and a set of coefficients $\beta_m(z)$ such that for all $h\varepsilon(1,\cdots,s)$

$$\sum_{j\varepsilon\text{Int}\cup\text{Ext}} \alpha_j'(z)\frac{\partial g_j}{\partial z_h}(z) = \sum_{m=1}^{N} \beta_m(z)\frac{\partial Z_m}{\partial z_h}(z) \ . \tag{A.12}$$

The sum on the left-hand side of (A.12) is restricted to the indices $j\varepsilon\text{Int}\cup\text{Ext}$, and for any such j

$$g_j(z) = p_j^2(k,q) - m_j^2 \ . \tag{A.13}$$

Furthermore, the $\alpha'_j(z)$ for $j\varepsilon$Int are given by

$$\alpha_j'(z) = \alpha_j \qquad j\varepsilon\text{Int} \ . \tag{A.14}$$

With these restrictions imposed the remaining n coefficients $\alpha_j'(z)$, $j\varepsilon$Ext, and N coefficients $\beta_m(z)$, $m = 1,\cdots,N$, are uniquely defined functions of $z\varepsilon\left[V \cap \phi^{-1}U\right]$.

Proof The $g_j(z)$ in (A.13) do not depend on the parameters α_j. Neither do the $Z_m(z) \equiv Z_m(k)$. Thus if z_h is any one of the parameters α_j then (A.12) holds trivially: both sides vanish. If z_h is any component of one of the q_ℓ then the right-hand side of (A.12) vanishes because the $Z_m(z)$ depend only on the k_e, not the q_ℓ, and the left-hand side vanishes by virtue of (A.1), (A.2), and (A.13). Thus (A.12) holds also for these z_h. Finally, consider those cases in which the z_h are components of the k_e. In these cases (A.12) can be written, in the form

$$\sum_{j\varepsilon \mathrm{Ext}} \alpha_j'(z) \frac{\partial g_j^{ex}(k)}{\partial k_e^{\mu}} - \sum_{m=1}^{N} \beta_m(z) \frac{\partial Z_m(k)}{\partial k_e^{\mu}} = - \sum_{j\varepsilon \mathrm{Int}} \alpha_{j'} \frac{\partial g_j(z)}{\partial k_e^{\mu}} , \tag{A.15}$$

where the z_k of (A.12) are now the $4(n-1)$ components of the $n-1$ vectors k_e. For fixed $z = (k,q,\alpha)$ there are $4(n-1)$ unknowns, namely the $N = 3n - 4$ coefficients $\beta_m(z)$ together with the n coefficients $\alpha_j'(z)$ for $j\varepsilon \mathrm{Ext}$. Thus, for fixed z, (A.15) is a set of $4(n-1)$ linear equations for $4(n-1)$ unknowns. These equations have a unique solution provided

$$\mathrm{rank}\left(\frac{\partial(g_j^{ex}, Z_m)}{\partial k_e^{\mu}} (k) \right) = 4(n-1) . \tag{A.16}$$

This rank condition is just (A.10), which holds for all $z\varepsilon\phi^{-1}U$. Thus (A.12-14) has a unique solution for each $z\varepsilon\left[V \cap \phi^{-1}U\right]$ QED.

Lemma A2 For any fixed $z\varepsilon\left[V \cap \phi^{-1}U\right]$ the unique set of $\beta_m(z)$ defined in lemma A1 satisfies the N equations

$$\beta_m(z) = 0 \qquad m = 1,\cdots,N \tag{A.17}$$

if and only if $z\varepsilon V^x$.

Proof If z lies in $V \cap \phi^{-1}U$ then the unique set of $\beta_m(z)$ defined in lemma A1 are defined by (A.15), and the points $z\varepsilon V^x$ are defined by (A.7). But, given (A.15), condition (A.7) is equivalent to (A.17).

Lemma A3 Let $\sigma \subset V$ be any analytic submanifold of $\mathbb{C}^s$ that lies in V. Let $Z|\sigma : \sigma \to \mathbb{C}^N$ be the restriction of the mapping $Z(z)$ to the submanifold σ. Let z be any point on $\sigma \cap (V - V^x) \cap \phi^{-1}U$. Then the rank of the mapping $Z|\sigma$ at z is less than N:

$$\mathrm{rank}_z\, Z|\sigma < N . \tag{A.18}$$

Proof Let z be any point on $\sigma \cap (V - V^x) \cap \phi^{-1}U$. Equation (A.12) holds for all $k\varepsilon(1,\cdots,s)$. Thus it holds also if the z_h are replaced by a coordinate set [25] $(z_1',\cdots,z_s')$ such that the coordinates $(z_1',\cdots,z_t')$ are local analytic coordinates of the (t-dimensional) analytic submanifold σ at z. Every function $g_j(z')$ vanishes identically on σ, near $z' = z'(z)$, since σ lies in $V \equiv V(g)$. Thus, for every j,

$$\frac{\partial g_j}{\partial z_h'}(z') = 0 \qquad \text{for } k = 1,\cdots,t \tag{A.19}$$

at $z' = z'(z)$. Hence the left-hand side of (A.12), with the z_h replaced by the z_h', vanishes for $h = 1,\cdots,t$ at $z' = z'(z)$. Thus the right-hand side also vanishes there:

$$\sum_{m=1}^{N} \beta_m(z') \frac{\partial Z_m}{\partial z'_h}(z') = 0 \qquad \text{for } h = 1,\cdots,t \qquad (A.20)$$

at $z' = z'(z)$. Lemma A2 and the requirement of this lemma $z \in (V - V^x) \cap \phi^{-1}U$ ensure that the $\beta_m(z')$ are not all zero at $z' = z'(z)$. But (A.20) with some $\beta_m(z') \neq 0$ at $z' = z'(z)$ is equivalent to the condition (A.18): each says that the rank of the matrix $(\partial Z_m/\partial z'_i)$ at $z' = z'(z)$, where the z'_h are local analytic coordinates of σ at $z' = z'(z)$, is less than N. QED. ∎ Any point on any analytic subvariety is either a manifold point or a nonmanifold point. A <u>manifold point</u> of an analytic subvariety is a point having a neighborhood such that the restriction of the subvariety to the neighborhood is an analytic submanifold. The remaining points of the variety are the nonmanifold points.

Almost every point of any analytic subvariety is a manifold point. In fact, the nonmanifold points of any analytic subvariety are confined to an analytic subvariety of lower dimension [26]. This latter subvariety can be similarly decomposed into manifold and nonmanifold points. By this process any analytic subvariety of $\mathbb{C}^s$ can be decomposed into a finite set of disjoint analytic submanifolds σ.

For algebraic subvarieties a similar but more detailed result holds:

<u>Lemma A4</u> Let t be any integer. Any algebraic subvariety of $\mathbb{C}^t$ can be decomposed into a finite set of disjoint analytic submanifolds σ each of the form $\sigma_{ij} = V_i - V_j$, where V_i is an irreducible algebraic subvariety of $\mathbb{C}^t$, V_j is a proper algebraic subvariety of

V_i, and the dimensions of V_i and V_j satisfy $\dim V_j < \dim V_i = \dim \sigma_{ij}$.

<u>Proof</u> Any algebraic subvariety of $\mathbb{C}^t$ can be decomposed into a finite union of irreducible algebraic subvarieties V_i [27,28]. Each irreducible algebraic subvariety V_i of $\mathbb{C}^t$ has a well-defined dimension $\dim V_i$, which is a non-negative integer $d \leq t$. The dimension of any algebraic subvariety is the maximum of the dimensions of its irreducible components, and the following property holds [27,28]: if V_i is irreducible and V_j is a proper subvariety of V_i (i.e., $V_j \subset V_i$ but $V_j \neq V_i$), then $\dim V_j < \dim V_i$.

Let V_i be any irreducible component of any algebraic subvariety of $\mathbb{C}^t$. The set V_i is defined as the set of common zeros of some set of polynomials $F \equiv (f_1,\cdots,f_m)$. Let $\operatorname{rank}_z F$ be the rank of $(f_1,\cdots,f_r)$ at z. Let r_i be the maximum of $\operatorname{rank}_z F$ over $z \in V_i$. Let

$$V_j \equiv \{z \in V_i : \operatorname{rank}_z F < r_i\} .$$

This set V_j is a proper subset of V_i and is defined by the zeros of certain determinants, which are polynomials. Thus V_j is a proper algebraic subvariety of V_j, and hence $\dim V_j < \dim V_i$. Evidently $\operatorname{rank}_z F = r_i$ on σ_{ij}, and hence σ_{ij} is a $(t - r_i)$-dimensional analytic submanifold of $\mathbb{C}^t$ [29,25]. Examination of the definitions shows that $\dim \sigma_{ij} = \dim V_i$.

The algebraic subvariety V_j can be decomposed into its irreducible components and the same procedure applied. The dimension of V_j is the maximum of the dimensions of its irreducible components Thus the dimension decreases at each step, and the process terminates

after some finite number of analytic submanifolds σ of the required form are obtained. QED.

Definition Let the measure of a subset of $\mathbb{C}^N$ be the real 2N-dimensional measure of this set considered as a subset of $\mathbb{R}^{2N}$.

Lemma A5

$$\text{meas } Z[(V - V^x) \cap \phi^{-1}U] = 0 \quad . \tag{A.21}$$

Proof The set V is an algebraic subvariety of $\mathbb{C}^s$. Thus, according to lemma A4, it can be decomposed into a finite set of analytic submanifolds σ.

Consider any one of these submanifolds $\sigma \subset V$. Sard's theorem [30], generalized to complex mappings [31], asserts that

$$\text{meas } Z[\{z\varepsilon\sigma : \text{rank}_z Z|\sigma < N\} = 0 \quad . \tag{A.22}$$

Lemma A3 asserts that

$$\text{rank}_z Z|\sigma < N \qquad \text{at all } z \varepsilon \sigma \cap (V - V^x) \cap \phi^{-1}U$$

Thus meas $Z[\sigma \cap (V - V^x) \cap \phi^{-1}U] = 0$. But V is a finite union of sets σ. Thus meas $Z[(V - V^x) \cap \phi^{-1}U] = 0$. QED.

Lemma A6 The set $\phi(V - V^x)$ lies in a codimension-one algebraic subvariety V^k of $\mathcal{M}$.

Proof Chevalley's theorem [32] ensures that the image under the algebraic mapping ϕ of a constructible set in $\mathbb{C}^s$ is a constructible set in $\mathbb{C}^{4(n-1)}$. For any t a constructible set in $\mathbb{C}^t$ is a subset of $\mathbb{C}^t$ that can be constructed as a finite union of sets $\sigma_{ij} \equiv V_i - V_j$, where V_i and V_j are algebraic subvarieties of $\mathbb{C}^t$, V_i is irreducible, and V_j is a proper subvariety of V_i. The set $V - V^x$ is constructible in $\mathbb{C}^s$. Thus $\phi(V - V^x)$ is constructible

in $\mathbb{C}^{4(n-1)}$. Let $\sigma_{ij} = V_i - V_j$ be any one of the finite number of sets from which $\phi(V - V^x)$ is constructed. By repeated use of lemma A4 one can arrange that σ_{ij} is also an analytic submanifold of $\mathbb{C}^{4(n-1)}$ of dimension $\dim \sigma_{ij} = \dim V_i$. Since σ_{ij} lies in $\mathcal{M}$, $\dim \sigma_{ij} \leqslant \dim \mathcal{M} = N$. If $\dim \sigma_{ij} = N$ then σ_{ij} contains a set that is isomorphic to an open set in $\mathbb{C}^N$ [33]. This set must intersect $\mathcal{M}'$, since $\mathcal{M} - \mathcal{M}'$ is a set of dimension less than N. Thus if $\dim \sigma_{ij} = N$ then $\sigma_{ij} \cap \mathcal{M}'$ must contain a set σ' that is isomorphic to an open set in $\mathbb{C}^N$. Let k be a point in σ', and consider the local analytic coordinates $Z_1, \cdots, Z_N$ of $\mathcal{M}'$ in a neighborhood $U \subset \sigma'$ of k. The set $U \cap \sigma'$ is isomorphic to an open set in $\mathbb{C}^N$, and hence must be mapped onto an open set in $\mathbb{C}^N$ by the mapping Z that maps $U \subset \mathcal{M}'$ onto $Z(U) \subset \mathbb{C}^N$. But this contradicts the conclusion of lemma A5 that meas $Z[(V - V^x) \cap \phi^{-1}U]$ is zero. Thus $\dim \sigma_{ij} = \dim V_i < N$. Hence the constructible set $\phi(V - V^x)$ is a finite union of sets $\sigma_{ij} = V_i - V_j$ of dimension $\dim \sigma_{ij} = \dim V_i < N$. Each set σ_{ij} lies in the mass shell $\mathcal{M}$. Hence the closure of each σ_{ij} lies in the closed set $\mathcal{M}$. But σ_{ij} is dense in V_i and its closure is V_i. Thus $\phi(V - V^x)$ lies in a finite union of irreducible algebraic varieties V_i each of which lies in $\mathcal{M}$ and has dimension $\dim V_i < N$. Thus $\phi(V - V^x)$ lies in a single algebraic subvariety of $\mathcal{M}$ of dimension less than N. Any algebraic variety of dimension less than N is contained in an algebraic variety of dimension N-1 [27]. QED.

Lemma A7 Let V_i be any irreducible component of V. Let $V_i^x \equiv V_i \cap V^x$. Then $\phi(V_i)$ lies in the V^k of lemma A6 unless $V_i \equiv V_i^x$.

Proof Suppose $V_i \not\equiv V_i^x$. Then V_i^x is a proper subvariety of V_i, and $\dim V_i^x < \dim V_i$. But then $V_i - V_i^x$ is dense in V_i, and V_i lies in the closure of $V_i - V_i^x$. But then $\phi(V_i)$ lies in the closure of $\phi(V_i - V_i^x)$, since the mapping ϕ is continuous. Thus $\phi(V_i)$ lies in the closed set V^k that contains $\phi(V_i - V_i^x)$. QED.

Lemma A8 $\phi(V)$ lies in a codimension-one algebraic subvariety of $\mathcal{M}$ unless for some i the set V coincides with V_i^x in some neighborhood U of some point $\bar{z}\varepsilon V_i^x$, and, moreover, $\phi(V_i^x)$ contains almost all points of $\mathcal{M}$.

Proof The image $\phi(V_i^x)$ of any V_i^x is constructible and hence either lies in an algebraic subvariety of dimension less than N, or contains almost all points of the mass shell. The former sets $\phi(V_i^x)$ can be combined with V^k to give an algebraic variety $V^{k'}$ of dimension less than N. If any V_i remains then this V_i must coincide with V_i^x, by lemma A7. But there are points on V_i that lie on no other irreducible component of V. Any such point $\bar{z}\varepsilon V_i \equiv V_i^x$ has a neighborhood U such that $V \cap U = V_i \cap U = V_i^x \cap U$. QED.

The above lemma says that $\phi(V) = L(D)$ lies in a codimension-one algebraic subvariety of $\mathcal{M}$ unless the conditions (A.5) that the external trajectory lines ℓ_j pass through a common point are redundant near some point of V: i.e., the conditions (A.5) are implied by the condition (A.6).

The quantities $\beta_m(z)$ that occur in (A.12) have a simple but important interpretation, which is now described. Let the $p_1,\cdots,p_n$ be the external p_j. The mass-shell constraints ensure that at least one component of each p_j is nonzero. Near real points the

energy components p_j^0 are nonzero. In the following discussion it is assumed that these energy components do not vanish, but minor changes would allow the other cases to be encompassed.

For each point $k\varepsilon\mathcal{M}'$ the external $p_j(k)$ are not all parallel. Thus for some $j < n$ the vector p_j is not parallel to p_n. Let the labelling be such that p_{n-1} is not parallel to p_n. Then for some space component, which is taken to be the $\mu = 3$ component,

$$\frac{p_{n-1}^3}{p_n^0} \neq \frac{p_n^3}{p_n^0} \quad . \tag{A.23}$$

If the labels are arranged in this way then the set $(Z_1,\cdots,Z_N)$ can be taken to be the set consisting of all of the space components of all of the $n-1$ vectors k_e except for k_{n-1}^3. For with this choice the quantity

$$\det\left(\frac{\partial(g_j^e, Z_m)}{\partial k_e^\mu}\right) = 2^n(p_n^0\, p_{n-1}^3 - p_{n-1}^0\, p_n^3)\prod_{j=1}^{n-2} p_j^0 \tag{A.24}$$

is nonzero, by virtue of condition (A.23), and our convention about p_j^0. This same condition also ensures that the projections onto (x^0,x^3) space of the external trajectory lines ℓ_{n-1} and ℓ_n of the space-time representation of the point z of V must intersect at some point $(\bar{x}^0,\bar{x}^3)$. Let the origin of time be chosen so that $\bar{t} \equiv \bar{x}^0 = 0$. Let the origin of space be the point at which the trajectory line of particle n intersects the plane $t \equiv x^0 = 0$. Then for each index $j\varepsilon\mathrm{Ext}$ let x_j be the three-vector from the origin of space to the point at which the trajectory line ℓ_j of external particle j intersects the plane $t = 0$.

In general, each of the three components x_j^{μ} $(\mu = 1,2,3)$ of each vector x_j is a complex number. However, the components x_n^1, x_n^2, x_n^3, and x_{n-1}^3 all vanish, by virtue of the choice of origin. The remaining $3n-4$ coordinates x_e^{μ} $(\mu = 1,2,3)$ are, apart from a factor of two, the coefficients $\beta_m(z)$. In particular, if $Z_m = k_{e(m)}^{\mu(m)} = p_{e(m)}^{\mu(m)}$ then

$$\beta_m(z) = 2x_{e(m)}^{\mu(m)}(z) \quad , \tag{A.25}$$

where the argument z of $x_e^{\mu}(z)$ signifies that x_e^{μ} depends on the point $z\varepsilon V$, since, of course, the space-time representation of D depends on $z\varepsilon V$. For each $j\varepsilon Ext$ the parameter $\alpha_j'(z)$ is the complex number such that $\Delta_j = \alpha_j' p_j$ is the space-time vector to the point $(0,x_j)$ from the point v_j on the external trajectory j where this trajectory joins D. These determinations of the $\alpha_j'(z)$ and of the $\beta_m(z)$ ensure that (A.15), and hence (A.12), holds.

The identification (A.25) means that the N-vector $\beta(z)$ with components $\beta_m(z)$, is determined in a simple way by the locations of the external trajectories ℓ_j of the space-time representation corresponding to $z\varepsilon V$. This vector $\beta(z)$ is zero, as noted in lemma A2, if and only if the external trajectory lines all pass through a common point.

The N-vector $\beta(z)$ discussed above is also essentially the normal to the Landau surface at the image of z:

<u>Lemma A9</u> Suppose σ is an analytic submanifold of $\mathbb{C}^s$ that lies in V. Suppose z is a point of $\sigma \cap (V - V^x) \cap \phi^{-1}U$ such that the restriction $Z|\sigma$ of the mapping $Z(z)$ to σ has rank $N-1$ at z. [Rank $> N-1$ is excluded by (A.18)]. Then there is a σ-neighborhood

of z, $U_z \subset \left[\sigma \cap (V-V^x) \cap \phi^{-1}U\right]$, such that $Z(U_z)$ is an analytic submanifold of $\mathbb{C}^N$ of dimension $N-1$. This $N-1$ dimensional analytic submanifold $Z(U_z)$ lies in $L(D)$. The normal to $Z(U_z)$ at $Z(z)$ is well-defined and nonzero, and it is equal to $\beta(z)$, apart from a nonzero scale factor.

<u>Proof</u> The rank of $Z|\sigma$ on $\sigma \cap (V - V^x) \cap \phi^{-1}U$ is, by virtue of Lemma A3, at most $N-1$. Since the rank can decrease only on the zeros of certain determinants the rank must be $N-1$ in some σ-neighborhood of z. Thus there is a σ-neighborhood of z, $U_z \subset [\sigma \cap (V - V^x) \cap \phi^{-1}U]$, such that $Z(U_z)$ is an $N-1$ dimensional analytic submanifold of $\mathbb{C}^N$ [29]. The normal to any codimension-one analytic submanifold of $\mathbb{C}^N$ is well defined and nonzero. Let $(z',\cdots,z_s')$ be a set of local analytic coordinates [25] such that $z_1',\cdots,z_t'$ are local coordinates of σ near the point z. Then the set of t N-vectors T^h, with $h = 1,\cdots,t$, whose components are $T_m^h = \partial Z_m/\partial z_h(z)$ span an $N-1$ dimensional subspace of $\mathbb{C}^N$, since the rank of $Z|\sigma$ at z is $N-1$. This $N-1$ dimensional subspace of $\mathbb{C}^N$ is the $N-1$ dimensional tangent space to $Z(U_z)$ at z. But then (A.20) is the condition that the N-vector $\beta(z)$ be a nonzero multiple of the normal to $Z(U_z)$ at z. QED.

This result that the positions of the external trajectory lines ℓ_j determine the normal to the Landau surface was derived earlier [15] by another method, for positive-α surfaces. That earlier method involves nonalgebraic functions that develop singularities when any of the internal α_j vanish. The present purely algebraic method extends the earlier results to the points z where one or more $\alpha_j = 0$.

Definition Let ϕ_R be the restriction of ϕ to real $z \varepsilon V$.

Lemma A10 Let D be any Landau diagram, and let V be the corresponding variety in $\mathbb{C}^s$. Suppose $\phi_R(V^x)$ is confined to a codimension-two algebraic subvariety of $\mathcal{M}$. Then there is a codimension-two algebraic subvariety $W(D)$ of $\mathcal{M}$ with the following four properties:

(1) $\mathcal{M} - \mathcal{M}'$ lies in $W(D)$.

(2) If z is a real point of V that is mapped by ϕ into $L(D^\sigma) - W(D)$, and if $r(z)$ is the space-time representation of D that corresponds to z, then the external lines ℓ_j of $r(z)$ do not all intersect at a common point; i.e., $\beta(z) \neq 0$.

(3) The set $L(D^\sigma) - W(D)$ is either empty or a codimension-one analytic submanifold of $\mathcal{M}'$ that is confined to a codimension-one algebraic subvariety of $\mathcal{M}$.

(4) If r and r' are any two real representations of D that correspond to the same point $k \varepsilon L(D^\sigma)-W(D)$ then r and r' are externally similar, i.e., there is a positive or negative scale change and an overall space-time translation that brings the external trajectory lines ℓ'_j of r' into coincidence with the corresponding lines ℓ_j of r.

Proof The required $W(D)$ is

$$W(D) = W_1(D) \cup W_2(D) \cup W_3(D) \cup W_4(D) \quad , \qquad (A.26)$$

where $W_1(D)$ is $\mathcal{M} - \mathcal{M}'$; $W_2(D)$ is the codimension-two algebraic subvariety of $\mathcal{M}$ that by hypothesis contains $\phi_R(V^x)$; $W_3(D)$ is the union of the algebraic subvarieties V_i and V_j of dimension less

than $N-1$ that arise in the decomposition, via the Chevalley theorem and lemma A4, of the constructible set $\phi(V - V^x)$ into analytic submanifolds $\sigma_{ij} = V_i - V_j$; and $W_4(D)$ is the closure of the image in $\mathcal{M}$ of the set of points $z \varepsilon V$ such that $\text{rank}_z\, \phi|\sigma_z < N-1$. Here σ_z is the analytic submanifold $\sigma_{ij} \subset V_i \subset V$ that arises in the decomposition of V by means of lemma A4, and that contains z, and $\phi|\sigma_z$ is the restriction of ϕ to σ_z.

The set $W_1(D)$ is an algebraic subvariety of $\mathcal{M}$ of dimension less than $N-2$. Its presence in $W(D)$ ensures property (1).

The set $W_2(D)$ is a codimension-two algebraic subvariety of $\mathcal{M}$ by hypothesis. Its presence in W ensures property (2), because the conditions that z be real point of V and that the external trajectories ℓ_j of $r(z)$ have a common point is precisely the condition that z belong also to V^x.

The set $W_3(D)$ is a finite union of algebraic subvarieties of $\mathcal{M}$ of dimension less than $N-1$, and is thus an algebraic subvariety of $\mathcal{M}$ of dimension less than $N-1$. The remaining sets V_i and σ_{ij} in the decomposition of $\phi(V - V^x)$ have dimension $N-1$, by virtue of lemma A6. And they include all points of $L(D^\sigma) - W(D)$, since $\phi_R(V^x)$ lies in $W(D)$.

The set $W_4(D)$ is

$$W_4(D) = \text{Closure}\ \phi\{z \varepsilon V : \text{rank}_z\, \phi|\sigma_z < N-1\} \quad . \qquad (A.27)$$

The set $\{z \varepsilon V : \text{rank}_z\, \phi|\sigma_z < N-1\}$ is a constructible set, and hence so, by virtue of the Chevalley theorem [32], is its image $\phi\{z \varepsilon V : \text{rank}_z\, \phi|\sigma_z < N-1\}$ in $\mathcal{M}$. But arguments analogous to those of lemma A5 show that the image under ϕ of the set

$\{z : \text{rank}_z\ \phi|\sigma_z < N-1\}$ cannot contain any set $\sigma_{ij} = V_i - V_j$ of dimension $N-1$. Hence the closure of $\phi\{z\varepsilon V : \text{rank}_z\ \phi|\sigma_z < N-1\}$ is a finite union of sets V_i of $\dim V_i < N-1$. Thus $W_4(D)$ is an algebraic subvariety of $\mathcal{M}$ of dimension less than $N-1$.

Each point $k \varepsilon L(D^\sigma)-W(D)$ is a point near which $L(D^\sigma)$ is a codimension-one analytic submanifold of $\mathcal{M}$, by property (3). Hence the normal to $L(D^\sigma)$ at any point of $L(D^\sigma) - W(D)$ is well defined. On the other hand, every real point z that maps to any point $k \varepsilon L(D^\sigma)-W(D)$ is a point $z\varepsilon V - V^x$ where $\text{rank}_z\ \phi|\sigma_z = N-1$. Thus, by virtue of lemma A9, the vector $\beta(z)$ associated with any real z that maps to any $k \varepsilon L(D^\sigma)-W(D)$ is a multiple of the well-defined normal to $L(D^\sigma)$ at k. Therefore all of the vectors $\beta(z)$ that are associated in this way with any given $k \varepsilon L(D^\sigma)-W(D)$ are nonzero multiples of each other. Since all of the representations $r(z)$ that generate any fixed point $k\varepsilon L(D^\sigma)$ have their corresponding external trajectory lines parallel, this equality (up to scale change) of the N-vectors $\beta(z)$ guarantees that these representations $r(z)$ are all externally similar. Thus property (4) holds. QED.

The results obtained above refer to a single Landau surface $L(D)$. Let B be any bubble diagram and let $F^B(k)$ be the corresponding bubble diagram function [See eq. (3.4)]. Let $L(B)$ be

$$L(B) = \bigcup_{D^\sigma \subset B} L(D^\sigma) \quad . \tag{A.28}$$

The structure theorem says that the singularities of $F^B(k)$ (at real k) are confined to $L(B)$. [See (4.6)]

For any given B there is an infinite set of $D^\sigma \subset B$. One trivial way in which this set becomes infinite is illustrated in fig. A.1.

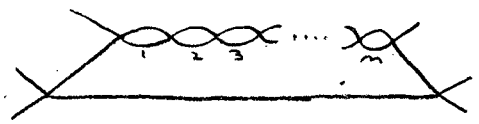


Fig. A.1. A typical nonbasic diagram.

As n runs from 2 to infinity the diagram in this figure generates an infinite set of diagrams. If the multiple loops in all these diagrams are formed from a single fixed pair of particles, and if the signs σ_j are all plus, or are all minus, then each of these diagrams gives the same surface $L(D^\sigma)$. These D^σ are examples of nonbasic diagrams, which can, as we shall see, be ignored.

A <u>nonbasic diagram</u> D^σ is a diagram that has a contractible part. A <u>contractible part</u> of D^σ is a part of D^σ that: (1) is connected to the rest of D^σ at exactly two vertices, v_1 and v_2; (2) contains more than two vertices; (3) contains no external line; and (4) has one common fixed sign σ_j, either plus or minus, for each line contained in the part. Such a part is contractible in the sense that the Landau equations associated with D^σ are either insoluable, in which case $L(D^\sigma)$ is the empty set, or they can be replaced by the Landau equations of an associated basic diagram $D_\beta{}^\sigma$. A basic diagram is a diagram with no contractible part. If D^σ is nonbasic then the associated basic diagram $D_\beta{}^\sigma$ is obtained from D^σ by contracting to the point v_2 all of the lines of the contractible

part that do not touch v_1. This replacement does not alter the conditions imposed by the part. In particular, for every representation r of D^σ there will be a representation r' of D_β^σ that is exactly the same as r, except for the internal structure of the contractible part. In particular, r' will be externally equivalent to r.

A contractible part of a $D^\sigma \subset B$ must be a part of the diagram D_b^+ or D_b^- that replaces some plus or a minus bubble b of B. Otherwise some sign would be a "sign" $\sigma_j = \pm$, hence not a well-defined sign plus or minus. Thus the restriction of $D^\sigma \subset B$ to basic diagrams is equivalent to the restriction of the constituent diagrams D_b^+ and D_b^- to basic diagrams $D_{b\beta}^+$ and $D_{b\beta}^-$.

Since every representation r of a nonbasic diagram D^σ is externally equivalent to a representation r' of a basic diagram D_β^σ one may write

$$L(B) = \bigcup_{D_\beta^\sigma \subset B} L(D^\sigma) \quad , \tag{A.29}$$

where D_β^σ is a basic diagram.

Lemma A11 Let $\mathcal{R}$ be any bounded region in k space. Then only a finite set of basic D_β^+ give surfaces $L(D_\beta^+)$ that intersect $\mathcal{R}$. And only a finite set of basic D_β^- give surfaces $L(D_\beta^-)$ that intersect $\mathcal{R}$.

Proof This was proved in ref. [35].

Remark Because of the mass-shell and positive-energy $(p_j^0 > 0)$ constraints a region $\mathcal{R}$ in k space is bounded if and only if the total energy of the process, $\frac{1}{2}\sum p_j^0$, is bounded.

Lemma A12 Let $\mathcal{R}$ be any bounded region in k space. Let B be any fixed bubble diagram. Then only a finite number of basic $D_\beta^\sigma \subset B$ give surfaces $L(D_\beta^\sigma)$ that intersect $\mathcal{R}$.

Proof The flow-diagram ordering condition of section VI.A, together with the conservation-law condition (4.2b), and the positivity-energy condition (4.2g), ensures that the total energy entering any bubble b of B is no greater than the total energy entering B. That is, a bound in k space implies a bound on the energy entering each individual bubble. But the Landau equations corresponding to D_β^σ cannot be satisfied unless the Landau equations for each of the constituent parts $D_{b\beta}^+$ and $D_{b\beta}^-$ can be satisfied. Thus the bound, mentioned above, on the energy entering each bubble b, together with lemma A11, implies that for each b the corresponding $D_{b\beta}^+$ or $D_{b\beta}^-$ must be one of a certain finite set of diagrams. However, only a finite set of diagrams D_β^σ can be constructed from the finite sets of $D_{b\beta}^\pm$'s inserted in all possible ways into the finite set of bubbles b of B.

Definitions

$$D_\eta^G \equiv \{D^\sigma : D^\sigma \subset B \in \mathcal{B}_\eta^G\} \quad . \tag{A.30}$$

$$\mathcal{D} \equiv \bigcup_{G,\eta} D_\eta^G \quad . \tag{A.31}$$

Let $\mathcal{R}$ be a region in k space and let $N_\beta(\mathcal{R})$ be the number of basic $D_\beta^\sigma \in \mathcal{D}$ such that $L(D_\beta^\sigma) \cap \mathcal{R} \neq \phi$.

Lemma A13 If $\mathcal{R}$ is bounded then $N_\beta(\mathcal{R})$ is finite.

Proof The number of pairs (G,η) is finite. For each (G,η) the set $\mathcal{B}_\eta^G$ is constructed so that in any bounded region $\mathcal{R}$ only a

finite set of $B \in \mathcal{B}_\eta^G$ satisfy the conservation-law, positive-energy, and mass-shell conditions associated with the explicit lines of B. These conditions are included among the Landau equations associated with the diagrams $D_\beta^\sigma \subset B$. Thus only a finite set of $B \in \mathcal{B}_\eta^G$ can have $D_\beta^\sigma \subset B$ that give surfaces $L(D_\beta^\sigma)$ that intersect any bounded $\mathcal{R}$. But then the finiteness of $N_\beta(\mathcal{R})$ follows from lemma A12. QED.

Definition

$$L \equiv \bigcup_{D^\sigma \in \mathcal{D}} L(D^\sigma) \quad . \tag{A.32}$$

Lemma A14 Let $\mathcal{R}$ be any bounded region in k space. Suppose for each $D_\beta^\sigma \in \mathcal{D}$ that $L^x(D_\beta^\sigma)$ is confined to a corresponding codimension-two algebraic subvariety of $\mathcal{M}$. Then there is a codimension-two algebraic subvariety $W \equiv W(\mathcal{R})$ of $\mathcal{M}$ such that

(1) $\mathcal{M} - \mathcal{M}'$ lies in W.

(2) if r is a representation of some $D^\sigma \in \mathcal{D}$ and if r corresponds to a point $k \in (L - W) \cap \mathcal{R}$ then the external trajectory lines ℓ_j of r do not all pass through a common point.

(3) The set $(L - W) \cap \mathcal{R}$ is either empty or a codimension-one analytic submanifold of $\mathcal{M}'$ that is confined to a codimension-one algebraic subvariety of $\mathcal{M}$.

(4) if r is a representation of some $D^\sigma \subset \mathcal{D}$ and r' is a representation of some $D^{\sigma'} \subset \mathcal{D}$ and if both r and r' correspond to the same point $k \in (L - W) \cap \mathcal{R}$, then r and r' are externally similar.

Proof It is sufficient to consider only the representations of basic diagrams D_β^σ, since every representation r of a nonbasic diagram is externally equivalent to a representation of a basic diagram. Let

$$L(\mathcal{R}) = \bigcup_{\substack{D_\beta^\sigma \subset \mathcal{D} \\ L(D_\beta^\sigma) \cap \mathcal{R} \neq \phi}} L(D_\beta^\sigma) \quad . \tag{A.33}$$

This set, which coincides with L in $\mathcal{R}$, is constructible, since it is a finite union of constructible sets $L(D_\beta^\sigma)$. Thus the lemma follows from essentially the same arguments that give lemma A10, but with $L(\mathcal{R})$ in place of $L(D^\sigma)$. QED.

Assumption A of section VI.E is now invoked: for each pair $(D^\sigma, D^{\sigma'})$ of Landau diagrams it is assumed that there is a codimension-two algebraic subvariety $W(D^\sigma, D^{\sigma'})$ of $\mathcal{M}$ such that if r and r' are externally inequivalent but externally similar representations of D^σ and $D^{\sigma'}$, respectively, then they must correspond to a common point k on $W(D^\sigma, D^{\sigma'})$.

Lemma A14' Lemma A14 can be modified in the following way:

(1) add assumption A,

(2) eliminate the assumption about $L^x(D_\beta^\sigma)$,

(3) change the final word from "similar" to "equivalent".

Proof It is again sufficient to deal with the finite set of basic diagrams D_β^σ that give $L(D_\beta^\sigma) \cap \mathcal{R} \neq \phi$. The set of pairs of these D_β^σ and $D_\beta^{\sigma'}$ is also finite. Thus one can replace the W of lemma A14 by its union with the union of the finite set of sets $W(D_\beta^\sigma, D_\beta^{\sigma'})$, and then infer from assumption A that the externally

similar r and r' of lemma A14 are also externally equivalent. To eliminate the assumption about $L^x(D_\beta^\sigma)$ one may note that the constructions of section VI ensures that for every representation of every $D^\sigma \varepsilon \mathcal{D}$ there are at least two vertices V_r and V_s that do not coincide, because a relation $\omega_s - \omega_r \varepsilon V^{-\eta}$ holds. But then for every representation r of any $D_\beta^\sigma \varepsilon \mathcal{D}$ such that the external trajectory lines ℓ_j all pass through a common point there is a representation r' of a $D^{\sigma'} \neq D_\beta^\sigma$ such that r' is externally similar to r, but not externally equivalent to it. This $D^{\sigma'}$ has its external vertices v_j' all lying at the common point of all of the ℓ_j. To form $D^{\sigma'}$ one must add to D_β^σ an internal line that connects the original external vertex v_j of D_β^σ to the corresponding external vertex v_j' of $D^{\sigma'}$. If the mass associated with this new internal line equals the mass m_j of the associated external line ℓ_j, then the validity of the Landau equations for $D^{\sigma'}$ will be implied by the validity of those for D_β^σ. Thus the existence of a representation r of D_β^σ will ensure the existence of the externally similar, but not externally equivalent, r'. Assumption A then asserts that this coincidence of the external trajectory lines ℓ_j of r with the corresponding external trajectory lines ℓ_j' of the externally inequivalent r' can be satisfied only over a codimension-two algebraic subvariety $W(D_\beta^\sigma, D_\beta^{\sigma'})$ of $\mathcal{M}$. Augmenting the W of lemma A14 also by the finite union of sets $W(D_\beta^\sigma, D^{\sigma'})$ one finds that the assumption about $L^x(D_\beta^\sigma)$ is now satisfied. QED.

This result, lemma A14', gives the properties (a), (b), and (c) listed in section VI.E.

Assumption A asserts that the conditions imposed on a set of external trajectory lines ℓ_j by the requirement that they

be simultaneously compatible with two essentially different solutions of the Landau equations can hold only over a codimension-two algebraic subvariety of $\mathcal{M}$. The statement of this assumption given in the main text might appear to be slightly weaker than the one given just before lemma A14', since the former requires only that the intersection of the set Y with any codimension-one analytic subvariety of $\mathcal{M}$ be confined to a codimension-two analytic subvariety. However, the conditions on Y that the representations r and r' be externally similar but not externally equivalent define a constructible set in the composite (z,z') space corresponding to the pair $(D^\sigma, D^{\sigma'})$, and hence, by the Chevalley theorem, also a constructible set in k space. But for a constructible set Y the condition that its intersection with every codimension-one analytic subvariety be confined to a codimension-two analytic subvariety implies that Y be confined to a codimension-two algebraic subvariety. For, by virtue of lemma A4, any constructible set Y can be confined to a codimension-two algebraic subvariety unless it contains a codimension-one analytic submanifold.

Assumption A plays two roles in this work. The first role is to ensure that $\phi_R(V^x)$ is confined to a codimension-two algebraic subvariety of $\mathcal{M}$. The points of $\mathcal{M}$ where some pair of four-vectors $p_j \varepsilon p$ are parallel is a codimension-three subset of $\mathcal{M}$. For the remaining points of $\mathcal{M}$ the conditions on V^x include, in addition to the conditions that define V, also at least the four extra conditions that the (unique) initial vertex where two or more initial lines meet lie at the same space-time points as the (unique) final vertex where two or more final lines meet. One would expect these four or more extra conditons to reduce the corresponding region

in $\mathcal{M}$ by at least four, to a set of codimension five or more. Assumption A asserts that the reduction is, in any case, to a set of codimension two.

The second role of assumption A is to ensure that externally inequivalent representations of diagrams $D^{\sigma} \varepsilon \mathbb{D}$ are externally similar only over a codimension-two subset of $\mathcal{M}$. Two different Landau surfaces generally intersect on a codimension-two algebraic subvariety. The extra condition that the externally inequivalent representations be externally similar imposes at least three extra conditions, and hence one would expect the corresponding region in $\mathcal{M}$ to be reduced to a set of codimension five or more. Assumption A asserts that this region lies, in any case, in a subset of $\mathcal{M}$ of codimension two.

APPENDIX B. COMBINATORIC IDENTITIES

Let H be any finite set. Suppose that

$$A_H = \sum_{H' \subset H} (-1)^{n(H')} B^{H'} \tag{B.1}$$

and

$$B^{H'} = \sum_{H'' \subset H'} (-1)^{n(H'')} C_{H''} , \tag{B.2}$$

where the sums run over all distinct subsets H' of H and H'' of H', respectively, and $n(F)$ is the number of elements of F. Substitution of (B.2) into (B.1) gives

$$A_H = \sum_{H'' \subset H} X_{HH''} C_{H''} , \tag{B.3}$$

where

$$X_{HH''} = \sum_{H'' \subset H' \subset H} (-1)^{n(H')-n(H'')} . \tag{B.4}$$

Clearly $X_{HH''} = 1$. Comparison of the expression for the remaining quantities $X_{HH''}$ with the binomial expansion of $(1 - 1)^{n(H)-n(H'')}$ shows that X is the unit matrix:

$$X_{HH''} = \delta_{HH''} \equiv \begin{cases} 1 & \text{for } H = H'' \\ 0 & \text{for } H \neq H'' \end{cases} . \tag{B.5}$$

This result implies (2.5). It also implies that the quantities $T_H^{\ G}$ defined for every G and H by

$$T_H^{\ G} = \sum_{G' \subset G} (-1)^{n(G')} T_{HG'} \tag{B.6}$$

satisfy

$$T_{HH'} = \sum_{H'' \subset H'} (-1)^{n(H'')} T_H^{\ H''} \quad . \tag{B.7}$$

The substitution of (B.7), with $H = \phi$, into (B.6) gives

$$T_H^{\ G} = \sum_{G' \subset G} (-1)^{n(G')} \sum_{H'' \subset HG'} (-1)^{n(H'')} T^{H''}$$

$$= \sum_{G' \subset G} (-1)^{n(G')} \sum_{G'' \subset G'} (-1)^{n(G'')} \sum_{H' \subset H} (-1)^{n(H')} T^{G''H'}$$

$$= \sum_{G''} \sum_{G'' \subset G' \subset G} (-1)^{n(G')-n(G'')} \sum_{H' \subset H} (-1)^{n(H')} T^{G''H'}$$

$$= \sum_{H' \subset H} (-1)^{n(H')} T^{GH'} \quad , \tag{B.8}$$

where (B.5) is used to get the last line. This result is (4.17b). The result (4.17c) follows from repeated application of the identity

$$T_H^{\ G} = T_H^{\ Gk} + T_{Hk}^{\ \ G} \quad , \tag{B.9}$$

which follows from (B.6), which is (4.17a), via the equation

$$T_H^{\ Gk} = \sum_{G' \subset G} (-1)^{n(G)} T_{HG'} + \sum_{G' \subset G} (-1)^{n(G)+1} T_{HkG'}$$

$$= T_H^{\ G} - T_{Hk}^{\ \ G} \quad . \tag{B.10}$$

FOOTNOTES AND REFERENCES

* Work supported in part by the Western Illinois University Research Council.

† Work supported by the U. S. Atomic Energy Commission.

1. A. H. Mueller, Phys. Rev. D2 (1970) 2963.
2. C. I. Tan, Phys. Rev. D5 (1972) 1476.
3. H. P. Stapp, Phys. Rev. D3 (1971) 3177.
4. K. E. Cahill and H. P. Stapp, Optical Theorems and Steinmann Relations, Lawrence Berkeley Laboratory Report LBL-2428.
5. C. E. DeTar and J. H. Weis, Phys. Rev. D4 (1971) 3141; I. G. Halliday, Nucl. Phys. B33 (1971) 285; J. H. Weis, Phys. Rev. D5 (1972) 1043; A. R. White, Nucl. Phys. B39 (1972) 477; A. R. White, Nucl. Phys. B50 (1972) 96; J. W. Weis, Phys. Rev. D6 (1972) 2823.
6. J. H. Weis, Phys. Letters 43B (1973) 487.
7. J. Coster and H. P. Stapp, J. Math. Phys. 10 (1969) 371.
8. ibid, Appendix A.
9. J. Coster and H. P. Stapp, J. Math. Phys. 11 (1970) 1441.
10. ibid, Appendix A.
11. J. Coster and H. P. Stapp, J. Math. Phys. 11 (1970) 2743, Sec. 3.
12. ibid, Sec. 2F.
13. D. Iagolnitzer and H. P. Stapp, Commun. Math. Phys. 14 (1969) 15.
14. S. Coleman and R. F. Norton, Nuovo Cimento 38 (1965) 438.
15. C. Chandler and H. P. Stapp, J. Math. Phys. 9 (1969) 1548.
16. H. P. Stapp, J. Math. Phys. 9 (1969) 1548, eq. (3.5).
17. ibid, Sec. 3.
18. F. Pham, Ann. Inst. Henri Poincare 6A (1967) 89, or ref. [7], eq. (3.14).

19. H. Epstein, J. Math. Phys. 1 (1960) 524, Appendix; J. Bros, Axiomatic Field Theory, in High Energy Physics and Elementary Particles, International Atomic Energy Agency, Vienna, 1965, p. 90.
20. J. Wright and H. P. Stapp, Nuovo Cimento 57A (1968) 475, Appendix D.
21. D. S. Kershaw, Phys. Rev. 50 (1972) 1976.
22. R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, The Analytic S Matrix (Cambridge, 1966), eqs (2.2.11), (2.2.5), and (1.5.30).
23. Ref. [22], p. 116.
24. C. Chandler, private communication.
25. R. C. Gunning and H. Rossi, Analytic Functions of Several Complex Variables (Prentice-Hall, Englewood Cliffs, New Jersey, 1965), p. 17.
26. Ref. [25], p. 111, 113, 115, 116, 141.
27. David Mumford, Introduction to Algebraic Geometry, Preliminary Version of First Three Chapters (Mathematics Department of Harvard University, 1966), p. 15, 75.
28. F. S. Macaulay, The Algebraic Theory of Modular Systems (Cambridge Tracts in Mathematics and Mathematical Physics 19, Cambridge University Press, 1916), p. 28.
29. Ref. [25], p. 160.
30. S. Sternberg, Lectures on Differential Geometry (Prentice-Hall, Englewood Cliffs, New Jersey, 1965), p. 47.
31. Ref. [20], Appendix B, Lemma B.
32. Ref. [27], p. 97; Ref. [20], Appendix B.
33. Ref. [25], p. 17, 111, 116.

34. Ref. [25], p. 160.

35. H. P. Stapp, J. Math. Phys. 8 (1967) 1606.

LEGAL NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

TECHNICAL INFORMATION DIVISION
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720