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NUMERICAL STUDY OF VISCOUS FLOWS PAST A CIRCULAR CYLINDER: APPLICATION OF CHORIN'S METHOD

Bijan Davari

December 1973

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NUMERICAL STUDY OF VISCOUS FLOWS PAST A CIRCULAR CYLINDER: APPLICATION OF CHORIN'S METHOD

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December 1973

ABSTRACT

A. J. Chorin's Method of numerical solution of the Navier Stokes Equations for slightly viscous flows is applied to the problem of an impulsively started circular cylinder. The drag of the cylinder is computed over a range of $Re = 10$ to $Re = 10^5$, and the validity of the results is discussed.

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I. INTRODUCTION

The theoretical investigation of viscous flow past a circular cylinder, especially at high values of Reynolds Numbers has always been interesting because it could be extended (by conformal mapping) to the flow past an airfoil.

The analytical solutions of the Navier Stokes Equations are only available for special cases. Hence, it has always been desirable to find methods of solving these Equations numerically. But all the numerical methods presently available for solving the Navier Stokes Equations require grids of smaller mesh size and bigger computational effort as Reynolds Number increases.

A. J. Chorin (1973) presented a numerical method of solving the Navier Stokes Equations for slightly viscous flows in two-space dimensions (ref.[1]). This method has the following advantages:

1) It is grid free, therefore it could be used at high values of Reynolds Number.
2) It simulates, numerically, the physical process of vorticity generation and dispersal.

In the present report, Chorin's Method shall be applied to the calculation of the flow past an impulsively started circular cylinder. The drag of the cylinder for different Reynolds Numbers shall also be computed, and the computed results shall be compared with the experimental data.

II. PRINCIPLE OF THE METHOD

(Summary)

Chorin assumed that, at each time, one could partition the vorticity of the flow into sum of a finite number of blobs with the property that the stream function of a vortex blob of strength unity is:

$$\psi^0(r) = \begin{cases} \frac{1}{2\pi} \log r & r \geq \sigma \\ \frac{1}{2\pi} \frac{r}{\sigma} + \text{constant} & r < \sigma \end{cases}$$

where $\vec{r}$ represents the position of the point in the plane with respect to the vortex blob, $r = |\vec{r}|$, and σ is a cut-off length. It can be shown that one can take

$$\sigma = h/2\pi$$

where h is the computational element of length.

Now, vortices are subject to

i) convection effects

ii) diffusion effects.

To simulate these two effects, we allow vortices to move. The displacement of each vortex blob due to convection effects may be obtained by computing the velocity of the vortex blob. The latter will be the sum of the following:

a) Velocity due to the potential flow,

b) Velocity created by the vortices present in the flow.

The diffusion effects may be approximated by random walk of the vortex blobs. In fact, one can show that if the density and the structure of the vortex blobs at a time t_0 approximates the vorticity density at that time, and that if these vortex blobs are displaced randomly, their displacement having Gaussian distribution with zero mean variance $2k/Re$, then their density at a time $t_0 + nk$ will approximate the vorticity density that we would obtain by solving the diffusion equation:

$$\frac{\partial \xi}{\partial t} = \frac{1}{Re} \Delta \xi$$

where k is the time increment. Vorticity must be created at boundaries in order to satisfy the boundary conditions there.

III. WORKING FORMULAE

1) Calculation of the Vorticity Created at the Boundary

The vorticity created at each time-step, at M points on the boundary is computed using the equation

$$\xi = \frac{\partial u_s}{\partial n} - \frac{\partial u_n}{\partial s}$$

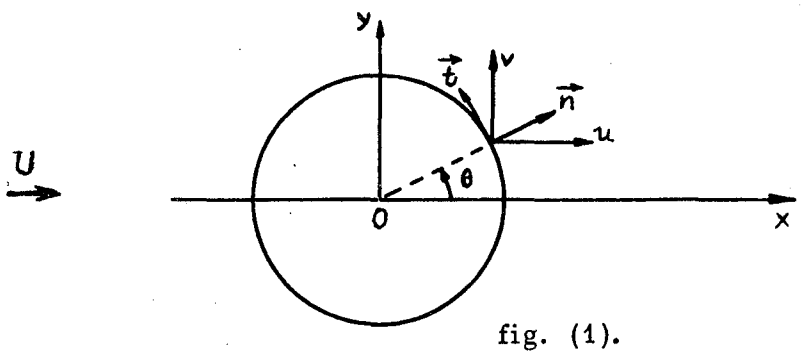


fig. (1).

The condition of zero-tangential velocity on the body states:

$$\frac{\partial u_n}{\partial s} = 0 \qquad \text{on the body.}$$

That is,

$$\xi_{(\text{boundary})} = \frac{\partial u_s}{\partial n} .$$

Therefore,

$$\bar{\xi}_{\text{boundary}} = \iint \frac{\partial u_s}{\partial n} \, dn\, ds$$

where the integration is carried over an element of surface and since $u_s = 0$ at the body,

$$\bar{\xi}_{\text{boundary}} \simeq \Delta u_s \cdot \Delta s = u_{s_1} \cdot h$$

where u_{s_1} is the tangential velocity at the vicinity of the body. Furthermore

$$u_{s_1} = v \cos\theta - u \sin\theta \qquad \text{(fig. 1)}$$

So

$$\bar{\xi}_{\text{body}} = (v \cos\theta - u \sin\theta)\, h . \qquad (1)$$

2) Random Displacement of a Vortex Blob

A random number generator subroutine exists in the computer's "System Library". It generates floating point random numbers uniformly distributed in the interval (0,1). To obtain the required Gaussianly distributed random walk, the following equations were used:

Let $\vec{E}$ be the vector displacement, η_1 and η_2 its components. Let

$$|\vec{E}| = q_r$$

$$\eta_1 = q_r \cos\theta_r \qquad \eta_2 = q_r \sin\theta_r \tag{2}$$

θ_r and q_r are given by

$$\theta_r = 2\pi\, q_2$$

$$q_r = (-2\mu^2 \log q_1)^{1/2}$$

where q_1 and q_2 are two uniformly distributed random numbers given by the generating subroutine, $\mu = 2k/Re$.

3) Velocity at a Given Point Created by a Vortex Blob

Suppose a vortex blob of strength $\bar{\xi}_j$ is placed at $\vec{r}_j$. The velocity vector created by this vortex blob at a point $\vec{r}$ is

$$\left.\begin{aligned} u(\vec{r},\vec{r}_j) &= \frac{1}{2\pi}\bar{\xi}_j \frac{y_j - y}{R_j^2} \\ v(\vec{r},\vec{r}_j) &= -\frac{1}{2\pi}\bar{\xi}_j \frac{x_j - x}{R_j^2} \end{aligned}\right\} \quad \text{if } R_j \geq \sigma$$

$$\left.\begin{aligned} u(\vec{r},\vec{r}_j) &= \frac{1}{2\pi}\bar{\xi}_j \frac{y_j - y}{\sigma\, R_j} \\ v(\vec{r},\vec{r}_j) &= -\frac{1}{2\pi}\bar{\xi}_j \frac{x_j - x}{\sigma\, R_j} \end{aligned}\right\} \quad \text{if } R_j \leq \sigma \tag{3}$$

where $\vec{R}_j = \vec{r}_j - \vec{r}$, and $R_j = |\vec{R}_j|$. It should be noted that the velocity created by a vortex blob $(\vec{r}_j; \bar{\xi}_j)$ is continuous at $R_j = \sigma$, while the stream function as defined by

$$\psi(r, r_j) = \begin{cases} \frac{1}{2\pi} \bar{\xi}_j \log (R_j) & \text{if } R_j \geq \sigma \\ \frac{1}{2\pi} \bar{\xi}_j \frac{R_j}{\sigma} & \text{if } R_j < \sigma \end{cases}$$

is discontinuous at $R_j = \sigma$. However, the latter may be made continuous by adding a constant whenever it is needed.

4) Potential Flow

The stream function of the inviscid flow past a circular cylinder with uniform velocity U at infinity upstream is given by ([2])

$$\psi(x, y) = - Uy \left(1 - \frac{a^2}{x^2 + y^2}\right) \tag{4}$$

where a is the radius of the cylinder. The velocity components are

$$\begin{aligned} u_p &= - \frac{\partial\psi}{\partial y} = U(1 - a^2(x^2 - y^2)/(x^2 + y^2)^2) \\ v_p &= \frac{\partial\psi}{\partial x} = - U(2a^2xy/(x^2 + y^2)^2) . \end{aligned} \tag{5}$$

Note that the condition of zero-normal velocity at the body is readily satisfied by the potential solution given above.

5) Flow Induced by a Vortex Blob in Presence of the Cylinder; Method of Images

Let there be a point vortex of strength $\bar{\xi}_0$ at a point (x_0, y_0) outside a cylinder of radius a. The motion due solely to the vortex can be modeled by replacing the cylinder by the images of the vortex (see [2]). The image system consists of a point vortex of strength $- \bar{\xi}_0$ at the inverse-point of (x_0, y_0), and a vortex of strength $+ \bar{\xi}_0$ at the center of the cylinder.

The inverse point of (x_0, y_0) has coordinates:

$$\begin{aligned} x_0' &= a^2 x_0/(x_0^2 + y_0^2) \\ y_0' &= a^2 y_0/(x_0^2 + y_0^2) \end{aligned} \tag{6}$$

Now, the flow induced by all vortex blobs in presence of the cylinder will be approximated by the flow induced by the vortex blobs and their images which shall be considered to be vortex blobs themselves. The advantage of this type of modeling is that it automatically satisfies the condition

$$\vec{u}_\xi \cdot \vec{n} = 0$$

at all points on the cylinder. Here $\vec{u}_\xi$ represents the velocity due to the effect of vortex blobs and their images. The image at the center of the cylinder will be a vortex of strength

$$\bar{\xi}_c = \sum_{i=1}^{N} \bar{\xi}_i$$

where $\bar{\xi}_i$ are strengths of vortex blobs outside the cylinder, and N is their number.

6) Computation of the Velocity Field

The velocity components at a given point (x, y) in the flow are given by

$$u(x,y) = u_p(x,y) + u_c(x,y) + \sum_{i=1}^{N} u_i(x,y) + \sum_{i=1}^{N} u_i'(x,y)$$

$$v(x,y) = v_p(x,y) + v_c(x,y) + \sum_{i=1}^{N} v_i(x,y) + \sum_{i=1}^{N} v_i'(x,y)$$

where u_p, v_p are velocities due to the potential flow

u_c, v_c are due to image of the vortices at center

u_i, v_i are velocities/due to the vortex blob of strength $\bar{\xi}_i$ placed at (x_i, y_i)

u_i', v_i' are velocities due to the image-vortex of strength $-\bar{\xi}_i$ placed at (x_i', y_i').

obtained by Equations (3) and (5).

Difficulty arises when the velocity created at the center of the vortex blob i

due to the presence of this vortex blob itself ("self-induced" velocity), is to be computed. To overcome this difficulty, we consider the vortex blob $(x_0, y_0; \bar{\xi}_0)$ alone in presence of the cylinder, and compute its "self-induced velocity". Let

$$z_0 = x_0 + i\, y_0, \quad \text{and} \quad \left.\frac{dw}{dz}\right|_{z = z_0} = u_0 - i\, v_0$$

where z_0 denotes the position, and u_0, v_0 are the velocities of the vortex. w is the complex velocity potential. Then

$$u_0 - i\, v_0 = -2\, i\, \frac{\partial \chi_0}{\partial z_0}$$

where

$$\chi_0 = \frac{1}{2}\, \frac{\bar{\xi}_0}{2\pi}\, g\,(z_0, \bar{z}_0;\, z_0, \bar{z}_0)$$

is Routh's Stream Function (see [2] for details), and

$$g(z, \bar{z}; z_0, \bar{z}_0) = -\frac{1}{2} \log\,(1 - a^2/z\,\bar{z}_0)\,(1 - a^2/\bar{z}\, z_0)$$

$\underline{a}$ is the radius of the cylinder. Therefore,

$$\begin{aligned} u_0 &= -\frac{1}{2\pi}\, \bar{\xi}_0\, a^2 y_0 / ((x_0^2 + y_0^2 - a^2)\,(x_0^2 + y_0^2)) \\ v_0 &= \frac{1}{2\pi}\, \bar{\xi}_0\, a^2 x_0 / ((x_0^2 + y_0^2 - a^2)\,(x_0^2 + y_0^2)). \end{aligned} \tag{8}$$

It follows that the velocity vector of a <u>vortex blob</u> $(x_i, y_i; \bar{\xi}_i)$ is given, using Equations (7) and (8):

$$\begin{aligned} \vec{u}(x_i, y_i) = {} & \vec{u}_p(x_i, y_i) + \vec{u}_c(x_i, y_i) + \vec{u}_0\,(x_i, y_i) - \vec{u}_{c_i}(x_i, y_i) \\ & + \sum_{\substack{j=1 \\ j \neq i}}^{N} (\vec{u}_j\,(x_i, y_i) + \vec{u}_j\,(x_i', y_i')) \end{aligned} \tag{9}$$

where $\vec{u}_j$ and $\vec{u}_j'$ are given by Equations (3).

7) Computation of the Drag

To compute the drag, Equations (21) through (24) of [1] (restated below) are used. Drag coefficient on a cylinder can be written

$$C_D = C_\nu + C_p$$

where C_ν and C_p the skin drag and the form drag are given by

$$C_\nu = -\frac{1}{Re}\int_{\partial D} \xi_\partial \sin\theta \, d\theta \tag{10}$$

$$C_p = \int_{\partial D} p_\partial \cos\theta \, d\theta \tag{11}$$

p_∂ and ξ_∂ are the pressure and the vorticity on the boundary ∂D, and

$$p_\partial = -\frac{1}{Re}\int_{\theta_0}^{\theta} \frac{\partial\xi}{\partial n} \, ds + \text{constant} \tag{12}$$

where $\frac{\partial\xi}{\partial n}$ is approximated as is shown in Chorin's Article (ref. [1], page 792). The constants in the above Equation are chosen as follows

$$\theta_0 = 0$$

$$p_\partial(\theta_0) = 0 \, .$$

8) Remarks

The values of U and a are normalized throughout the computations:

U = 1 uniform velocity at infinity upstream,

a = 1 radius of the cylinder,

and the Reynolds number is based on the radius of the cylinder:

$$Re = 1/\nu \, .$$

(The Reynolds Number usually used in literature is

$$Re' = \frac{2}{\nu} = 2 \, Re.$$

This has been considered in the comparison of the computed results with experimental data).

Because of the randomness of the solution, the average drag is computed at each time T

$$C_{D_{av}} = \frac{1}{T} \int_0^T C_D(t)\, dt \tag{13}$$

IV. COMPUTATIONAL ALGORITHM

(Summary)

At each time-step:

1) The velocity components on the cylinder are computed using the Equations (7).

2) The vorticity created at the points on the cylinder is computed by Equation (1).

3) The motion of each vortex blob is computed as follows:

i) velocity components of the vortex blob are computed using Equations (9); call them $u(x_i, y_i)$, $v(x_i, y_i)$.

ii) random walk of the vortex blob is computed using Equations (2).

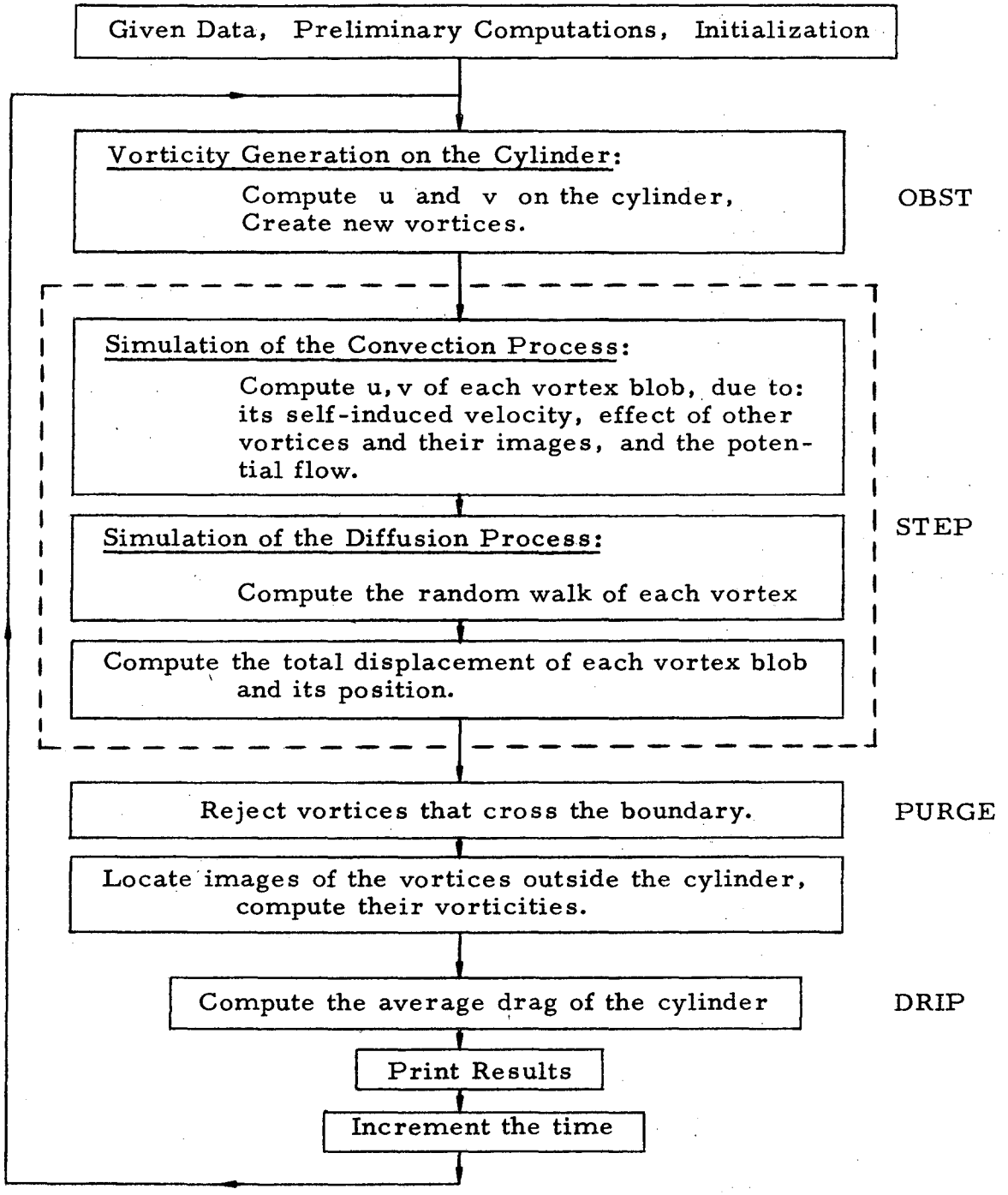
iii) the new position of the vortex blob is $(x_i + dx_i,\ y_i + dy_i)$, where

$$\begin{cases} dx_i = u(x_i, y_i)\, dt + \eta_1 \\ dy_i = v(x_i, y_i)\, dt + \eta_2 \end{cases} \tag{14}$$

4) The average drag is computed using Equations (10) through (13).

5) To represent completely the physical process of vorticity generation and dispersal, one must simulate the disappearance of those vortices that cross the boundary.

Summarized Flow-Chart



V. CONSOLIDATION OF VORTICES

As the Reynolds Number increases, fewer vortex blobs disappear by crossing the cylinder (the standard deviation of the random walk is smaller), and therefore, more vortices remain in the fluid. For example, the number of vortex blobs for Re = 10^3 after 150 time-steps could attain the value of 1800 or 2000. For this value of Reynolds Number, the computing time needed for calculation of the flow until 150 time-steps (when the average drag is stabilized) is about 12 minutes of CPU time on the CDC 7600 of the LBL. It is, therefore, needed to reduce the number of vortex blobs in the flow, and hence, to reduce the computing time (while trying not to violate the principle of the method). This would allow further computations to be done more economically.

A simple way of reducing the number of vortices is to consolidate those vortices which are far enough from the cylinder. Since one expects the formation of an asymmetric vortex street behind the cylinder, a reasonable way of reducing the number of vortices is as follows. Consider the grid shown in fig. (2).

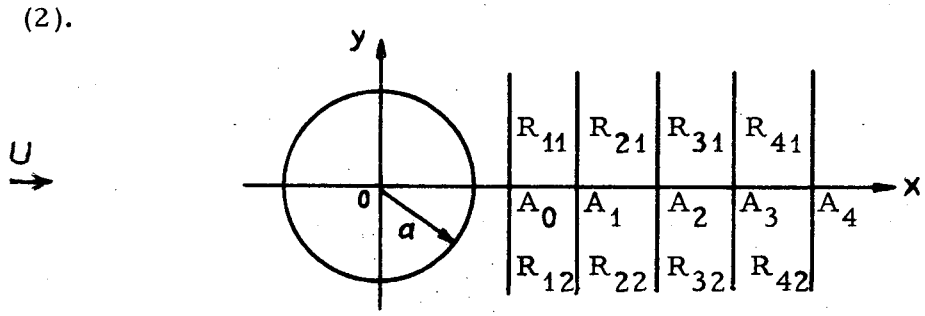


fig. (2).

All vortex blobs in the region R_{ij} may be consolidated at their "center of gravity". Let $(x_{ijk}, y_{ijk}; \bar{\xi}_{ijk})$ denote vortices whose centers fall in the region R_{ij}. The position and vorticity of their "center of gravity" are given by:

$$xcg_{ij} = (\sum_k |\bar{\xi}_{ijk}| x_{ijk})/(\sum_k |\bar{\xi}_{ijk}|)$$

$$ycg_{ij} = (\sum_k |\bar{\xi}_{ijk}| y_{ijk})/(\sum_k |\bar{\xi}_{ijk}|) \qquad (15)$$

$$\overline{\xi cg}_{ij} = \sum_k \bar{\xi}_{ijk}$$

The new vortex structure will therefore be:

- vortex blobs with $x \leq A_0$ are retained.
- vortex blobs of each region R_{ij} are rejected
- a new vortex blob is added in each region R_{ij}, its position and vorticity given by Equations (15).

At low Reynolds Numbers, the computed drag was found to be rather insensitive to the values of $A_0, A_1, \cdots$, perhaps because the diffusion effects are dominant, and the vortices (whose number is rather small) move very far in the flow, and their effects become negligible as time goes on. At high Reynolds Numbers, however, the results were seen to depend on the values of $A_0, A_1, \cdots$. After some experimentations, the following values were picked for $Re = 10^3$:

$$A_0 = 2a, \quad A_1 = 2.5\ a$$

$$A_2 = 3a, \quad A_3 = 3.5\ a$$

$$A_4 = 10a$$

where a is the radius of the cylinder. All computations have been carried out using these values of A_i, for all Reynolds Numbers. However, two additional computations were done at $Re = 10^4$, and $Re = 10^5$ without "consolidation". Their results are seen to be the same as when the "consolidation" was used. The subroutine that consolidates the vortices is called "CONSOL".

The consolidation of the vortices resulted in a reduction of CPU time from 12 minutes to 2 minutes (on CDC 7600) for $Re = 10^3$. Now, the question arises whether the consolidation violates the principle of the method. As mentioned in Chapter II, the density and the structure of the vortex blobs are shown to approximate those of the flow at a given time before the consolidation was introduced. The consolidation may violate this fact, and since it reduces the number of vortices, the results sometimes yield larger oscillations. Nevertheless, on the average, the drag is not affected too much by the fact that the consolidation is introduced since the vorticity structure near the body is unaltered.

VI. ADDITIONAL REMARKS ON THE COMPUTER PROGRAM

1) The results, for the very first time-steps, are very sensitive to the random number generator used. However, as could be expected, this effect dies out at larger time-steps.

2) To display the flow pattern, a grid is chosen and at each mesh-point of this grid, the value of the stream function is computed. A contour mapping subroutine readily available in the CDC 7600's library is used to draw the streamlines of prescribed values.

The stream function ψ at a point (x, y) in the plane is given by

$$\psi(x, y) = \psi_p(x, y) + \sum_{i=1}^{N} (\psi_i(x, y) + \psi_{0i})$$

where ψ_p is the stream function of the potential flow,

ψ_i is the stream function of the vortex of strength $\bar{\xi}_i$ located at (x_i, y_i), and its 2 images,

ψ_{0i} is a constant.

ψ_{0i} is chosen such that the stream function due to the vortex $\underline{i}$ and its images takes the value of 0 at a point (x_{0i}, y_{0i}) on the cylinder. Figure (3) shows how the point (x_{0i}, y_{0i}) is chosen.

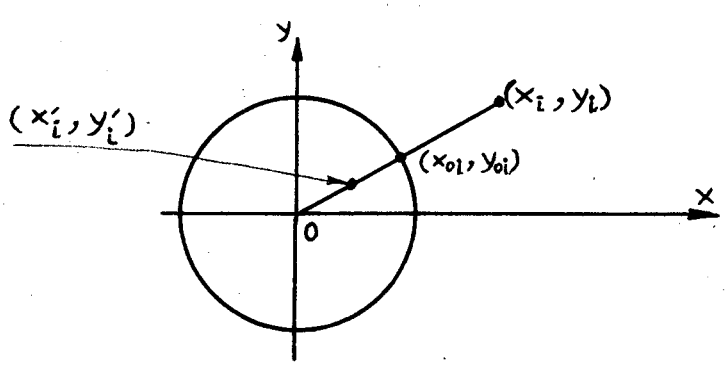


fig. (3).

ψ_p and ψ_i are given by

$$\psi_p(x,y) = -Uy\left(1 - \frac{a^2}{x^2+y^2}\right)$$

$$\psi_i(x,y) = \begin{cases} \frac{1}{2\pi}\left(\log(R_i) + c\right) & \text{if } R_i \geq \sigma \\ \frac{1}{2\pi}\frac{R_i}{\sigma} & \text{if } R_i \leq \sigma \end{cases}$$

where

$$R_i = \left[(x - x_i)^2 + (y - y_i)^2\right]^{1/2} .$$

the constant c is chosen such that ψ_i becomes continuous at $R_i = \sigma$. Its value is:

$$c = 1 - \log \sigma .$$

Some flow patterns are shown fig. (5) through fig. (12). The regions of high vorticity density are apparaent in these figures. This is where the solution is most reliable.

Note the imaginary streamlines present inside the cylinder, since the method of images replaces the real cylinder by the images of the vortices.

3) Since the velocity components at a given time, can be computed for each point, other flow variables (such as the pressure on the cylinder) can easily be evaluated at any time and position, but will contain statistical errors.

VII. RESULTS AND DISCUSSION

1) Effect of (M, k) on the Computed Drag

For a given Reynolds Number ($Re = 10^3$) the average drag is computed for different values of the couple (M, k). It is seen that for most values of (M, k) the average drag reaches a stabilized value (1.00 ± 0.05) in excellent agreement with the experimental data (1.00). However, for certain values of (M, k) the average drag does not attain a stabilized value (even within 300 time-steps), and fluctuates between 0.7 and 1.30 (Table 1). Nevertheless, it may be said that for a given Re, the results are independent of the number of points on the boundary M, from which the vorticity diffuses, and of the time-increment k.

Table 1

M \ k	0.15	0.20	0.25
15	0.7 (OS)	0.9 (OS)	0.8 (OS)
20	1.05 (S)	1.01 (S)	0.8 (OS)
25	1.35 (OS)	1.02 (S)	0.96 (S)

(S = Stabilized Value; OS = Oscillatory Value, last value given by computer is taken)

2) Drag Coefficient for Different Reynolds Numbers

The stabilized average drag for different values of Reynolds Number from $Re = 10$ to $Re = 10^5$ are computed and plotted fig. (4). In all computations the values of M and k are taken as $M = 20$ and $k = 0.20$. The computed results are in excellent agreement with experimental data over the whole range of Reynolds Numbers from 10 to 10^3. For $Re > 10^3$, the drag coefficient drops smoothly to reach a value of about 0.12 at $Re = 4 \times 10^4$. This early occurence of the "Drag Crisis" compared to the experimental results (which indicate a "Drag Crisis" at Reynolds Number of about 4×10^5) may be attributed to the "roughness" of the body. In fact, the effect of roughness on the value of critical Reynolds Number of a circular cylinder is significant (see [3], p.622). The present Method produces a "computational roughness" on the cylinder because it allows vortices to walk randomly. The transition from a laminar to a turbulent boundary layer, therefore, occurs at a lower Reynolds number than for an ideal zero-roughness cylinder. On the other hand, the roughness

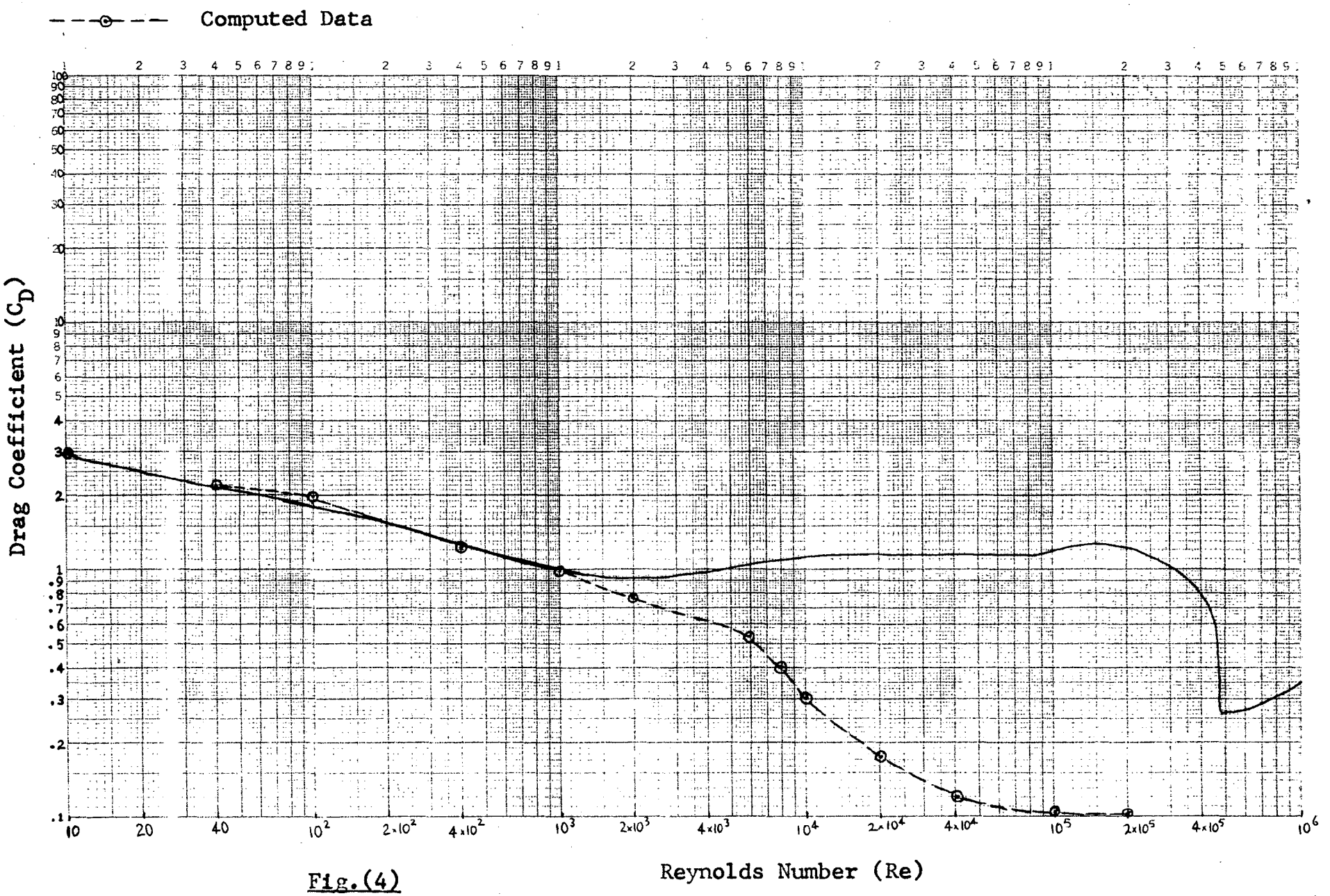

Experimental Data (Schlichting H., Boundary-Layer Theory; McGraw-Hill 1968, p.17)
Computed Data
Drag Coefficient (C_D)
Reynolds Number (Re)
10
20
40
10^2
2×10^2
4×10^2
10^3
2×10^3
4×10^3
10^4
2×10^4
4×10^4
10^5
2×10^5
4×10^5
10^6

Fig.(4)

should have resulted in a larger drag at supercritical Reynolds Numbers, and a much smaller drop in the drag at critical Reynolds Number (see [3], p. 622). The computations show a large but smooth drop in drag taking place over a range of $Re = 10^3$ to $Re = 10^5$.

The reason for this discrepancy is that the validity of the Method after the transition occurs is questionable. Also, there is no assurance that this method is capable of simulating the flow with a turbulent boundary layer. Furthermore, three-dimensional effects have not been taken into consideration in the present calculations, and their effect should be substantial.

Remark: In all the above-mentioned computations, the consolidation of vortices was used.

3) There is a general method of finding the flow induced by vortices outside a body of any shape which satisfies the zero-normal velocity condition on the body. This method, as presented in [1], consists of distributing a single layer source on the surface of the body and finding the strengths of the sources by solving an integral equation. The latter may be approximated by a system of linear algebraic equations which can be solved by usual numerical methods. This way of finding the flow induced by vortices outside the body was applied to the present problem. The drag crisis was found to occur at $Re = 10^4$ (later than that of the present method, and still sooner the experimental drag crisis). However, the results were found to depend very much on the values of M and k.

4) The flow patterns presented in figures (7) to (22) (Appendix B) show an interesting feature of the flow past an impulsively started circular cylinder. As can be seen in these figures, a periodic circulation is formed on the cylinder which corresponds to the periodic lift often observed during experimentations. The streamlines presented are those corresponding to the stream functions $\psi = 0$, $\psi = \pm 0.05$, and $\psi = \pm 0.1$ to $\psi = \pm 2$. (by increments of $\Delta\psi = \pm 0.1$).

VIII. SUGGESTED FURTHER WORK

Further investigations of the following topics may be interesting.

1) The effect of changes in M and k at different Reynolds Numbers (more detailed investigation).

2) It can be shown (ref. [1], page 791) that error increases as Reynolds Number decreases. But, it has been seen that even at low enough Reynolds Numbers (Re = 10, for example), the computed drag is in excellent agreement with the experiment. The problem is to find the exact pattern of the error as function of Re (and other parameters of the computational scheme, eventually).

3) The computer program can easily be adapted to the problem of flow past an impulsively started spinning cylinder.

4) The method may also be applied to the flow past a thin or thick airfoil, at any angle of attack.

IX. REFERENCES

1. Chorin, A. J., "Numerical Study of Slightly Viscous Flows." Journal of Fluid Mechanics (1973) vol. 57, part 4, pp. 785- 796.

2. Milne-Thomson, L. M., Theoretical Hydrodynamics, 5th Edition, 1968, Macmillan, pages 159, 364, and 365.

3. Schlichting, H., Boundary-Layer Theory, 6th Edition, 1968, McGraw Hill, pages 17, 18, and 622.

APPENDIX A

The Computer Program

```
      PROGRAM VORTEX(INPUT,OUTPUT,TAPE98=101,PLOT,TAPE99=PLOT)
      COMMON/VOR/ AVD,AVDN,CURLC,CURL(2000),CURLIM(2000),CUT,C2,DS,DSH
     1      ,DT,M,MB,N,PI,RAD,RAV,RE,RUNA,RUNN,SMALL,TPI,XAVD,JAM,JCON
     2      ,X(2000),XIM(2000),XX(40),Y(2000),YIM(2000),YY(40),UMEAN
      COMMON/DISP/XB(150),YB(150),PSID(50),MDIS,DXB,DYB,K11,K22,DPSID

      PI=3.1415926536
 3    TPI=2.*PI
 5    SMALL=1.E-10
 6    IPR=10

C**** IPR DETERMINES THE FREQUENCY OF PRINTS

 7    UMEAN=1.
11    RAD=1.

C**** UMEAN IS THE MAINSTREAM VELOCITY
C**** RAD=RADIUS OF THE CYLINDER

11    NSTP=150
12    RE=500.
14    RE1=2.*RE

C**** HERE RE=1/2 REYNOLDS NUMBER USUALLY USED IN LITERATURE

14    M=20
15    DT=0.2
17    MB=M
20    CM=MB
21    CUT=1./CM
22    C2=CUT*CUT
22    DS=TPI/CM
24    DSH=(DS/2.)**2

C**** DT IS THE TIME INCREMENT(CALLED K IN THE  PAPER)
C**** M IS THE NUMBER OF POINTS DIFFUSED FROM BOUNDARY
C**** MB IS THE NUMBER OF POINTS ON THE BOUNDARY
C**** CUT=CUT-OFF LENGTH (CALLED SIGMA IN THE PAPER)
C**** DS IS ELEMENT OF LENGTH (CALLED H IN THE PAPER)
C**** DSH=(H/2)**2

26    DO 10 I=1,MB
30    CIM=I-1
31    XX(I)=RAD*COS(TPI*CIM/CM)
40    YY(I)=RAD*SIN(TPI*CIM/CM)
47 10 CONTINUE

C**** XX AND YY ARE THE COORDINATES OF POINTS ON THE BOUNDARY
C**** (NOTE THAT THE RADIUS RAD OF THE CYLINDER IS 1.)


C**** INITIALIZATION


C**** N IS THE NUMBER OF VORTICES
```

```
 51          N=0
 51          AVD=0.
 52          AVDN=0.
 53          RUNN=0.
 53          RUNA=0.
 54          CURLC=0.
 54          NN=100
 56          DO 20 I=1,NN
 64          X(I)=0.
 65          Y(I)=0.
 65          CURL(I)=0.
 66          XIM(I)=0.
 66          YIM(I)=0.
 66          CURLIM(I)=0.
 66       20 CONTINUE
 67          N11=0
 70          N9=0

      C**** X AND Y ARE COORDINATES OF VORTEX BLOBS
      C**** CURL = VORTICITY OF THE BLOB (X,Y)(CALLED XI-BAR IN PAPER)
      C**** XIM,YIM ARE COORDINATES OF IMAGES OF VORTICES (INVERSE POINTS)
      C**** CURLIM IS THE VORTICITY OF (XIM,YIM)
      C**** CURLC IS SUM OF VORTICITIES OF 2ND IMAGES AT THE CENTER)


 71          PRINT 9004
 74          PRINT 9003
100          PRINT 9001,M,DT
110          PRINT 9002,RE1
116          PRINT 9003
122          PRINT 9005


      C******************
      C*    TIME STEP   *
      C******************

126          TIME=0.
127          DO 80 ISTP=1,NSTP
131          TIME=TIME+DT
132          IYES=(ISTP/IPR)*IPR

      C**** IYES = ISTP  IF  ISTP=IN*IPR ,  IN=INTEGER
      C**** (RESULTS ARE PRINTED)


137          CALL OBST

      C**** OBST GENERATES VORTICES ON THE BOUNDARY

140          IF(N.EQ.0)GO TO 30
141          CALL STEP
142       30 CONTINUE

      C**** STEP ADVANCES THE VORTICES
```

```
142          N10=N
143          IF(N.LE.100) GO TO 40
146          CALL CONSOL
147       40 CONTINUE
147          N9=N9+N10-N

     C**** CONSOL CONSOLIDATESTHE VORTICES THAT LIE IN CERTAIN REGIONS
     C**** FAR ENOUGH FROM THE BODY


     C**** ALL VORTICES CROSSING THE BOUNDARY MUST DISAPPEAR

151          N8=N
152          CALL PURGE
153          N11=N11+N8-N

     C**** COMPUTATION OF IMAGES AND THEIR VORTICITIES

155          CURLC=0.
156          IF(N.LE.0) GO TO 51
157          DO 50 I=1,N
167          CIMG=RAD**2/(X(I)**2+Y(I)**2)
171          XIM(I)=CIMG*X(I)
172          YIM(I)=CIMG*Y(I)
174          CURLIM(I)=-CURL(I)
174          CURLC=CURLC+CURL(I)
176       50 CONTINUE
177       51 CONTINUE

177          JAM=1
200          IF(ISTP.EQ.IYES)JAM=0

203          CALL DRIP

     C**** DRIP COMPUTES THE PRESSURE ON THE BOUNDARY, THE SKIN DRAG, THE
     C**** FORM DRAG, AND THE TOTAL DRAG (EQUATIONS21 THROUGH 24)

204          IF(ISTP.NE.IYES) GO TO 70

206          SUMV=0.
207          DO 60 I=1,N
214          SUMV=SUMV+CURL(I)
215       60 CONTINUE

     C**** SUMV IS THE TOTAL VORTICITY

216          PRINT 9006,TIME,RAV,XAVD,N,SUMV,N9,N11

     C**** RAV IS THE AVERAGE OF TOTAL DRAG  FROM (TIME) TO (TIME+IPR*DT)
     C**** XAVD IS THE AVERAGE OF TOTAL DRAG FROM 0 TO TIME+IPR*DT
     C**** N9 = TOTAL NUMBER OF VORTICES BY CONSOLIDATION (CONSOL)
     C**** N11 = TOTAL NUMBER OF VORTICES DISAPPEARED (IN PURGE)

240          RUNN=0.
241          RUNA=0.
242       70 CONTINUE
```

```
    242      80 CONTINUE
    245      90 CONTINUE

           9001 FORMAT(1H.,48X,1H.,/,1H.,5X,*M =*,I3,10X,*DT =*,F3.2,20X,1H.)
           9002 FORMAT(1H.,48X,1H.,/,1H.,5X,*REYNOLDS NUMBER = *,E9.2,16X,1H.,/,
                1.,48X,1H.)
           9003 FORMAT(50(1H.))
           9004 FORMAT(1H1)
           9005 FORMAT(///,11X,*TIME*,8X,*RAV*,7X,*XAVD*,8X,*N*,8X,*SUMV*,8X,*N9
                16X,*N11*,/)
           9006 FORMAT(10X,F6.2,5X,F6.3,5X,F6.3,5X,I4,5X,F7.3,5X,I4,5X,I4)

    245         CALL EXIT
    246         END


PROGRAM LENGTH INCLUDING I/O BUFFERS

02204

FUNCTION ASSIGNMENTS

STATEMENT ASSIGNMENTS
30      - 000143     40      - 000150     51      - 000200     70      - 000243
9001    - 000271     9002    - 000300     9003    - 000311     9004    - 000314

BLOCK NAMES AND LENGTHS
VOR     - 027507/01  DISP    - 000544/02

VARIABLE ASSIGNMENTS
AVD     - 000000/01  AVDN    - 000001/01  CIM     - 000356     CIMG    - 000367
CURLC   - 000002/01  CURLIM  - 003723/01  CUT     - 007643/01  C2      - 007644/01
DT      - 007647/01  I       - 000355     IPR     - 000351     ISTP    - 000363
M       - 007650/01  MB      - 007651/01  N       - 007652/01  NN      - 000357
N11     - 000360     N8      - 000366     N9      - 000361     PI      - 007653/01
RAV     - 007655/01  RE      - 007656/01  RE1     - 000353     RUNA    - 007657/01
SUMV    - 000370     TIME    - 000362     TPI     - 007662/01  UMEAN   - 027506/01
XB      - 000000/02  XIM     - 013606/01  XX      - 017526/01  Y       - 017576/01
YY      - 027436/01

START OF CONSTANTS-000251        TEMPS--000336    INDIRECTS-000346

7600 COMPILATION -- RUN76 LEVEL 7B       73/11/18.

ROUTINE COMPILES IN 044500
```

```
      SUBROUTINE OBST
      COMMON/VOR/ AVD,AVDN,CURLC,CURL(2000),CURLIM(2000),CUT,C2,DS,DSH
     1      ,DT,M,MB,N,PI,RAD,RAV,RE,RUNA,RUNN,SMALL,TPI,XAVD,JAM,JCON
     2      ,X(2000),XIM(2000),XX(40),Y(2000),YIM(2000),YY(40),UMEAN

C**** EVALUATION OF VORTICITY ON THE BOUNDARY

      DO 40 I=1,M

C**** U,V DUE TO THE POTENTIAL FLOW AROUND THE CYLINDER

      U=UMEAN*(1.-(XX(I)**2-YY(I)**2))
    7 V=-2.*UMEAN*XX(I)*YY(I)
C**** (NOTE THAT THE RADIUS RAD OF THE CYLINDER = 1.)

   13 IF(N.EQ.0)GO TO 30
   14 DO 20 J=1,N

C**** U,V AT XX(I),YY(I) DUE TO VORTICES OUTSIDE THE OBSTACLE

   16 CX=X(J)-XX(I)
   23 CY=Y(J)-YY(I)
   24 R2=CX**2+CY**2
   26 IF(R2.LT.SMALL) GO TO 10
   31 IF(R2.LE.C2)R2=SQRT(R2)*CUT
   40 U=U+CURL(J)*CY/(R2*TPI)
   44 V=V-CURL(J)*CX/(R2*TPI)
   50   10 CONTINUE

C**** U,V AT XX(I),YY(I) DUE TO VORTICES INSIDE THE OBSTACLE
C**** (IMAGES OF REAL VORTICES OUTSIDE THE OBSTACLE)

   52 CX=XIM(J)-XX(I)
   54 CY=YIM(J)-YY(I)
   56 R2=CX**2+CY**2
   60 IF(R2.LT.SMALL) GO TO 20
   63 IF(R2.LE.C2)R2=SQRT(R2)*CUT
   72 U=U+CURLIM(J)*CY/(R2*TPI)
   76 V=V-CURLIM(J)*CX/(R2*TPI)
  102   20 CONTINUE

C**** U,V DUE TO THE VORTICITY (CURLC) AT THE CENTER OF THE CYLINDER

  107 CX=-XX(I)
  110 CY=-YY(I)
  111 R2=CX**2+CY**2
  113 U=U+CURLC*CY/(R2*TPI)
  116 V=V-CURLC*CX/(R2*TPI)
  122   30 CONTINUE

C**** GENERATION OF NEW VORTICES ON THE OBSTACLE

  123 K=N+I
  124 X(K)=XX(I)
  127 Y(K)=YY(I)
  130 CURL(K)=(V*XX(I)-U*YY(I))*DS
```

```
135      40 CONTINUE
140         N=N+M

141         RETURN
141         END
```

SUBPROGRAM LENGTH

00200

FUNCTION ASSIGNMENTS

STATEMENT ASSIGNMENTS

10	- 000051	20	- 000103	30	- 000123

BLOCK NAMES AND LENGTHS

VOR - 027507/01

VARIABLE ASSIGNMENTS

CURL	- 000003/01	CURLC	- 000002/01	CURLIM	- 003723/01	CUT	- 007643/01
C2	- 007644/01	DS	- 007645/01	I	- 000170	J	- 000173
N	- 007652/01	R2	- 000176	SMALL	- 007661/01	TPI	- 007662/01
V	- 000172	X	- 007666/01	XIM	- 013606/01	XX	- 017526/01
YY	- 027436/01						

START OF CONSTANTS-000144 TEMPS--000147 INDIRECTS-000161

7600 COMPILATION -- RUN76 LEVEL 7B 73/11/18.

ROUTINE COMPILES IN 044100

```
              SUBROUTINE STEP
              COMMON/VOR/ AVD,AVDN,CURLC,CURL(2000),CURLIM(2000),CUT,C2,DS,DSH
             1       ,DT,M,MB,N,PI,RAD,RAV,RE,RUNA,RUNN,SMALL,TPI,XAVD,JAM,JCON
             2       ,X(2000),XIM(2000),XX(40),Y(2000),YIM(2000),YY(40),UMEAN
              DIMENSION DX(2000),DY(2000)

              NPP=N-M
   3          TSIG=4.*DT/RE
   6          DO 10 I=1,N
  13          DX(I)=0.
  14          DY(I)=0.
  14       10 CONTINUE

      C**** EVALUATION OF THE VELOCITY  AT (X(I),Y(I))

  15          DO 60 I=1,N

      C**** U,V DUE TO THE POTENTIAL FLOW AROUND THE CYLINDER

  16          R2=X(I)**2+Y(I)**2
  22          CPOT=(RAD**2)/(R2**2)
  24          U=UMEAN*(1.-CPOT*(X(I)**2-Y(I)**2))
  31          V=-2.*UMEAN*CPOT*X(I)*Y(I)
  34          IF(NPP.LE.0) GO TO 50
  36          RU=R2-RAD**2

  41          DO 40 J=1,NPP

  42          IF(I.NE.J) GO TO 20

      C**** U,V DUE TO THE PRESENCE OF THE VORTEX BLOB (X(I),Y(I)) ALONE
      C**** OUTSIDE THE CYLINDER
      C**** NOTE THAT IF THE VORTEX BLOB IS ON THE CYLINDER, ITS INDUCED
      C**** VELOCITY ON ITSELF IS NEGLECTED

  43          IF(RU.LE.SMALL) GO TO 40
  51          RU1=CURL(I)*RAD**2/(RU*R2*TPI)
  54          U=U+RU1*Y(I)
  57          V=V-RU1*X(I)
  61          U=U+CURL(I)*Y(I)/(R2*TPI)
  64          V=V-CURL(I)*X(I)/(R2*TPI)
  67          GO TO 40
  67       20 CONTINUE

      C**** EFFECT OF OTHER VORTICES

  71          CX=X(J)-X(I)
  73          CY=Y(J)-Y(I)
  75          R2=CX**2+CY**2
  77          IF(R2.LE.SMALL) GO TO 30
 102          IF(R2.LE.C2)R2=SQRT(R2)*CUT
 111          U=U+CURL(J)*CY/(R2*TPI)
 115          V=V-CURL(J)*CX/(R2*TPI)
 121       30 CONTINUE

      C**** EFFECT  OF  OBSTACLE (U,V DUE TO IMAGES)
```

```
123          CX=XIM(J)-X(I)
125          CY=YIM(J)-Y(I)
127          R2=CX**2+CY**2
131          IF(R2.LE.SMALL) GO TO 40
134          IF(R2.LE.C2)R2=SQRT(R2)*CUT
143          U=U+CURLIM(J)*CY/(R2*TPI)
147          V=V-CURLIM(J)*CX/(R2*TPI)
153       40 CONTINUE
157          CX=-X(I)
160          CY=-Y(I)
161          R2=CX**2+CY**2
163          IF(R2.LE.SMALL) GO TO 50
170          U=U+CURLC*CY/(R2*TPI)
173          V=V-CURLC*CX/(R2*TPI)
175       50 CONTINUE

     C**** RANDOM WALK OF THE VORTEX AT (X(I),Y(I))

175          Q1=RGEN(0.)
177          Q2=RGEN(0.)

     C**** RGEN GENERATES FLOATING POINT NUMBERS UNIFORMLY DISTRIBUTED
     C**** IN THE INTERVAL (0,1)

201          QR=SQRT(-TSIG*ALOG(Q1))

     C**** TOTAL  WALK OF THE VORTEX AT (X(I),Y(I))

207          DX(I)=U*DT+QR*COS(TPI*Q2)
220          DY(I)=V*DT+QR*SIN(TPI*Q2)
230       60 CONTINUE

232          DO 70 I=1,N

     C**** NEW POSITION OF THE VORTEX (X(I),Y(I))

240          X(I)=X(I)+DX(I)
241          Y(I)=Y(I)+DY(I)
243       70 CONTINUE

244          RETURN
244          END
```

SUBPROGRAM LENGTH

10163

FUNCTION ASSIGNMENTS

STATEMENT ASSIGNMENTS

20	- 000070	30	- 000122	40	- 000154	50	- 000176

BLOCK NAMES AND LENGTHS

PURGE

```
                SUBROUTINE PURGE
                COMMON/VOR/ AVD,AVDN,CURLC,CURL(2000),CURLIM(2000),CUT,C2,DS,DSH
               1       ,DT,M,MB,N,PI,RAD,RAV,RE,RUNA,RUNN,SMALL,TPI,XAVD,JAM,JCON
               2       ,X(2000),XIM(2000),XX(40),Y(2000),YIM(2000),YY(40),UMEAN

        C**** ALL VORTICES CROSSING THE BOUNDARY MUST DISAPPEAR

                RAD2=RAD**2
      2         NAFTER=N
      4         NAFN=N-1
      6         DO 40 I=1,N
      7      10 CONTINUE
      7         IF(I.GT.NAFTER) GO TO 40
     13         DIST=X(I)**2+Y(I)**2
     15         IF(DIST.GE.RAD2) GO TO 40
     20         IF(I.EQ.NAFTER) GO TO 30
     22         DO 20 J=I,NAFN
     30         X(J)=X(J+1)
     31         Y(J)=Y(J+1)
     32         CURL(J)=CURL(J+1)
     33      20 CONTINUE
     34      30 CONTINUE
     35         NAFTER=NAFTER-1
     36         NAFN=NAFTER-1

     37         GO TO 10
     37      40 CONTINUE
     42         N=NAFTER
     43         RETURN
     43         END
```

SUBPROGRAM LENGTH

00063

FUNCTION ASSIGNMENTS

STATEMENT ASSIGNMENTS

10	- 000010	30	- 000035	40	- 000040

BLOCK NAMES AND LENGTHS

VOR - 027507/01

VARIABLE ASSIGNMENTS

CURL	- 000003/01	CURLIM	- 003723/01	DIST	- 000061	I	- 000060
NAFN	- 000057	NAFTER	- 000056	RAD	- 007654/01	RAD2	- 000055
XX	- 017526/01	Y	- 017576/01	YIM	- 023516/01	YY	- 027436/01

START OF CONSTANTS-000046 TEMPS--000047 INDIRECTS-000053

7600 COMPILATION -- RUN76 LEVEL 7B 73/11/18.

ROUTINE COMPILES IN 044000

DRIP

```
      SUBROUTINE DRIP
      COMMON/VOR/ AVD,AVDN,CURLC,CURL(2000),CURLIM(2000),CUT,C2,DS,DSH
     1      ,DT,M,MB,N,PI,RAD,RAV,RE,RUNA,RUNN,SMALL,TPI,XAVD,JAM,JCON
     2      ,X(2000),XIM(2000),XX(40),Y(2000),YIM(2000),YY(40),UMEAN
      REAL MU
      DIMENSION SURF(40),SURFT(40)
      CM=M

C**** COMPUTATION OF DRAG

C**** MU=STANDARD DEVIATION OF THE RANDOM NUMBERS USED IN STEP
C**** OUT1 DETERMINES REGION (AJ)
C**** OUT2 DETERMINES REGION (AJ+)

 2    MU=SQRT(2.*DT/RE)
10    OUT1=(1.+MU)**2
11    OUT2=(1.+2.*MU)**2
14    DO 23 J=1,M
21    SURF(J)=0.
22    SURFT(J)=0.
22 23 CONTINUE
23    DO 20 I=1,N
24    R2=X(I)**2+Y(I)**2
26    IF(R2.GT.OUT2)GOTO20
32    IF(R2.GT.OUT1)GOTO21
35    CLOSE=10.
36    DO 22 J=1,M
37    DIST=(X(I)-XX(J))**2+(Y(I)-YY(J))**2
44    IF(DIST.GT.CLOSE)GOTO22
50    CLOSE=DIST
50    JP=J
51 22 CONTINUE
54    SURF(JP)=SURF(JP)+CURL(I)
57 21 CONTINUE
57    CLOSE=10.
61    DO 24 J=1,M
62    DIST=(X(I)-XX(J))**2+(Y(I)-YY(J))**2
67    IF(DIST.GT.CLOSE)GOTO24
73    JP=J
73    CLOSE=DIST
75 24 CONTINUE
100   SURFT(JP)=SURFT(JP)+CURL(I)
103 20 CONTINUE

C**** JP DESIGNATES THE POINT (XX(JP),YY(JP)) ON THE BOUNDARY,NEAREST
C**** TO THE POINT (X(I),Y(I))
C**** SURF(J)= SUM OF VORTICES WHOSE CENTERS FALL WITHIN REGION (AJ)
C**** SURFT(J)=SUM OF VORTICES WHOSE CENTERS FALL WITHIN REGION (AJ+)

106   CC1=PI*(OUT1-1.)/CM
111   CC2=PI*(OUT2-OUT1)/CM
113   DO 25 J=1,M
122   SURF(J)=SURF(J)/CC1
123   SURFT(J)=SURFT(J)/CC2
124 25 CONTINUE
```

DRIP

```
C**** SURF(J) HERE APPROXIMATES XI(QJ)
C**** SURFT(J) HERE APPROXIMATES XI(QJ)+DXI(QJ)

125        DTETA=TPI/CM
127        PP=0.
127        FORM=0.
130        SKIN=0.

C**** PP   IS PRESSURE
C**** FORM IS FORM DRAG
C**** SKIN IS SKIN DRAG

131        DO 9 J=1,M
132        JP=J+1
133        IF(J.EQ.M)JP=1

C**** (SURFT(J)-SURF(J))/MU APPROXIMATES D(XI(QJ))/DN (SEE PARAGRAPH
C**** AFTER EQUATION 23)

C**** EQUATION (23)

136        PP=PP-DS*(SURFT(J)-SURF(J))/(MU*RE)
144        IF(J.EQ.M)PP=PP/2.

C**** EQUATION (22)

150        FORM=FORM-PP*DTETA*0.5*(XX(J)+XX(JP))

C**** EQUATION (21)

156        SKIN=SKIN-SURF(J)*YY(J)*DTETA/RE
162      9 CONTINUE
165        TOT=FORM+SKIN
167        AVD=AVD+TOT
170        AVDN=AVDN+1.
172        XAVD=AVD/AVDN
173        RUNA=RUNA+TOT
174        RUNN=RUNN+1.

C**** EQUATION (24)

176        RAV=RUNA/RUNN

C**** TOT IS TOTAL DRAG
C**** RAV IS THE AVERAGE OF TOTAL DRAG  FROM (TIME) TO (TIME+IPR*DT)
C**** XAVD IS THE AVERAGE OF TOTAL DRAG FROM 0 TO TIME+IPR*DT


177        RETURN
177        END
```

SUBPROGRAM LENGTH

00372

```
      SUBROUTINE CONSOL
      COMMON/VOR/ AVO,AVON,CURLC,CURL(2000),CURLIM(2000),CUT,C2,DS,DSH
     1       ,DT,M,MB,N,PI,RAD,RAV,RE,RUNA,RUNN,SMALL,TPI,XAVO,JAM,JCON
     2       ,X(2000),XIM(2000),XX(40),Y(2000),YIM(2000),YY(40),UMEAN
      DIMENSION AX(10,10),AY(10,10),AD(10,10),AC(10,10),XCG(10,10)
     1,YCG(10,10),XIN(1000),YIN(1000),CURLIN(1000),FAR(10)

      N10=N
      NDIV=4
      AO=2.*RAD
      FAR(1)=2.5*RAD
      FAR(2)=3.*RAD
      FAR(3)=3.5*RAD
      FAR(4)=10.*RAD
      NK=2
      IN=0
      NJ=0
      DO 99 J=1,10
      DO 99 K=1,10
      AX(J,K)=0.
      AY(J,K)=0.
      AC(J,K)=0.
      AD(J,K)=0.
      XCG(J,K)=0.
      YCG(J,K)=0.
   99 CONTINUE

      DO 103 I=1,N
      ACURL=ABS(CURL(I))
      PRX=ACURL*X(I)
      PRY=ACURL*Y(I)
      IF(X(I).GT.AO)GO TO 100
      IN=IN+1
      XIN(IN)=X(I)
      YIN(IN)=Y(I)
      CURLIN(IN)=CURL(I)
      GO TO 103
  100 CONTINUE
      DO 102 J=1,NDIV
      IF(X(I).GT.FAR(J)) GO TO 102
      K=2
      IF(Y(I).GT.0.) K=1
      AX(J,K)=AX(J,K)+PRX
      AY(J,K)=AY(J,K)+PRY
      AD(J,K)=AD(J,K)+ACURL
      AC(J,K)=AC(J,K)+CURL(I)
      NJ=J
      GO TO 103
  102 CONTINUE
  103 CONTINUE

      IF(NJ.LE.0) GO TO 107
      DO 104 J=1,NJ
      DO 104 K=1,NK
      IF(AD(J,K).LT.SMALL) GO TO 104
      XCG(J,K)=AX(J,K)/AD(J,K)
```

```
CONSOL
    132          YCG(J,K)=AY(J,K)/AD(J,K)
    134      104 CONTINUE
    141          DO 105 J=1,NJ
    143          DO 105 K=1,NK
    153          I=J+NJ*(K-1)
    156          X(I)=XCG(J,K)
    157          Y(I)=YCG(J,K)
    161          CURL(I)=AC(J,K)
    162      105 CONTINUE
    166          NI=NK*NJ
    170          IF(IN.LE.0) GO TO 107
    172          DO 106 L=1,IN
    203          X(L+NI)=XIN(L)
    204          Y(L+NI)=YIN(L)
    205          CURL(L+NI)=CURLIN(L)
    206      106 CONTINUE
    207          N=NI+IN
    211          N9=N+1
    213          DO 108 K=N9,N10

    220          X(K)=0.
    221          Y(K)=0.
    221          CURL(K)=0.
    221      108 CONTINUE
    222      107 CONTINUE
    222          RETURN
    223          END


SUBPROGRAM LENGTH

07320

FUNCTION ASSIGNMENTS

STATEMENT ASSIGNMENTS
100     - 000061     102     - 000111     103     - 000114     104     - 000135

BLOCK NAMES AND LENGTHS
VOR     - 027507/01

VARIABLE ASSIGNMENTS
AC      - 000723     ACURL   - 007312     AD      - 000557     AX      - 000247
CURL    - 000003/01  CURLIM  - 003723/01  CURLIN  - 005317     FAR     - 007267
J       - 007307     K       - 007310     L       - 007316     N       - 007652/01
NJ      - 007306     NK      - 007304     N10     - 007301     N9      - 007317
RAD     - 007654/01  SMALL   - 007661/01  X       - 007666/01  XCG     - 001067
XX      - 017526/01  Y       - 017576/01  YCG     - 001233     YIM     - 023516/01

START OF CONSTANTS-000226        TEMPS--000234     INDIRECTS-000240

7600 COMPILATION -- RUN76 LEVEL 78      73/11/18.

ROUTINE COMPILES IN 044400
```

DISPL

```
              SUBROUTINE DISPL
              COMMON/VOR/ AVD,AVDN,CURLC,CURL(2000),CURLIM(2000),CUT,C2,DS,DSH
             1       ,DT,M,MB,N,PI,RAD,RAV,RE,RUNA,RUNN,SMALL,TPI,XAVD,JAM,JCON
             2       ,X(2000),XIM(2000),XX(40),Y(2000),YIM(2000),YY(40),UMEAN
              COMMON/CONCOM/IMIN,IMAX,ISTOP,JMIN,JMAX,JSTOP,MAX,K1,PPXMAX,PPYMAX
              COMMON/DISP/PSID(50)
              DIMENSION XB(110),YB(110),PSI(110,110)
              DIMENSION PSIC(2000)
              IMAX=101
  2           IMIN=1
  3           ISTOP=1
  4           JMAX=51
  5           JMAX=26
  6           JMIN=1
  7           JSTOP=1
 10           MAX=110
 11           K1=31
 12           PPXMAX=18.
 13           PPYMAX=9.
 15           PPYMAX=4.5
 16           DXB=.2
 20           DYB=.2
 20           DPSID=.1
 22           DO 502 I=1,26
 30           DI=I-13
 31           YB(I)=DYB*DI
 32       502 CONTINUE
 33           DO 501 I=1,101
 40           DI=I-26
 41           XB(I)=DXB*DI
 42       501 CONTINUE
 43           DO 505 I=1,40
 50           CI=I-20
 51           PSID(I)=CI*DPSID
 52       505 CONTINUE
 53           PSID(41)=.05
 54           PSID(42)=-.05
 55           K1=42

     C**** COMPUTATION OF STREAM-FUNCTIONS

 57           CONST=(ALOG(CUT)-CUT)*CUT
 62           DO 4 K=1,N
 65           PSIC(K)=0.
 65           R2=X(K)**2+Y(K)**2
 70           R1=SQRT(R2)
 72           R=R1+1.
 74           PS=ALOG(R)
 76           RIM2=XIM(K)**2+YIM(K)**2
101           RIM1=SQRT(RIM2)
103           RIM=RIM1+1.
105           PSIM=ALOG(RIM)
107           PSIC(K)=(PS-PSIM)*CURL(K)
113         4 CONTINUE
115           DO 2 I=1,IMAX
116           DO 2 J=1,JMAX
```

DISPL

```
122          PSI(I,J)=0.
122          RB2=XB(I)**2+YB(J)**2
125          RB1=SQRT(RB2)
127          PSCEN=(R+CONST)/CUT
132          IF(RB1.GE.CUT) PSCEN=ALOG(RB1)
140          IF(N.LT.1) GO TO 5

       C**** STREAM FUNCTIONS OF VORTICES AND THEIR IMAGES

143          DO 3 K=1,N
144          RK=SQRT(X(K)**2+Y(K)**2)
152          RTEST=RK-1.
153          CX=XB(I)-X(K)
156          CY=YB(J)-Y(K)
161          R2=CX**2+CY**2
164          R=SQRT(R2)
165          IF((RTEST.LT.CUT).AND.(R.LT.CUT)) GO TO 3
175          PS=(R+CONST)/CUT
177          IF(R.GE.CUT) PS=ALOG(R)
205          CX=XB(I)-XIM(K)
210          CY=YB(J)-YIM(K)
213          RIM2=CX**2+CY**2
216          RIM=SQRT(RIM2)
217          PSIM=(RIM+CONST)/CUT
222          IF(RIM.GE.CUT) PSIM=ALOG(RIM)
227          PSI(I,J)=PSI(I,J)+(PS-PSIM)*CURL(K)-PSIC(K)
237        3 CONTINUE
242          PSI(I,J)=PSI(I,J)+ PSCEN*CURLC
247        5 CONTINUE


       C**** STREAM FUNCTION OF POTENTIAL FLOW

247          IF(RB2.LE.SMALL) RB2=SMALL
256          PSIP=-UMEAN*YB(J)*(1.-RAD**2/RB2)
263          PSI(I,J)=PSI(I,J)/TPI+PSIP
265        2 CONTINUE
272          SUMPS=0.
273          DO 10 K=1,N
300       10 SUMPS=SUMPS+PSIC(K)
302          PRINT 9010,CURLC,SUMPS
      9010 FORMAT(2(10X,F10.4))

312          CALL CONMAP(XB,YB,PSI,PSID,JCON)

       C**** GIVEN A GRID WITH NODE POINTS XB(I,J),YB(I,J) , WITH STREAM-
       C**** FUNCTION PSI(I,J) AT EACH NODE POINT, CONMAP DRAWS STREAM-LINES
       C**** WHOSE VALUES ARE THE PRESCRIBED PSID(K)

316          RETURN
317          END
```

SUBPROGRAM LENGTH

NOMENCLATURE

The definitions of some variables used in the computer program are given below. Variables not mentioned below, are defined in the program immediately before they are used.

AVD	Sum of total drag from t = 0 to t = TIME.
AVDN	Number of time-steps over which the average drag is computed (here, AVDN = ISTP).
CURL(I)	a N-vector denoting the vorticities of the vortices outside the cylinder placed at (x_i, y_i) $(= \bar{\xi}_i)$.
CUT	Cut-off length ($= \sigma$).
C2	$= \sigma^2$
DS	Element of length (= h).
DSH	$= (h/2)^2$.
DT	Time-increment (= k).
DX(I), DY(I)	N-vectors denoting the total displacement of vortices
FAR(I)	Represents regions of consolidation of vortices ($= A_i$).
FORM	Form drag ($= C_p$).
IPR	Frequency of print of the results.
ISTP	Time-increment index.
IYES	IYES = ISTP if ISTP = IPR $\times$ an integer (results are printed if IYES = ISTP).
M	Number of points on the boundary where vorticity is generated and diffused.
MB	Number of points on the boundary where calculations are done (here, M = MB).
MU	Standard deviation of the Gaussianly distributed random displacement of vortices. ($= \mu$).
N	Total number of vortices present in the flow (outside the cylinder) at t = TIME.

NPP	Number of vortices outside the cylinder, excluding the vortices just created on the cylinder (= N-M).
NSTP	Total number of time-steps.
N11	Total number of vortices disappeared by crossing the boundaries (from t = 0 to t = TIME).
N9	Reduction of number of vortices by consolidation (from t = 0 to t = TIME).
OUT2	$= \frac{1}{\pi}$ (area of the circle of radius $(1 + 2\mu)$) representing the regions A_j^+.
PI	$= \pi$ (= 3.1415926536).
PP	Pressure at a point on the cylinder.
QR	Length of the random displacement vector of each vortex blob due to diffusion process.
Q1, Q2	Two random numbers uniformly distributed in the interval (0, 1).
RAD	Radius of the cylinder (= a). <u>In this program</u>, <u>RAD = 1</u>.
RAV	$= \frac{1}{IPR \times DT} \int_{TIME-IPR \times DT}^{TIME} (\text{total drag})\, dt$
RE	Reynolds Number used in this computation
RE1	Reynolds number usually used in literature (RE1 = 2×RE).
RUNA	Total drag from (TIME - IPR×DT) to (TIME).
RUNN	Number of time-steps from (TIME - IPR×DT) to (TIME).
SMALL	An arbitrary small positive number.
SKIN	Skin drag (= C_ν).
SUMV	Total vorticity in the flow (at each time-step).
TIME	Time at which calculations are done.
TOT	Total drag (= C_D).
TPI	$= 2\pi$

TSIG	$= 2\,\mu^2$.
U, V	Components of velocity vector at the point (x, y).
UMEAN	Mainstream velocity (UMEAN = 1. in this computations).
X(I), Y(I)	Coordinates of the position of vortices outside the cylinder.
XX(I), YY(I)	Coordinates of the points on the cylinder (a MB-vector).
XIM(I), YIM(I)	Coordinates of the position of the images of vortices.
XCG, YCG	Coordinates of the consolidated vortices in the regions R_{ij}.
XAVD	$= \frac{1}{\mathrm{TIME}} \int_0^{\mathrm{TIME}} (\text{total drag})\,dt \quad (= C_{D\,av})$.
OUT1	$= \frac{1}{\pi}$ (area of the circle of radius $(1 + \mu)$) representing the regions A_j.

APPENDIX B

- Some Results
- Flow Patterns

(The direction of flow in all figures is from left to right)

M = 20 DT =.20

REYNOLDS NUMBER = 4.00E+01

TIME	RAV	XAVD	N	SUMV	N9	N11
2.00	3.487	3.487	77	-2.490	0	123
4.00	2.284	2.886	98	-2.538	23	279
6.00	2.564	2.778	109	-.575	59	432
8.00	2.198	2.633	129	1.958	95	576
10.00	2.212	2.549	117	4.380	151	732
12.00	2.870	2.602	114	1.914	208	878
14.00	2.447	2.580	124	1.141	258	1018
16.00	2.501	2.570	145	-1.312	299	1156
18.00	2.585	2.572	117	.036	356	1327
20.00	2.759	2.591	119	.318	401	1480
22.00	2.422	2.575	141	-.077	439	1620
24.00	2.560	2.574	127	-3.090	495	1778
26.00	2.280	2.552	137	-.951	548	1915
28.00	2.068	2.517	127	.894	605	2068
30.00	2.027	2.484	125	-1.321	662	2213

M = 20 DT =.20

REYNOLDS NUMBER = 1.00E+02

TIME	RAV	XAVD	N	SUMV	N9	N11
2.00	2.400	2.400	82	-5.081	0	118
4.00	2.370	2.385	109	-2.352	25	266
6.00	1.960	2.243	128	-.814	57	415
8.00	1.971	2.175	137	1.851	96	567
10.00	2.498	2.240	132	2.205	152	716
12.00	2.353	2.259	156	-.166	205	839
14.00	1.441	2.142	154	-.883	254	992
16.00	2.335	2.166	155	-6.236	308	1137
18.00	2.169	2.166	156	-7.066	372	1272
20.00	1.853	2.135	159	-2.949	421	1420
22.00	2.450	2.164	159	-7.101	476	1565
24.00	2.884	2.224	160	-3.174	530	1710
26.00	2.169	2.219	175	-3.821	571	1854
28.00	.511	2.097	165	-5.885	632	2003
30.00	1.976	2.089	183	-6.844	673	2144

M = 20 DT = .20

REYNOLDS NUMBER = 4.00E+02

TIME	RAV	XAVD	N	SUMV	N9	N11
2.00	1.649	1.649	96	-.669	-1	105
4.00	1.208	1.429	157	-1.743	14	229
6.00	1.671	1.510	203	-2.723	56	341
8.00	1.123	1.413	204	.146	124	472
10.00	.092	1.149	218	4.566	190	592
12.00	.969	1.119	201	2.332	263	736
14.00	-.120	.942	194	-1.440	327	879
16.00	1.022	.952	196	-.938	391	1013
18.00	1.460	1.008	205	-3.773	447	1148
20.00	1.503	1.058	192	-1.928	531	1277
22.00	.787	1.033	189	.890	606	1405
24.00	1.443	1.067	212	-6.848	667	1521
26.00	1.263	1.082	208	-5.922	736	1656
28.00	2.179	1.161	210	-1.531	821	1769
30.00	1.369	1.175	181	-1.353	910	1909

.....

M = 20 DT = .20

RE = 500.0

COEF =SQRT(2. DT/RE)= .028

SKIN=4./SQRT(2. RE)= .1265

RATIO = 1.6 DT NB/TDI= 1.019

.....

TIME	RAV	XAVD	N	SUMV		
2.00	1.080	1.080	110	7.140	-3	93
4.00	1.161	1.121	160	3.557	17	223
6.00	1.376	1.206	176	-2.011	66	358
8.00	1.520	1.284	201	-1.633	122	477
10.00	1.074	1.242	225	-1.389	174	601
12.00	1.144	1.226	249	-1.564	225	726
14.00	1.364	1.245	272	-2.615	286	842
16.00	.794	1.189	245	-5.133	380	975
18.00	1.393	1.212	218	-3.561	481	1101
20.00	1.204	1.211	216	-5.848	556	1228
22.00	.391	1.136	225	-3.925	630	1345
24.00	1.131	1.136	232	-4.063	715	1453
26.00	.028	1.051	231	-7.644	799	1570
28.00	.722	1.027	263	-8.149	859	1678
30.00	.351	.982	278	-1.628	914	1808
32.00	1.236	.998	269	-2.430	994	1937
34.00	.797	.986	267	-4.940	1096	2037
36.00	.735	.972	287	-7.494	1164	2149
38.00	1.309	.990	259	-7.917	1258	2283
40.00	1.488	1.015	292	-6.073	1331	2377

M = 20 DT =.20

REYNOLDS NUMBER = 2.00E+03

TIME	RAV	XAVD	N	SUMV	N9	N11
2.00	.385	.385	118	4.199	-6	88
4.00	.554	.470	179	3.333	19	202
6.00	.593	.511	209	6.064	60	331
8.00	1.568	.775	241	5.067	113	446
10.00	1.414	.903	240	4.534	201	559
12.00	.455	.828	225	2.709	302	673
14.00	.418	.770	236	10.322	375	789
16.00	.542	.741	242	-1.027	464	894
18.00	.462	.710	254	-1.005	548	998
20.00	.177	.657	288	-8.062	619	1093
22.00	.773	.667	271	-4.712	748	1181
24.00	1.281	.719	266	-9.321	855	1279
26.00	.936	.735	277	-6.282	939	1384
28.00	1.323	.777	279	-7.579	1029	1492
30.00	.636	.768	313	-4.110	1098	1589

M = 20 DT =.20

REYNOLDS NUMBER = 4.00E+03

TIME	RAV	XAVD	N	SUMV	N9	N11
2.00	.785	.785	121	1.395	-6	85
4.00	.438	.612	200	.737	18	182
6.00	.272	.499	238	-5.876	100	262
8.00	-.374	.280	259	1.941	185	356
10.00	.827	.390	276	5.793	263	461
12.00	.447	.399	295	6.038	340	565
14.00	.323	.388	294	5.995	442	664
16.00	-.263	.307	325	-9.843	511	764
18.00	.423	.320	327	-13.760	596	877
20.00	.551	.343	287	-14.120	736	977
22.00	.525	.360	295	-6.627	834	1071
24.00	.379	.361	309	-7.172	917	1174
26.00	.156	.345	309	-7.296	1010	1281
28.00	.515	.357	360	-6.868	1079	1361
30.00	.908	.394	358	-18.309	1185	1457

M = 20 DT = .20

REYNOLDS NUMBER = 6.00E+03

TIME	PAV	XAVD	N	SUMV	N9	N11
2.00	.707	.707	135	1.236	-1	66
4.00	1.204	.956	220	.303	17	163
6.00	.644	.852	264	-1.124	81	255
8.00	.666	.805	243	-3.744	214	343
10.00	.170	.678	253	-3.440	313	434
12.00	.629	.670	286	-9.727	373	541
14.00	.797	.688	306	-12.063	434	660
16.00	.705	.690	321	-15.895	530	749
18.00	.049	.619	304	-7.803	650	846
20.00	.384	.595	311	-4.932	754	935
22.00	.627	.598	314	-5.423	860	1026
24.00	.379	.580	311	-3.532	987	1102
26.00	-.115	.527	311	-3.439	1081	1208
28.00	.549	.528	306	-14.000	1194	1300
30.00	1.013	.561	291	-13.693	1293	1416

M = 20 DT = .20

REYNOLDS NUMBER = 8.00E+03

TIME	RAV	XAVD	N	SUMV	N9	N11
2.00	.664	.664	135	1.423	-7	72
4.00	.335	.500	213	5.847	18	169
6.00	.940	.646	273	6.273	73	254
8.00	.716	.664	317	-3.932	144	339
10.00	.776	.686	342	-6.207	221	437
12.00	.618	.675	329	-9.842	328	543
14.00	.454	.643	317	-10.105	449	634
16.00	.324	.603	300	-11.027	585	715
18.00	-.071	.528	313	-7.515	670	817
20.00	.699	.545	310	-11.723	771	919
22.00	.662	.556	335	-10.610	871	994
24.00	.754	.573	334	-7.395	968	1098
26.00	.325	.553	342	-16.956	1071	1187
28.00	.094	.521	347	-9.308	1186	1267
30.00	-.235	.470	358	-12.132	1273	1369

M = 20 DT =.20

REYNOLDS NUMBER = 1.00E+04

TIME	RAV	XAVD	N	SUMV	N9	N11
2.00	.650	.650	141	.412	-10	69
4.00	.365	.508	232	-1.538	0	168
6.00	.201	.406	271	-6.136	72	257
8.00	.148	.341	297	-4.271	170	333
10.00	.034	.280	318	-3.907	279	403
12.00	.400	.300	312	-1.887	407	481
14.00	.553	.336	323	-6.714	513	564
16.00	.383	.342	318	-13.305	625	657
18.00	.601	.371	323	-20.850	736	741
20.00	.383	.372	337	-11.956	853	810
22.00	.128	.350	344	-13.327	957	899
24.00	-.275	.298	355	-9.577	1051	994
26.00	.575	.319	339	-5.851	1194	1067
28.00	.259	.315	378	-2.708	1276	1146
30.00	-.092	.288	386	-5.751	1384	1230

M = 20 DT =.20

REYNOLDS NUMBER = 2.00E+04

TIME	RAV	XAVD	N	SUMV	N9	N11
2.00	.199	.199	131	1.588	-12	81
4.00	.258	.228	215	-.787	18	167
6.00	-.280	.059	288	5.241	82	230
8.00	.515	.173	355	5.391	136	309
10.00	.108	.160	380	-2.301	230	390
12.00	-.034	.128	396	-.749	342	462
14.00	.085	.121	384	-3.234	488	528
16.00	.587	.180	351	-10.457	629	620
18.00	.080	.169	364	-11.523	722	714
20.00	.176	.169	398	-12.171	817	785
22.00	.062	.160	421	-14.702	910	869
24.00	-.264	.124	415	-11.712	1040	945
26.00	.650	.165	389	-17.102	1189	1022
28.00	.367	.179	393	-15.558	1301	1106
30.00	.334	.189	392	-9.921	1410	1198

M = 20 DT =.20

REYNOLDS NUMBER = 4.00E+04

TIME	RAV	XAVD	N	SUMV	N9	N11
2.00	.168	.168	144	5.743	2	54
4.00	.518	.343	238	-.598	27	135
6.00	.195	.294	291	-1.198	101	208
8.00	.043	.231	332	-.445	190	278
10.00	-.123	.160	349	6.102	298	353
12.00	.009	.135	379	2.072	400	421
14.00	-.196	.088	384	.739	520	496
16.00	.093	.089	419	1.669	623	558
18.00	.251	.107	433	-1.578	732	635
20.00	.208	.117	445	-2.295	850	705
22.00	.096	.115	458	-7.064	977	765
24.00	.276	.128	466	-17.292	1097	837
26.00	.348	.145	468	-21.838	1219	913
28.00	.033	.137	426	-17.798	1377	997
30.00	-.037	.126	444	-17.308	1494	1062

M = 20 DT =.20

REYNOLDS NUMBER = 2.00E+05

TIME	RAV	XAVD	N	SUMV	N9	N11
4.00	.067	.067	328	.833	0	72
8.00	-.042	.012	639	14.966	0	161

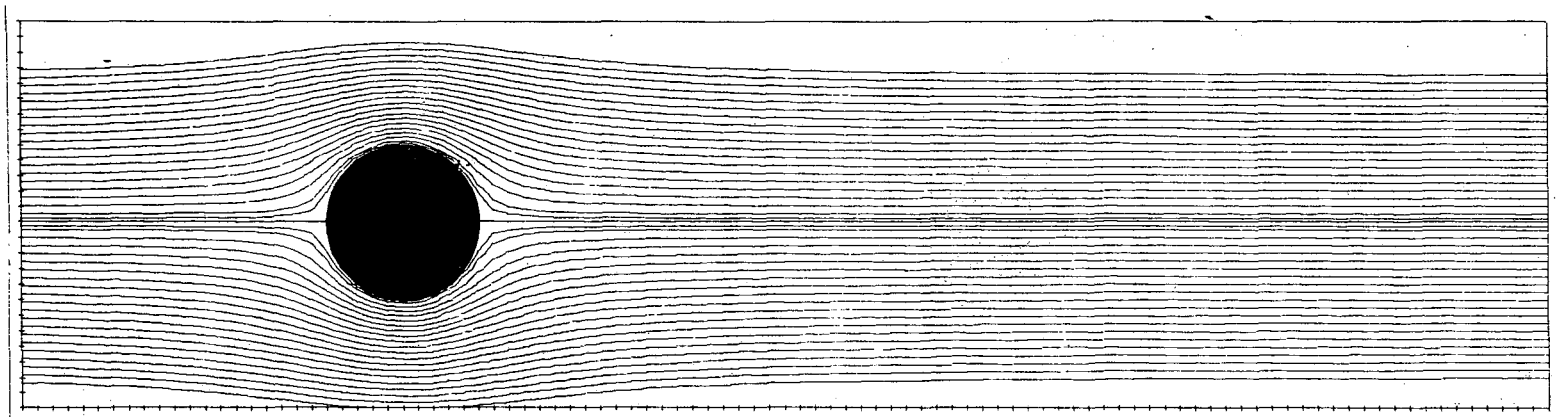


fig. (5). Re = 1000, t = 0. (potential flow)

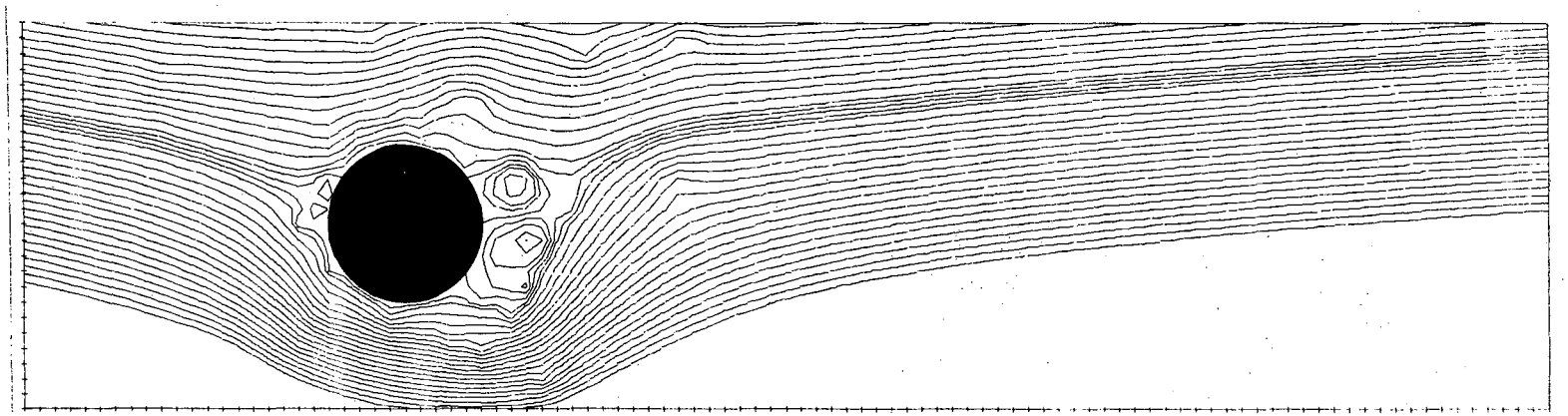


fig. (6). Re = 1000, t = 2.

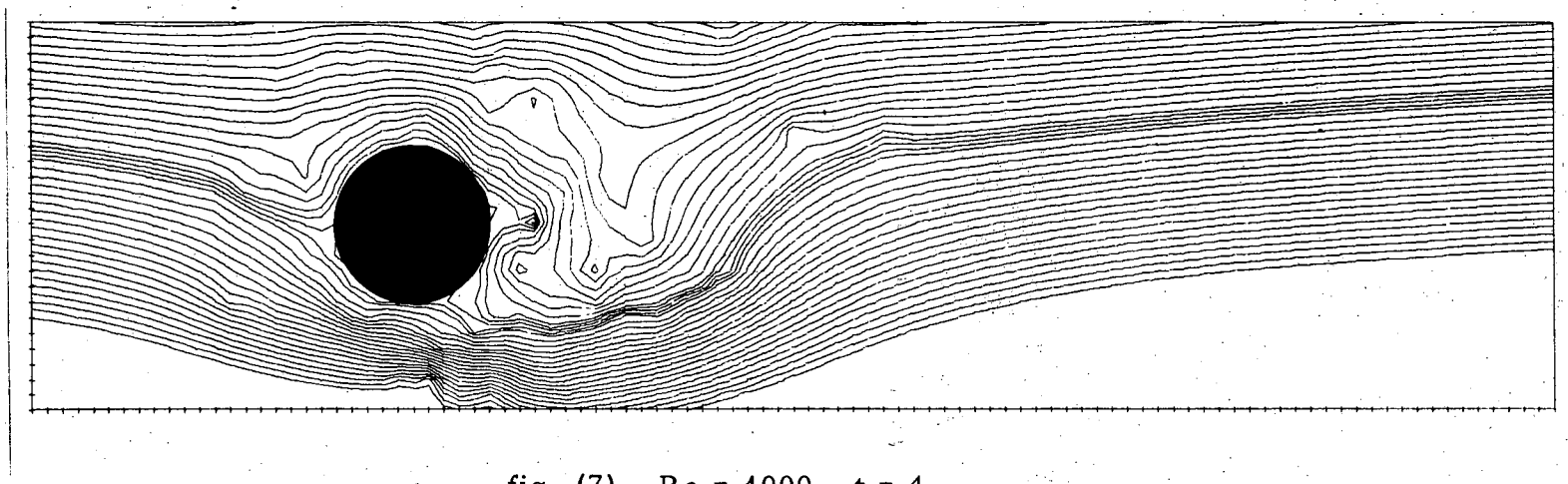


fig. (7). Re = 1000, t = 4.

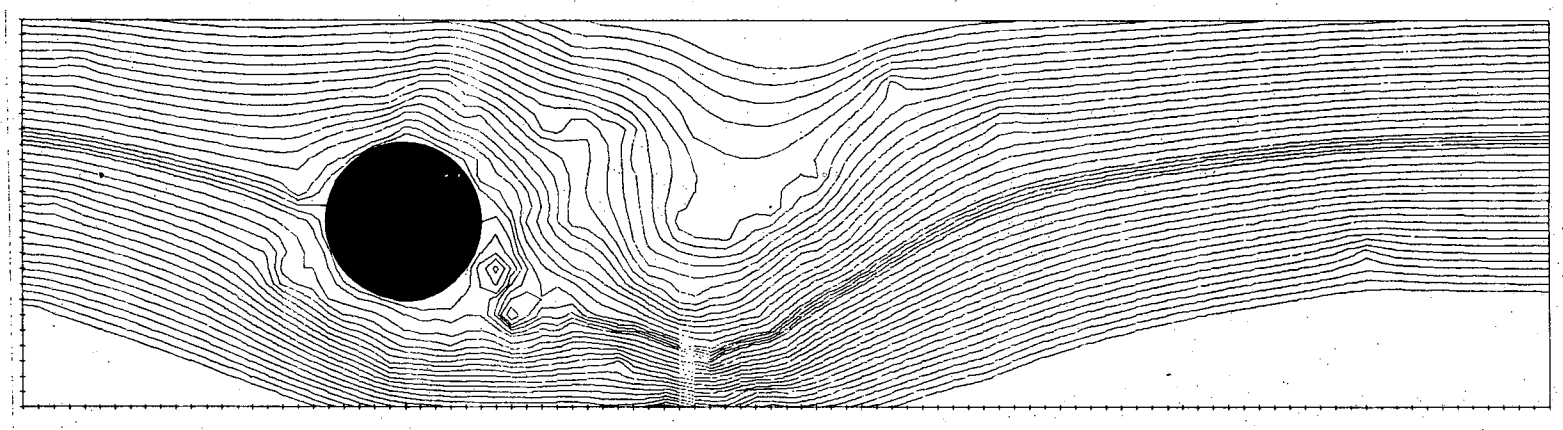


fig. (8). Re = 1000, t = 6.

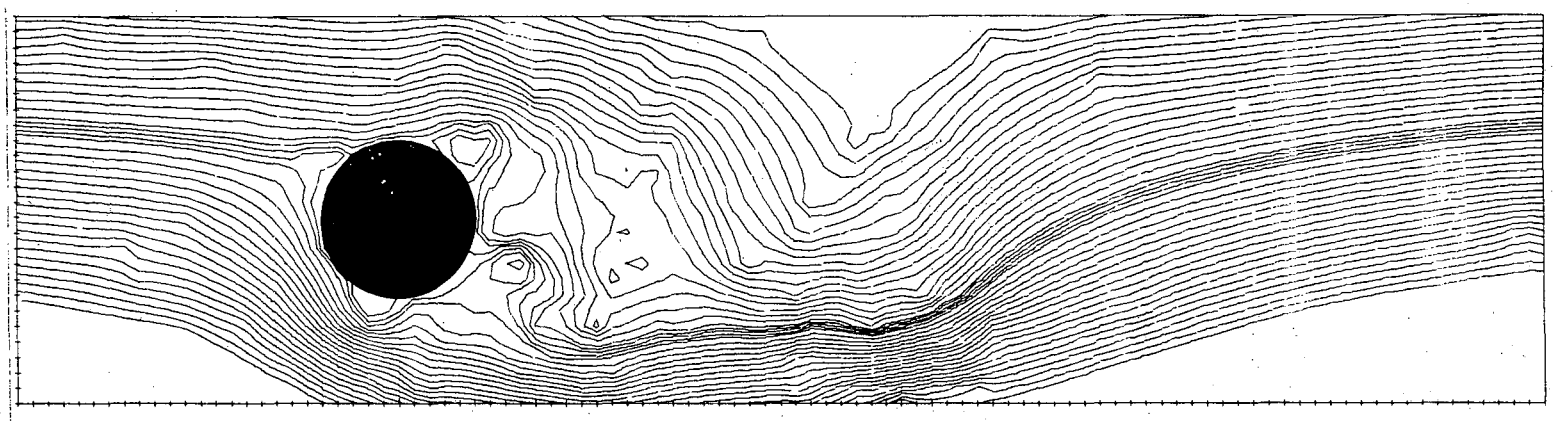


fig. (9). Re = 1000, t = 8.

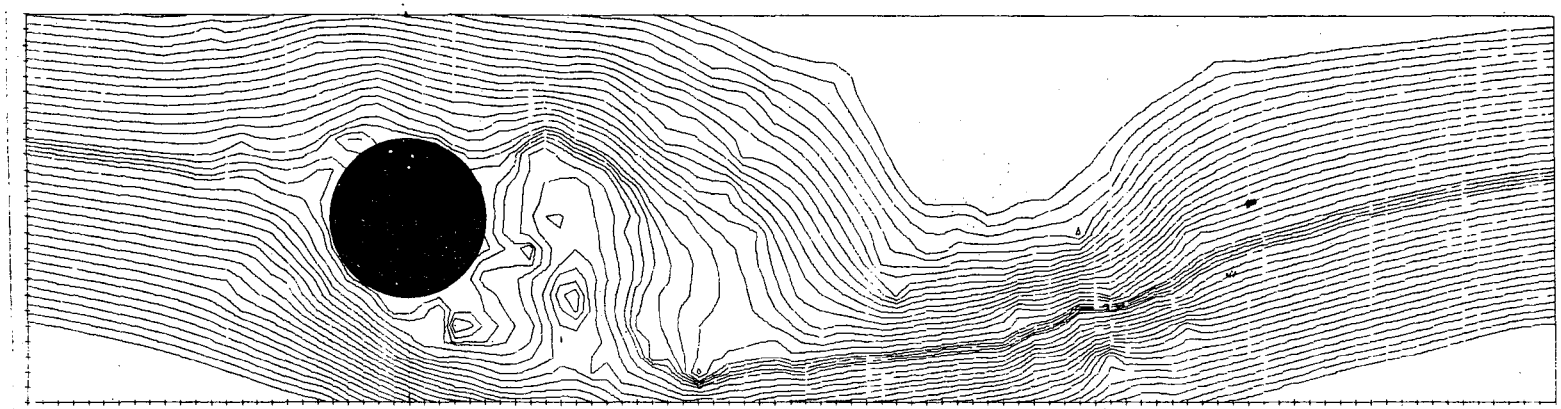


fig. (10). Re = 1000, t = 10.

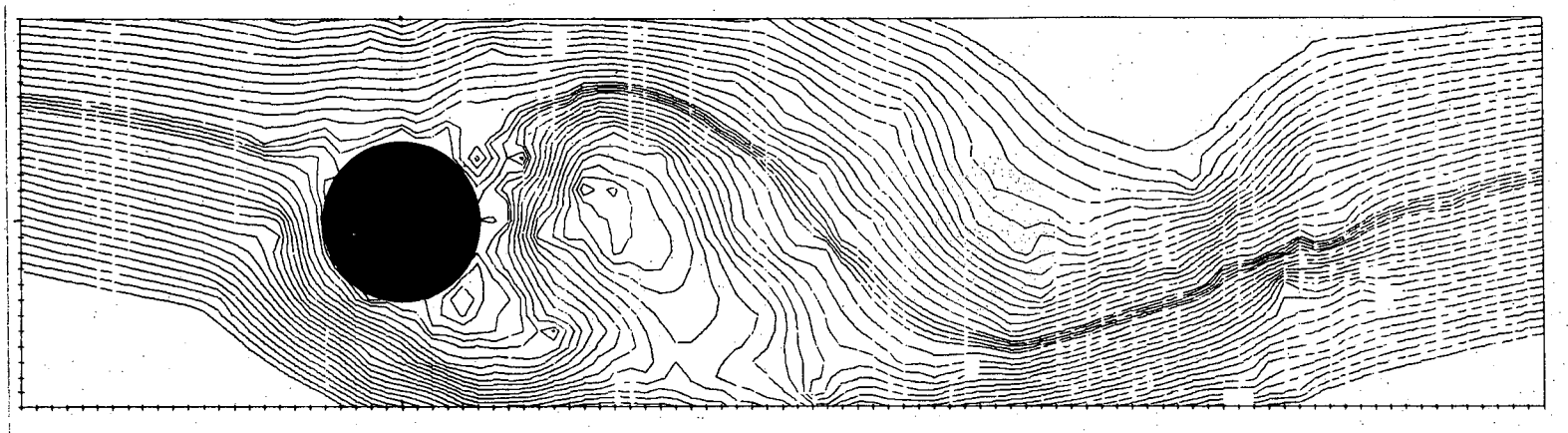


fig. (11). Re = 1000, t = 12.

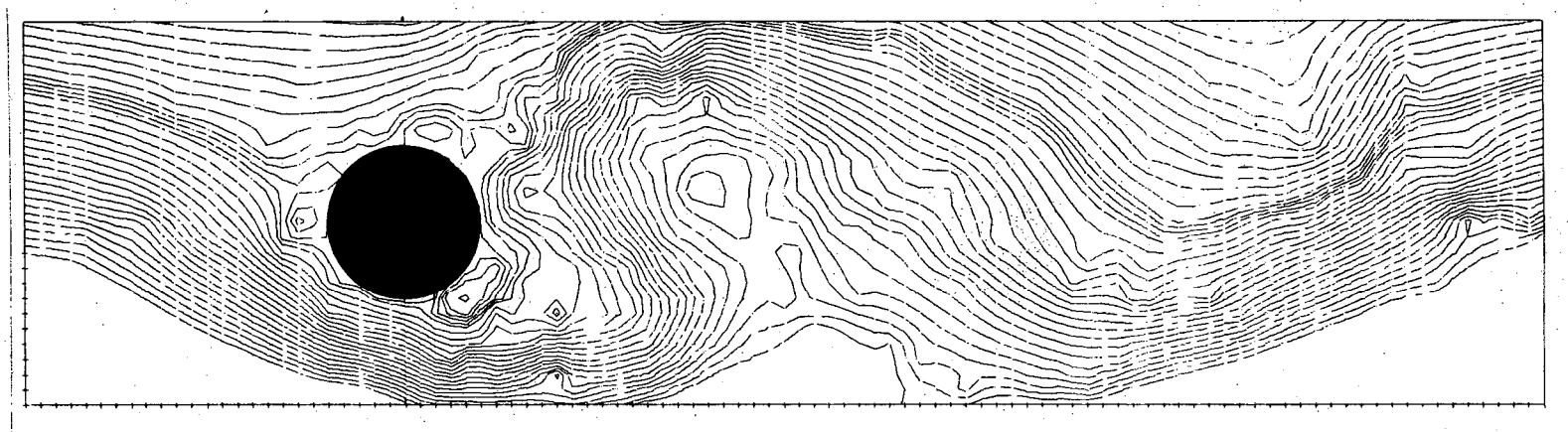


fig. (12). Re = 1000, t = 14.

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