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CANONICAL INTEGRATION AND ANALYSIS OF PERIODIC MAPS USING NON-STANDARD ANALYSIS AND LIE METHODS*

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Canonical Integration and Analysis of Periodic Maps using Non-Standard Analysis and Lie Methods

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Abstract

We describe a method and a way of thinking which is ideally suited for the study of systems represented by canonical integrators. Starting with the continuous description provided by the Hamiltonian, we replace it by a succession of preferably canonical maps. The power series representation of these maps can be extracted with a computer implementation of the tools of Non-Standard Analysis and analyzed by the same tools. For a nearly integrable system, we can define a Floquet ring in a way consistent with our needs. Using the finite time maps, the Floquet ring is defined only at the locations s_i where one perturbs or observes the phase space. At most the total number of locations is equal to the total number of steps of our integrator. We can also produce pseudo-Hamiltonians which describe the motion induced by these maps.

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1. Introduction: The equation of motion

The work presented here is motivated by the study of single particle motion in a complex circular storage ring. If we denote by $s(0 < s < 1)$, the position around this ring, we can defined a map $\mathcal{G}(s; s+1)$ which describes completely our system. The map $\mathcal{G}(s; s+1)$ propagates any function of phase space f belonging to the set $\mathcal{V}$ of analytic functions into its new form after one iteration (or one turn). It is well known that the map $\mathcal{G}$ preserves the Poisson bracket. It is a symplectic map:

$$f \in \mathcal{V} \xrightarrow{\mathcal{G}} \mathcal{G}f \in \mathcal{V} \; ; \; \mathcal{G} \in End(\mathcal{V}) \tag{1.1a}$$

$$\mathbf{x} \in \mathbb{R}^{2N} \xrightarrow{f} f(\mathbf{x}) \in \mathbb{R} \tag{1.1b}$$

$$\mathbf{x} = (q_1, p_1, \ldots, q_N, p_N) \tag{1.1c}$$

where $\mathbf{q}$ are the positions and $\mathbf{p}$ the momenta. We choose canonical variables which obey the famous Poisson bracket condition

$$[q_i, p_j] = \delta_{ij} \, . \tag{1.1d}$$

As we just said, the Poisson bracket is preserved by $\mathcal{G}$:

$$(f, g) \in \mathcal{V}^2 \xrightarrow{[\,,\,]} [f, g] \in \mathcal{V} , \tag{1.2a}$$

$$\mathcal{G}[f, g] = [\mathcal{G}f, \mathcal{G}g] \, . \tag{1.2b}$$

Furthermore, we assume that our starting point is a Hamiltonian $H(t)$ describing the motion between t and $t + dt$:

$$\frac{d}{dt} f(\mathbf{x})|_{\mathbf{x}=\mathbf{x}(t)} = [-H(t), f(\mathbf{x})]|_{\mathbf{x}=\mathbf{x}(t)} \, . \tag{1.3}$$

We can derive a differential equation for the map $\mathcal{G}(s; s+t)$ in terms of the Lie operator $: -H(t) :$ associated with $H(t)$ [1, 2]:

$$\frac{d}{dt} \mathcal{G}(s; s+t) = \mathcal{G}(s; s+t) \; : -H(t) : , \tag{1.4a}$$

$$\mathcal{G}(s; s) = \mathcal{E} \; ; \; \mathcal{E} = Identity , \tag{1.4b}$$

$$: f : g = [f, g] \; ; \; : f : \in End(\mathcal{V}) . \tag{1.4c}$$

The fundamental problem of particle optics in a periodic system is the understanding of the effect of repeated iteration of $\mathcal{G}$. In other words, we would like to understand the map $\mathcal{G}(s; s+n) = \mathcal{G}(s; s+1)^n$ as the integer n gets larger and larger.

The goal of this paper is to present a new powerful set of techniques which allow us to perform some standard normalization transformations on the map $\mathcal{G}(s; s+1)$ without compromising on the actual complexity of the original Hamiltonian $H(t)$. To put this in perspective, we will first describe the usual way these studies are done by accelerator physicists.

2. Conventional Approach for the Study of $\mathcal{G}(s; s+1)$

A) Ray Tracing.

The simplest way to include all the information contain in $H(t)$ is to integrate Eq.(1.3) for the 2N projection functions Π_i defined as follows:

$$\mathbb{R}^{2N} \xrightarrow{\Pi_i} \mathbb{R}; \;\; \Pi_i \in \mathcal{V}\,, \tag{2.1a}$$

$$\Pi_i(\mathbf{x}) = x_i\,. \tag{2.1b}$$

Of course, only a finite set of initial conditions can be traced and therefore it is hard to extract a lot of information from ray tracing. In this brute force approach, the simulation can be done with a Hamiltonian $H(t)$ which can have all the pieces simulating field errors, position errors or any other messy effects. Traditionally and accidentally, accelerator physicists have used low order explicit symplectic (or canonical) integrators to simulate their complex machines.

What if analytical results are needed? How do they proceed? The answer is simple: they go back to the Hamiltonian $H(t)$. We briefly describe this approach is the next paragraph.

B) Normalization of H(t).

During a normal form process, one attempts to transform the Hamiltonian $H(t)$ into a "simpler" Hamiltonian $K(t)$ by a periodic canonical transformation $\mathcal{A}(t)$. This process requires the knowledge of $\mathcal{A}(t)$ for all t's. For example, let us suppose that we have the following relations:

$$\mathbf{z} = \mathcal{A}(t)^{-1}\mathbf{x} = \exp(: -w(\mathbf{x}; t) :)\mathbf{x}\,, \tag{2.2a}$$

$$\mathbf{z} = \textit{new variables}\,. \tag{2.2b}$$

Then the new Hamiltonian is given by a relation first derived by Cary (1978) [3]:

$$K(t) = \mathcal{A}(t)\left(H(t) + \mathrm{iexp}(: -w(t) :) \frac{\partial w}{\partial t}\right), \tag{2.3a}$$

$$\mathrm{iexp}(\lambda) \;=\; \frac{(\exp(\lambda) - 1)}{\lambda}\,. \tag{2.3b}$$

Corresponding to $K(t)$, a map $\mathcal{N}(t)$ can be derived and it obeys Eq.(1.4). In fact, using the definition of $\mathcal{A}$ and remembering that our maps transform functions of phase space, we get:

$$\mathcal{N}(s; s+1) \;=\; \mathcal{A}(s)\; \mathcal{G}(s; s+1)\; \mathcal{A}(s)^{-1}\,. \tag{2.4}$$

If the system under study is of sufficient complexity, the direct computation of $K(t)$ and $\mathcal{A}(t)$ is not feasible without approximating $H(t)$. Hence a Birkhoff (or Deprit) type normalization of the map described by $H(t)$ is rarely done in a way consistent with the results of the integrator used in tracing rays in phase space.

C) Conclusion on Old Methods.

Perturbation theory on the Hamiltonian is not feasible on complex systems. In addition, since we are interested in the behavior of $\mathcal{G}(s; s+1)$ at a finite number of locations s, the complete solution required by Eq.(2.7) contains a lot of useless information even when obtained.

What can be done to maintain the generality of the integrator and at the same time perform normalization algorithms on the very model used by the integrator without further approximations? This we will answer in the next sections.

3. A New Approach for the Study of $\mathcal{G}(s; s+1)$

What we have done and will now describe is to adapt old methods and develop new ones in order to match exactly the hard and easy aspects of the systems we intend to study. The tools that will be invoked in this section are not necessarily more complicated than the old tools, but they are more suited to the study of complex periodic systems.

We list the important concepts in our approach in the chronological order in which they appear in a calculation.

i) An explicit integrator must be written to described the system. Whenever possible, we prefer to use an explicit canonical integrator. Integrators of this type have been derived for Lie groups for up to fourth order in the time step. Without further approximations, rays can now be traced.

ii) The following is true for any symplectic map and in particular for the one turn map $\mathcal{G}$, given any function $f \in \mathcal{V}$:

$$(\mathcal{G}(s; s+1)f)(\mathbf{x}) = f(\mathbf{z}) \tag{3.1a}$$

$$\mathbf{z}(s) = (z_1(s), ..., z_{2N}(s)) = ((\mathcal{G}(s; s+1)\Pi_1)(\mathbf{x}), ..., (\mathcal{G}(s; s+1)\Pi_{2N})(\mathbf{x})) . \tag{3.1b}$$

we need only to compute the 2N functions $z_i(\mathbf{x})$ to know the action of the full map $\mathcal{G}$ on any function belonging to $\mathcal{V}$. Property (3.1) can be viewed as a consequence of Hamilton's equations Eq.(1.3) or a consequence of the differential character of Lie operators (they obey Leibnitz rule: $: f : (gh) = (: f : g)h + g(: f : h)$).

Furthermore, since our goal is to perform a perturbative calculation ordered by the degrees in the power series expansion of H around a periodic orbit $\mathbf{z}^0(s)$ ($\mathbf{z}^0(s+1) = \mathbf{z}^0(s)$), we will need to obtain the power series of the 2N functions z_i to a predetermined order N_o around the periodic orbit.

This particular calculation can be done with the new software implementation of the powerful tools of Differential Algebra which itself is an application of Non-Standard Analysis. Essentially, we create a new field $\mathcal{R}$ which is an extension of $\mathbb{R}$. An analytic function over the field $\mathbb{R}$ is known completely if it is known for one single point of the field $\mathcal{R}$ (super-Cauchy theorem!). Hence our integrator is instructed to compute the projection functions Π_i by following the periodic orbit $\mathbf{z}^0(s)$ in that super-field. In practice (and in FORTRAN![5]), one has augmented the integrator introduced in item i) with a map extraction algorithm which will provide a power series expansion of the functions $z_i(s)$ around any orbit and in particular around a periodic orbit. These tools are limited in the order N_o and in the number of degrees of freedom 2N by the power of our computers only. The coefficients of the resulting power series are exactly computed for (and by) our integrator and are therefore very accurate as compared to those obtained by numerical differentiation. In fact the only source of inaccuracies in the computation of these coefficients is in the cumulative truncation error produced by a large number of operations.

To sum up we have cheated the gigantic size of our task twice: firstly we noticed that the action of the infinite dimensional Lie group of Classical Dynamics can be studied by its action on the projection functions Π_i and secondly we constructed a super-field in which these functions need to be evaluated on one element only.

In addition, once a particular ordering of the Lie operators [4]has been chosen to represent a symplectic map then the following statements are true:

Ideally two different elements of the super-field $\mathcal{R}$ representing symplectic maps will lead to different symplectic maps. In practice two sets of symplectic functions Π_i lead to the same Lie representation if their power series are identical to the highest order considered (i.e.N_o). Hence the functions Π_i belong to an equivalence class which is independent of the reconstruction process. For a non-canonical integrator, this is not true. For each process there exists a infinite number of non-symplectic functions leading to the same Lie operators. Hopefully the process is chosen so as to introduce an error which is not greater than the violation of the symplectic condition in the integrator.

iii) The production of the map and the analysis of the one period map are now independent procedures. The analysis and normalization of this map are best achieved by computing from the functions $z_i(s)$ the Lie generators of the original map $\mathcal{G}(s; s+1)$. This is not surprising since the process leading from $H(t)$ to $K(t)$ was perform on these Hamiltonians which are just the infinitesimal generators of $\mathcal{G}$ and $\mathcal{N}$ as indicated by Eq.(1.4). This process can also be handled by the software tools in the Differential Algebra package.

iv) Finally, if $\mathcal{G}(s; s+1)$ represents a quasi-integrable system, which we want to perturb at a few selected locations by a very nonlinear force, we can set up a Hamiltonian-Free context for that study. More precisely, we can derive a infinite number of pseudo-Hamiltonians which describe the quasi-integrable system between the locations of the additional nonlinear force.

In sections 4,5,6 and 7 we will describe in a moderate amount of details the four items on this list which allow us to integrate complex systems and properly analyse them.

4. Canonical Integration in the Symplectic Group

A) Explicit Integration in a Lie Group

We will now describe a very ideal situation which occurs most of the time in the study of medium to large periodic systems. In these systems, for reasons outside the range of this paper, the Hamiltonian $H(t)$ and its Lie operators can be broken in two terms which are exactly solvable in terms of simple functions:

$$H(t) = H_1(t) + H_2(t)\,. \tag{4.1a}$$

$$: -H(t) : = : -H_1(t) : + : -H_2(t) : \,. \tag{4.1b}$$

In other words, we have an explicit representation for the action of $\mathcal{M}_i$ on the coordinate projection functions Π_i. Of course, $\mathcal{M}_i(t)$ is a solution of Eq.(1.4a):

$$\frac{d}{dt}\mathcal{M}_i(s;s+t) \;=\; \mathcal{M}_i(s;s+t) \;\; : -H_i(t) : , \tag{4.2}$$

How to combine the two solvable maps $\mathcal{M}_i$ over a time step Δt and approximate the full map $\mathcal{G}(s;s+\Delta t)$ is the fundamental question behind "two-map" explicit symplectic integrators. Needless to say that the problem stated here can be generalized to any Lie group by simply stating that the three operators $: -H :$, $: -H_1 :$ and $: -H_2 :$ belong to the Lie Algebra of the group under study.

First, it is often possible to get rid of the t-dependence by a temporary extension of phase space:

$$: -h : = : -H_1(t) - p_t : + : -H_2(t) - p_t : + : p_t : , \tag{4.3a}$$

$$=: -h_1 : + : -h_2 : + : p_t : \,. \tag{4.3b}$$

The new "time-like" variable τ is related to t by the relation:

$$\frac{dt}{d\tau} = \frac{\partial h}{\partial p_t} = 1; \text{hence} \Rightarrow \;\; \tau = t\,. \tag{4.4}$$

Using the τ-indepence of $: -h :$, we can immediately write an exact formal solution for the maps $\mathcal{G}(s;s+\Delta t)$ and $\mathcal{M}_i(s;s+\Delta t)$:

$$\mathcal{G}(s;s+\Delta t) := \exp(-\Delta t : h :)\,. \tag{4.5a}$$

$$\mathcal{M}_i(s;s+\Delta t) := \exp(-\Delta t : h_i :)\,. \tag{4.5b}$$

We use the symbol $:=$ to indicate that the maps are not really equal since the original one acts on functions of only 2N variables while the new map acts on the extended phase space of 2N+2

variables. Making use of the Campbell-Baker-Hausdorff formula and of Eqs.(4.5a) and (4.5b), we can derive four approximations of $\mathcal{G}(s; s+\Delta t)$:

$$\exp(-\Delta t : h :) = \exp(-\Delta t : h_1 :) \exp(-\Delta t : h_2 :) \exp(\Delta t : p_t :) + ...\text{order}(\Delta t^2)\,,$$

$$\mathcal{G}(s; s+\Delta t) = \mathcal{M}_1(s; s+\Delta t)\mathcal{M}_2(s+\Delta t; s+2\Delta t) + ...\text{order}(\Delta t^2)\,; \tag{4.6a}$$

$$\exp(-\Delta t : h :) = \exp(-\frac{\Delta t}{2} : h_1 :) \exp(\frac{\Delta t}{2} : p_t :)$$

$$\exp(-\Delta t : h_2 :) \exp(\frac{\Delta t}{2} : p_t :) \exp(-\frac{\Delta t}{2} : h_1 :) + ...\text{order}(\Delta t^3)\,,$$

$$\mathcal{G}(s; s+\Delta t) = \mathcal{M}_1(s; s+\frac{\Delta t}{2})\mathcal{M}_2(s; s+\Delta t)\mathcal{M}_1(s+\frac{\Delta t}{2}; s+\Delta t) + ...\text{order}(\Delta t^3)\,; \tag{4.6}$$

A fourth order integrator can also be derived, however due to its lengthy expression, we give only the time independent result [6]:

$$\mathcal{G}(\Delta t) = \mathcal{M}_1(s_1)\mathcal{M}_2(d_1)\mathcal{M}_1(s_2)\mathcal{M}_2(d_2)\mathcal{M}_1(s_2)\mathcal{M}_2(d_1)\mathcal{M}_1(s_1) + ...\text{order}(\Delta t^5)\,; \tag{4.7a}$$

$$s_1 = \frac{1}{2(2-\beta)}\Delta t\,, \tag{4.7b}$$

$$s_2 = \frac{(1-\beta)}{2(2-\beta)}\Delta t\,, \tag{4.7c}$$

$$d_1 = \frac{1}{2-\beta}\Delta t\,, \tag{4.7d}$$

$$d_2 = \frac{-\beta}{2-\beta}\Delta t\,, \tag{4.7e}$$

$$\beta = 2^{\frac{1}{3}}\,. \tag{4.7f}$$

B) Implicit Integration in a Lie Group

For the symplectic group it is possible to write an implicit symplectic integration scheme using a characteristic function. This works all the time even when the Hamiltonian $H(t)$ precludes the existence of an explicit solution. For example, we can reproduce the effect of $\mathcal{G}(s; s+\Delta t)$ on the projection functions Π_i to first order in Δt by tracking using the characteristic function:

$$F_1 = \mathbf{q}(s)\cdot\mathbf{p}(s+\Delta t) - \Delta t\; H(\mathbf{q}(s), \mathbf{p}(s+\Delta t); s)\,. \tag{4.8}$$

In general F_k is a solution of the Hamilton-Jacobi equation [7]:

$$\frac{\partial F_k}{\partial t} + H(\mathbf{q}, \frac{\partial F_k}{\partial \mathbf{q}}; t) = 0 + ...\text{order}(H^{k+1})\,. \tag{4.9}$$

The map extraction techniques described in the next section are still applicable to a characteristic function integrator. First, one computes the central trajectory by solving the implicit set of

equations produced by F_k and then F_k is expanded around this trajectory and the resulting power series is partially inverted using the the Differential Algebra tools of section 5.

C) Conclusion

We have seen how one produces simple symplectic integrators of the explicit and implicit kind. Furthermore, if only a map is needed for analysis or tracking, it can be obtained from a non-canonical integrator. The resulting power series can be "symplectified" by a procedure which extracts a characteristic function or a Lie operator representation of $\mathcal{G}$. These computations are all done by the tools of the next two sections.

5. Non-Standard Analysis and its Application to Map Extraction

In this section we will discuss a very powerful tool for the advertised study of the production and analysis of maps in their power series representation. For our practical purposes, it will allow a very straightforward computation of derivatives and thus Taylor series expansions for very complicated functions of arbitrarily many variables to arbitrary order on a computer. It is based on a rigorous treatment of infinitely small quantities, differentials, and will allow the computation of derivatives from the difference quotient using differential differences.

Contrary to the other existing methods for rigorous treatment of such differentials (see for instance references [8, 9]), our method is fully constructive and can be implemented on a computer.

We start by defining a special collection of subsets of the real numbers, the collection of almost-finite sets $\mathcal{F}$. We call a set almost-finite, if for any real number r there are only finitely many elements of the set which are smaller than r. From this definition it is clear that all finite subsets of the real numbers are in $\mathcal{F}$. Furthermore, it is easy to convince oneself that each subset of the positive numbers is in $\mathcal{F}$. However, the set of all integers is not in $\mathcal{F}$, since there are infinitely many integers smaller than zero.

It is easy to show that a subset of any set in $\mathcal{F}$ is also in $\mathcal{F}$, and that with two sets their union and their intersection are in $\mathcal{F}$. Furthermore, with two sets $M, N \in \mathcal{F}$ the set $M + N = \{x + y | x \in M, y \in N\}$ is in $\mathcal{F}$, and there are only finitely many ways to write an element of $M + N$ as a sum of two elements of M and N, respectively. Finally we note that each set in $\mathcal{F}$ has a smallest element.

We are now ready to define a new, very large set, which will turn out to be a generalization of the real numbers and also contain "infinitely small" numbers. We define $\mathcal{R}$ to be the set of functions on the real numbers that are zero everywhere except on a set which is almost-finite, i.e. the set of non-zeroes of such functions belongs to $\mathcal{F}$.

On $\mathcal{R}$ we define an addition by just adding the functions. The resulting sum function is again in $\mathcal{R}$ because the set of non-zeroes of the sum function is contained in the union of the non-zeroes of the functions to be summed, and is hence in $\mathcal{F}$ according to the above reasoning. Hence the set $\mathcal{R}$ is closed under addition.

We also define a multiplication in the following way. For two functions $f, g \in \mathcal{R}$ let N_f, N_g denote the set of non-zeroes. We define the product function $f \cdot g$ in the following way. In case $x \notin N_f + N_g$, we say $(f \cdot g)(x) = 0$. In case $x \in N_f + N_g$, we define $(f \cdot g)(x)$ in the following way:

$$(f \cdot g)(x) = \sum_{x_f + x_g = x} f(x_f) \cdot g(x_g) \tag{5.1}$$

There are only finitely many terms in the sum because as stated above, there are only finitely many ways to write an element of $N_f + N_g$ as a sum of an element of N_f and an element of N_g. Since $N_f + N_g \in \mathcal{F}$, we can infer that the product function $f \cdot g$ is again in $\mathcal{R}$.

It can be shown that with this definition of an addition and a multiplication, the set $\mathcal{R}$ becomes a field. Most of the properties of fields can be shown quite easily to be fulfilled; the only exception is probably the existence of multiplicative inverses. For details consult reference [10]. We note that

the neutral element of addition is just the function identical to zero, and the neutral element of multiplication is the function which vanishes everywhere except at $x = 0$ where it has the value 1.

On this new field we introduce an ordering relation in the following way. As customary for functions, we say $f = g$, if the two functions agree for all values of x. In case $f \neq g$, we look at the smallest x where $f(x) \neq g(x)$ and denote it by $x_{\neq}$. It exists since the set of x values where f and g disagree is contained in $N_f \cup N_g$ and is hence in $\mathcal{F}$ and thus has a smallest value. We say now that $f > g$ if $f(x_{\neq}) > g(x_{\neq})$ and $f < g$ if $f(x_{\neq}) < g(x_{\neq})$.

It follows that for arbitrary f, g we have exactly one of the conditions $f = g$, $f < g$ or $f > g$. Hence the set $\mathcal{R}$ is well ordered. Furthermore, one obtains $f < g \Rightarrow f + h < g + h$, and $f < g, h > 0 \Rightarrow f \cdot h < g \cdot h$.

After we have discussed the basic properties of this new set $\mathcal{R}$, we owe the reader a justification for the introduction and usefulness of this new structure. First we note that the set of real numbers can be embedded into $\mathcal{R}$ by identifying the real number r with the function which is zero everywhere except at zero where it takes the value r. Denoting this embedding map from the real numbers into $\mathcal{R}$ by π, we can easily verify that $\pi(x + y) = \pi(x) + \pi(y)$, $\pi(x \cdot y) = \pi(x) \cdot \pi(y)$ and that $x < y \Rightarrow \pi(x) < \pi(y)$. So π preserves the field operations $+$ and $\cdot$ and the ordering $<$.

Hence our new structure contains all the real numbers. However, it contains much more than the real numbers, especially infinitely small and infinitely large quantities as we will show now. First we denote with d the element of $\mathcal{R}$ which vanishes everywhere except at 1 where it has the value 1. By using the definition of the multiplication, we can convince ourselves that its multiplicative inverse d^{-1} is the function that vanishes everywhere except at -1 where it has the value 1. Using our ordering relations, we can conclude that for every positive real number r we have the properties $0 < d < r$, and $d^{-1} > r$. Hence d lies between zero and any arbitrary positive real number and is thus "infinitely small". On the other hand, d^{-1} is "infinitely large". Hence the number d is what physicists like to think of as a differential.

As we will see, the field $\mathcal{R}$ allows us indeed a fully rigorous treatment of infinitely small and infinitely large quantities and naturally allows the introduction of deltafunctions. But most importantly for our purposes, it is helpful for the computation of derivatives and hence Taylor series expansions. We shall illustrate this with a little example. Consider the function $f(x) = x^2 + 2$. Then obviously, its derivative at $x = 2$ is 4. Having infinitely small quantities at our disposal, it is natural to try to obtain the derivative as the difference quotient

$$\frac{f(2+d) - f(x)}{d} \tag{5.2}$$

Using the arithmetic on $\mathcal{R}$ and in particular the field properties, we obtain

$$\frac{f(2+d) - f(x)}{d} = \frac{((2+d)^2 + 2) - 6}{d} = \frac{6 + 4d + d^2 - 6}{d} = 4 + d \tag{5.3}$$

Thus we obtained the exact value of the derivative, up to the infinitely small quantity d. If all we are interested in is the real derivative, we are done because we simply have to extract the "real part" from the result. So in a way we have pushed numerical differentiation techniques to the

extreme; usually the error in representing the derivative by a difference quotient becomes smaller and smaller the smaller the value of Δx. In our case, Δx became infinitely small, and so did the error.

Using the extended real numbers, one can construct extended complex numbers $\mathcal{C}$ in the usual way by introducing ordered pairs and operations on them. As it turns out, very many functions on the complex numbers can be generalized to the extended field $\mathcal{R}$ in a natural manner. In particular, this holds for all analytic functions. We shall sketch this and refer to reference[10]for details. Suppose, f is analytic in a neighborhood of z. Then f can be expressed as a power series around z with a nonzero radius of convergence. It can be shown that this power series converges even for every element in $\mathcal{C}$ to an element of $\mathcal{C}$ once the argument lies within the radius of convergence.

One of the most striking properties of such analytic functions on the extended complex numbers $\mathcal{C}$ shall be illustrated now. As we know from basic Cauchy theory, every analytic function in a simply connected region in the plane is uniquely determined by its values along a closed curve completely inside the region, because then the Cauchy formula allows its computation everywhere as an integral. Hence, because of analyticity, the function needs to be given only on a certain "one-dimensional subset" of the region. For extensions of analytic functions into $\mathcal{C}$ it turns out that they are uniquely given by their value at one suitable point, namely $z_0 + d$. This is true because of the existence of the power series:

$$f(z_0 + d) = f(z_0) + d \cdot f'(z_0) + d^2 \cdot \frac{f''(z_0)}{2} + ... \tag{5.4}$$

Evaluating $f(z_0+d)$ in $\mathcal{C}$ and noting that all the derivatives are complex, we obtain that $f(z_0+d)$ (as an element of $\mathcal{C}$) vanishes everywhere except for positive integer powers of d and the values at these points are the derivatives (times the factorials).

So for practical purposes, it suffices to compute an analytic function at only one point to obtain complete information about it and especially all its derivatives. We also note that this procedure can be extended to power series of several variables, which is of importance for our computation of map expansions.

In addition to the elementary operations addition and multiplication we introduce a "left shift" operation ∂ that decreases all the non-zeroes x_i by one and multiplies the values at the non-zeroes f_i with the nonzero. It turns out that this operator is a "derivative" in that $\partial(f \cdot g) = (\partial f) \cdot g + f \cdot (\partial g)$. With this derivative, we obtain a Differential Algebra in the sense of reference[11]. Furthermore, if the f is a power series evaluated at the point $z_0 + d$, the operation ∂ transforms f into the derivative of the power series evaluated at $z_0 + d$.

Using this "derivative", we can compute Poisson brackets and hence can do all the manipulation of functions of phase space required in our Lie Algebraic treatment.

Contrary to the other methods to introduce Non-Standard Analysis ([8, 9]), here it is possible to implement the arithmetic on the computer. In order to do that every element is characterized by the values of non-zeroes x_i and the corresponding f_i up to a certain depth of x_i. Then the operations addition, multiplication and "derivative" can be implemented following the definitions of the operations.

If the objective is to compute only power series of functions of one variable, this is even quite straightforward. The situation becomes more difficult, however, in the case of power series of many variables in which we are interested here. In this case an efficient implementation of the multiplication requires a quite elaborate storing and retrieval technique. For details we refer to reference [5].

6. Normal Form Procedures on a power Series Map

A) A First Order Calculation on the Lie Representation

As mentioned in Eq.(2.4), the normal form procedures transform the map $\mathcal{G}(s; s+1)$ into a simpler map $\mathcal{N}(s; s+1)$ (dropping s from now on):

$$\mathcal{N} \;=\; \mathcal{A}\,\mathcal{G}\,\mathcal{A}^{-1}. \tag{6.1a}$$

$$\mathcal{A} = \exp(:F_\Omega:)\ldots\exp(:F_\omega)\ldots\exp(:F_0:). \tag{6.1b}$$

This factorization of $\mathcal{A}$ was first used by Dragt and Finn in a modification of the Deprit normal form algorithm[12](see also reference[3]for a good review of these various methods). We will also assume that $\mathcal{N}$ is a product of a linear map $\mathcal{R}$ and a nonlinear correction $\mathcal{N}_\Omega$:

$$\mathcal{N} \;=\; \mathcal{R}\,\mathcal{N}_\Omega. \tag{6.2a}$$

$$\mathcal{N}_\Omega = \exp(:T_0:)\ldots\exp(:T_\Omega:). \tag{6.2b}$$

In a perturbative process carried order by order in the canonical variable $\mathbf{x}$, the Lie polynomials T_ω and F_ω are of degree $\omega+2$. The highest order Ω will be just N_0+1 where N_0 is the highest degree of the power series of the projection functions Π_i. The map $\mathcal{R}$ will be a generalized rotation and has the following Lie operator:

$$\mathcal{R} = \exp(:f_2:), \tag{6.3a}$$

$$f_2 = \sum_{k=1}^{N} -\frac{\mu_k}{2}\,(q_k^2 + (\epsilon_k - \bar{\epsilon}_k)p_k^2) = \sum_{k=1}^{N} f_2^k \tag{6.3b}$$

$$\begin{cases} \epsilon_k = 1, \bar{\epsilon}_k = 0 \;\; ; & \text{for stable motion in } k^{th} \text{ plane.} \\ \epsilon_k = 0, \bar{\epsilon}_k = 1 \;\; ; & \text{for unstable motion in } k^{th} \text{ plane.} \end{cases} \tag{6.3c}$$

Essentially we assume that the linear part of $\mathcal{G}$ can be normalized and the result is hyperbolic or elliptic motion for all the planes (q_i, p_i). To give a flavour of the calculations, we now perform a first order determination of $\mathcal{A}$ on a map $\mathcal{M}$ of the form:

$$\mathcal{M} = \mathcal{R}\exp(:\alpha f:), \tag{6.4a}$$

$$\mathcal{M} \in End(\mathcal{V})\;;\; \mathcal{R} \in End(\mathcal{V}). \tag{6.4b}$$

Consider a canonical transformation $\mathcal{A}$ whose purpose is to modify $\mathcal{M}$ into a new factorized representation $\mathcal{N}$ defined to first order in α. Using a Lie representation for $\mathcal{A}$, we get for $\mathcal{N}$

$$\mathcal{N} = \mathcal{A}\mathcal{M}\mathcal{A}^{-1}$$

$$
\begin{aligned}
&= \exp(:\alpha F:)\mathcal{R}\exp(:\alpha f:)\exp(:-\alpha F:) \\
&= \mathcal{R}\exp(:\alpha\mathcal{R}^{-1}F:)\exp(:\alpha f:)\exp(:-\alpha F:) \\
&= \mathcal{R}\exp\left(:\alpha\{-(\mathcal{E}-\mathcal{R}^{-1})F+f\}+O(\alpha^2):\right) \\
&\qquad (\mathcal{E} = \text{identity map} \in End(\mathcal{V}))
\end{aligned} \tag{6.5}
$$

Denoting by $\mathcal{T}$ the operator $\mathcal{E}-\mathcal{R}^{-1}$, it is clear from Eq. (6.5) that one must study the range and the kernel of $\mathcal{T}$ in order to specify what possible linear terms in α can remain in Eq. (6.5). Suppose f is decomposed as follows:

$$
f = f_r + f_o \quad ; \quad f_r \perp Ker\,\mathcal{T} \tag{6.6}
$$

Then, we can select $\mathcal{A}$ or F such that $\mathcal{N}$ becomes $\exp(:\alpha f_o + O(\alpha^2):)$. The function F is just given by:

$$
F = \mathcal{T}^{-1} f_r\,. \tag{6.7}
$$

From this short discussion one sees the central importance of the map $\mathcal{R}$. The eigenvectors of $\mathcal{R}$ of unit eigenvalue will constitute the kernel $Ker\ \mathcal{T}$ so critical to the inversion of $\mathcal{T}$. In the case of Eq.(6.3) the operators $:f_2^k:$ form a semi-simple algebra, we purposely neglect the case $\epsilon_k = \bar{\epsilon}_k = 0$. Its inclusion would complicate the discussion, since it is not true anymore that the vector space of polynomial functions is a direct sum of the range $Im\mathcal{T}$ and the kernel $Ker\ \mathcal{T}$ if the Lie Algebra of the $:f_2^k:$ has a nilpotent component. We now find the range and the kernel of $\mathcal{T}$ by constructing a suitable eigenbasis for the study of $\mathcal{R}$ in the case described by Eq.(6.3).

The evaluation of $\mathcal{T}^{-1}f_r$ requires a decomposition of f_r in eigenvectors of $:f_2:$. These eigenvectors are easy to obtain, the answer is given by:

$$
:f_2^k: h_k^{\pm} = \mp(i\epsilon_k + \bar{\epsilon}_k)\mu_k\, h_k^{\pm} = \mp\lambda_k h_k^{\pm} \tag{6.8a}
$$

$$
h_k^{\pm} = q_k \pm (i\epsilon_k + \bar{\epsilon}_k)p_k \tag{6.8b}
$$

$$
:f_2^k: = -\frac{\mu_k}{2}\, h_k^+ h_k^-\,. \tag{6.8c}
$$

Using this new basis, we can easily find the kernel $Ker\ \mathcal{T}$. Let us define a new vector as follows:

$$
|\mathbf{m},\mathbf{n}\rangle = (h_1^+)^{m_1}(h_1^-)^{n_1}\ldots(h_N^+)^{m_N}(h_N^-)^{n_N}\,. \tag{6.9}
$$

Using the differential property of the operator $:f_2:$, we can compute the eigenvalue of $|\mathbf{m},\mathbf{n}\rangle$

$$
:f_2: |\mathbf{m},\mathbf{n}\rangle \;=\; (\mathbf{n}-\mathbf{m})\cdot\boldsymbol{\lambda}\,|\mathbf{m},\mathbf{n}\rangle\,. \tag{6.10}
$$

Assuming that the λ_ks are irrational and prime amongst themselves, we conclude that

$$
|\mathbf{m},\mathbf{n}\rangle \in Ker\mathcal{T} \implies \mathbf{n}-\mathbf{m} \;=\; \mathbf{0}\,. \tag{6.11}
$$

Providing that one can easily change basis to the $|\mathbf{m},\mathbf{n}\rangle$, the computation of $\mathcal{T}^{-1}f_r$ is trivial:

$$f_r = \sum_{\mathbf{m},\mathbf{n}} A_{\mathbf{m},\mathbf{n}} \, |\mathbf{m},\mathbf{n}\rangle \tag{6.12a}$$

$$\mathcal{T}^{-1} f_r = \sum_{\mathbf{m},\mathbf{n}} \frac{A_{\mathbf{m},\mathbf{n}}}{1 - \exp\Big((\mathbf{m}-\mathbf{n}) \cdot \boldsymbol{\lambda} \Big)} \, |\mathbf{m},\mathbf{n}\rangle \, . \tag{6.12b}$$

B) Conclusion: To Higher Order with the Differential Algebra Tools

In reality, we have learned that one starts the normal form process on the representation of the map $\mathcal{G}(s; s+1)$ on the projection functions Π_i as given by their power series expansions obtained around the periodic orbit. Technically we do not have a Lie representation of the map.

However, it should be plausible to the reader that we can perform a normalization procedure on the map to an arbitrary order if the following statements are true (of course they are!):

1) The Differential Algebra software can take derivatives and integrals of polynomial functions. Hence it can extract all the Lie operators $\exp(: f_k :)$ from a map whose power series representation of the projection functions Π_i is known to the $(k-1)^{th}$ degree. Through Differential Algebra we can also multiply the coefficient of the monomials of a polynomial by an arbitrary function of the exponents of the monomials, allowing us to compute the effect of $\mathcal{T}^{-1}$ in Eq.(6.12).

2) The Differential Algebra software can compose functions represented by power series and therefore can perform the change of basis necessary for the production of $\mathcal{N}$. In the Π_i representation, it can compose maps to an arbitrary order avoiding the use of the Campbell-Baker-Hausdorff or the Zassenhaus formulae.

Going into the details of these operations would too lengthy and the work can be found elsewhere [13]. These calculations are mathematically equivalent to a normalization process on the Hamiltonian $H(t)$. However by separating the map extraction process from the normalization algorithm we can perform the perturbative calculations exactly for the symplectic integrator used.

7. The Floquet Representation and its Hamiltonian-Free Description

A) The Old Way: Normalizing the Hamiltonian

We assume that the Hamiltonian $H(t)$ consists of an quasi-integrable part $H_0(t)$ and a residual part $V(t)$. In the standard approach, one attempts to normalized the Hamiltonian H_0 using Eq.(2.3) :

$$K_0(t) = \mathcal{A}(t)\left(H_0(t) + \mathrm{iexp}(: -w(t) :) \frac{\partial w}{\partial t}\right), \tag{7.1a}$$

$$K_0(t) = K_0(\mathbf{J}, t), \tag{7.1b}$$

$$\mathbf{J} = \frac{1}{2}(h_1^+ h_1^-, \ldots, h_N^+ h_N^-), \tag{7.1c}$$

$$\mathcal{A}(t+1) = \mathcal{A}(t). \tag{7.1d}$$

The motion produced by the new Hamiltonian $K_0(t)$ will be a simple J-dependent rotation-expansion. Defining the action Φ in the usual manner,

$$h_k^{\pm} = (2J_k)^{\frac{1}{2}} \exp(\mp(i\epsilon_k + \bar{\epsilon}_k)\Phi_k) \tag{7.2}$$

we then get the usual tori for the motion of the resulting map $\mathcal{N}(s; s+t)$:

$$\mathcal{N}(s; s+t) = \exp(\int_s^{s+t} : -K_0(\tau) : d\tau) = \exp(: -\Gamma(\mathbf{J}; s, s+t) :), \tag{7.3a}$$

$$\Phi(s+t) = \Phi(s) + \frac{\partial}{\partial \mathbf{J}}\Gamma(s, s+t) = \Phi(s) + \Delta\Phi(s; s+t). \tag{7.3b}$$

In general the map $\mathcal{G}(s; s+t)$ which describes the motion of $H_0(t)$ from s to $s+t$ can be decomposed into three factors:

$$\mathcal{G}(s; s+t) = \mathcal{A}(s)^{-1}\, \mathcal{N}(s; s+t)\, \mathcal{A}(s+t). \tag{7.4}$$

As indicated by Eq.(7.4), a function is transformed into the Floquet space at location s by $\mathcal{A}(s)^{-1}$, then it is rotated into position $s+t$ in the Floquet space where it is finally extracted by $\mathcal{A}(s+t)$. The reader will notice that one regains Eq.(2.4) by letting $t = 1$. If we assume that the action of $\mathcal{A}^{-1}$ on the projection functions Π_i produces analytic functions (i.e. expandable in power series which are asymptotic at the very least), then one can show that:

$$\mathcal{N}(s; s+1) = \mathcal{N}(s+t, s+t+1), \text{ for all t }, \tag{7.5a}$$

and that:

$$\mathcal{N}(s; s+1) \text{ is the same for all choices of } \mathcal{A}(s) \tag{7.5b}$$

The function $\Gamma(s, s+1)$ associated to the Lie operator of $\mathcal{N}(s; s+1)$ is a global invariant of the full system.

Finally we can write an expression for the full transformed Hamiltonian $K(t)$:

$$K(t) \;=\; K_0(t) + \mathcal{A}(t)^{-1}V(t)\,. \tag{7.6}$$

The freedom that we have in selecting $\mathcal{A}(t)$ is normally exploited in trying to modify $V(t)$ as little as possible in Eq.(7.6).

B) Conclusion on Hamiltonian Normalization

Three things can be said about the procedure outlined in the previous section:

1) The solution of Eq.(7.6) requires solving for w (i.e $\mathcal{A}(t)$) at every location around our complex system. Usually this is impractical for a complex system as we pointed out already.

2) Extremely nonlinear perturbation such as $V(t)$ are often localized in the variable t. Therefore only the maps between these locations are necessary. Hence the steps outlined in section (7.A) produce unnecessary information. In fact, if canonical integrators are used to represent $H_0(t)$ as well as $V(t)$, the map is better described by tools which do not assume any differentiability in the variable t.

3) If, for whatever reason, one needs a normalized Hamiltonian which describes correctly the map from one localized perturbation to the next then there exists an infinite number of these Hamiltonians. Not surprisingly the exact Hamiltonian $K_0(t)$ is one of them; but it is one of the most complicated choice.

In the next section, we describe an approach ideal suited for studying a finite number of localized perturbations. Ultimately, as we pointed out in section 4, an integrator is best described as a finite product of maps .

C) Floquet transformation on the Map

To the extent that a symplectic integrator is a discontinuous set of operations (so is any integrator!), normalization of the Hamiltonian is not very suited to the study of the system produced by the integrator. In addition, as we just mentioned in section (7.B), this is even more so if a localized perturbation is added.

In this last section, we will develop an approach to the study of a full periodic system based on the maps alone[14, 15]. We will therefore assume the existence of a map $\mathcal{G}(s; s+t)$ known at few locations where we intend to perturb our system. For complete generality, we need only to consider two locations s_1 and s_2 as pictured on Fig. 1. First, we assume the existence of the maps $\mathcal{A}(s_i)$:

$$\text{There exists } \mathcal{A}_i \text{ such that } \mathcal{N} \;=\; \mathcal{A}_i^{-1}\mathcal{G}_i\mathcal{A}_i\,, \tag{7.7a}$$

$$\mathcal{N} \;=\; \exp(-:\Gamma(\mathbf{J}):)\;;\; \mathcal{G}_i = \mathcal{G}(s_i, s_i+1)\,. \tag{7.7b}$$

As we said earlier, the uniqueness of $\mathcal{N}$ is guaranteed by the good behaviour of the power series of $(\mathcal{A}\Pi_i)(\mathbf{x})$ near the origin. In the perturbation theory of section 5, this is implicitly assumed if the power series is to have any sense at all.

Tranformation between Phase Space

and

Normalized Phase Space

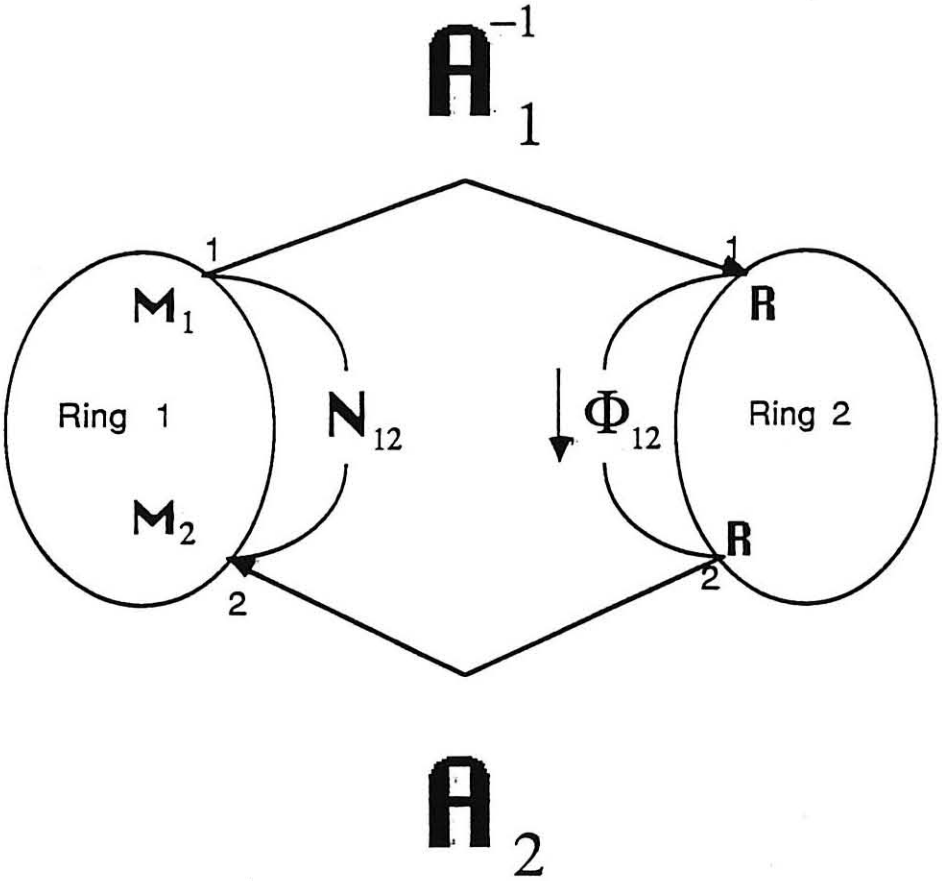

FIGURE 1

To proceed further, we write $\mathcal{G}_1$ in terms of $\mathcal{G}_2$ and use the integrability relation of Eq.(7.7) twice:

$$\mathcal{G}(s_1, s_1+1) = \mathcal{G}(s_1, s_2)\ \mathcal{G}(s_2, s_2+1)\ \mathcal{G}(s_1, s_2)^{-1}\,, \tag{7.8a}$$

$$\mathcal{G}(s_1, s_1+1) = \mathcal{G}(s_1, s_2)\ \mathcal{A}_2^{-1}\ \mathcal{N}\ \mathcal{A}_2\ \mathcal{G}(s_1, s_2)^{-1}\,, \tag{7.8b}$$

$$\mathcal{N} = \mathcal{A}_1\mathcal{G}(s_1, s_2)\mathcal{A}_2^{-1}\ \mathcal{N}\ \mathcal{A}_2\mathcal{G}(s_1, s_2)^{-1}\mathcal{A}_1^{-1}\,. \tag{7.8c}$$

Finally, using the Lie representation of $\mathcal{N}$:

$$\mathcal{N} = \mathcal{B}(s_1, s_2)\ \mathcal{N}\ \mathcal{B}(s_1, s_2)^{-1}\,, \tag{7.9a}$$

$$\mathcal{N} = \exp(- : \Gamma(\mathcal{B}(s_1, s_2)\mathbf{J}) :)\,, \tag{7.9b}$$

$$\text{where } \mathcal{B}(s_1, s_2) = \mathcal{A}_1\mathcal{G}(s_1, s_2)\mathcal{A}_2^{-1}\,. \tag{7.9c}$$

From Eq.(7.9b) we conclude that $\mathcal{B}$ must leave $\mathbf{J}$ invariant. Hence it must have the form:

$$\mathcal{B}(s_1, s_2) = \exp(- : \Gamma(\mathbf{J}; s_1, s_2) :)\,. \tag{7.10}$$

The function $\Gamma(\mathbf{J}; s_1, s_2)$ is identical to its Hamiltonian counterpart of Eq.(7.3).

Something must be said about the choice of $\mathcal{A}_i$: technically the only constraint is that $\mathcal{A}_i$ must be uniquely defined for each position s_i in order to insure periodicity. In addition, we often restrict ourselves to $\mathcal{A}_i$'s which are internally defined; by this we mean that our construction method is an injection from the set of possible $\mathcal{G}(s_i, s_i+1)$ (finite set for an integrator) to the infinite set of possible $\mathcal{A}_i$. This insures that the phase advance between two "matched" locations is the same for all possible injective construction (Matched locations have the same one turn map $\mathcal{G}$). Again however, the nature of the perturbation added to the map will dictate the choice of $\mathcal{A}$.

Finally, we add two localized perturbations at positions s_1 and s_2, given by their Lie operators $: V_1 :$ and $: V_2 :$ respectively. The new maps $\mathcal{G}^{total}(s_1, s_1+1)$ and $\mathcal{N}^{total}(s_1, s_1+1)$ are just:

$$\mathcal{G}^{total}(s_1, s_1+1) \;=\; \exp(- : V_1 :)\ \mathcal{G}(s_1, s_2)\exp(- : V_2 :)\ \mathcal{G}(s_2, s_1)\,, \tag{7.11a}$$

$$\mathcal{N}^{total}(s_1, s_1+1) \;=\; \exp(- : \mathcal{A}_1 V_1 :)\exp(- : \mathcal{B}(s_1, s_2)\mathcal{A}_2 V_2 :)\ \mathcal{N}\,. \tag{7.11b}$$

The various factors of $\mathcal{N}^{total}(s_1, s_1+1)$ have a simple interpretation. The first two factors represent the perturbations V_1 and V_2. One notices that each perturbation is modified by two maps: firstly, as we mentioned, the Lie operator is brought into the Floquet representation by $\mathcal{A}$; secondly the operator has its phase Φ advanced by the appropriate amount by the map $\mathcal{B}(s_1, s_2)$. The third factor is the unperturbed Floquet map $\mathcal{N}$. One might view the full action of $\mathcal{N}^{total}(s_1, s_1+1)$ on an arbitrary function f as follows: first f is distorted into a new function by the perturbations and then it is transformed by the original map $\mathcal{N}$.

As we said earlier, it is sometimes useful to produce a Hamiltonian which describes the map $\mathcal{N}^{total}(s_i, s_i+1)$ for $(i=1,2)$. Of course, the exact Hamiltonian $H(t)$ does this for all possible s:

$$H(t) \;=\; H_0(t) + \sum_{i=1,2} \delta_p(s-s_i)V_i\,, \tag{7.12a}$$

$$\delta_p(s) = \delta_p(s+m) \;\;;\;\; m = \text{integer}\,. \tag{7.12b}$$

However, this defeats the purpose of using maps. Indeed we assumed that the map is known only between s_1 and s_2. Nevertheless it is easy to write a infinite number of Hamiltonians which reproduce the motion of $\mathcal{N}^{total}(s_i, s_i+1)$ for $(i=1,2)$:

$$K_\kappa(t) = \kappa(\mathbf{J},t) + \sum_{i=1,2} \delta_p(s-s_i)\mathcal{A}_i V_i\,, \tag{7.13a}$$

$$\text{where } \kappa \text{ is such that: } \int_{s_1}^{s_2} \kappa(\mathbf{J},t)\,dt = \Gamma(\mathbf{J},s_1,s_2)$$

$$\text{and } \int_{s_1}^{s_1+1} \kappa(\mathbf{J},t)\,dt = \Gamma(\mathbf{J},s_1,s_1+1)\,. \tag{7.13b}$$

The Hamiltonians of Eq.(7.13) have the proper phase advance $\mathcal{B}$ between the the points s_1 and s_2 and the correct $\mathcal{A}$ at these locations. The function κ can be chosen to be very simple (or extremely complex).

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