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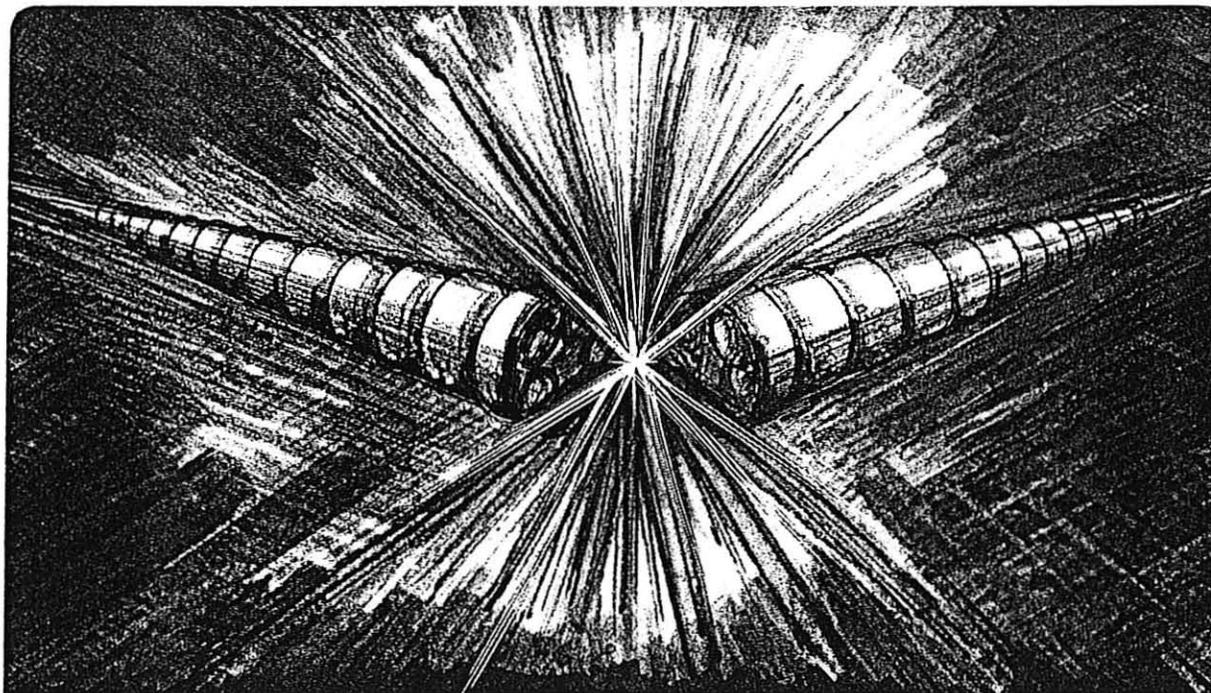
Accelerator & Fusion Research Division

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Lugano, Switzerland, April 11-16, 1988

Theory and Analysis of Nonlinear Dynamics and Stability in Storage Rings—A Working Group Summary

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- A WORKING GROUP SUMMARY*

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ABSTRACT

A summary and commentary of the available theoretical and analytical tools and recent advances in the nonlinear dynamics, stability and aperture issues in storage rings are presented.

I. INTRODUCTION

In this report we summarize the efforts and discussions of the Working Group on Theoretical and Analytical Studies at the second ICFA Advanced Beam Dynamics Workshop on "Aperture Limitations in Storage Rings," held in Lugano, Switzerland, April 11-16, 1988. The working group identified several major issues to be addressed during the workshop. These are:

1. Comparison and contrast of different analytical methods used to date for determining Dynamic Aperture in Storage Rings.
2. A model lattice cell to compare these different methods.
3. Other approaches to accurate analytic computation of dynamical distortions to very high orders of nonlinearity.
4. Remainder estimation, long-term weak diffusion rates and all that (Nekhoroshev's theorem).
5. A strategy for a thorough beam stability and aperture analysis from design considerations.

In addition, questions regarding the dependence of the suitable methods on the goals of achieving a certain stability criterion, simple criteria and "scaling laws" for "stable aperture," the definition of "linear aperture" and "SMEAR," and approach to the "border of stability" from above

(diffusion rates in the chaotic region by Melnikov's method) were raised and vigorously discussed and debated during the workshop. Various members of the group gave expository talks on many of these issues throughout the workshop. We discuss issues 1 through 4 below, followed by a summary of the major conclusions. Issue 5 is discussed in detail in a companion paper in these proceedings [1]. Nonlinear dynamics with damping and noise is a specialized topic and although discussed during the workshop, is not addressed in this summary. The presence of strong radiation damping in an electron storage ring allows questions regarding long-term stability beyond the damping time to be answered comfortably anyway.

II. ANALYTICAL METHODS FOR DETERMINING DYNAMIC APERTURE - A BRIEF COMMENTARY

We are concerned with different analytical methods dealing with amplitude limitations of stable motion in storage rings. All the methods studied in detail so far approach the limit of stability from 'below' i.e. from the side where the motion is still stable because the initial amplitude is not yet too large. A summary describing these methods is provided by G. Guignard [2] in these workshop proceedings. In a companion paper in these same proceedings, G. Guignard [3] also provides some comments on the advantages and limitations of these various methods. The reader should refer to these papers for a complete exposition.

Most of these methods are based on perturbation techniques of some kind. There are methods using perturbation theory in the Hamiltonian formalism e.g. Poincaré-von Zeipel-Moser procedure, Deprit's algorithm using Lie Transforms, etc. There are perturbation treatments by iterations on the equations of motion directly and successive linearizations thereof. There is the secular perturbation theory based on Lindstedt-Poincaré technique, using power series expansion of the solution in the nonlinear perturbation strength parameter and removing secular terms in each order of perturbation by selecting frequencies appropriately; and others. All these methods strive to compute perturbatively, to as high an order as feasible, nonlinear distortions of the phase-space at an amplitude of motion as large as possible e.g. distortion ΔI of the invariant action (or equivalently emittance), nonlinear detuning (i.e. amplitude-dependent tune shift), resonance widths, etc. The border of stability is then conjectured, in a somewhat ad hoc manner, to be associated with that value of the action J_0 or amplitude x_0 , for which one or more of the following conditions are satisfied: (i) the relative nonlinear distortion of action at amplitude x_0 , computed to certain order, becomes greater than or equal to unity, $\Delta I(x_0)/J_0 \geq 1$; (ii) the nonlinear action becomes negative, $J_0 = I_0 + \Delta I(x_0) \leq 0$; (iii) the nonlinear action becomes infinitely large; (iv) the perturbation solution fails to converge for amplitudes $x \geq x_0$, etc. We should note that the

agreement of any one of these conjectures with the onset of large-scale global chaos supposedly associated with the dynamic aperture cannot be proven rigorously. These methods thus do not provide estimates of the true dynamic aperture in the strictest sense. Indeed for an arbitrary and general nonlinear lattice the limit arrived at this way could be very far from the true border of stability. These limits however have practical value in the sense that they do indicate the amplitude at which the phase-space topology gets highly complicated and, conservatively speaking, thus provide a necessary, although not sufficient condition for unstable motion. Under certain fortuitous circumstances, these limits may even be arbitrarily close to the dynamic aperture, as numerical tracking has suggested in a few simple cases [2,4].

In most of these methods, the analytical developments become cumbersome after a few low order calculations in the perturbation series, linearizations, iterations, etc. The convergence of the perturbation procedure, even when good, can not be proved mathematically. In many cases the generalization to magnetic multipole elements other than sextupoles has not been achieved and in some cases the extensions to two-dimensions even remains to be done. Often the validity of these methods is also restricted to a limited range of the linear-optics parameter space e.g. tunes etc. The Hagel-Moshhammer approach [4] using techniques of secular perturbation theory has been the most successful one, leading to reasonable approximate estimation of the dynamic aperture for systems as complex as the LEP collider.

Amongst nonperturbative methods, one can consider applying the numerically derived Residue Criterion of Green and McKay. However, the validity of this criterion has not been demonstrated beyond one-dimensional 'standard' and 'quadratic' maps. It can be applied exactly and directly to the stability of longitudinal synchrotron motion. For one-dimensional transverse betatron motion, it at best confirms Chirikov's resonance overlap criterion for the onset of chaos. Application of this method for extracting analytic expressions for the stability limit in a realistic, general, nonlinear two-dimensional map representing a storage ring is not obvious.

Another nonperturbative method, proposed by Gabella, Ruth and Warnock [5], is based on the direct solution of the two-dimensional Hamilton-Jacobi equation. This method allows computation of distorted invariant tori at amplitudes close to the dynamic aperture and looks for the singularities of the implicit equations defining the orbits via the generating function. There are indications that the breakdown of the generating function via a singularity is related to the residue criterion and can be generalized to higher dimensions. This seems to be one of the most promising analytic Hamiltonian approaches so far.

One can also attempt to approach the border of stability from 'above' i.e., from the region of large-scale global chaos in phase space. One may look at various stochastic diffusion processes e.g., Arnold diffusion, etc. using existing mathematical techniques such as Melnikov's method, etc. Not much progress has been made along these directions in storage ring dynamics. Comments on these approaches are provided by H. Mais [6] in his contributions to these proceedings.

III. A MODEL LATTICE FOR COMPARING DIFFERENT ANALYTICAL METHODS TO CALCULATE THE DYNAMIC APERTURE

For convenience and standardization, it was felt that one should focus on a model standard lattice cell on which to compare, contrast and numerically test the different analytical methods for calculating the Dynamic Aperture. The Lattice should be modestly nonlinear and yet be simple enough in structure with relatively known dynamic properties in order for it to be useful.

A FODO-lattice with superimposed quadrupoles and sextupoles separated by drift spaces of equal lengths was chosen as such a model. The lattice configuration is shown in Fig. 1.

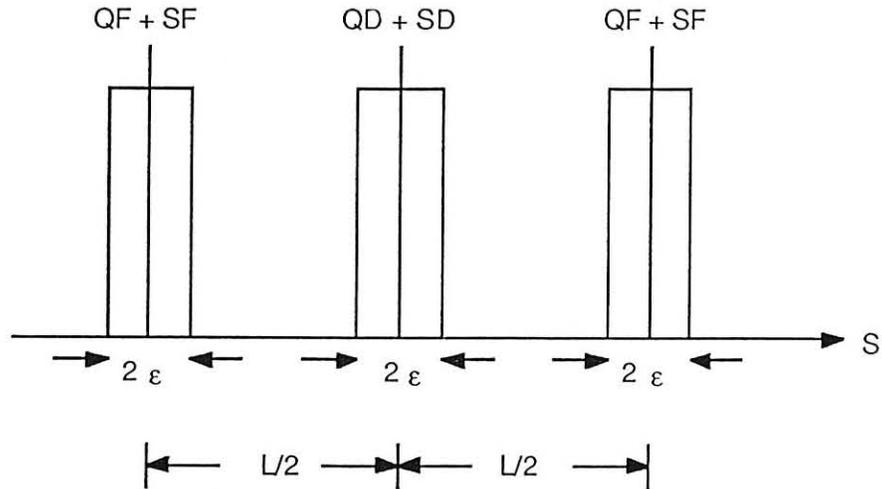


Fig. 1

The integrated strengths of each quadrupole (QF or QD) is chosen to be:

$$(K\ell) = \frac{4}{L} \sin\left(\frac{\mu}{2}\right) \quad (1)$$

where $\mu = 2\pi Q$ is the phase advance across the cell. The sextupole strengths could be chosen either arbitrarily or such as to correct the natural chromaticities ξ_x, ξ_y to vanishing values. In the latter case, the sextupole strengths should be chosen as:

$$(K' \ell)_F = \frac{(K \ell)}{\eta_F} \quad (2)$$

$$(K' \ell)_D = \frac{(K \ell)}{\eta_D} \quad (3)$$

where $\eta_{F,D} = \frac{L^2 n}{8 \sin^2\left(\frac{\mu}{2}\right)} \left[2 \pm \sin\left(\frac{\mu}{2}\right) \right]$ and $n = 1/\rho$.

As an example of the parameter values, one can consider the identical LEP cell lattice:

$$L = 79 \text{ m}; \quad \mu = \pi/3; \quad \rho = 3100 \text{ m}.$$

For comparison with numerical tracking, one could use initial or starting condition $\dot{x}(0) = \dot{y}(0) = 0$ and $x(0) = x_0, y(0) = y_0$. One could then ask what is the maximum value of the (x_0, y_0) pair, (x_0^{\max}, y_0^{\max}) , for which the motion is stable and bounded?

It was also suggested during the workshop that one could use any one of the high-brightness, low-emittance lattices envisaged for the many third generation synchrotron radiation sources around the world (Berkeley, Trieste, Taiwan, etc.) as test cases. These lattices are sufficiently nonlinear at modest amplitudes to put the analyses to a real test of their computational power and practicality.

IV. ANOTHER ANALYTIC APPROACH - HAMILTONIAN-FREE ANALYSIS ON EXACT MAPS

We have noted that analytic developments using the Hamiltonian approach become increasingly cumbersome as we go to higher orders in perturbation theory at large amplitudes of motion. It is indeed an ambitious goal to be able to derive explicit analytic expressions for the Dynamic Aperture (i.e. border of stability) for any arbitrary nonlinear two (and three) dimensional Hamiltonian system, such as is represented by a real storage ring. We shy away from such a goal of trying to reach the border of stability in one leap. Instead we ask a more practical and modest question: Do we have analytic tools to understand the phase space topology and compute with

enough accuracy dynamical distortions (e.g. distortion of invariant tori of action, nonlinear resonance widths, amplitude-dependent tune-shifts etc.) at large amplitudes close to the Dynamic Aperture for a real three-dimensional storage ring lattice? The answer is yes and this tool is based on extracting exact "maps" for the accelerator lattice (rather than a Hamiltonian) and doing analysis (perturbative or otherwise) on these maps.

Hamiltonian formalism and exact equations of motion derived from them, are natural and useful tools for simple situations like a few objects moving in a rather "smooth" potential well. The Hamiltonian corresponding to a storage ring is a rather complicated beast, involving periodic delta-function-like objects corresponding to localized electromagnetic elements and time-dependence as well. A storage ring is intrinsically "modular", affecting a large number of "almost discrete" transformations on a particle's phase space coordinates. A description of beam dynamics with "maps" seems more natural. A map M transforms a point z_i of phase space at azimuth s_i on the reference orbit into a point z_f at azimuth s_f :

$$z_f = M(z_i; s_f, s_i) . \quad (4)$$

A full turn map $M(s)$ at azimuth s for a storage ring of circumference C corresponds to $s_i = s$, $s_f = s + C$; a multi-turn (n -turn) map $M_n(s)$ to $s_f = s + nC$. Insofar as one does not care to look at the phase-space topology at any other azimuth between s_i and s_f , the same "map" can be derived from a large number of Hamiltonians belonging to a certain "class", up to a certain order of nonlinearities of the map.

Maps have been used as standard tools to describe linear storage ring lattices for a long time. In the particular case of linear motion, maps are amenable to representations in terms of matrices. Indeed, transport matrices through linear elements involving Twiss parameters are all too familiar in the standard theory of betatron motion in circular accelerators, one of the greatest triumphs of Courant-Snyder formalism. These matrices (i.e. maps) summarize the results of integrating the equations of motion and provide stability analysis directly. In the same spirit one could foresee great virtue in generalizing this formalism to the nonlinear case, using nonlinear maps. For this purpose, one needs to construct these maps in a sufficiently simple, reliable and accurate way. We will discuss the construction of maps later in the section. Once an accurate map is obtained, one could either use it to perform numerical tracking by iteration (thus saving computation time, since most of the work has already been done in extracting the map thus effectively eliminating redundant integrations) or could simply use a full-turn map to do analysis, perturbative or otherwise, on it to extract nonlinearly distorted dynamical quantities such as distortions of invariant

tori (i.e. action or emittance), nonlinear tune-shifts, resonance analysis, etc. We turn to this latter issue of Hamiltonian-free analysis on exact maps for the moment .

A full-turn map allows the study of phase-space topology (e.g. invariant surfaces, etc.) without explicit reference to the underlying Hamiltonian. This is achieved either through the Normal Form Analysis [7,8,9] based on perturbation theory or through the solution of a functional equation for the invariant surface by an appropriate iterative method [5,10].

The Normal Form analysis is based on perturbation theory on exact maps and phase advances. It is the nonlinear map analogue of the linear Courant-Snyder solution:

$$\begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} = U(s) R[\psi(s)] U^{-1}(0) \begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix} \quad (5)$$

where

$$U(s) = \begin{pmatrix} \sqrt{\beta(s)} & 0 \\ \gamma(s) & \frac{1}{\sqrt{\beta(s)}} \end{pmatrix}; \quad \gamma(s) = \dot{\beta}(s)/2\sqrt{\beta(s)}, \quad (6)$$

$$\beta(0) = \beta(C) \Rightarrow U(0) = U(C); \quad \psi(C) = 2\pi\nu \quad (7)$$

and R is the rotation matrix corresponding to phase advance $\psi(s)$, ν being the total tune and $\beta(s)$ the lattice beta-function.

Using the complex notation $z = x + ip$ for phase space coordinates under the transformation $z' = M(z, z^*)$, the Normal Form analysis uses symplectic transformations close to identity:

$$\{z, z^*\} \rightarrow \{\xi, \xi^*\} : z = \Phi(\xi, \xi^*) = \sum_{n \geq 1} \Phi_n(\xi, \xi^*) \quad (8)$$

where Φ_n 's are homogeneous polynomials of degree n in ξ, ξ^* to bring the map M to the Normal Form:

$$M = \Phi e^{i\Omega(\xi, \xi^*)} \Phi^{-1} \quad (9)$$

analogous to Eq. (5) for the linear case, where Ω is also of degree $\leq n+1$ and is determined recursively, together with Φ . Simply stated, this analysis transforms the phase-space to

coordinates where the map is a pure rotation, with nonconstant amplitude-dependent angle however (Ω depends on $|\xi|^2 = \xi\xi^*$):

$$\xi' = e^{i\Omega(\xi\xi^*)}\xi \quad (10)$$

It can be best visualized pictorially as in Fig. 2.

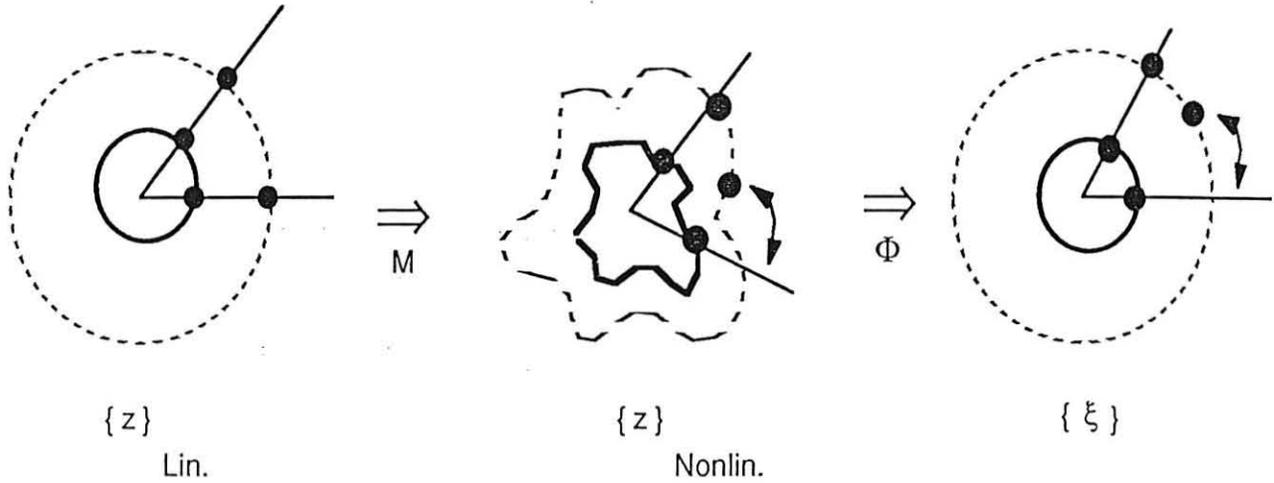


Fig. 2

After t -iterations, the map simply reads:

$$M^t = \Phi e^{it\Omega(\xi\xi^*)}\Phi^{-1} \quad (11)$$

The analysis extends up to an arbitrary order in Φ . If M is only an order N symplectic truncation, the normal form does not change up to the same order.

From Normal Forms, one can compute a whole host of dynamical quantities related to nonlinear phase-space distortions in a straightforward way. For example perturbative "tune-shifts" and distortion of invariant action or "SMEAR" can be simply computed as:

$$\text{Tune Shift: } \delta\nu = \frac{1}{2\pi} \Omega(\rho) - \nu_0 \quad (12)$$

$$\text{SMEAR: } \sigma = \frac{\langle (R - \langle R \rangle)^2 \rangle^{1/2}}{\langle R \rangle} \quad (13)$$

where $\langle R \rangle = \frac{1}{2\pi} \int_0^{2\pi} R(\theta) d\theta$ (14)

and $R(\theta) = |z| = |\Phi(\xi, \xi^*)| = |\Phi(\rho e^{i\theta}, \rho e^{-i\theta})|$ (15)

$$\rho = |\xi| = |\Phi^{-1}(z, z^*)| = |z| + \dots$$
 (16)

One can also compute σ_n , the sum of the strengths of all n-th order resonances. One can write the full-turn map, explicitly delineating the linear map M_L , as:

$$M = M_L e^{\hat{g}_3} e^{\hat{g}_4} \dots e^{\hat{g}_n} \dots$$
 (17)

where $g_n = \sum_{\substack{j+k=n \\ |l+m| \leq n}} A_{jk,lm} J_x^{j/2} J_y^{k/2} 2^{(j+k)/2} e^{i(l\psi_x + m\psi_y)}$ (18)

One can then construct resonance strengths at a certain amplitude $A_0 = \epsilon_0^{1/2}$ as follows:

$$\sigma_n = \sum_{\substack{j+k=n \\ |l+m| \leq n}} |A_{jk,lm} A_{jk,lm}^*| \epsilon_0^{n/2}$$
 (19)

where ϵ_0 is a typical emittance of the particle. The σ_n 's are a full figure of merit of nonlinearity, containing tune-shifts, resonances and nonlinear distortion. They are also a measure of the resonance widths and are "invariant" under the linear map. What is more, one can mathematically subtract the nonlinear detuning or amplitude-dependent tune-shift, thus obtaining a quantity σ_n^* , which is a measure of "pure distortion" of invariant tori contributed by all n-th order resonances:

$$\sigma_n^* = [\sigma_n - \text{Tune shifts}] \quad \text{"Nonlinear Distortion"} \quad (20)$$

While σ_n is affected by the full nonlinear map, σ_n^* is affected by the "Coherent Nonlinear Map". This is depicted in Fig. 3 below. Quantities $\{\sigma_n\}$ and $\{\sigma_n^*\}$ allow us to disentangle linear vs. nonlinear and nonlinear detuning vs. nonlinear distortion effects in a machine in a most effective way.

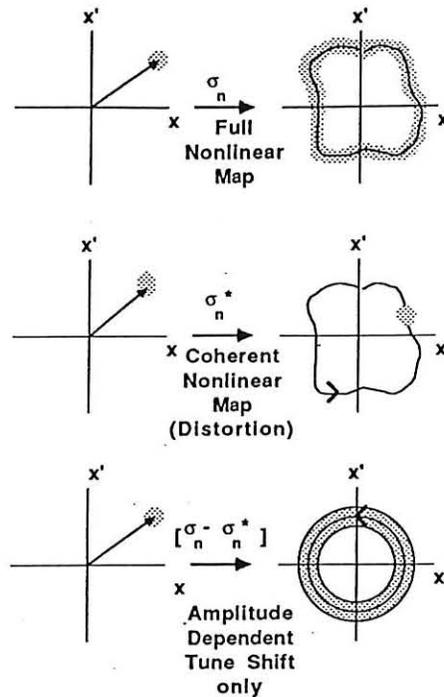


Fig. 3

The alternative nonperturbative method [5,10] of solving the functional equation for the invariant surface exploits the power of Newton's method to study invariant surfaces very close to unstable regions of phase space. From such a calculation again, one obtains nonlinear tune-shifts and Fourier analysis of the invariant surface to reveal the spectrum of contributing resonances.

Both these Hamiltonian free analysis tools on exact maps are powerful and complement each other. While it may be sufficient to use the Normal Form Analysis in many cases yielding quick and accurate quantitative phase-space analysis at large amplitudes, solving the functional equation for invariant surfaces may have a wider range of utility however, since it does not depend on perturbation theory.

Modular maps on a storage ring provide the advantage that any point in the ring is just as good for analytic manipulations on maps: any other point in the ring can be obtained by simple phase-advances, etc. One could also combine full-turn maps with maps describing localized effects such as the beam-beam interaction, r.f. kicks, undulators, etc., thus allowing studies of the effect of variations of the local effect in an otherwise unchanged full-turn lattice. One can even envision combining noise or dissipation (as in electron storage rings) with the symplectic maps considered here. Maps are also smooth functions of tunes, nonlinear magnet strengths, particle

energy etc., in general. One can thus store maps for a few values of these parameters and use some kind of interpolation to reach other parameter values, thus allowing an efficient way to explore tune space, etc. All these favorable attributes, combined with the ability to track particles and existence of the above two powerful analysis tools to find invariant surfaces and compute accurately nonlinear phase-space distortions in greatly reduced computation time even at large amplitudes approaching the border of stability, point to great promise in the use of analysis based on exact maps and should allow us to take a significantly long leap forward in quantitatively thorough beam stability studies....provided we have tools to extract maps to any desired accuracy efficiently. We turn to this point next.

In spite of the recognition of the usefulness of nonlinear maps, their full power has not been fully exploited until recently, due to the difficulties associated with extraction of accurate maps. Representing the map $M(z; s_f, s_i)$ as a truncated Taylor series in the components of z :

$$M: z_f = z_i + Lz_i^2 + Tz_i^3 + \dots + Wz_i^N \quad (21)$$

one needs accurate computation of the Taylor coefficients, which involve increasingly higher order derivatives $W = [\partial^{N-1} z_f / \partial z_i^{N-1}]_0$ on any reference closed orbit. Such computations become increasingly inaccurate with N , involving high order ratios of vanishing numbers. This has tended to limit accuracy, since even with the help of Lie algebraic methods and symbolic manipulation, it was not practical to compute these coefficients beyond the first few orders. A significant recent development has resulted in a superbly improved technique [9] for calculation of derivatives, exact up to the computing machine precision, using Differential Algebra. This innovation, by M. Berz, finally can provide maps of the desired accuracy and can work to arbitrary order, limited only by computer storage and time.

In some extremely nonlinear cases with large amplitudes, one may hit a practical limitation even with differential algebraic techniques - it may simply be not feasible to compute a sufficient number of Taylor coefficients, the storage and computing time beginning to increase catastrophically beyond a certain order. Considerable progress is being made in a complementary and alternative approach [10] involving a global approximation of a map with "spline functions" (in 'action') and orthogonal basis functions (Fourier analysis in 'angle'). The coefficients in this representation can be obtained in an elementary way by running a tracking code for one turn starting from a set of initial values on a suitable mesh in action-angle space and then by proper fitting with the above functions. No derivatives of the map M are required. If the tracking code is symplectic, the resulting map will typically be symplectic to good accuracy. A small modification

makes the map symplectic to any required precision. This method of Warnock et al. [10] although at a development stage, also seems quite promising with respect to accuracy, simplicity and computation time.

V. REMAINDER ESTIMATION, LONG-TERM WEAK DIFFUSION RATES AND ALL THAT

At every stage of Normal Form Analysis as a function of amplitude, the Birkhoff asymptotic perturbation series fails to converge beyond a certain number of symplectic normal form transformations. There is a "remainder" that is left over at the penultimate convergence stage, which is a measure of the remaining fluctuation in the Hamiltonian or action ($\Delta J/J$) and is a measure of the weak diffusion rate at that amplitude resulting from the infinitely many thin stochastic layers in the phase space enclosed by that amplitude. Typically, inverse of these residual fluctuations is a broad measure of the lifetime τ_{life} of the particle resulting from these diffusion mechanisms. The situation is illustrated pictorially in Fig. 4.

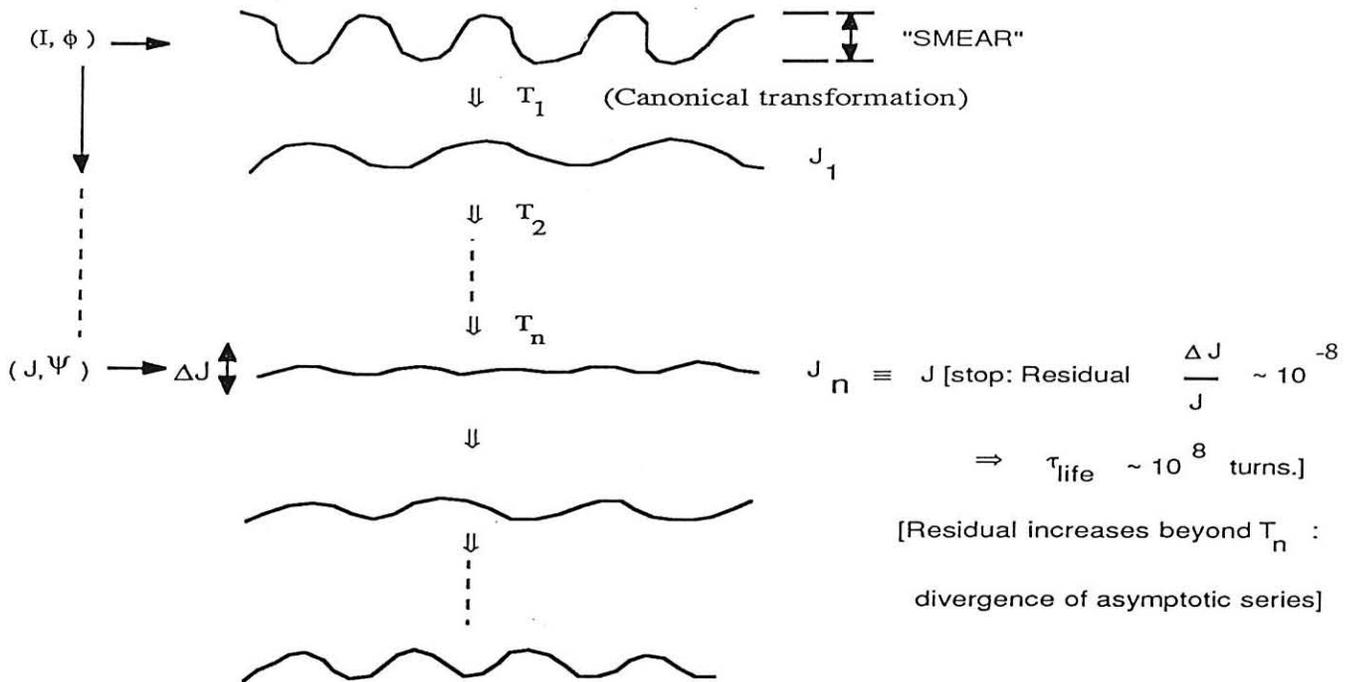


Fig. 4

If the remainder could be estimated accurately, we could estimate the weak diffusion rate accurately as well. Rigorous results in multidimensions exist only for autonomous (time-independent) Hamiltonian maps. No such rigorous result exists for nonautonomous (time-dependent) Hamiltonian maps, as for storage rings. However in the one-dimensional case in a phase plane, similar estimates hold. There are still arbitrary constants in these estimates and the results have to be used with extreme care. However, these estimates may still be useful for relative "scaling" of rates with amplitudes (emittances), etc. The mathematics is intricately related to Nekhoroshev's theorem and is exposed in the contribution of Bazzani and Turchetti [8] in these proceedings.

VI. CONCLUSIONS

We find that a global analytic expression and simple "scaling laws" for the Dynamic Aperture of an arbitrary nonlinear storage ring are hard, if not impossible, to find. Under some fortunate and very special circumstances for particular lattices, certain analytic methods using low order perturbation theory in the Hamiltonian formalism as outlined in Section II can, at best, point to a limiting amplitude of motion, around and beyond which the motion and the phase space structure itself get pathologically complicated. These limiting amplitudes, when analytically expressed (and if believed to be relevant and applicable), are cumbersome enough to be of little practical use. Often, they have to be numerically evaluated and simple "scaling" with parameters is not obvious, except under special circumstances. For simple situations with modest nonlinearity however, they have practical value in the sense that they provide a first guess at the pathological region of phase space.

Considerable progress has been made however, thanks to some powerful newly-developed computational tools as outlined in Section III of this report, in the ability to perform nonlinear computational analysis, perturbative or otherwise, at impressively large three dimensional amplitudes on any nonlinear lattice for which a tracking code exists without compromising unduly on accuracy, faithfulness and economy of time. These analytical methods allow one to penetrate deep into the nonlinear phase space, with accurate knowledge of the nonlinear optical distortions (distortion of invariant surfaces, nonlinear resonance spectrum, their strengths and widths, amplitude-dependent tune-shifts, etc.) at every stage, ultimately reaching amplitudes so close to the real border of stability or Dynamic Aperture that they can be accepted as the limiting amplitude for all practical purposes. Significant milestones in this development have been techniques to extract exact full-turn maps and technique to analyze these maps. In the former category, the algorithm to extract exact maps by computing derivatives with Differential Algebra is a significant innovation by

M. Berz and is surely to be recognized as an unusually powerful tool. It will certainly find even more widespread application in the future. The other method of map extraction from an arbitrary tracking code using 'spline' and 'Fourier' fitting, as proposed by Warnock et al. recently, also holds significant promise and complements Berz's method in extremely nonlinear situations where storage and computation time become catastrophic. In the latter category of 'analysis of maps', E. Forest and others have resurrected the Normal Form analysis, known since Birkhoff, to a level where it provides computational capability orders of magnitude superior to any other methods that were used before. In parallel, the nonperturbative solution of the functional equation for the generating function, proposed by Warnock et al., provides another attractive and promising alternative, to be explored further in the future. These developments have been a significant step towards the thoroughness of beam stability analysis and computational analytic estimates of the border of stability, without involving the practically difficult long-term tracking.

The question of ultimate long-term stability (after 10^{10} turns, say) was vigorously discussed during the workshop. Both the numerically tracked short-term Dynamic Aperture and the above analyses have little to do with long-term stability. There exist no strict mathematical theorems applicable to a three dimensional storage ring. Estimation of the 'remainder' and weak diffusion rates, in the spirit of Nekhoroshev's theorem as discussed in Section IV, holds a very weak promise in this direction. This issue and possible alternatives to long-term tracking are discussed in companion papers [1] and [11] in these proceedings. A strategy for a thorough beam stability and aperture analysis for a storage ring from design considerations, taking into account the methods and recent developments outlined in this report, is discussed in reference [1].

ACKNOWLEDGMENT

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