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Quantum Electrodynamics at Large Distances. *

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Abstract

Traditional momentum–space methods for dealing with infrared divergences depend upon the neglect of certain terms that become dominant when the primary scattering function is evaluated on a singularity. Consequently, those methods lead to predictions that are highly inaccurate in the mesoscopic and macroscopic domains, and in fact violate the physically mandated correspondence–principle connection to classical physics in the macroscopic limit. These deficiencies can be remedied by a more precise treatment in which the infrared part of the problem is solved exactly, in the following sense: photons are divided into hard and soft photons, and a Feynman coordinate–space perturbation series is set up for the hard photons and massive particles alone, with all soft photons

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temporarily omitted. Each contribution to this series, before the integration over the locations $(x_1, ..., x_n) = x$ of the n vertices, is represented by a spacetime diagram $D(x)$. The soft photons are then separated into classical photons and quantum photons. For processes with no charged external particles the contributions from all numbers of classical photons can be summed exactly. This sum is a unitary operator which acting on the photon vacuum gives a quantum coherent state. This coherent state corresponds to the soft–photon part of the classical electro–magnetic field radiated by a classical charged particle moving along the charged–particle trajectory defined by the spacetime diagram $D(x)$. The bulk of the virtual and real soft–photon effects are contained in this well–defined unitary operator, and the remaining factor should contain no infrared divergences. This expected convergence property is confirmed here, to all orders in the remaining soft photons (i.e., the quantum photons), for the case in which the charged–particle lines of $D(x)$ form a triangle. It is also shown that strongest singularity on the triangle–diagram surface has the usual $\log \varphi$ character, and that the discontinuity around it is given by the Cutkosky formula. This property allows one to obtain, from the factorization property of the Cutkosky formula, the scattering functions for processes with charged initial or final particles. These analyticity properties also ensure the validity of the classically prescribed large–scale behavior. In the course of establishing these result a detailed formalism is developed that is suited to practical calculations of the consequences of quantum electrodynamics in the mesoscopic and macroscopic domains.

1. Introduction

Quantum systems composed of particles of nonzero mass enjoy three important general properties: conservation of energy-momentum, cluster decomposition, and pole factorization. Conservation of energy-momentum needs no explanation. Cluster decomposition is the property whereby if a system is divided into two parts, and these are, in a succession of cases, shifted further and further apart in any spatial direction, then these parts become dynamically independent in the limit of large separation: in this limit the individual parts separately conserve energy-momentum, and each part is separately invariant under translations in space and time. These properties ensures that if several parts of the quantum system are located at large distances from each other then these parts can be considered separately. This feature is one of the pillars of physics: it entails —insofar as the effects of massless particles can be neglected—that one can ignore the parts of the universe that are far away from a system being studied.

Pole factorization is the first correction to cluster-decomposition, and is almost as important. It is the feature according to which the largest correction to the cluster decomposition property can be understood in terms of the idea that a small physical object, which moves in accordance to the laws of classical physics, is emitted from one of the two partial systems and absorbed by the other. The energy-momentum transferred between the two systems is precisely that carried by such a particle. Moreover, the only information carried from one partial system to the other is information that can be carried by the degrees of freedom of this object. These pole factorization properties are crucial to the entire idea that the world can, in some sense, be considered to be built out of "particles". The experimentalist makes essential use of these properties when he interprets his experiment at an accelerator facility in terms of the idea that the accelerator is emitting particles that may eventually hit a target. And the theorist is using the idea when he attempts to express the general interaction in terms of diagrams whose lines represent "particles". That diagrammatic procedure is essentially an expansion around the large-distance limit.

In view of the basic importance of the pole-factorization property to both the interpretation of quantum theory and our basic computational procedures it would seem worthwhile to see whether this property carries over to quantum

electrodynamics, which is, after all, our premier quantum theory. It is apparent that this property does in fact hold true in nature, to a good approximation, even for charged particles. For example, a beam of electrons emerging from an accelerator behaves at large distances approximately like a beam of classical particles. Basic physical theory ought to explain why this is so, just as it did in the case where all particles had nonzero mass.

There are also important practical reasons why quantum electrodynamics ought to be developed so as to be able to resolve questions pertaining to large-distance behaviour. The first has to do with the extension of the scope of the theory to cover cases that are not in the realm of atomic physics proper, and, in particular, to the case of the physics of materials. This field is, of course, a highly developed subject, but there is nonetheless a problem of principle, namely the need to disentangle in a rational way the quantum and classical aspects of the properties of materials. The quantum properties of materials are enmeshed in a matrix of classical properties, and this relationship should emerge in a logical way from quantum theory itself, rather than being put in essentially by intuition. This problem of the quantum/classical connection is closely connected to another problem, namely the technical one of coping with infrared divergences in situations where the basic momentum-space scattering function is not free of singularities. These three problems—the large-distance behavior, the emergence of classical structure, and the infrared divergences at singularities—are, from a technical point of view, intimately interwoven. The task of the present work is the resolution of the technical problems that lie at the core of this tangled complex of physical problems.

For systems involving no massless particles quantum theory covers immediately the macroscopic domain. Consider a multiparticle system in which stable particles travel large distances between collisions mediated by short–range interactions. If the distances between collisions are large, in comparison to the range of the interactions, then the quantum formalism immediately entails the accuracy of a certain classical idea: the intermediate stable particles can be conceived to propagate between the collisions as classical particles. This result follows directly from the fact that the asymptotic behavior of the scattering amplitude in coordinate space is controlled by its singularities in momentum space, and in particular by the Landau singularity corresponding to the relevant

multiple–scattering diagram. The discontinuity around this singularity surface is given by the Cutkosky rule: it is essentially the product of the various relevant collision amplitudes. The fact that the stable particles are propagated by Feynman propagators, which have pole singularities in momentum space, leads precisely to the correct geometric fall-off in coordinate space, namely the fall-off associated with the straight-line propagation of classical point particles between the short-range collisions. Hence the validity of this classical concept in the macroscopic regime emerges directly from the Landau-Cutkosky singularity structure of scattering amplitudes in the physical region—if all particles are massive.

Straight-forward perturbation calculations in quantum electrodynamics indicate that these pertinent singularities in momentum space are stronger than what they would be in theories where all particles are massive, and that the long-distance fall-off of the charged-particle propagator is correspondingly slower than what classical concepts demand. Such a result, if really correct, would seriously jeopardize the consistency of quantum theory, which is normally thought to be compatible with the idea that charged electrons propagate over large distances in an approximately classical way.

This large apparent disparity between classical concepts and asymptotic properties in quantum theory arises from the presence of massless particles, and is closely connected to the problem of the infrared divergence of quantum electrodynamics. Yet from a strictly mathematical point of view there is no infrared divergence here. One can do the calculations and exhibit the form of the pertinent scattering functions, and the integrals are infrared convergent (Appendix A).The problem arises not from a *divergence* of the actually occurring integrals, but rather from the *form* of the singularity: the form of the singularity is stronger than what it is expected to be. Consequently, when one tries to compute the 'residue of the pole' of a singularity that is stronger than a pole one naturally gets a nonsensical result. The problem, therefore, from a strictly mathematical point of view, is to show that the *form* of the singularity is what is required on physical grounds: if the form of the singularity can be shown to be correct then the infrared finiteness of the result will follow from general mathematical principles.

The problem of deducing the correct form of the pertinent singularities was

partly solved in an earlier work. In the context of a coordinated-space, diagrammatic, perturbative expansion of the hard-photon part of the electromagnetic interaction a certain 'classical part' of the soft-photon interaction was defined, and summed to all orders to give a unitary operator. This operator, acting on the photon vacuum, generates a quantum-mechanical coherent state that corresponds to a certain classical electromagnetic field. This latter field is precisely the soft (low-frequency) part of the electromagnetic field classically radiated by charged point particles moving on the spacetime trajectories specified by the hard-photon spacetime diagram.

The reaction of this classical part of the soft-photon field upon the electron is small: there are energy-momentum adjustments that tend to zero as the largest soft-photon energy-momentum magnitude $|k|$ tends to zero. Hence this part has no infrared divergence. The remaining 'quantum' part of the soft-photon interaction has extra numerator powers of k, and is expected to be likewise infrared convergent. One of our aims here is to confirm this expectation.

The standard detailed description of how to get infrared finite results from quantum electrodynamical computations was given by Yennie, Frautschi, and Suura,[1] and by Grammer and Yennie.[2] However, those treatments do not cover points where the basic scattering amplitude under consideration is singular. But these are precisely the points that control the large–distance behavior.

Efforts have been made[3,4,5,6] to extend the general methods of Yennie et.al., to cases where some internal particles are on mass shell. However, these extensions do not yield the correct results. The problem is that a key step in the standard procedure is to ignore terms with factors $(e^{iky} - 1)$, on the grounds that for finite y these terms vanish as $k \to 0$. But this argument does not hold in the macroscopic limit, where the appropriate limit is $y \to \infty$, with k small but finite. Moreover, in the present context it is not possible to dismiss terms simply because they are finite. For our problem is to determine the dominant contributions to the asymptotic behavior. Consequently, all terms, even finite ones, must be carefully controlled, in order to single out the dominant large-distance contributions.

Described in more detail the failure of traditional infrared methods to give the correct forms of singularities arises in this way. To avoid infrared divergences the soft photons corresponding to the classical bremsstrahlung radiation asso-

ciated with the deflections of the charged particles is separated out for special treatment, leaving a remainder that can be calculated by standard perturbative methods. However, what is traditionally separated out is not the actual bremsstrahlung radiation itself, but rather the bremsstrahlung radiation that would be produced if all the deflections were to occur at some single point, conventionally taken to be the origin of spacetime. This approximation involves, typically, replacing a factor e^{ikx} by unity, where x is the point where the deflection occurs. If x were confined to a bounded region then this approximation would become increasingly accurate as k, the photon four–momentum, approaches zero. However, the character of momentum–space singularities is controlled by the *rate* at which the coordinate–space functions fall off in the limit $|x| \to \infty$. Thus, for any fixed nonzero k, the approximation is not accurate in the regime that controls the character of the singularity.

This effect can be understood also in more physical terms. If a charged particle is deflected at a point x, and travels to a point y were it is again deflected, then there will be bremsstrahlung radiation associated with each of these two deflections. The approximation of the actual bremsstrahlung by the bremsstrahlung that would be obtained if both deflections were to occur at the origin is, for each nonzero value of k, a poor approximation in the asymptotic regime in which x and y tend to infinity in different directions. Thus the infrared photons that are separated out for special treatment in this traditional approximation are not the physically relevant infrared photons in the regime that is controlling the singularity structure. To obtain the physically relevant infrared photons one must include all the neglected terms in the expansion of e^{ikx}.

The way to correct this inaccuracy is evident: one should place the sources of the separately–treated soft photons at the points where the deflections occur. However, the calculations are normally done in momentum space, where these points do not appear.

A procedure for placing the sources of the specially treated soft photons at precisely the points where the deflections occur has been constructed in ref. 7. The photons are first divided into hard photons, which have $|k| > \delta$, and soft photons, which have $|k| \leq \delta$, where $|k|$ is the Euclidean norm of k, and δ is very small on the scale of the electron mass. The calculation of the hard–photon

contributions is set up in coordinate space, as a Feynman-type perturbation calculation. The coordinate variables x_i specify the points where the hard photons interact with the charged particles. Prior to the integrations over the variables x_i each contribution is represented by a spacetime diagram D in which vertices located at the points x_i are connected by line segments representing charged particles. The propagation of a charged particle to a point x from a point y is represented by a Feynman propagator $P(x,y)$.

Holding the hard–photon vertices x_i fixed one introduces the contributions associated with soft photons. But the points where soft photons interact are not held fixed. Instead, one transforms the soft–photon variables into momentum space. At this stage, with the hard–photon vertices x_i held fixed, one separates out the soft photons corresponding to the classical bremsstrachlung radiation arising from the deflections of the charged particles at the various hard–photon vertices x_i. This separation is achieved as follows. The interaction of a soft photon having momentum k_j with a charged–particle running from a hard–photon vertex y to a hard–photon vertex x is separated into two parts by means of the formula

$$\gamma_\mu = C_\mu + Q_\mu, \tag{1.1}$$

where

$$C_\mu = z_\mu \not{k}_j (zk_j + i0)^{-1}. \tag{1.2}$$

Here γ_μ is the usual Dirac matrix associated with the photon interaction, $\not{k}_j = k_j^\nu \gamma_\nu$, and z is the coordinate difference

$$z_\mu = (x_\mu - y_\mu). \tag{1.3}$$

The soft photons that couple into the hard–photon diagram D are classified as "classical" or "quantum" in the following way: any soft photon that has a quantum coupling Q_μ on at least one end is a quantum photon; all other soft photons are classical photons.

Holding fixed the hard–photon interaction points x_i one can now explicitly sum to all orders the contributions of all classical photons. One finds[7], for the case of processes with no external charged particles, that this sum defines a unitary operator $U(D)$ that generates precisely the soft photons corresponding to the classical bremsstrahlung radiation from the classical spacetime process

represented by the spacetime diagram D. Specifically, the operator $U(D)$ acting upon the soft–photon vacuum generates the soft–photon coherent state that corresponds to the classical electromagnetic field radiated at the deflection points x_i. Thus the separation $\gamma_\mu = C_\mu + Q_\mu$ achieves the objective of separating out precisely the soft photons that correspond to the classical radiation field. The source of each classical photon is, for each term in the perturbative expansion of the hard–photon field, located at exactly the point x_i where the deflection that produces the classical photon takes place.

This operator $U(D)$ appears in the equation

$$F'(D') = U(D)F(D'), \tag{1.4}$$

where $F(D')$ is an operator representing the process associated with the diagram D' consisting of the hard–photon diagram D plus some specified set of lines representing soft *quantum* photons, and $F'(D')$ is the operator that represents this process D' together with all classical–photon corrections to it.

It was shown in ref. 7 that if the function $F(D')$, expressed in momentum space, has the normal analytic structure (i.e., the one in which stable particles correspond to poles) then the coordinate–space function $F'(D')$ will have the correct physically mandated fall off and factorization properties. These properties allow the functions representing processes with charged initial or final particles to be defined by identifying appropriate factors in processes of the kind discussed above, which involve only neutral external particles.

The argument just mentioned depends, as stated, on the assumption that the function $F(D')$, expressed in momentum space, is well defined (i.e., is not infrared divergent) and possesses the requisite analyticity properties. Since the bulk of the very soft photon contributions should be contained in the classical part represented by $U(D)$ these required properties of $F(D')$ are expected to hold.

A principal aim of this work is to confirm that the functions $F(D')$ do have the requisite finiteness and analyticity properties for an important class of diagrams D'. These are the diagrams D' obtained by inserting quantum–photon lines into a diagram D formed from three charged–particle line segments, joined together to form a triangle, plus external neutral–particle lines . In the course of deriving this result we shall develop and apply the basic machinery needed

for general calculations of the singularity structure, and hence large-distance behavior, in quantum theories involving (massless) photons.

The plan of the paper is as follows. In the following section 2 rules are established for writing down the functions $F(D')$ directly in momentum space. These rules are expressed in terms of operators that act on the original momentum-space Feynman function corresponding to diagram D'. These operators convert each original coupling γ_μ either to C_μ or to Q_μ, according to the operator chosen.

It is advantageous always to sum together the contributions corresponding to all ways in which a photon can couple with C–type coupling into each individual line $\overline{xy}$ of D. This sum can be expressed as a sum of just two terms. In one term the photon is coupled into the endpoint x, and in the other term it is coupled into the end point y. Thus all C–type couplings become converted into couplings at the hard–photon vertices of D. This conversion introduces an important property. The charge–conservation (or gauge) condition $k^\mu J_\mu = 0$ normally does not hold in quantum electrodynamics for individual diagrams: one must sum over all ways in which the photon can be inserted into the diagram. But in the form we use, with each quantum vertex Q coupled into the interior of a line of D, but each classical vertex C coupled into a hard–photon vertex of D, the charge–conservation equation (gauge invariance) holds for each diagram separately.

In section 3 the modification of the charged–particle propagator caused by inserting a single quantum vertex Q_μ is studied in more detail. The propagator is re–expressed as a sum of three terms. The first two are "meromorphic" terms having poles at $p^2 - m^2$ and $p^2 + 2pk + k^2 - m^2$, respectively, in the variable p^2. Because of the special form of the quantum coupling Q_μ each residue is of first order in k, relative to what would have been obtained with the usual coupling γ_μ.This extra power of k will lead to the infrared convergence of the residues of the pole singularities.

The third term is a nonmeromorphic contribution. It is a difference of two logarithms, and this difference has a power of k that renders the contribution infrared finite.

In section 4 the results just described are used to study the function corresponding to a diagram D' that is formed by inserting into the triangle diagram

D a single quantum photon that has Q–type interactions at each end. In order to treat in a rigorous way the contribution from the neighborhood of the point $k = 0$ we introduce polar coordinates $k = r\Omega$, $\Omega\tilde{\Omega} \equiv \Omega_0^2 + \vec{\Omega} \cdot \vec{\Omega} = 1$. For the meromorphic contributions it is found that the integrand of the integral that defines the residue behaves like rdr near the end point $r = 0$, and that the compact domain of integration in the variable Ω can be distorted away from all singularities. This shows that there is no infrared divergence. The two meromorphic contributions from each line lead to four contributions to $F(D')$. One of them gives the normal $\log \varphi$ singularity on the Landau triangle–diagram surface $\varphi = 0$, and the other three give weaker singularities. The contributions from the nonmeromorphic contributions also give weaker singularities.

The aim of the remaining sections is basically to prove that the analogous results hold for all diagrams D' constructed from D by the addition of quantum–photon lines, and to construct an efficient general machinery for computing the physical-region singularity structure, or equivalently the large-distance behavior.

In section 5 we examine the generalized propagator that corresponds to propagation between two hard–photon vertices x and y with an arbitrary number of Q–type insertions. The meromorphic part is exhibited explicitly: there is one pole term for each of the original energy denominators. The residues factorize, and each factor (unless it is unity) has *one* factor of k_i beyond what would occur if the couplings were the original γ_μ couplings. This single extra factor of k_i in each residue factor will lead to infrared convergence of the meromorphic parts.

This infrared convergence result, for our general diagram D', is proved in sections 6 and 7, subject to the assumption that, in analogy to what occurred in the simple case, the Ω contours can be distorted so as to avoid all singularities. This distortion assumption reduces the problem to one of counting powers of r. However, it is not sufficient to merely count overall powers of r. One must show that, for every possible way in which the variables k_i can tend to zero, there is convergence of every sub-integral. This argument can be regarded as a systematization and confirmation of the argument for infrared convergence given by Yennie, Frautschi, and Suura.

The distortion property assumed in sections 6 and 7 is proved in section 8. The proof is rather lengthy, but the fact that pinches of the contour of integration can be avoided is essential to any rigorous treatment of infrared

divergence problems. New methods are introduced.

In section 9 the results of the earlier sections are gathered together to give the result that singularity on the triangle diagram surface coming from the meromorphic parts of the contributions arising from the quantum photons are no stronger than logarithmic. In sections 10 and 11 the similar results for the non-meromorphic parts are obtained. Section 12 gives a comparison of the present treatment to other treatments of the infrared divergence problem. Appendices contain some technical details, and the evaluations of certain classes of integrals that are needed in the proofs.

One principal motivation for this refinement of the treatment of infrared photons is to establish in quantum electrodynamics two of the most secure and important features of relativistic quantum field theory obtained for massive particles, namely the physically mandated fall–off and factorization properties associated with the long–range propagation of stable particles, and the associated normal analytic structure in momentum space. These properties emerge in quantum–electrodynamics when one separates out, in the way discussed above, the classically describable soft–photon component.

The need for these sophisticated methods to establish the large-distance behavior in QED suggests the need for similar methods in the more difficult case of QCD, but we have made no effort in this direction.

A related motivation for the present work arises from the apparent need for an accurate treatment of the classical and macroscopic aspects of quantum electrodynamics in order to study what is perhaps the most important fundamental problem in contemporary physics, namely the problem of the interface between the quantum and classical domains. Quantum theory was originally designed to cover phenomena in the domain of atomic physics, and its orthodox interpretation depends upon the idea that the quantum system is first prepared and later detected by classically described devices, and that during the interval between its preparation and detection the quantum system does not act upon its classically described environment. This conception does not fit the situation encountered in, for example, quantum cosmology, in which the quantum system is the whole universe, which has no external system of observers and devices. Moreover, as emphasized by Bohr, this conception does not fit the situation that prevails in the study of biological systems, where "the incessant exchange

of matter which is inseparably connected with life will even imply the impossibility of regarding the organism as a well–defined system of material particles like the system considered in any account of the ordinary physical and chemical properties of matter."[8] Finally, contemporary technological advances in the fields of microfabrication and cryogenics are bringing to science a wealth of data in which quantum effects are important in systems that are not adequately conceived of as systems that are prepared and later detected by classically described devices, and that are, in the interim, not acting upon their environment. If one is to develop a form of quantum theory that is better able to cope with systems that lie at the interface between the quantum and classical domains then it will probably be necessary to have an accurate treatment of the classical and macroscopic aspects of quantum theory itself. For without an accurate evaluation of these effects one cannot study carefully the way in which quantum concepts give way to classical ones as one moves from small systems to large ones.

2. Basic Momentum–Space Formulas

The separation of the soft–photon interaction into its quantum and classical parts is defined in Eq. (1.1). This separation is defined in a mixed representation in which hard photons are represented in coordinate space and soft photons are represented in momentum space. In this representation one can consider a "generalized propagator". It propagates a charged particle from a hard–photon vertex y to a hard–photon vertex x with, however, the insertion of soft–photon interactions.

Suppose, for example, one inserts the interactions with two soft photons of momenta k_1 and k_2 and vector indices μ_1 and μ_2. Then the generalized propagator is

$$\begin{aligned} &P_{\mu_1,\mu_2}(x,y;k_1,k_2) \\ &\quad = \int \frac{d^4p}{(2\pi)^4} e^{-ipx+i(p+k_1+k_2)y} \\ &\quad \times \frac{i}{\not p - m + i0}\gamma_{\mu_1}\frac{i}{\not p + \not k_1 - m + i0}\gamma_{\mu_2}\frac{i}{\not p + \not k_1 + \not k_2 - m + i0}. \end{aligned} \tag{2.1}$$

The generalization of this formula to the case of an arbitrary number of inserted soft photons is straightforward. The soft–photon interaction γ_{μ_j} is separated into its parts Q_{μ_j} and C_{μ_j} by means of (1.1), with the x and y defined as in (1.3).

This separation of the soft–photon interaction into its quantum and classical parts can be expressed also directly in momentum space. Using (1.2) and (1.3), and the familiar identities

$$\frac{1}{\not p - m}\not k\frac{1}{\not p + \not k - m} = \frac{1}{\not p - m} - \frac{1}{\not p + \not k - m}, \tag{2.2}$$

and

$$\left(-\frac{\partial}{\partial p^\mu}\right)\frac{1}{\not p - m} = \frac{1}{\not p - m}\gamma_\mu\frac{1}{\not p - m}, \tag{2.3}$$

one obtains for the (generalized) propagation from y to x, with a single classical interaction inserted, the expression (with the symbol m standing henceforth for $m - i0$)

$$
\begin{aligned}
P_\mu(x,y;C,k) &= \int \frac{d^4p}{(2\pi)^4}\left(\frac{i}{\not p - m}\,\not k\frac{i}{\not p + \not k - m}\right)\frac{z_\mu}{zk+io}e^{-ipz+iky} \\
&= \int \frac{d^4p}{(2\pi)^4}\left(\frac{i}{\not p - m}\,\not k\frac{i}{\not p + \not k - m}\right)\frac{1}{zk+io}\left(\frac{i\partial}{\partial p^\mu}\right)e^{-ipz+iky} \\
&= \int \frac{d^4p}{(2\pi)^4}e^{-ipz+iky}\frac{1}{zk+io}\left(-i\frac{\partial}{\partial p^\mu}\right)\left(\frac{i}{\not p - m}\,\not k\frac{i}{\not p + \not k - m}\right) \\
&= \int \frac{d^4p}{(2\pi)^4}e^{-ipz+iky}\lim_{\epsilon\to 0}(-i)\int_0^\infty d\lambda e^{i\lambda(zk+i\epsilon)} \\
&\quad \times\left(-i\frac{\partial}{\partial p^\mu}\right)\left(\frac{i}{\not p - m}\,\not k\frac{i}{\not p + \not k - m}\right) \\
&= \int \frac{d^4p}{(2\pi)^4}\lim_{\epsilon\to 0}(-i)\int_0^\infty d\lambda\; e^{-i(p-\lambda k)z+iky-\epsilon\lambda} \\
&\quad \times\left(-i\frac{\partial}{\partial p^\mu}\right)\left(\frac{i}{\not p - m}\,\not k\frac{i}{\not p + \not k - m}\right) \qquad (2.4a) \\
&= \int \frac{d^4p}{(2\pi)^4}\lim_{\epsilon\to 0}(-i)\int_0^\infty d\lambda\; e^{-i(p-\lambda k)z+iky-\epsilon\lambda} \\
&\quad \times\frac{\partial}{\partial p^\mu}\left(\frac{i}{\not p - m} - \frac{i}{\not p + \not k - m}\right) \\
&= \int \frac{d^4p}{(2\pi)^4}\lim_{\epsilon\to 0}(-i)\int_0^\infty d\lambda\; (e^{-i(p-\lambda k)z} - e^{-i(p-k-\lambda k)z})e^{-\epsilon\lambda} \\
&\quad \times e^{iky}\frac{\partial}{\partial p^\mu}\left(\frac{i}{\not p - m}\right) \\
&= \int \frac{d^4p}{(2\pi)^4}\lim_{\epsilon\to 0}(-i)\int_0^1 d\lambda\; e^{-i(p-\lambda k)z}e^{-\epsilon\lambda}\times e^{iky}\frac{\partial}{\partial p^\mu}\left(\frac{i}{\not p - m}\right) \\
&= \int \frac{d^4p}{(2\pi)^4}e^{-ipz+iky}\int_0^1 d\lambda\left(-i\frac{\partial}{\partial p^\mu}\right)\left(\frac{i}{\not p + \lambda\not k - m}\right) \qquad (2.4b) \\
&= \int \frac{d^4p}{(2\pi)^4}e^{-ipz+iky}\int_0^1 d\lambda\left(\frac{i}{\not p + \lambda\not k - m}\gamma_\mu\frac{i}{\not p + \lambda\not k - m}\right). \qquad (2.4c)
\end{aligned}
$$

Comparison of the result (2.4b) to (2.1) shows that the result in momentum

space of inserting a single quantum vertex j into a propagator $i(\not{p}-m)^{-1}$ is produced by the action of the operator

$$\widehat{C}_{\mu_j}(k_j) = \int_0^1 d\lambda_j O(p \to p + \lambda_j k_j) \left(-i\frac{\partial}{\partial p^{\mu_j}}\right) \tag{2.5}$$

upon the propagator $i(\not{p}-m)^{-1}$ that was present *before* the insertion of the vertex j. One must, of course, also increase by k_j the momentum entering the vertex at y. The operator $O(p \to p + \lambda_j k_j)$ replaces p by $p + \lambda_j k_j$.

Suppose that there were already a soft-photon insertion on the charged -particle line L so that the propagator before the insertion of vertex j were

$$P_{\mu_1}(p; k_1) = \frac{i}{\not{p}-m}\gamma_{\mu_1}\frac{i}{\not{p}+\not{k}_1-m}. \tag{2.6}$$

And suppose the vertex j is to be inserted in all possible ways into this line (i.e., on both sides of the already-present vertex 1). Then the same argument as before, with (2.2) replaced by its generalization[9]

$$\begin{aligned}
&\frac{1}{\not{p}-m}\ \not{k}_j\frac{1}{\not{p}+\not{k}_j-m}\gamma_{\mu_1}\frac{1}{\not{p}+\not{k}_j+\not{k}_1-m} \\
&\quad + \frac{1}{\not{p}-m}\gamma_{\mu_1}\frac{1}{\not{p}+\not{k}_1-m}\ \not{k}_j\frac{1}{\not{p}+\not{k}_j+\not{k}_1-m} \\
&\quad = \frac{1}{\not{p}-m}\gamma_{\mu_1}\frac{1}{\not{p}+\not{k}_1-m} \\
&\quad - \frac{1}{\not{p}+\not{k}_j-m}\gamma_{\mu_1}\frac{1}{\not{p}+\not{k}_j+\not{k}_1-m},
\end{aligned} \tag{2.7}$$

shows that the effect in momentum space is again given by the operator $\widehat{C}_{\mu_j}(k_j)$ defined in (2.5).

This result generalizes to an arbitrary number of inserted classical photons, and also to an arbitrary generalized propagator: the momentum-space result of inserting in all orders into any generalized propagator $P_{\mu_1\cdots\mu_n}(p; k_1, \cdots k_n)$ a set of N classically interacting photons with $j = n+1, \cdots, n+N$ is

$$\prod_{j=n+1}^{n+N} \widehat{C}_{\mu_j}(k_j) P_{\mu_1,\cdots,\mu_n}(p;k_1,\cdots,k_n) = \int_0^1 \cdots \int_0^1 d\lambda_{n+1} \ldots d\lambda_{n+N} \prod_{j=1}^{N} \left(-i\frac{\partial}{\partial p^{\mu_{n+j}}}\right)$$
$$P_{\mu_1,\cdots,\mu_n}(p+a;k_1,\cdots,k_n) \tag{2.8}$$

where $a = \lambda_{n+1}k_{n+1} + \cdots + \lambda_{n+N}k_{n+N}$. The operations are commutative, and one can keep each $\lambda_j = 0$ until the integration on λ_j is performed.

To obtain the analogous result for the quantum interactions we introduce the operator $\widehat{D}_{\mu_j}(k_j)$ whose action is defined as follows:

$$\begin{aligned}
&\widehat{D}_{\mu_j}(k_j)\frac{i}{\not p - m} = \frac{i}{\not p - m}\gamma_{\mu_j}\frac{i}{\not p + \not k_j - m},\\
&\widehat{D}_{\mu_j}(k_j)\frac{i}{p - m}\gamma_{\mu_1}\frac{i}{\not p + \not k_1 - m}\\
&\quad = \frac{i}{\not p - m}\gamma_{\mu_j}\frac{i}{\not p + \not k_j - m}\gamma_{\mu_1}\frac{i}{\not p + \not k_j + \not k_1 - m},\\
&\quad + \frac{i}{\not p - m}\gamma_{\mu_1}\frac{i}{\not p + \not k_1 - m}\gamma_{\mu_j}\frac{i}{\not p + \not k_j + \not k_1 - m},\\
&\text{etc.}
\end{aligned} \tag{2.9}$$

That is, $\widehat{D}_{\mu_j}(k_j)$ acts on any generalized propagator by inserting in all possible ways an interaction with a photon of momentum k_j and vector index μ_j. Then one may define

$$\widehat{Q}_{\mu_j}(k_j) = \widehat{D}_{\mu_j}(k_j) - \widehat{C}_{\mu_j}(k_j). \tag{2.10}$$

Then the result in momentum space of inserting in all possible ways (i.e., in all possible orders) into any generalized propagator P of the kind illustrated in (2.1) a set of J quantum interactions and a set of J' classical interactions is

$$\prod_{j' \epsilon J'} \widehat{C}_{\mu'_j}(k_{j'}) \prod_{j \epsilon J} \widehat{Q}_{\mu_j}(k_j) P. \tag{2.11}$$

Consideration of (2.3) and (2.9) shows that the operators $\widehat{C}_i$ and $\widehat{Q}_i$ appearing in (2.11) all commute, provided we reserve until the end all integrations over the variables λ_i, in order for the action of the operators $\widehat{D}_i$ to be well defined.

One may not wish to combine the results of making insertions in all orders. To obtain the result of inserting the classical interaction at just one place, identified by the subscript $j\epsilon\{1,\cdots,n\}$, into a (generalized) propagator $P_{\mu_1}\cdots_{\mu_n}(p;k_1,\cdots,k_n)$, abbreviated now by P_{μ_j}, one begins as in (2.4) with $k_j^{\sigma_j}\ P_{\sigma_j}$ in place of the quantity appearing in the bracket. However, one does not introduce (2.2), which led to the restriction of the integration to the range $1 \geq \lambda_j \geq 0$. Then, provided $k_j^2 \neq 0$, equation (2.4a) gives for the result in momentum space the result produced by the action of

$$\tilde{C}_{\mu_j}(k_j) \equiv \int_0^\infty d\lambda_j O(p_i \to p_i + \lambda_j k_j)\left(-\frac{\partial}{\partial p^{\mu_j}}\right) \tag{2.12}$$

upon $k_j^{\sigma_j} P_{\sigma_j}$.

For $k_j^2 \neq 0$ this integral converges at the upper endpoint. The indefinite integral can then be defined so that it vanishes at $\lambda = \infty$. We define $\tilde{C}_{\mu_j}(k_j)$ at $k_j^2 = 0$ by then using uniformly only the contribution from the lower endpoint $\lambda = 0$, as was entailed from the start by the initially finite value of ϵ in (2.4). (Strictly speaking, one should use a Pauli-Villars regulator to define the integral in p space—then no special treatment is needed for $k_j^2 = 0$)

To obtain a form analogous to (2.12) for the quantum interaction one may use the identity

$$\begin{aligned} k_j^{\rho_j} \int_0^\infty d\lambda_j \left(-\frac{\partial}{\partial p^{\rho_j}}\right) P_{\mu_j}(p+\lambda_j k_j) \\ = \int_0^\infty d\lambda_j \left(-\frac{\partial}{\partial \lambda_j}\right) P_{\mu_j}(p+\lambda_j k_j) \\ = P_{\mu_j}(p). \end{aligned} \tag{2.13}$$

Then the momentum–space result produced by the insertion of a quantum coupling in $P_{\mu_1\cdots\mu_n}(p;k_1,\cdots k_\mu) = P_{\mu_j}$ at the vertex identified by μ_j is generated by the action of

$$\tilde{Q}_{\mu_j}(k_j) \equiv (\delta_{\mu_j}^{\sigma_j} k_j^{\rho_j} - \delta_{\mu_j}^{\rho_j} k_j^{\sigma_j})\tilde{C}_{\rho_j}(k_j) \tag{2.14}$$

upon P_{σ_j} .

An analogous operator can be applied for each quantum interaction. Thus the generalized momentum–space propagator represented by a line L of D into which n quantum interactions are inserted in a fixed order is

$$
\begin{aligned}
&P_{\mu_1\cdots\mu_n}(p;Q,k_1,Q,k_2,\cdots Q,k_n) = \\
&\prod_{j=1}^{n}\left[\int_0^\infty d\lambda_j(\delta^{\sigma_j}_{\mu_j}k^{\rho_j}_j - \delta^{\rho_j}_{\mu_j}k^{\sigma_j}_j)\left(-\frac{\partial}{\partial p^{\rho_j}}\right)\right] \\
&\Big(\frac{i}{\not p+\not q-m}\gamma_{\sigma_1}\frac{i}{\not p+\not q+\not k_1-m}\gamma_{\sigma_2}\frac{i}{\not p+\not q+\not k_1+\not k_2-m} \\
&\cdots\times\gamma_{\sigma_n}\frac{i}{\not p+\not q+\not k_1+\cdots\not k_n-m}\Big),
\end{aligned}
\tag{2.15}
$$

where

$$a = \lambda_1 k_1 + \lambda_2 k_2 + \cdots \lambda_n k_n. \tag{2.16}$$

If some of the inserted interactions are classical interactions then the corresponding factors $(\delta^{\sigma_j}_{\mu_j}k^{\rho_j}_j - \delta^{\rho_j}_{\mu_j}k^{\sigma_j}_j)$ are replaced by $(\delta^{\rho_j}_{\mu_j}k^{\sigma_j}_j)$.

These basic momentum–space formulas provide the starting point for our examination of the analyticity properties in momentum space, and the closely related question of infrared convergence.

One point is worth mentioning here. It concerns the conservation of charge condition $k^\mu J_\mu(k) = 0$. In standard Feynman quantum electrodynamic this condition is not satisfied by the individual photon–interaction vertex, but is obtained only by summing over all the different positions where the photon interaction can be coupled into a diagram. This feature is the root of many of the difficulties that arise in quantum electrodynamics.

Equation (2.14) shows that the conservation – law property holds for the individual *quantum* vertex: there is no need to sum over different positions. The classical interaction, on the other hand, has a form that allows one easily to sum over all possible locations along a generalized propagator, even before multiplication by k^μ. This summation converts the classical interaction to a sum of two interactions, one located at each end of the line associated with the generalized propagator. (See, for example, Eq. (7.1) below). We always perform this summation, which simplifies calculations. Then the classical parts of the interaction are shifted to the hard–photon interaction points. This shift bring the

classical interaction into conformity with the classical idea that electromagnetic radiation comes from places where a charged particle is deflected, not from places where it is moving along a straight line in spacetime.

At each such point of deflection there are, for each deflected charged particle, two parts of the classical interaction, one from the incoming leg and one from the outgoing leg. The sum of these two classical interactions satisfies the conservation law property. Thus in this formulation the conservation–law property holds at each vertex separately: there is no need to sum over different diagrams. To confirm these properties of the classical interaction one can use (7.1) for the meromorphic part, or, more generally, (2.8) and the fact that $k_j^{\mu_j}\partial/\partial p^{\mu_j}$ is equivalent to $\partial/\partial\lambda_j$ therein.

3. The Quantum Vertex

Suppose a single quantum interaction is inserted into a line of D. Then the associated generalized propagator is given by (2.11), (2.10), (2.9), (2.5) and (2.3):

$$
\begin{aligned}
&P_\mu\ (p;\widehat{Q},k) \\
&\quad = \frac{i}{\not p - m}\gamma_\mu \frac{i}{\not p + \not k - m} \\
&\quad - \int_0^1 d\lambda \frac{i}{\not p + \lambda \not k - m}\gamma_\mu \frac{i}{\not p + \lambda \not k - m}.
\end{aligned} \tag{3.1}
$$

The first term in (3.1) is

$$
\begin{aligned}
&\frac{i}{\not p - m}\ \gamma_\mu \frac{i}{\not p + \not k - m} \\
&\quad = -\frac{(\not p + m)}{p^2 - m^2}\gamma_\mu \frac{(\not p + \not k + m)}{(p+k)^2 - m^2} \\
&\quad = -(\not p + m)\gamma_\mu(\not p + \not k + m) \\
&\quad \times \left(\frac{1}{p^2 - m^2}\ \frac{1}{2pk + k^2} - \frac{1}{2pk + k^2}\ \frac{1}{(p+k)^2 - m^2}\right) \\
&\quad = -\Big[\frac{-(p^2 - m^2)\gamma_\mu + (\not p + m)(2p_\mu + \gamma_\mu \not k)}{(2pk + k^2)(p^2 - m^2)} \\
&\quad - \frac{-((p+k)^2 - m^2)\gamma_\mu + (2p_\mu + 2k_\mu - \not k\gamma_\mu)(\not p + \not k - m)}{(2pk + k^2)(p^2 - m^2)}\Big] \\
&\quad = -\Big[\frac{2p_\mu}{(\not p - m)(2pk + k^2)} - \frac{2p_\mu + 2k_\mu}{(\not p + \not k - m)(2pk + k^2)} \\
&\quad + \frac{1}{(\not p - m)} \times \frac{\gamma_\mu \not k}{(2pk + k^2)} + \frac{\not k \gamma_\mu}{(2pk + k^2)} \times \frac{1}{(\not p + \not k - m)}\Big],
\end{aligned} \tag{3.2}
$$

where $\{\gamma_\mu, \not p\}_+ = 2p_\mu$ has been used, and pk represents $pk + i0$.

The second term in (3.1) can be computed from standard integral tables. Then it can be cast into a form similar to (3.2) by first considering it to be a function of the variable $t = p^2 - m^2$, with pk and k^2 regarded as parameters, next separating it into its meromorphic and nonmeromorphic parts in this variable t, and finally evaluating its meromorphic part as a sum of poles times residues.

This gives for the meromorphic part

$$\left[\int_0^1 d\lambda \, \frac{1}{\not{p}+\lambda\not{k}-m}\gamma_\mu\frac{1}{\not{p}+\lambda\not{k}-m}\right]_{Mero}$$
$$= \left[\frac{(\not{p}+m)\gamma_\mu(\not{p}+m)}{2pk\,(p^2-m^2)} - \frac{(\not{p}+\not{k}+m)\gamma_\mu(\not{p}+\not{k}+m)}{2(p+k)k\,((p+k)^2-m^2)}\right]_{Mero}$$
$$= \frac{2p_\mu}{2pk\,(\not{p}-m)} - \frac{2p_\mu+2k_\mu}{2(p+k)k\,(\not{p}+\not{k}-m)}, \tag{3.3}$$

where a term not depending on (p^2-m^2) has been dropped from the last line.

The singularities of this function at $pk=0$ and $(p+k)k=0$ are artifacts of the separation into meromorphic and non meromorphic parts: their sum does not have singularities at generic points on these surfaces. Thus we may replace pk by $pk+i0$ in both the meromorphic and non meromorphic parts and introduce the identities

$$\frac{1}{2pk} = \frac{1}{2pk+k^2}\left(1+\frac{k^2}{2pk}\right) \tag{3.4a}$$

and

$$\frac{1}{2pk+2k^2} = \frac{1}{2pk+k^2}\left(1-\frac{k^2}{2pk+2k^2}\right). \tag{3.4b}$$

Then the combination of (3.2) and (3.3) gives

$$P_\mu\,(p;Q,k)_{Mero}$$
$$= \frac{1}{2pk+k^2}\left[\frac{1}{\not{p}-m}\left(\frac{2p_\mu k^2}{2pk} - \gamma_\mu\,\not{k}\right)\right.$$
$$\left. + \left(\frac{(2p_\mu+2k_\mu)k^2}{2pk+2k^2} - \not{k}\gamma_\mu\right)\frac{1}{\not{p}+\not{k}-m}\right]. \tag{3.5}$$

This function is of zeroth order in $|k|$, whereas the individual contributions (3.2) and (3.3) are each of order $|k|^{-1}$.

The result (3.3) can be obtained also directly by inspection of the integral appearing on the left-hand side, written in the form

$$\int_0^1 d\lambda \frac{(\not{p}+\lambda\,\not{k}+m)\gamma_\mu(\not{p}+\lambda\,\not{k}+m)}{(p^2-m^2+2pk\lambda+k^2\lambda^2)^2}.$$

The singularities of this integral lying along the surface $p^2=m^2$ arise from the endpoint $\lambda=0$ of the domain of integration. Thus the analytic character of these

singularities is controlled by the character of the integrand in an arbitrarily small neighborhood of this endpoint. Positive powers of λ in the numerator diminish the contributions from this endpoint, and lead to singularities on $p^2 = m^2$ that are, in form, not as strong as the singularity coming from the terms that are of zeroth order in λ. Thus to find the strongest singularity we may set the λ's appearing in the numerator to zero. For similar reasons we can set the λ^2 terms in the denominator equal to zero, provided the coefficient $2pk$ of the first power of λ is nonzero. Thus the strongest singularity of the integral arising from the lower endpoint is

$$\int_0^\infty d\lambda \, \frac{(\not{p}+m)\gamma_\mu(\not{p}+m)}{(p^2-m^2+2pk\lambda)^2} = \frac{(\not{p}+m)\gamma_\mu(\not{p}+m)}{2pk(p^2-m^2)}. \tag{3.6}$$

This is just the result obtained from the full calculation. The other term in (3.3) comes from the other endpoint, $\lambda = 1$. Because the strongest or dominant singularities coming from the two endpoints are poles any other singularities coming from these endpoints belong to the nonmeromorphic part.

The full nonmeromorphic part of $P_\mu(p;Q,k)$ is, by direct calculation,

$$\begin{aligned}
&P_\mu\,(p;Q,k)_{NonMero} \\
&= \Bigg[(\not{p}+m)\gamma_\mu(\not{p}+m)\left(\frac{-2k^2}{-d}\right) \\
&+ ((\not{k}\gamma_\mu(\not{p}+m) + (\not{p}+m)\gamma_\mu\,\not{k})\left(\frac{2pk}{-d}\right) \\
&+ \not{k}\gamma_\mu\,\not{k}\left(\frac{-2(p^2-m^2)}{-d}\right)\Bigg] \\
&\times \Bigg[\frac{1}{\sqrt{-d}}\log\left(\frac{1-\frac{\sqrt{-d}}{2pk+2k^2}}{1+\frac{\sqrt{-d}}{2pk+2k^2}}\right) \\
&- \frac{1}{\sqrt{-d}}\log\left(\frac{1-\frac{\sqrt{-d}}{2pk}}{1+\frac{\sqrt{-d}}{2pk}}\right) + \frac{2}{2pk+2k^2} - \frac{2}{2pk}\Bigg]
\end{aligned} \tag{3.7}$$

where $-d = (2pk)^2 - 4k^2(p^2-m^2) = (2(p+k)k)^2 - 4k^2((p+k)^2-m^2)$. The two non-log terms in the final square bracket cancel the pole singularity in

$t = p^2 - m^2$ at $d = 0$ that would otherwise arise from the small$-d$ behavior of the log terms.

The singularity surfaces of $P_\mu(p; Q, k)$ are shown in Fig. 1.

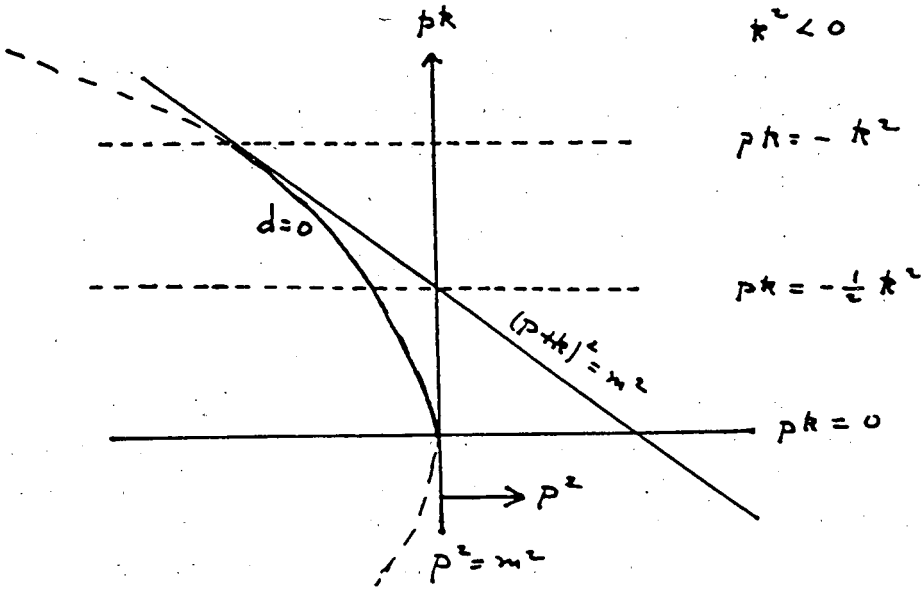

Fig. 1 The singularities of $P_\mu(p; Q, k)$ are confined to the surfaces $p^2 - m^2 = 0$, $(p+k)^2 - m^2 = 0$, and the branch of $d = 0$ lying between $pk = 0$ and $pk = -k^2$.

The singularities of $P_\mu(p; Q, k)$ are confined to the surfaces $p^2 - m^2 = 0$, $(p+k)^2 - m^2 = 0$, and to the portion of the surface $d = 0$ that lies between $pk = 0$ and $pk = -k^2$. Except at points of contact between two of these three surfaces the function $P_\mu(p; Q, k)$ is analytic on the three surfaces $2pk = 0, 2pk + k^2 = 0$, and $2pk + 2k^2 = 0$, and has the form $d^{-3/2}$ on the singular branch of the surface $d = 0$. It has both pole and logarithmic singularities on the surfaces $p^2 - m^2 = 0$ and $(p + k)^2 - m^2 = 0$. The $i0$ rule associated with $d = 0$ matches the $i0$ rules at $p^2 = m^2$ and $(p + k)^2 = m^2$ at their points of contact.

The meromorphic and nonmeromorphic parts of $P_\mu(p; Q, k)$ each separately have singularities on the surfaces $2pk = 0$, $2pk + k^2 = 0$ and $2pk + 2k^2 = 0$.

The results of this section may be summarized as follows: the insertion

of a single quantum interaction into a propagator $i(\not{p} - m)^{-1}$ associated with D converts it into a sum of three terms. The first is a propagator $i(\not{p} - m)^{-1}$ multiplied by a factor that is zeroth order in $r = |k|$. The second is a propagator $i(\not{p} + \not{k} - m)^{-1}$ multiplied by a factor that is zeroth order in r. The third is a vertex–type term, which has logarithmic singularities on the two surfaces $p^2 - m^2 = 0$ and $(p + k)^2 - m^2 = 0$. This latter term has a typical vertex–correction type of analytic structure even though it is represented diagrammatically as (the nonmeromorphic part of) a simple vertex insertion.

4. Triangle Diagram Process

In the introduction we described a hard–photon process associated with a triangle diagram D. In this section we describe the corrections to it arising from a single soft photon that interacts with D in the way shown in Fig. 2.

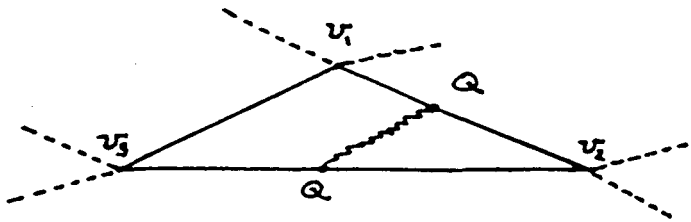

Fig. 2 Diagram representing a soft–photon correction to a hard–photon triangle diagram process. Hard and soft photons are represented by dashed and wiggly lines, respectively.

Each external vertex v_i of Fig. 1 represents the <u>two</u> vertices upon which the two external hard photons are incident, together with the charged–particle line that runs between them. The momenta of the various external photons can be chosen so that the momentum–energy of this connecting charged–particle line is far from the mass shell, in the regime of interest. In this case the associated propagator is an analytic function. We shall, accordingly, represent the entire contribution associated with each external vertex v_i by the single symbol V_i, and assume only that the corresponding function is analytic in the regime of interest. The analysis will then cover also cases outside of quantum–electrodynamics.

In Fig. 2 the two solid lines with Q–vertex insertions represent generalized propagators. We consider first the contributions that arise from the meromorphic or pole contributions to these two generalized propagators.

Each generalized propagator has, according to (3.5), two pole contributions, one proportional to the propagator $i(\not p - m)^{-1}$, the other proportional to $i(\not p + \not k - m)^{-1}$. This gives four terms, one corresponding to each of the four diagrams in Fig. 3. Each line of Fig. 3 represents a propagator $i(\not p_i - m)^{-1}$ or $i(\not p_i + \not k - m)^{-1}$, with $i = 1$ or 2 labelling the two relevant lines. The singularities on the Landau triangle–diagram surface $\varphi = 0$ arise from a conjunction of three

such singularities, one from each side of the triangle.

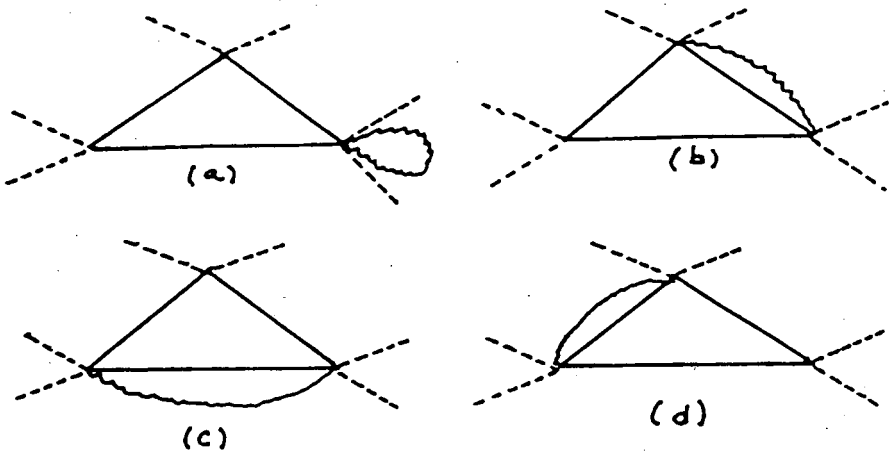

Fig. 3 Diagrams representing the four contributions that arise from inserting into each of the two generalized propagators represented in Fig. 2 the sum of the two meromorphic terms given by (3.5).

The diagram (a) represents, by virtue of (3.5), the function

$$
\begin{aligned}
F_a \;=& \int \frac{d^4p}{(2\pi)^4} \int_{|k|\le\delta} \frac{d^4k}{(2\pi)^4} i(k^2+i0)^{-1} \\
& \mathrm{Tr}\Big\{ \frac{i(\not p+m)}{p^2-m^2} V_1 \frac{(\not p_1+m)}{p_1^2-m^2} \Big(\frac{2p_{1\mu}k^2(2p_1k)^{-1} - \gamma_\mu \not k}{2p_1k+k^2} \Big) \\
& V_2 \Big(\frac{2p_{2\mu}k^2(2p_2k)^{-1} - \not k\gamma_{\mu 2}}{2p_2k+k^2} \Big) \frac{(\not p_2+m)}{p_2^2-m^2} V_3 \Big\}
\end{aligned}
\tag{4.1}
$$

where $p_1 = p+q_1$, $p_2 = p-q_3$, $p_ik = p_ik+i0$, and q_i is the momentum–energy carried out of vertex v_i by the external hard photons incident upon it. The vector $p \equiv p_3$ is the momentum–energy flowing along the internal line that runs from v_1 to v_3.

To give meaning to the function $(k^2+i0)^{-1}$ at the point $k=0$ we introduce polar coordinates, $k = r\Omega$, and write[10]

$$
\int_{|k|\le\delta} \frac{d^4k}{k^2+i0} f(k) = \int_0^\delta 2rdr \int d^4\Omega \frac{\delta(\Omega_0^2+\bar\Omega^2-1)}{\Omega^2+i0} f(r\Omega) \tag{4.2}
$$

Then F_a becomes

$$F_a = \int \frac{d^4p}{(2\pi)^4} \int_0^\delta 2rdr \int \frac{d^4\Omega}{(2\pi)^4} \frac{i\delta(\Omega_0^2 + \vec{\Omega}^2 - 1)}{\Omega^2 + i0}$$

$$\mathrm{Tr}\Big\{ \frac{i(\not{p} + m)}{p^2 - m^2} V_1 \frac{(\not{p}_1 + m)}{p_1^2 - m^2} \left(\frac{2p_{1\mu}\Omega^2(2p_1\Omega)^{-1} - \gamma_\mu \not{\Omega}}{2p_1\Omega + r\Omega^2} \right)$$

$$V_2 \left(\frac{2p_{2\mu}\Omega^2(2p_2\Omega)^{-1} - \not{\Omega}\gamma_\mu}{2p_2\Omega + r\Omega^2} \right) \frac{(\not{p}_2 + m)}{p_2^2 - m^2} V_3 \Big\}. \tag{4.3}$$

where $p_i\Omega$ represents $p_i\Omega + i0$.

The integrand of this function behaves near $r = 0$ like rdr. Hence the integral is infrared finite.

We are interested in the form of the singularity at interior points of the positive–α branch of the Landau triangle–diagram surface $\varphi = 0$. Let $q = (q_1, q_2, q_3)$ be such a point on $\varphi(q) = 0$. The singularity at q is generated by the pinching of the contour of integration in p–space by the three surfaces $p_i^2 - m^2 = 0$. This pinching occurs at a point in the domain of integration where the three vectors (p_1, p_2, p_3) lie at a point $(p_1(q), p_2(q), p_3(q))$ that is determined uniquely by the value q on $\varphi = 0$.[11] At this point none of these vectors is parallel to any other one. Consequently, in view of the $i0$ rules described in connection with Fig. 1., it is possible, in a sufficiently small p–space neighborhood of $(p_1(q), p_2(q), p_3(q))$, for sufficiently small δ, to shift the contour of integration in Ω space simultaneously into the regions Im $p_1\Omega > 0$ and Im $p_2\Omega > 0$, and to make thereby the denominator factors $p_i\Omega$ and $p_i\Omega + r\Omega^2$,for $i \in \{1, 2\}$, all simultaneously nonzero, for all points on the Ω contour. In this way the factors in (4.3) that contain these denominator functions can all be made analytic in all variables in a full neighborhood of the pinching point. Consequently, these factors can, for the purpose of examining the character of the singularity along $\varphi = 0$ be incorporated into the analytic factor V_2.

The computation of the form of the singularity on $\varphi = 0$ then reduces to the usual one: the singularity has the form $\log\varphi$, and the discontinuity is given by the Cutkosky rule, which instructs one to replace each of the three propagator–poles $i(p_i^2 - m^2)$ by $2\pi\delta(p_i^2 - m^2)$.

This gives most of what we need in this special case: it remains only to be shown that the remaining singularities on $\varphi = 0$ are weaker in form than $\log\varphi$.

If one were to try to deal in the same way with the function represented by Fig. 2, but with the original vertices γ_μ rather than Q_μ, then (3.2) would be used instead of (3.5) and the integration over r in the expression replacing (4.3) would become infrared divergent. The definition of $k^2 + i0$ embodied in (4.2) is insufficient in this case. A proper treatment (see Appendix A) shows that the dominant singularity on the surface $\varphi = 0$ would in this case be $(\log\varphi)^2$.

The diagram (b) of Fig. 3 represents the function

$$\begin{aligned} F_b &= \int \frac{d^4p}{(2\pi)^4} \int_0^\delta 2rdr \int \frac{d^4\Omega}{(2\pi)^4} \; \frac{i\delta(\Omega_0^2 + \vec{\Omega}^2 - 1)}{\Omega^2 + i0} \\ &\quad \mathrm{Tr}\left\{ \frac{i(\not p + m)}{p^2 - m^2} V_1 \left(\frac{(2p_{i\mu} + 2r\Omega_\mu)\Omega^2(2p_1\Omega + 2r\Omega^2)^{-1} - \not\Omega\gamma_\mu}{2p_1\Omega + r\Omega^2} \right) \right. \\ &\quad \times \frac{(\not p_1 + r\not\Omega + m)}{(p_1 + r\Omega)^2 - m^2} V_2 \\ &\quad \left. \times \left(\frac{2p_{2\mu}\Omega^2(2p_2\Omega) - \gamma_\mu \not\Omega}{2p_2\Omega + r\Omega^2} \right) \frac{(\not p_2 + m)}{p_2^2 - m} V_3 \right\} \end{aligned} \tag{4.4}$$

where $p_i\Omega$ represents $p_i\Omega + i0$. This integral also is free of infrared divergences. It is shown in Appendix D that its singularity on $\varphi = 0$ has the form $\varphi^2 \log\varphi$. The same result is obtained for diagrams (c) and (d) of Fig. 3.

The remaining contributions to the process represented in Fig. 2 involve the nonmeromorphic parts of at least one of the two generalized propagators. These nonmeromeorphic parts are given by (3.7). This expression gives logarithmic singularities on $p_i^2 - m^2 = 0$ and $(p_i + r\Omega)^2 - m^2 = 0$, for $i = 1$ and 2. It gives singularities also on $p_i\Omega = 0$ and $p_i\Omega + r\Omega^2 = 0$, and a $d_i^{-3/2}$ singularity on the portion of the surface $d_i = 0$ that lies between $p_i\Omega = 0$ and $p_i\Omega + r\Omega^2 = 0$.

For p in a small neighborhood of the fixed pinching point one can again, for sufficiently small δ, distort the Ω contour simultaneously into the upper–half planes of both $p_1\Omega$ and $p_2\Omega$, and thereby avoid simultaneously the zeros of $p_i\Omega$, $p_i\Omega = r\Omega^2$, and also those of

$$d_i(2r)^{-2} = (p_i\Omega)^2 - (p_i^2 - m^2)\Omega^2.$$

Thus for every point on the Ω contour the nonmeromorphic part of the propagator associated with line i takes, near the pinching point, the form

$$A_i \frac{1}{r} \log \frac{(p_i + r\Omega)^2 - m^2}{p_i^2 - m^2}, \tag{4.5}$$

where A_i is analytic in all variables.

If we combine the two factors (4.5), one from each end of the photon line, then the two displayed powers of r^{-1} join with rdr to give dr/r. Consequently, if each of the two logarithmic factors in (4.5) were treated separately then an infrared divergence would ensue. However, the entire (4.5), taken as a unit, is of zeroth order in r, and it gives no such divergence. It is therefore necessary in the treatment of the nonmeromorphic part to keep together those contributions coming from various logarithmic singularities, such as the two logarithmic singularities of (4.5), that are naturally tied together by a cut. By contrast, in the meromorphic part it was possible to treat separately the contributions from the two different pole singularities associated with each of the two sides $i = 1$ and $i = 2$ of the triangle: for the meromorphic part each of the four terms indicated in Fig. 3 is separately infrared convergent.

The product of the two factors (4.5) gives an integrand factor of the form

$$I = \frac{dr}{r}(\log \frac{(p_1 + r\Omega)^2 - m^2}{p_1^2 - m^2})(\log \frac{(p_2 + r\Omega)^2 - m^2}{p_2^2 - m^2}). \tag{4.6}$$

The dominant singularity on $\varphi = 0$ generated by this combination is shown in Appendix E to be of the form $\varphi^2(\log\varphi)^2$. If one combines the nonmeromorphic part from one end of the soft–photon line with the meromorphic part from the other end then the resulting dominant singularity on $\varphi = 0$ has the form $\varphi \log \varphi$. Replacement of *one* of the two Q–type interactions in Fig. 2 by a C–type interaction does not materially change things. The results are described in Appendix F.

We now turn to the generalization of these results to processes involving arbitrary numbers of soft photons, each having a Q–type interaction on at least one end.

5. Residues of Poles in Generalized Propagators

Consider a generalized propagator that has only quantum–interaction insertions. Its general form is, according to (2.15),

$$\prod_{j=1}^{n}\left[\left(\delta^{\sigma_j}_{\mu_j}k_j^{\rho_j}-\delta^{\rho_j}_{\mu_j}k_j^{\sigma_j}\right)\int_0^\infty d\lambda_j\left(-\frac{\partial}{\partial p^{\rho_j}}\right)\right]$$
$$\left(\frac{i}{\not p+\not a-m}\gamma_{\sigma_1}\frac{i}{\not p+\not a+\not k_1-m}\gamma_{\sigma_2}\frac{i}{\not p+\not a+\not k_1+\not k_2-m}\right.$$
$$\left.\cdots\times\gamma_{\sigma_n}\frac{i}{\not p+\not a+\not k_1\cdots+\not k_n-m}\right) \tag{5.1}$$

where

$$a=\lambda_1k_1+\cdots+\lambda_nk_n. \tag{5.2}$$

The singularities of (5.1) that arise from the multiple end–point $\lambda_1=\lambda_2=\cdots\lambda_n=0$ lie on the surfaces

$$p_i^2=m^2, \tag{5.3}$$

where now (in contrast to earlier sections)

$$p_i=p+k_1+k_2+\cdots+k_i. \tag{5.4}$$

At a point lying on only one of these surfaces the strongest of these singularities is a pole. As the first step in generalizing the results of the preceding section to the general case we compute the residues of these poles.

The Feynman function appearing in (5.1) can be decomposed into a sum of poles times residues. At the point $a=0$ this gives

$$\frac{i(\not p+m)\gamma_{\mu_1}i(\not p+\not k_1+m)\gamma_{\mu_2}\cdots\gamma_{\mu_n}i(\not p+\cdots+\not k_n+m)}{(p^2-m^2)((p+k_1)^2-m^2)((p+\cdots+k_n)^2-m^2)}$$
$$=\sum_{i=0}^{n}\frac{N_{1i}}{D_{1i}}\frac{i(\not p_i+m)}{p_i^2-m^2}\frac{N_{2i}}{D_{2i}}, \tag{5.5}$$

where for each i the numerator occurring on the right–hand side of this equation is identical to the numerator occurring on the left–hand side. The denominator factors are

$$D_{1i}=\prod_{j<i}(2p_ik_{ij}+(k_{ij})^2+i0), \tag{5.6a}$$

and

$$D_{2i} = \prod_{j>i}(2p_i k_{ij} + (k_{ij})^2 + i0), \tag{5.6b}$$

where

$$k_{ij} = \pm[(k_1 + \cdots + k_j) - (k_1 + \cdots + k_i)]. \tag{5.7}$$

The $\pm$ sign in (5.7) is specified below Eq. (8.3c). Since the singularities in question arise from the multiple endpoint $\lambda_1 = \cdots \lambda_n = 0$ it is sufficient for the determination of the analytic character of the singularity to consider an arbitrarily small neighborhood of this endpoint. We shall consider, for reasons that will be explained later, only points in a closed domain in the variables k_j upon which the parameters $p_i k_j$ and $2p_i k_{ij} + (k_{ij})^2$ are all nonzero. Then the factors D_{1i}^{-1} and D_{2i}^{-1} are analytic functions of the variables λ_j in a sufficiently small neighborhood of the point $\lambda_1 = \cdots = \lambda_n = 0$. Hence a power series expansion in these variables can be introduced.

The dominant singularity coming from the multiple end point $\lambda_1 = \cdots = \lambda_n = 0$ is obtained by setting to zero all the λ_j coming from either the numerators N_{1i} and N_{2i} or the power series expansion of the factors D_{1i}^{-1} and D_{2i}^{-1}. Then the only remaining λ_j's are those in the pole factor $((p_i + a)^2 - m^2)^{-1}$ itself.

Consider, then, the term in (5.1) coming from the ith term in (5.5). And consider the action of the first operator, $j = 1$, in (5.1). This integral is essentially the one that occurred in section 3. Comparison with (2.3), (3.6), and (3.3) shows that the dominant singularity on $p_i^2 - m^2 = 0$ is the function obtained by simply making the replacement

$$\int_0^\infty d\lambda_j \left(-\frac{\partial}{\partial p^{\rho_j}}\right)(O(p \to p + \lambda_j k_j)) \to p_{i\rho_j}(p_i k_j)^{-1}. \tag{5.8}$$

Each value of j can be treated in this way. Thus the dominant singularity of the generalized propagator (5.1) on $p_i^2 - m^2 = 0$ is

$$\prod_{j=1}^{n} \left[\left(\delta_{\mu_j}^{\sigma_j} k_j^{\rho_j} - \delta_{\mu_j}^{\rho_j} k_j^{\sigma_j}\right) p_{i\rho_j}(p_i k_j)^{-1}\right]$$
$$\times \frac{N_{1i} i(\not{p}_i + m) N_{2i}}{D_{1i}(p_i^2 - m^2) D_{2i}}. \tag{5.9}$$

The numerator in (5.9) has, in general, a factor

$$
\begin{aligned}
& i(\not{p}_i - \not{k}_i + m)\gamma_{\sigma_i} i(\not{p}_i + m)\gamma_{\sigma_{i+1}} i(\not{p}_i + \not{k}_{i+1} + m) \\
&= i(\not{p}_i - \not{k}_i + m)\gamma_{\sigma_i} i((\not{p}_i + m) i(2p_{i\sigma_{i+1}} + \gamma_{\sigma_{i+1}} \not{k}_{i+1}) \\
&\quad + i(\not{p}_i - \not{k}_i + m)\gamma_{\sigma_i}\gamma_{\sigma_{i+1}}(p_i^2 - m^2) \\
&= i(2p_{i\sigma_i} - \not{k}_i\gamma_{\sigma_i}) i(\not{p} + m) i(2p_{i\sigma_{i+1}} + \gamma_{\sigma_{i+1}} \not{k}_{i+1}) \\
&\quad + i(p_i^2 - m^2)\gamma_{\sigma_i}(2p_{i\sigma_{i+1}} + \gamma_{\sigma_{i+1}} \not{k}_{i+1}) \\
&\quad + i(\not{p}_i - \not{k}_i + m)\gamma_{\sigma_i}\gamma_{\sigma_{i+1}}(p_i^2 - m^2)
\end{aligned}
\tag{5.10}
$$

The last two terms in the last line of this equation have factors $p_i^2 - m^2$. Consequently, they do not contribute to the residue of the pole at $p_i^2 - m^2 = 0$. The terms in (5.10) with a factor $2p_{i\sigma_{i+1}}$, taken in conjunction with the factor in (5.9) coming from $j = i + 1$, give a dependence $2p_{i\rho_j} 2p_{i\sigma_j}$. This dependence upon the indices ρ_j and σ_j is symmetric under interchange of these two indices. But the other factor in (5.9) is antisymmetric. Thus this contribution drops out. The contribution proportional to $p_{i\sigma_i}$ drops out for similar reasons.

Omitting these terms that do not contribute to the residue of the pole at $p_i^2 - m^2$ one obtains in place of (5.10) the factor

$$
(-i \not{k}_i\gamma_{\sigma_i}) i(\not{p}_i + m)(i\gamma_{\sigma_{i+1}} \not{k}_{i+1}) \tag{5.11}
$$

which is first–order in both $\not{k}_i$ and $\not{k}_{i+1}$.

The above argument dealt with the case in which $i \neq 0$ and $i \neq n$: i.e., the propagator i is neither first nor last. If $i = 0$ then there is no factor $k_i = k_0$ in (5.11): in fact no such k_j is defined. If $i = n$ then there is no factor $k_{i+1} = k_{n+1}$ in (5.11): in fact no such k_j is defined in the present context. Thus one or the other of the two $k-$ dependent factors drops out if propagator i is the first or last one in the sequence.

This result (5.11) is the generalization to the case $n > 1$ of the result for $n = 1$ given in (3.5). To obtain the latter one must combine (5.11) with (5.9). The effect of (5.11) is to provide, in conjunction with these pole singularities, a "convergence factor" for the factors lying on either side of each pole factor in the pole–residue decomposition (5.5). That these "convergence factors" actually lead to infrared convergence is shown in the following sections.

6. Infrared Finiteness of Scattering Amplitudes.

Let D be a hard–photon diagram. Let D' be a diagram obtained from it by the insertion of soft photons. Suppose at first that each soft photon is connected on both ends into D by a Q–type interaction.

Each charged–particle line segment L of D is converted into a line L' of D' by the insertion of $n \geq 0$ soft–photon vertices. The line L' of D' represents a generalized propagator. Let the symbols L_i, with $i \in \{0, \ldots n\}$, represent the various line segments of L'.

In this section we shall be concerned only with the contributions coming from the pole parts of the propagator described in section 5. In this case each generalized propagator is expressed by (5.9) as a sum of pole terms, each with a factorized residue enjoying property (5.11).

One class of diagrams is of special interest. Suppose for each charged line L' of D' there is a segment L_i such that the cutting of each of these segments L_i, together perhaps with the cutting of some hard–photon lines, separates the diagram D' into a set of disjoint subdiagrams each of which contains precisely one vertex of the original diagram D. In this case the soft–photon part of the computation decomposes into several independent parts: all dependence on the momentum k_j of the soft photon j is confined to the functional representation of the subdiagram in which the line representing this photon is contained.

The purpose of this section is to prove infrared convergence for the special case of *separable* diagrams defined by two conditions. The first condition is that the diagram D' separate into subdiagrams in the way just described. We then consider for each line L' of D' a single term in the corresponding generalized propagator (5.9). The second condition is that in this term of (5.9) the factor $i(\not{p}_i + m)(p_i^2 - m^2)^{-1}$ correspond to the line segment of L_i that is cut to produce the separation into subdiagrams. Then each subdiagram will contain, for each charged–particle line that either enters it or leaves it, a half–line h that contains either the set of vertices $j \geq i$, or, alternatively, the set of vertices $j < i$, of that charged–particle line.

It is also assumed that the diagram D is simple: at most one line segment (i.e., edge) connects any pair of vertices of D.

The contributions associated with diagrams of this kind are expected to give

the dominant singularities of the full function on the Landau surface associated with D. If the functions associated with all the various subdiagrams are well defined when the momenta associated with all lines of D are placed on–mass–shell then the discontinuity of the full function across this Landau surface will be a product of these well defined functions. By virtue of the spacetime fall–off properties established in paper I these latter functions can then be identified with contributions to the scattering functions for processes involving charged external particles. The purpose of this section is to prove the infrared finiteness of these contributions to the scattering functions.

Each subdiagram can be considered separately. Thus it is convenient to introduce a new labelling of the set of, say, n soft photons that couple into the subdiagram under consideration. To do this the domain of integration $0 \leq |k_j| \leq \delta$, $j \in \{1, \ldots, n\}$, is first decomposed into $n!$ domains according to the relative sizes of the Euclidean magnitudes $|k_j|$. Then in each of these separate domains the vectors k_i are labelled so that $|k_1| \geq |k_2| \geq \ldots \geq |k_n| \geq 0$. A generalized polar coordinate system is then introduced:

$$\begin{aligned} k_1 &= r_1 \Omega_1 \\ k_2 &= r_1 r_2 \Omega_2 \\ &\vdots \\ k_n &= r_1 r_2 \cdots r_n \Omega_n. \end{aligned} \tag{6.1}$$

Here $|r_1| \leq \delta$, and $|r_j| \leq 1$ for $j = 2, \cdots n$, and $\Omega\tilde{\Omega} \equiv (\Omega_{j0})^2 + (\vec{\Omega}_j)^2 = 1$.

The factors in $D_i(a = 0)$, as defined in (5.6), are $2p_i k_{ij} + (k_{ij})^2$. However, the k_{ij} are no longer given by (5.7). With our new labelling the formula (5.7) becomes

$$k_{ij} = \sum_{j' \in J(i,j)} \pm k_{j'}, \tag{6.2}$$

where the signs $\pm$ are the same as the signs in (5.7): only the labelling of the vectors is changed.

Let $j(i,j)$ be the smallest number in the set of numbers $J(i,j)$. Then singling out this term in k_{ij} one may write

$$2p_i k_{ij} + (k_{ij})^2 = r_1 r_2 \cdots r_{j(i,j)} (2p_i \Omega_{j(i,j)} + R) \tag{6.3}$$

where R is bounded.

The zeros of the factors $(2p_i\Omega_{j(i,j)} + R)$ play an important role in the integration over Ω space. However, our objective in this section is to prove the convergence of the integrations over the radial variables r_j, under the condition that the Ω contours can be distorted so as to keep all of these Ω–dependent factors finite, and hence analytic. The validity of this distortion condition is proved in Section 8.

To prove infrared convergence under this condition it is sufficient to show, for each value of j, that if the differential dr_j is considered to be of degree one in r_j then the full integrand, including the differential dr_j, is of degree at least two in r_j. This will ensure that the integration over r_j is convergent near $r_j = 0$.

The power counting in the variables $r_{j'}$ is conveniently performed in the following way: the factor $|k_j|\, d\,|k_j|$ arising from $d^4k_j/k_j^2 + i0$ gives, according to (6.1), a factor that has, in each variable $r_{j'}$, the degree of $(r_1 \cdots r_j)^2$. This factor may be separated into two factors $(r_1 \cdots r_j)$, one for each end of the photon line. Then each individual generalized propagator can be considered separately: for each coupling of a photon j carrying momentum $k_j = r_1 \cdots r_j \Omega_j$ into a half–line h we assign to h one of the two factors $(r_1 \cdots r_j)$ mentioned above. Thus each half–line h will have one such numerator factor for each of the photon lines that is incident upon it, and this numerator factor can be associated with the vertex upon which the photon line is incident. On the other hand, (6.3) entails that there is a dominator factor $r_1 \cdots r_{j(i,j)}$ associated with the jth interval of h. Finally, if the photon incident upon the endpoint of h that stands next to the interval that was cut is labelled by e then there is an extra numerator factor $r_1 \cdots r_e$: it comes from the factor $\not{k}_{i+1}$ (or $\not{k}_i$) in (5.11).

We shall now show that these various numerator and denominator factors combine to produce for each j, and for each half–line upon which the soft photon j, is incident, a net degree in r_j of at least one, and for every other half–line a net degree of at least zero.

Consider any fixed j. To count powers of r_j we first classify each soft photon j' as "nondominant" or "dominant" according to whether $j' \geq j$ or $j' < j$. Any line segment of h along which flows the momentum $k_{j'}$ of a dominant photon j' will, according to (6.3), not contribute a denominator factor r_j.

Thus the denominator factors that do contribute a power of r_j can be displayed graphically by first considering the line h that starts at the initial vertex $j = e$, which stands, say, just to the right of the cut line–segment L_i, and that runs to the right. Soft photons are emitted from the succession of vertices on h, and some of these photons can be reabsorbed further to the right on h. In such cases the part of h that lies to the right of the vertex where a dominant photon is emitted but to the left of the point where it is reabsorbed may be contracted to a point: according to (6.3) none of these contracted line segments of h carry a denominator factor of r_j. If a dominant soft photon is emitted but is never reabsorbed on h then the entire part of the line h lying to the right of its point of emission can be contracted to this point.

If the line obtained by making these two changes in h is called h' then, by virtue of (6.3), there is exactly one denominator factor r_j for each line segment of h'.

Self–energy and vertex corrections are to be treated in the usual way by adding counterterms. Thus self–energy–diagram insertions and vertex–correction diagrams should be omitted: the residual corrections do not affect the power counting. This means that every vertex on h', excluding the last one on the right end, will be either:

1. An original vertex from which a single nondominant photon is either emitted or absorbed; or

2. A vertex formed by a contraction. Any vertex of the latter type must have at least two nondominant soft photons connected to it, due to the exclusion of self–energy and vertex corrections.

The first kind of vertex will contribute one power of r_j to the numerator, whereas the second kind of vertex will contribute at least two powers of r_j.

Every line segment of h' has a vertex standing immediately to its left. Thus each denominator power of r_j will be cancelled by a numerator power associated with this vertex. This cancellation ensures that each half–line will be of degree at least zero in r_j.

If the soft–photon e incident upon the left–hand end of h' is nondominant then one extra power of r_j will be supplied by the factor $\not{k}_e$ coming from (5.11).

If the soft photon e is dominant then there are two cases: either the left–most vertex of h' is the only vertex on h', in which case there are no denominator factors of r_j, but at least one numerator factor for each k_j vertex incident on h; or the left most vertex of h' differs from the rightmost one, and is formed by contraction, in which case at least two nondominant lines must be connected to it. These two lines deliver two powers of r_j to the numerator and hence the extra power needed to produce degree one in r_j.

This result for the individual half lines means that for the full subdiagram the degree in r_j is at least one for every j. Hence the function is infrared convergent.

The arguments of this section cover specifically the special class of separable diagrams D'. It is shown in section 8.6 how the argument is adapted to cover a general diagram D' constructed from a triangle diagram D.

7. Inclusion of the Classical Interactions

The power–counting arguments of the preceeding section dealt with processes containing only Q–type interactions. In that analysis the order in which these Q–type interactions were inserted on the line L of D was held fixed: each such ordering was considered separately.

In this section the effects of adding C–type interaction are considered. Each C–type interactions introduces a coupling $k^\sigma \gamma_\sigma = \not{k}$. Consequently, the Ward identities, illustrated in (2.7), can be used to simplify the calculation, but only if the contributions from all orders of its insertion are treated together. This we shall do. Thus for C–type interactions it is the operator $\widehat{C}$ defined in (2.5) that is to be used rather than the operator $\widetilde{C}$ defined in (2.12).

Consider, then, the generalized propagator obtained by inserting on some line L of D a set of n interactions of Q–type, placed in some definite order, and a set of N C–type interactions, inserted in all orders. The meromorphic part of the function obtained after the action of the n operators $\widetilde{Q}_j$ is given by (5.9). The action upon this of the N operators $\widehat{C}_j$ of (2.5) is obtained by arguments similar to those that gave (5.9), but differing by the fact that (2.5) acts upon the propagator present *before* the action of $\widehat{C}_j$, and the fact that now both limits of integration contribute, thus giving for each $\widehat{C}_j$ two terms on the right–hand side rather than one. Thus the action of N such $\widehat{C}_j$'s gives 2^N terms:

$$\begin{aligned}
&\left[\prod_{j=n+1}^{n+N} \widehat{C}_{\mu_j}(k_j) P_{\mu_1\cdots\mu_n}(p; Q, k_1, Q, k_2, \cdots Q, k_n)\right]_{Mero} \\
&= \sum_{\Theta=1}^{2^N} Sgn(\Theta) \sum_{i=0}^{n} \prod_{j=n+1}^{n+N} \left(\frac{i p^{\Theta}_{i\mu_j}}{p^{\Theta}_i k_j}\right) \\
&\times \left\{\prod_{j=1}^{n} \left[\left(\delta^{\sigma_j}_{\mu_j} k^{\rho_j}_j - \delta^{\rho_j}_{\mu_j} k^{\sigma_j}_j\right) \left(\frac{p^{\Theta}_{i\rho_j}}{p^{\Theta}_i k_j}\right)\right]\right\} \\
&\times \frac{N^{\Theta}_{1i}}{D^{\Theta}_{1i}} \frac{i(\not{p}^{\Theta}_i + m)}{(p^{\Theta}_i)^2 - m^2} \frac{N^{\Theta}_{2i}}{D^{\Theta}_{2i}},
\end{aligned} \tag{7.1}$$

where

$$\begin{aligned}
\Theta &= (\Theta_{n+1}, \cdots, \Theta_{n+N}), \\
\Theta_j &= +1 \text{ or } 0,
\end{aligned}$$

$$
\begin{aligned}
Sgn(\Theta) &= (-1)^{\Theta_{n+1}}(-1)^{\Theta_{n+2}}\cdots(-1)^{\Theta_{n+N}} \\
p_i^{\Theta} &= p_i + \Theta_{n+1}k_{n+1} + \cdots + \Theta_{n+N}k_{n+N}, \\
p_i &= p + k_1 + \cdots + k_i, \qquad (7.2)
\end{aligned}
$$

and the superscript Θ on the N's and D's means that the argument p_i appearing in (5.5) and (5.6) is replaced by p_i^{Θ}. Note that even though the action of $\widehat{C}_j$ and $\widetilde{Q}_j$ involve integrations over λ and differentiations, the meromorphic parts of the resulting generalized propagators are expressed by (7.1) in relatively simple closed form. These meromorphic parts turn out to give the dominant contributions in the mesoscopic regime, as we shall see.

The essential simplification obtained by summing over all orders of the C–type insertions is that after this summation each C–type interaction gives just two terms. The first term is just the function before the action of $\widehat{C}_j$ multiplied by $ip_{i\mu_j}(p_ik_j)^{-1}$; the second is minus the same thing with p_i replaced by $p_i + k_j$. Thus, apart from this simple factor, and, for one term, the overall shift in p_i, the function is just the same as it was before the action of $\widehat{C}_j$. Consequently, the power–counting argument of section 6 goes through essentially unchanged: there is for each classical photon j one extra denominator factor (p_ik_j) coming from the factor $ip_{i\mu_j}(p_ik_j)^{-1}$ just described, but the powers of the various r_i in this denominator factor are exactly cancelled by the numerator factor $(r_1\cdots r_j)$ that we have associated with the vertex $\widehat{C}_j$. Because of this exact cancellation the C-type couplings do not contribute to the power counting. Hence when C-type couplings are allowed the arguments of section 6 lead to the result that the meromorphic part of the function F associated with the quantum photons is of degree at least one in each of the variables r_j. Hence is infrared convergent.

8. Distortion of the Ω Contours

8.1 Landau Equations, Diagrams, and Distortion Conditions

In this section we show that it is possible to distort the Ω contours in such a way as to avoid all Ω–dependent singularities in any term in the pole–residue decomposition of the function corresponding to a simple triangle diagram with an arbitrary number of photon–line insertions, each having a quantum–type vertex on at least one end. The simple triangle diagram is shown in Fig. 4.

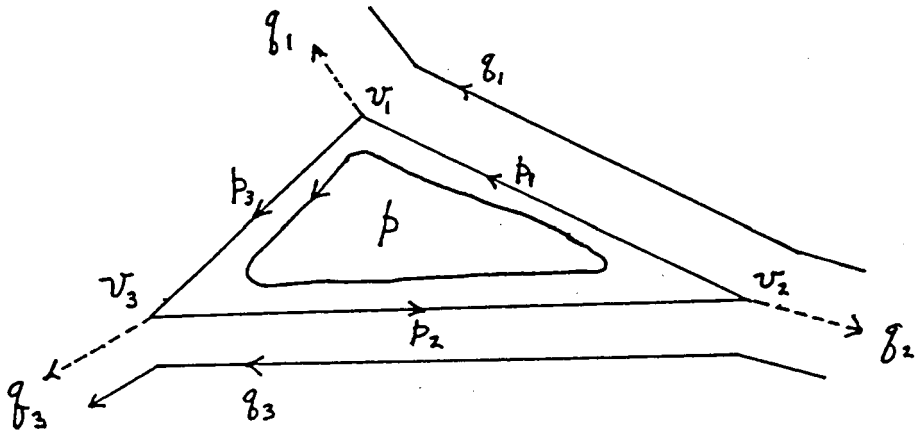

Figure 4. The basic charged–particle triangle diagram. The momentum–energy p_s flows along side s of the triangle in the direction of the arrow. Here $p_1^0 > 0$, $p_2^0 < 0$, and $p_3^0 > 0$.

The momenta $p_1, -p_2$, and p_3 represent the momenta flowing from v_2 to v_1, from v_2 to v_3, and from v_1 to v_3, respectively. Conservation of energy–momentum is represented by introducing a closed loop carrying momentum p, and two open paths carrying momenta q_1 and q_3, respectively, in the directions indicated by the arrows. Then $p_1 = p + q_1$, $\quad p_2 = p - q_3$, and $p_3 = p$.

The function associated with this Feynman diagram has a singularity on the positive–α Landau–Nakanishi triangle–diagram singularity surface $\varphi(q) = 0$, where $q = (q_1, q_2, q_3)$ and $q_3 \equiv -q_1 - q_2$. For each point q on this surface $\varphi = 0$ there is a uniquely defined set of three four–vectors $p_1(q), p_2(q)$, and $p_3(q)$ such that the singularity at q of the Feynman function $F_D(q)$ corresponding to the

diagram D of Fig. 4 arises from an arbitrarily small neighborhood

$$p \approx p(q) = p_1(q) - q_1 = p_2(q) + q_3 = p_3(q) \tag{8.1a}$$

in the domain of integration of the Feynman function. These three four–vectors $p_s(q)$ satisfy the mass–shell constraints

$$(p_s(q))^2 = m^2, \tag{8.1b}$$

and the (Landau–Nakanishi) loop equation

$$\alpha_1 p_1(q) + \alpha_2 p_2(q) + \alpha_3 p_3(q) = 0, \tag{8.1c}$$

where the α_s are nonnegative real numbers. This loop equation implies that for each q on $\varphi(q) = 0$ the three four–vectors $p_s(q)$ lie in some two–dimensional subspace of the four–dimensional energy–momentum space.

We shall consider a fixed *interior* point q of the surface $\varphi = 0$. In this case each of the three parameters α_s is nonzero, and each of the three four–vectors vectors $p_s(q)$ is nonparallel to each of the other two.

Consider now a diagram D' obtained by inserting some finite number of soft–photon lines i ($i \epsilon I$) into D. Each inserted line begins on a line of D and ends on a line of D. The bound δ on the Euclidean norms $|k_i|$ of the photon momenta is taken shall enough so that

$$n\delta < \delta' << m, \tag{8.2}$$

where n is the number of photon lines in the diagram.

Momentum–energy conservation is now maintained by introducing a separate closed loop for the momentum k_i of each photon line. Momentum k_i flows along the photon line segment i in the direction indicated by the arrow placed on that line segment. It then continues to flow through the diagram by flowing along certain charged-particle lines of the diagram D'. This continuation through D' is specified by the condition that it pass through at most one of the three vertices v_1, v_2, v_3.

The arrow on photon line i is chosen so that every term $p_s k_i$ that occurs in any Feynman denominator occurs with a plus sign. Consequently, the Feynman rule that m^2 represents $m^2 - i0$ is compatible with the rule that each $p_s k_i$

represents $p_s k_i + i0$. No condition is placed on the sign of the energy component k_i^0.

Each charged–particle line segment j has an arrow placed on it. The momentum flowing along the charged–particle segment j in the direction of this arrow is called $\sum_j$. It is the momentum p_s flowing along the side of the triangle upon which segment j lies, as defined in Fig. 4, plus the sum of photon momenta k_i carried by the photon loops that pass along this segment j.

Our interest here is in the functions that arise from inserting the pole–residue decomposition formula (5.5) into the generalized propagators corresponding to the three sides of the original triangle diagram D. For the simple diagram D' of Fig. 2 this decomposition gives a sum of the four terms represented by the four graphs of Fig. 5

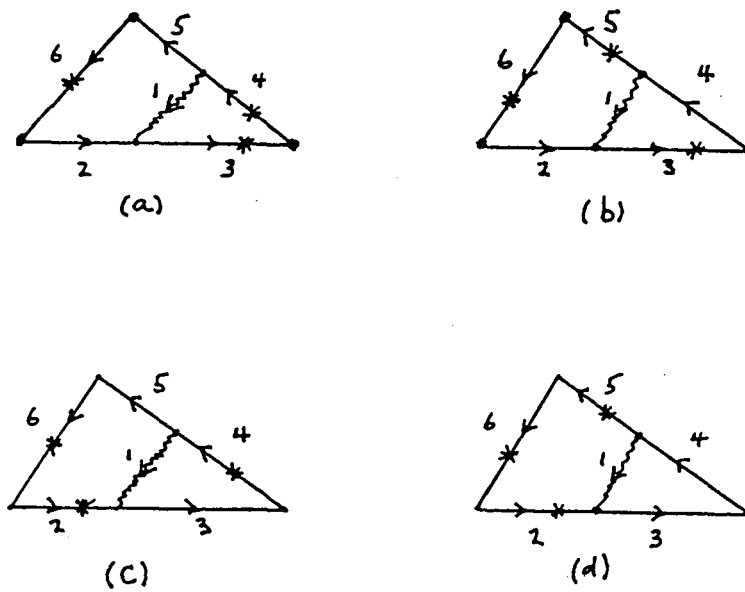

Figure 5. The graphs representing the four terms that arise from inserting the pole–residue decomposition formula (5.5) into the function represented by the diagram D' of Fig. 2.

The asterisk on a line segment indicates that it is the segment associated with the denominator $p_i^2 - m^2 + i0$ in the pole-residue decomposition (5.5). Each of the other charge–particle segments j is associated with a pole–residue denominator function

$$f_j = 2p_i\Omega_{ij} + \rho_{ij}\Omega_{ij}^2 + i0 \tag{8.3a}$$

where, in the notation of (6.3),

$$\rho_{ij} = r_1 r_2 ... r_{j(i,j)} \tag{8.3b}$$

and

$$\Omega_{ij} = (\Omega_{j(i,j)} + ...). \tag{8.3c}$$

Each of the remaining terms in the parentheses in (8.3c) is a product of some $\pm\Omega_k$ with a product of a non empty set of factors $r_h (h \geq 2)$.

Each of the pole-residue factors f_j is formed by first taking the difference $\sigma_{js}(\sum_j^2 - \sum_s^2)$, where $\sum_j$ is the momentum–energy flowing along segment j in the direction of the arrow on that segment, and $\sum_s$ is the momentum–energy flowing along the $*$ segment on the same side s of the charged–particle triangle, and then dividing out the common factors $r_h (h \geq 1)$. (The vector $\sum_s$ was called p_i in sections 5, 6 and 7.) The sign σ_{js} is the sign that makes the term $2p_s k_\ell$ in $\sigma_{js}(\sum_j^2 - \sum_s^2)$ that has the smallest value of ℓ appear with a positive sign.

The $i0$ rule for this difference is fixed in the following way: in order to make the pole–residue formula well defined each quantity $p_s k_i$ is replaced by $p_s k_i + i\epsilon_i$ with $\epsilon_i >> \epsilon_{i+1} > 0$, for the ordering (6.1). Thus each ϵ_i is taken to be much larger than the next smaller one, so that it dominates over any sum of smaller ones. This makes each difference of denominators that occurs in the pole-residue decomposition well defined, with a well–defined nonvanishing imaginary part. Then the sign of $\pm i0$ in $(\sum_j^2 - \sum_s^2)$ is fixed so as to agree with the sign of the imaginary part of the difference $\sum_j^2 - \sum_s^2$ defined in this way. This yields the $+i0$ shown in (8.3).

The full set of functions f_j whose zero's define the locations of the singularities of the four functions F represented by the diagrams of Fig. 5 are given in Fig. 6. The functions f_j for $j = (1, ..., 6)$ correspond to denominators $f_j + i0$. The function f_7 corresponds to the δ–function constraint $\delta(\Omega\tilde{\Omega} - 1)$, and f_8

corresponds to the Heaviside function $\theta(r)$.

(a)	(b)
$f_1 = \Omega^2$	$f_1 = \Omega^2$
$f_2 = 2p_2\Omega + r\Omega^2$	$f_2 = 2p_2\Omega + r\Omega^2$
$f_3 = (p_2 + r\Omega)^2 - m^2$	$f_3 = (p_2 + r\Omega)^2 - m^2$
$f_4 = (p_1 + r\Omega)^2 - m^2$	$f_4 = 2p_1\Omega + r\Omega^2$
$f_5 = 2p_1\Omega + r\Omega^2$	$f_5 = p_1^2 - m^2$
$f_6 = p_3^2 - m^2$	$f_6 = p_3^2 - m_2$
$f_7 = \Omega\widetilde{\Omega} - 1$	$f_7 = \Omega\widetilde{\Omega} - 1$
$f_8 = r$	$f_8 = r$

(c)	(d)
$f_1 = \Omega^2$	$f_1 = \Omega^2$
$f_2 = p_2^2 - m^2$	$f_2 = p_2^2 - m^2$
$f_3 = 2p_2\Omega + r\Omega^2$	$f_3 = 2p_2\Omega + r\Omega^2$
$f_4 = (p_1 + r\Omega)^2 - m^2$	$f_4 = 2p_1\Omega + r\Omega^2$
$f_5 = 2p_1\Omega + r\Omega^2$	$f_5 = p_1^2 - m^2$
$f_6 = p_3^2 - m^2$	$f_6 = p_3^2 - m^2$
$f_7 = \Omega\widetilde{\Omega} - 1$	$f_7 = \Omega\widetilde{\Omega} - 1$
$f_8 = r$	$f_8 = r$

Figure 6. The functions f_j whose zeros define the singularity surfaces of the four functions F represented by the graphs of Fig. 5. Here, and in what follows, the vectors p_s, $s = 1, 2, 3$, are the vectors defined beneath Fig. 4.

The necessary (Landau–Nakanishi) conditions[12–15] for a singularity (in the original real domain of definition) of one of these functions F is that there be a set of real numbers $\alpha_1, ..., \alpha_8$, not all zero, a real number $r \geq 0$, and a pair of real four–vectors Ω and p, with $p_1 = p + q_1$, $p_2 = p - q_3$, and $p_3 = p$, such that

$$\alpha_j f_j = 0 \qquad \text{all} \quad j\epsilon\{1, ..., 8\}, \tag{8.4a}$$

and

$$\sum_{i=j}^{8} \alpha_j \frac{\partial f_j}{\partial x_i} = 0 \qquad \text{all} \quad i\epsilon\{1, 2, 3\}, \tag{8.4b}$$

where $x_1 = \Omega, x_2 = r, x_3 = p$, and

$$\alpha_j \geq 0 \qquad j\epsilon\{1, ..., 6\}. \tag{8.4c}$$

Also,

$$f_7 = 0, \quad \text{and} \quad r << m. \tag{8.4d}$$

The contribution from the upper end points of the r integrals are neglected because these end points are artificially introduced, and hence do not represent singularities of the full function.

The Landau matrix $L_{ij} \equiv \frac{1}{2}\partial f_j/\partial x_i$ for the function represented by the graph of Fig. 5a is shown in Fig. 7. The Landau (loop) equations (8.4b) are formed by multiplying each row j of this matrix by α_j and requiring the sum of each of its columns to vanish.

f_j	$d\Omega$	dr	dp
$f_1 = \Omega^2$	Ω	0	0
$f_2 = 2p_2\Omega + r\Omega^2$	$p_2 + r\Omega$	$\frac{1}{2}\Omega^2$	Ω
$f_3 = (p_2 + r\Omega)^2 - m^2$	$r(p_2 + r\Omega)$	$(p_2 + r\Omega)\Omega$	$p_2 + r\Omega$
$f_4 = (p_1 + r\Omega)^2 - m^2$	$r(p_1 + r\Omega)$	$(p_1 + r\Omega)\Omega$	$p_1 + r\Omega$
$f_5 = 2p_1\Omega + r\Omega^2$	$p_1 + r\Omega$	$\frac{1}{2}\Omega^2$	Ω
$f_6 = p_3^2 - m^2$	0	0	p_3
$f_7 = \Omega\tilde{\Omega} - 1$	$\tilde{\Omega}$	0	0
$f_8 = r$	0	$\frac{1}{2}$	0

Figure 7. The Landau matrix L_{ij} corresponding to the graph in Fig. 5a. The σ_{j_s}'s are negative for $j = 2$ and $j = 5$.

There are two cases: $r \neq 0$, and $r = 0$. If $r \neq 0$ then the equation (8.4a) implies $\alpha_8 = 0$. If one forms the combination of columns $\Omega d\Omega - r dr$ and compares the entries to equation (8.4a), $\alpha_j f_j = 0$, then one finds that the only term in the resulting loop equations is $\alpha_7 \Omega\tilde{\Omega} = 0$, with $\Omega\tilde{\Omega} = 1$ This entails $\alpha_7 = 0$. If, on the other hand, $r = 0$ then the dr column of L_{ij} has an entry in row 8, and hence it cannot be used in this way. But for $r = 0$ this column does not contribute to $r dr$. So in either case the conclusion holds: $\alpha_7 = 0$, and the $\Omega\tilde{\Omega} = 1$ row does not contribute.

Similar arguments in the case of graphs with more lines show that one can always eliminate all of the rows corresponding to $\Omega_i\tilde{\Omega}_i - 1$. In the general case it is the combination of columns $\Omega_i d\Omega_i - r_i dr_i + r_{i+1} dr_{i+1}$ that is used to show the vanishing of the row corresponding to $\Omega_i\tilde{\Omega}_i = 1$. (See Appendix B.)

Consider now the function corresponding to the graph in Fig. 5d, and the corresponding set of functions f_j in Fig. 6d. This graph is a graph of the separable kind: cutting the three $*$ segments separates it into three disjoint parts.

If one considers the $d\Omega$ column with the $\Omega\tilde{\Omega} = 1$ row deleted then one im-

mediately concludes from a look at Fig. 6d, and from the nonparalled character of $p_2 + r\Omega$, and $p_1 + r\Omega$, and the impossibility of the simultaneous vanishing of f_1 and either f_3 or f_4, that the only solution of the implied Ω loop equation [and (8.4a)] is the trivial one in which all three contributions are zero—i.e., $\alpha_1 = \alpha_3 = \alpha_4 = 0$

In this situation we may invoke a basic lemma[16]: "For any sets of real numbers η_{ba} and λ_{ca} the system of equations

$$\sigma_b = \sum_a \eta_{ba}\delta_a \qquad \sigma_b > 0 \tag{8.5a}$$

$$0 = \sum_a \lambda_{ca}\delta_a$$

has a solution $\delta \equiv \{\delta_a\}$ if and only if the system of equations

$$\sum_b \alpha_b\eta_{ba} + \sum_c \beta_c\lambda_{ca} = 0 \qquad \alpha_b \geq 0,\ \sum \alpha_b > 0 \tag{8.5b}$$

has no solution (α, β)."

Identifying $(\eta_{ba}, \lambda_{ca})$ with the entries in the $d\Omega$ and dr columns of L_{ij}, with $b = j\epsilon\{1, ...6\}$ and $c = j\epsilon\{7, 8\}$, and identifying $\delta_a = \delta\Omega_a$, for $a\epsilon\{0, 1, 2, 3\}$, as an imaginary displacement of the four–vector contour–of–integration variable Ω, we find from this lemma, and the above–mentioned fact [that the only solution of these equations is the trivial one with every term equal to zero], that at every point in the space of integration variables p and Ω where some set of functions f_j vanishes there is a displacement of the contour in Ω space that shifts the contour away from every Ω–dependent vanishing f_j: by virtue of $(\partial f_j/\partial\Omega)\delta\Omega > 0$ [i.e., (8.5a)] every such function $f_j(\Omega)$ is shifted by this distortion into its upper–half plane.

We are particularly interested in the functions represented by separable graphs, i.e., by graphs that separate into three disjoint parts when the three $*$ segments are cut.

The argument just given for the graphs of Fig. 5d generalizes to all separable graphs: for all such graphs the Ω_i contours can be distorted away from their singularities. To obtain this result, consider first the example shown in Fig. 8

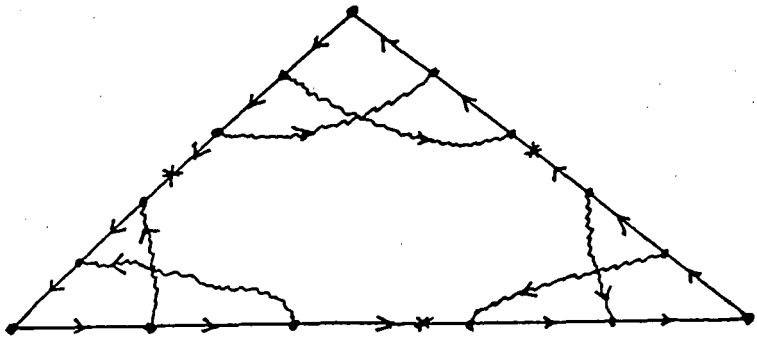

Figure 8. The graph representing a term in the pole–residue decomposition. This graph separates into three disjoint parts when one cuts the three $*$ segments.

Consider first the case where all $r_i \neq 0$. In this case the Landau equations are equivalent to the Landau equations that arise from using the k-space variables, instead of the (r, Ω) variables. Then the Landau equations associated with the function represented by the graph shown in Fig. 8 can be expressed in a simple geometric form: these equations are equivalent to the existence of a "Landau diagram" (a diagram in four–dimensional space) that has the form shown in Fig. 9.

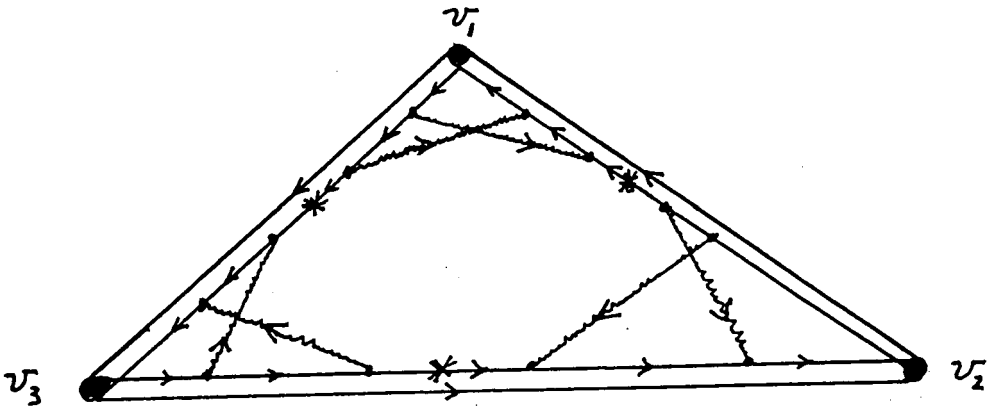

Figure 9. The Landau diagram associated with the graph of Fig. 8. We distinguish 'Landau diagrams' from 'graphs: the former are geometric, the latter topological.

This Landau diagram is a diagram in four-dimensional space (thought of as spacetime), and each segment of the diagram represents a four vector. The rules are these:

1. Each directed photon line segment i represents the vector

$$V_i = \alpha_i k_i, \tag{8.6a}$$

where k_i is the momentum flowing along segment i of the graph in the direction of the arrow, and $\alpha_i \geq 0$.

2. Each directed charged-particle segment j corresponding to a pole-residue factor f_j represents the vector

$$V_j = \beta_{js} \Sigma_j, \tag{8.6b}$$

where $\sum_j$ is the momentum flowing along segment j of the graph in the direction of the arrow on it, and

$$\sigma_{js} \beta_{js} = \alpha_j \geq 0, \tag{8.6c}$$

where the sign σ_{js} is defined below (8.3).

3. Each directed charged-particle line segment s corresponding to a pole denominator $(\sum_s^2 - m^2 + i0)$ is represented by a star (asterisk) line segment s, and it represents the vector.

$$V_s' = \alpha_s' \Sigma_s \quad , \tag{8.6d}$$

where $\sum_s$ is the momentum flowing along $*$ line segment s of the graph in the direction shown, and

$$\alpha_s' = \alpha_s - \sum_{j \in J(s)} \beta_{js}. \tag{8.6e}$$

Here α_s is the Landau parameter α corresponding to the function $f_s = \sum_s^2 - m^2 + i0$, and for each side s the set $J(s)$ is the set of indices j that label the pole-residue denominators that are associated with side s of the triangle graph.

4. Three line segments appear in the Landau diagram that are not images of segments that appear in the graph. They are the three *direct* line segments that directly connect pairs of vertices from the set $\{v_1, v_2, v_3\}$. The vector V_s associated with the direct segment s is

$$V_s = \alpha_s \Sigma_s + \sum_{j \in J(s)} \beta_{js}(\Sigma_j - \Sigma_s). \tag{8.6f}$$

It is equal to the sum of the vectors corresponding to the sequence of $*$ and non $*$ charged-particle line segments that connect the pair of vertices v_i between which the direct line segment s runs.

The p loop equation is represented by the closed loop formed by the three direct line segments V_s specified in (8.6f) The photon loop equation associated with the photon line carrying momentum k_i is formed by adding to $\alpha_i k_i$ the sum of the vectors corresponding to the charged-particle segments needed to complete a closed loop in the diagram (See Appendix C). Thus the existence of a (nontrivial) solution of the Landau equations is equivalent to the existence of a (nonpoint) Landau diagram having the specified topological structure, with its line segments equal to the vectors specified in (8.6).

Although Figs. 8 and 9 represent a separable case the rules described above general: they cover all cases in which all r_i are nonzero.

For each s we can use in the Landau diagram either V_s' or V_s. We shall henceforth use always V_s, the segment that directly connects a pair of vertices v_i, rather than V_s', and we shall place a star (asterisk) on each of these three direct line segments. These three direct line segments are geometrically more useful than the V_s''s because they display immediately the p loop equations, and also the relative locations of the three external vertices v_i, and because each one has only a single contribution, $\alpha_s \Sigma_s$, of well-defined sign and direction, in the limit $k_i \Rightarrow 0$, provided condition (8.8) holds.

We specify the way that photon loops pass through Landau diagrams: a photon loop shall pass through the star line s of a *Landau diagram* (i.e., along the direct line segment s) if and only if the corresponding loop in the *graph* passes through the star line s of the *graph.*

We are considering first only QQ type photons: these have Q–type vertices on both ends, and propagate according to the usual Feynman rule.

The positivity of the photon-line α_i's entails that each directed vector $\alpha_i k_i$ of Fig. 9 points in the positive (energy/time) direction (i.e., to the left) if the energy k_i^0 is positive, and in the negative direction (i.e., to the right) if the energy k_i^0 is negative. This fact entails that positive energy is carried by each nonzero (length) photon line segment of Fig. 9 *out of* the vertex that stands on its right–hand end and *into* the vertex that stands on its left–hand end. This result is true independently of the direction in which the arrow points, or of the sign of the energy component k_i^0.

In the general separable case some of the non $*$ segments may have $\alpha_j = 0$, and hence contract to points. Consequently several photons may emerge from, or enter into, a single vertex of the Landau diagram.

We now assert a main result: for almost all points on the triangle diagram Landau surface the only solutions to these Landau equations (for graphs with no self–energy parts) are those in which every non $*$ segment (charged-particle and photon) contracts to a point: i.e., every α_j vanishes except possibly for the three α_s's associated with the three $*$ segments.

8.2 Separable Case; All $r_i \neq 0$.

To prove this result for the separable case, and when all $r_i \neq 0$, let us consider any one of the three disjoint partial diagrams of non $*$ segments. Let V be the set of vertices of this partial diagram that lie on an end of at least one photon line that is not contracted to a point. Let V_R be any element of V such that every nonzero-length photon line incident upon V_R has its other end lying to the left of V_R. Let V_L be any element of V such that every nonzero-length photon line that is incident upon V_L has its other end lying to the right of V_L. Then the total momentum K carried into either V_R or V_L by all photons incident upon it satisfies $K \neq 0$ and $K^2 \geq 0$: these properties follow from the fact that each photon line of nonzero length incident upon V_R must carry a light-cone-directed momentum-energy with positive energy out of V_R, and each photon line of nonzero length incident upon V_L must carry a light-cone-directed momentum-energy with positive energy into V_L. However, one cannot satisfy $2pK + K^2 = 0$ with $p \simeq p_1, p_2$ or p_3, and with a small $K \neq 0$ satisfying $K^2 \geq 0$. Consequently the charged–particle line segments of the partial Landau diagram lying on the outer extremities of the two charged particle lines must contract to points, by virtue of (8.4a): the associated Landau parameter α_i must vanish. Recursive use of this fact entails that *all* of the lines in this partial diagram must contract to a single point.

The existence of zero–length photon lines whose ends do not lie in V does not disturb this argument, provided self–energy diagrams are excluded.

This result, that each non $*$ line contracts to a point, means that every entry in every Ω_i loop equation vanishes. Under this condition the lemma expressed by Eq. (8.5) shows that every Ω_i contour can be distorted away from every Ω_i–dependent singularity.

We next show that this result continues to hold when some or all of the r_i vanish.

8.3 Separable Case; Some $r_i = 0$

Let us first consider the simple example shown in Fig. 10.

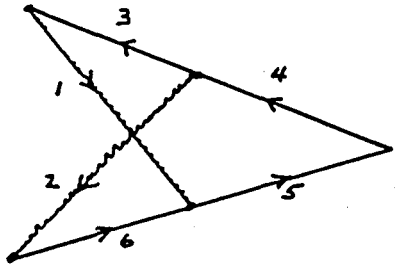

Figure 10. Part of the diagram of Fig. 9.

The Landau matrix for the diagram of Fig. 10 is shown in Fig. 11.

f_j	$d\Omega_1$	$d\Omega_2$
Ω_1^2	Ω_1	0
Ω_2^2	0	Ω_2
$2p_1\Omega_1 + r_1\Omega_1^2$	$p_1 + r_1\Omega_1$	0
$2p_1(\Omega_1 + r_2\Omega_2) + r_1(\Omega_1 + r_2\Omega_2)^2$	$p_1 + r_1\Omega_1 + r_1r_2\Omega_2$	$r_2(p_1 + r_1\Omega_1 + r_1r_2\Omega_2)$
$2p_2(\Omega_1 + r_2\Omega_2) + r_1(\Omega_1 + r_2\Omega_2)^2$	$p_2 + r_1\Omega_1 + r_1r_2\Omega_2$	$r_2(p_1 + r_1\Omega_1 + r_1r_2\Omega_2)$
$2p_2\Omega_2 + r_1r_2\Omega_2^2$	0	$p_2 + r_1r_2\Omega_2$

Figure 11. The Landau matrix corresponding to the diagram of Fig. 10. The rows corresponding to the conditions $\Omega_j\tilde{\Omega}_j = 1$ have been removed, by using the argument given in Appendix B.

If $r_1 \neq 0 \neq r_2$ then one can multiply the Ω_1^2 row by r_1, multiply the Ω_2^2 row by $r_1r_2^2$, multiply the last row by r_2, and divide the $d\Omega_2$ column by r_2. This brings the matrix into an equivalent one in which r_1 and r_2 occur only in the combinations $k_1 = r_1\Omega_1$ and $k_2 = r_1r_2\Omega_2$: this is the equivalent k form that was previously used for the case $r_1 \neq 0 \neq r_2$.

If $r_1 = 0$ and $r_2 \neq 0$ then one can perform the same transformations involving r_2, and bring the equations to the same form as before, except that the vector associated with the photon line segment 1 is now $\alpha_1\Omega_1$ instead of $\alpha_1 k_1$, and the vector associated with the photon line segment 2 is now $\alpha_2 r_2\Omega_2$ instead of $\alpha_2 k_2$. The vectors $r_1\Omega_1$ and $r_1r_2\Omega_2$ that occur summed with p_1 or p_2 become zero. Thus the situation is geometrically essentially the same as in the case $r_1 \neq 0 \neq r_2$, though slightly simpler: the small additions k_1 and k_2 to the vectors p_1 and p_2 now drop out. The important point is that the critical denominators $2pK + K^2$ of the earlier argument now take the form $2p\Omega$, with

$\Omega^2 \geq 0$ and $\Omega \neq 0$. Such a product cannot vanish. Thus the earlier $r_i \neq 0$ argument goes through virtually unchanged.

If $r_1 \neq 0$ and $r_2 = 0$ then the Ω_1 and Ω_2 loop equations can be considered separately. The earlier $r_i \neq 0$ argument of section 8.2 can be applied to the first part alone, and it shows that each line segment on the Ω_1 loop must contract to a point. Next the Ω_2 equation can be considered alone, with each segment along which the Ω_1 loop flows contracted to a point. Then the earlier $r_1 = 0$ arguments can be applied now to this Ω_2 part of the diagram (with r_2 in place of r_1): it shows that each of the segments along which Ω_2 flows also must contract to a point: the corresponding α_j must be zero.

The case $r_1 = r_2 = 0$ is not much different from the case just treated: r_1 enters Fig. 11 only in an unimportant way.

The generalization of this argument from the case of Fig. 11 to the general separable case is straightforward. Let r_g be the first vanishing element of the ordered set $r_1, r_2, ..., r_n$. Then the set of Ω columns of the Landau matrix separates into one part involving only the Ω_i columns for $i < g$, and a second part involving only the Ω_i columns for $i \geq g$. For the first part of this matrix the argument given above for the case with all $r_i \neq 0$ holds, and it entails that every line segment in this part must contract to a point. With all of the rows corresponding to these contracted segments omitted one may apply the $r_1 = 0$ argument (with r_g in place of r_1) to the part $i \geq g$, and proceed iteratively. This arguments leads to conclusion that the only solution to all of the Ω_i loop equations is the trivial one where every entry in every Ω column is zero. Hence the lemma expressed by Eq. (8.5) ensures that each Ω_i contour can be distorted away from all of its singularities, in the general separable case.

As one moves from the domain where all $r_i > 0$ to the various boundary points where some $r_i = 0$ two kinds of changes can occur. Certain conditions that particular vectors Ω_j be in the upper-half plane with respect to a variable like $(p_1 + r_1\Omega_1 + r_1 r_2 \Omega_2) \cdot \Omega_j$ becomes slightly simplified when an r_i becomes zero. Since the different conditions of this kind correspond to vectors p_1, p_2, and p_3 that are well separated, the passage to a point $r_i = 0$ causes no discontinuous change in the set of vectors that satisfy such conditions. The second kind of change is that some contributions to particular $d\Omega_j$'s may suddenly drop out if some r_i vanishes. (See Fig. 11 with $r_2 = 0$). These changes at the boundary

points of the region $r_i \geq 0$ do not entail any discontinuity in the distortion of the Ω contours on the boundary. The possibility of using a distortion in Ω space that is everywhere continuous in (r, Ω) follows from the continuousness of the gradients of the functions $f_i(r, \Omega)$, and the fact that at every point in the domain of integration the set of gradients of the set of vanishing f_i form a *convex* set: the Landau equations cannot be satisfied.

8.4 Nonseparable Case; All $r_i \neq 0$.

We consider next the functions represented by graphs such that the cutting of the three $*$ segments does not separate the graph into three disjoint parts. The same result about distortions of Ω_i contours can be obtained also for these functions.

To obtain this result we consider first, as before, the case in which all $r_i \neq 0$. Then we may use the k form of the Landau equations given in (8.6).

The argument proceeds as before, by making use of the vertices V_R and V_L. No such vertex can join together two pole-residue segments j of nonzero length: it is impossible to satisfy both $2pK_1 + K_1^2 = 0$ and $2pK_2 + K_2^2 = 0$ if $K_1 - K_2 = K$ satisfies $K^2 \geq 0$ and $K \neq 0$, and K_1 and K_2 are small compared to the timelike p. Likewise, neither V_R nor V_L can join a $*$ segment to a pole-residue segment j with $\alpha_j \neq 0$: one cannot satisfy $2pK + K^2 = (2p + K)K = 0$ if $K^2 \geq 0$ and $K \neq 0$, and K is much smaller than the timelike p. Consequently each of the vertices V_R and V_L must be confined to the set of external vertices v_i:

$$\{V_R, V_L\} \subset \{v_1, v_2, v_3\}. \tag{8.7}$$

In the nonseparable case some of the signs σ_{j_s} will be negative. Consequently some of the vectors corresponding to pole-residue factors f_j will point in the 'reversed' direction, because their β_{j_s}'s, defined in (8.6c), are negative. There are also some (sometimes-compensating) reversals of the ways that certain photon loops run. These latter reversals arise because we have used, in the Landau diagrams, the three line segments that directly connect the pairs in $\{v_1, v_2, v_3\}$, rather than the images of the three star lines of the original graph. For example, the graph of Fig. 5c gives a Landau diagram of the form shown in Fig. 12.

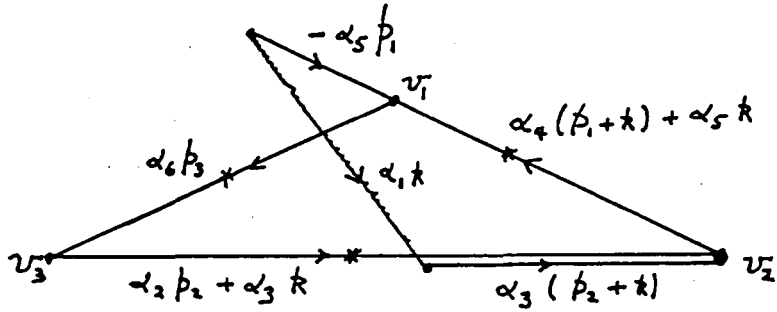

Figure 12. The Landau diagram corresponding to the graph (c) of Fig. 5. This diagram represents the equations obtained from Fig. 6c, with f_1 multiplied by r^2, f_3 and f_4 multiplied by r, and $r\Omega$ replaced by k. These changes recover the k form of the equations. The backward orientation of the vector $\alpha_5 p_1$ arises from the negative sign of $\sigma_{5,1}$. However, this vector is oriented against the direction of the photon loop. Consequently all contributions to this photon-loop equation proportional to any p_s have the form $\alpha_j p_s$: the two reversals of the line segment $j = 5$ compensate for each other.

A second example is the function represented by the graph shown in Fig. 13.

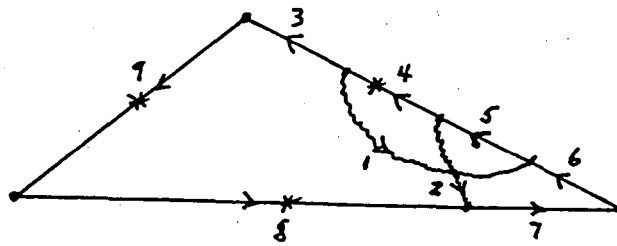

Figure 13. A graph representing a term in the pole-residue decomposition.

The functions f_j and the Landau matrix corresponding to the function represented by the graph in Fig. 13 are shown in Fig. 14, for $|k_1| > |k_2| > 0$

f_j	dk_1	dk_2	dp
$f_1 = k_1^2$	k_1	0	0
$f_2 = k_2^2$	0	k_2	0
$f_3 = 2p_1k_1 + k_1^2$	$p_1 + k_1$	0	k_1
$f_4 = (p_1 + k_1)^2 - m^2$	$p_1 + k_1$	0	$p_1 + k_1$
$f_5 = 2p_1k_2 + 2k_1k_2 + k_2^2$	k_2	$p_1 + k_1 + k_2$	k_2
$f_6 = 2p_1(k_1 - k_2) + k_1^2 - k_2^2$	$p_1 + k_1$	$-(p_1 + k_2)$	$k_1 - k_2$
$f_7 = 2p_2k_2 + k_2^2$	0	$p_2 + k_2$	k_2
$f_8 = p_2^2 - m^2$	0	0	p_2
$f_9 = p_3^2 - m^2$	0	0	p_3

Figure 14. The Landau matrix for the function represented by the graph in Fig. 13, for $|k_1| > |k_2| > 0$. The sign of σ_{j1} is minus for $j = 3$ and 6, and otherwise plus.

The Landau diagram corresponding to the Landau matrix in Fig. 14 is shown in Fig. 15

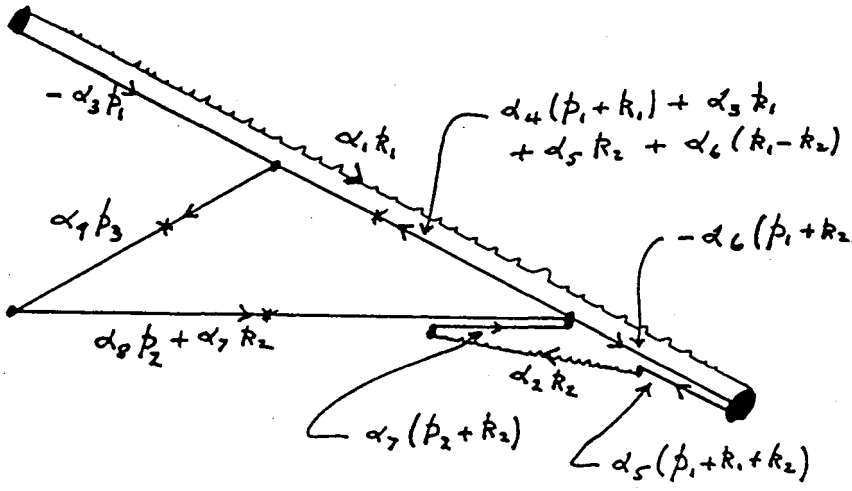

Figure 15. The 'Landau diagram' that represents the Landau equations associated with the Landau matrix shown in Fig. 14. This diagram is not a true Landau diagram, because, for example, the vector $\alpha_i k_i$ cannot be a light-cone vector. Moreover, condition (8.7) is not satisfied. Were it not for the non-negativity condition on α_3 one could satisfy the Landau equations with $\alpha_3 = -\alpha_4$, and $\alpha_1 = \alpha_2 = \alpha_5 = \alpha_6 = \alpha_7 = 0$.

The argument leading to (8.7) entails more than (8.7). It shows, in the present case where all $k_i \neq 0$, that each vertex of the diagram that does not lie in $\{v_1, v_2, v_3\}$ and that has at least one nonzero-length photon line segment incident upon it must have at least two nonzero-length photon lines incident upon it: each such vertex must lie on the right-hand end of at least one such photon line segment, and on the left-hand end of some other such photon line

segment. Consequently, every nonzero-length photon line must lie on a 'zig-zag' path of photon lines that begins at a vertex in the set $\{v_1, v_2, v_3\}$, moves always to the left, and ends on another vertex in $\{v_1, v_2, v_3\}$: only in this way can the conditions $K^2 \geq 0$ and $K \neq 0$ used in the derivation of (8.7) be overcome, if all k_i are different from zero.

Consider, then, an example with vertices labelled as in Fig. 16.

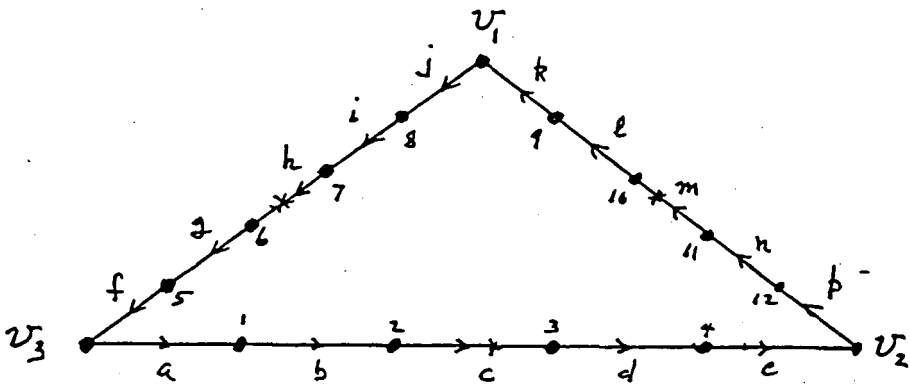

Figure 16. A triangle graph with photon vertices labelled by numbers, and charged–particle line segments labelled by letters. The segments h, c, and n are $*$ segments associated with the pole–residue decomposition (5.5). The photon lines have been suppressed.

Suppose $V_L = v_3$ and $V_R = v_1$ are the unique V_L and V_R. Then some sequence of photon lines of nonzero length must join together to give a zig–zag path from v_1 to v_3. Three examples are shown in Fig. 17.

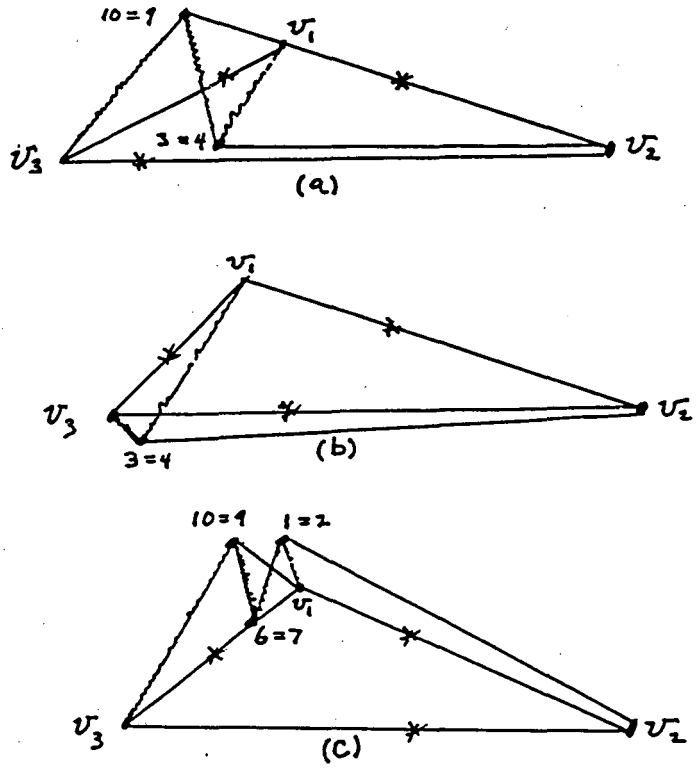

Figure 17. Three diagrams with zig–zag paths of photons connecting v_1 to v_3.

To analyse such diagrams we assume temporarily that for all pertinent solutions of the Landau equations

$$|\alpha_j| \leq |\alpha_s| B \text{ for } j \epsilon J(s), \tag{8.8}$$

where B is some fixed finite number. That is, we exclude temporarily the case where some α_j becomes unbounded, with the α_s bounded. Then as one lets the δ' in (8.2) tend to zero the vector V_s defined in (8.6f) and, for $j \epsilon J(s)$, the vectors V_j defined in (8.6b) all become increasingly parallel to p_s.

Consider then a sequence of bounds δ'_t, $t = 1, 2, ...$, that tend to zero, and a corresponding sequence of solutions S_t to the Landau equations in which: (1), $k_i \neq 0$, $i = 1, ..., n$; (2), $|k_i| \leq \delta'_t/n$, $i = 1, ..., n$; (3), some $\alpha_i k_i \neq 0$; and (4), condition (8.8) holds. If $q^t = (q_1^t, q_2^t, q_3^t)$ is the vector $q = (q_1, q_2, q_3)$ specified by S_t, then any accumulation point $\overline{q}$ of the set $\{q^t\}$ must be specified by a limiting diagram in which every charged-particle segment is parallel to one of the vectors $p_s, s = 1, 2, 3$, and in which some zig-zag path of light-cone vectors runs leftward from a vertex V_R of $\{v_1, v_2, v_3\}$ to a vertex V_L of $\{v_1, v_2, v_3\}$, but carries zero momentum-energy. The limit point $\overline{q}$ must therefore lie on the Landau triangle diagram singularity surface $\varphi(q) = 0$. However, the presence of the zig-zag photon line connecting two of the three vertices v_i imposes an extra condition, which define a codimension-one submanifold of $\varphi(q) = 0$. These submanifolds are finite in number (for any fixed diagram D'), and hence are nondense in the interior of $\varphi = 0$. If a point $q \epsilon \{\varphi = 0\}$ lies at a nonzero distance from each of these submanifolds then no solution of the kind specified above can occur, and hence for some sufficiently small neighborhood N of q, and for some sufficiently small δ', any solution to the Landau equations for $q \epsilon N$ satisfying $0 < |k_i| \leq \delta'/n$ for all i, and conditions (1) and (4), can have only zero-length photon lines: i.e., for all photon lines i

$$\alpha_i k_i = 0 \quad . \tag{8.9}$$

We are interested here in the singularity structure at a general point on $\varphi = 0$, rather than at special points where other singularity surfaces are relevant. Hence we may restrict our attention to a neighborhood N in $\varphi = 0$ where (8.9) holds.

Condition (8.9) says that every photon line segment i must have zero length. This condition entails the stronger result that every segment on every photon loop i in the Landau diagram must contract to a point.

To obtain this stronger result consider *in order* the loop equations corresponding to the sequence of variables $k_1, ..., k_n$, as defined in (6.1).

Consider first, then, the closed loop 1 in the Landau diagram. For each charged-particle segment on this loop the k_ℓ with smallest ℓ that flows along this loop 1 is k_1 itself. Consequently the orientations of all of the segments along this loop are unambiguously determined: for each $s \epsilon \{1, 2, 3\}$ every contribution

to the loop 1 that arises from a charged-particle segment on side s *adds* to the loop equation a vector that is very close to a non-negative multiple of p_s, just as in Figs. 12 and 15. Use can be made here of the facts[11,14,15] that the triple of four-vectors (v_1, v_2, v_3) specified by the three external vertices v_i constitute a normal to the Landau surface (in $q = (q_1, q_2, q_3)$ space) associated with the diagram, and that this surface can be tangent to the triangle diagram Landau surface $\varphi(q) = 0$ at a point q only if the directions of the three vectors V_s are the same as they are for the simple Landau diagram that corresponds to figure 4. Because we are staying away from exceptional points of lower dimension the three vectors V_s must be parallel to the three vectors p_s. Alternatively, one can use the condition (8.8), and take δ' sufficiently small, in order to deduce that V_s is approximately equal to $\alpha_s p_s$.

Each photon loop passes along at most *two* sides s of the triangle. Hence, on any single photon loop in the Landau diagram, each charged-particle segment points approximately in the direction of one or the other of at most *two* of the three vectors p_s. (See Figs. 12 and 15.) Hence the contraction to a point, demanded by (8.9), of the remaining segment of the loop, namely $\alpha_1 k_1$, forces every segment on loop 1 to contract to a point.

Consider next the loop 2. All segments along which k_1 runs have now been contracted out. Thus the k_ℓ with the smallest value of ℓ that flows along the surviving part of loop 2 is k_2 itself. Hence each segment on this loop also must contract to a point, by the same argument that was just used for loop 1. Proceeding step by step one finds that every segment on every photon loop must contract to a point.

In this nonseparable case with all $r_i \neq 0$ at least one photon line must pass along a star line. Hence at least one of the three star lines of the Landau diagram must also contract to a point. But then the other two sides of the triangle (v_1, v_2, v_3) must also contract to points, since, in accordance with the conditions imposed below Eq. (8.1), the three sides of the triangle connecting the three vertices v_i are nonparallel. But then every segment of the Landau diagram is forced to a point, and thus there is no solution of the Landau equations, in this nonseparable case with all $r_i \neq 0$.

This conclusion was derived under the assumption (8.8). However, that assumption is not necessary. Suppose we normalized the solutions by the re-

quiring that max $|v_i - v_j| = 1$, and drop (8.8). Then the direction of V_s is not constrained, but its Euclidean length is.

Consider, under these conditions, the sequence of loops i. A first part of loop 1 consists of either the zero, one, or two vectors V_s that are included on the loop. Their directions are indeterminate, but their magnitudes are at most unity. In fact the magnitude of the sum of these segments is at most unity.

A second part of this closed loop is the segment corresponding to the photon 1 itself. The length of this segment is limited by the fact that any nonzero-length photon line segment must lie on a zig-zag path that runs between two of the vertices v_i, and is composed of leftward pointing light-cone vectors. Since the Euclidean distance between the endpoints of this zig-zag path is bounded by unity, the individual segments along this path are likewise bounded. Thus these first two parts of loop 1 are bounded.

The third and final part of loop 1 is the sum of the contribution of the segments j associated with the pole-residue denominators f_j. All of these contributions to the loop are essentially of the form $\alpha_j p_s$, with all the α_j's positive, and s ranging over either *one or two* of its three possible values. (See Figs. 12 and 15). We can impose the condition that at the points $q \epsilon \{\varphi = 0\}$ under consideration the three vectors p_s are *far* from parallel. In this case the bound on the first two parts of the closed loop 1 imposes a comparable bound on the third part, and, in particular, a bound on the sum of the α_j corresponding to those segments j that lie on loop 1.

We then turn to loop 2. Bounds are established as before for all parts of loop 2 that are not pole-residue segments j, and also for all pole-residue segments j that lie on loop 1. Since the contributions from the pole-residue segments j that lie on loop 2 but not loop 1 have the form $\alpha_j p_s$, with $\alpha_j \geq 0$, and with s ranging over at most two of the three possible values, we can now establish upper bounds on the sum of these new α_j's. Proceeding in this way we establish bounds on all of the α_j's associated with all the pole-residue denominators f_j. Then for a sufficiently small δ' we can ensure that, for each value of s, the contribution to V_s, specified by (8.6f), that arises from the photon momenta k_i is small compared to this vector V_s itself. This is the result that in the earlier argument was obtained from (8.8), which we therefore no longer need.

8.5 Nonseparable Case; Some $r_i = 0$.

The results for the $k_i \neq 0$ case carry over to the general situation, provided the (r, Ω) variables are retained.

The argument for the case where some $r_i = 0$ proceeds much as in the case of separable diagrams. Let r_g be the first vanishing member of the ordered sequence $r_1, r_2, ..., r_n$. Then the Landau matrix separates into two parts. The first consists of the $d\Omega_i$ columns for $i < g$, plus the dp column; the second consists of the $d\Omega_i$ columns for $i \geq g$. By multiplying and dividing various rows and columns of the Landau matrix by appropriate nonzero factors r_i $(i < g)$ one can convert the $i < g$ part to the k form, with all k_j for $j \geq g$ set to zero. The $r_i \neq 0$ argument can then be applied to these $i < g$ Landau equations: they imply the vanishing of the α_j's corresponding to all segments j of the Landau diagram along which run the photon loops i with $i < g$.

The remaining columns, which give the $i \geq g$ part of the Landau equations, can be separated into *sectors*, where each sector begins with a column $d\Omega_i$ such that $r_i = 0$, and is followed by the set of columns $d\Omega_{i+1}, ..., d\Omega_{i+h}$ such that $r_{i+1}, ..., r_{i+h}$ are all nonzero. These latter r's can be changed to unity without altering the content of the Landau equations. We shall do this, purely for notational convenience.

The rows corresponding to the three pole denominators do not contribute to the $i \geq g$ equations because

$$\frac{1}{2}\frac{\partial}{\partial \Omega_j}[(p + r_1 ... r_j \Omega_j + ...)^2 - m^2] = 0 \text{ for } j \geq g,$$

due to $r_g = 0$.

One proceeds step-by-step, starting with the $i < g$ part, then considering the various individual sectors, in order of increasing values of i. The Landau equations for each one of the individual sectors can be expressed by a Landau diagram constructed in accordance with the rules (8.6), with, however, the following changes: (1), the three vectors V_s corresponding to the three direct line segments s are set to zero; (2), all the segments of the Landau diagrams that occur at earlier stages of the step-by-step process are contracted to points; and (3), the photon propagator contribution $\alpha_i k_i$ to each $d\Omega_i$ column that belongs

to the sector in question is replaced by $\alpha_i\Omega_i$.

The Landau diagram corresponding to a sector S has a 'spider' form: it consists of a single central vertex v, which represents the three coincident vertices v_i, plus a web of segments sprouting out from v. All segments of the Landau diagrams corresponding to the previously considered sectors are contracted to the single point v, together with all of the segments that constitute the part $i < g$. All charged-particle segments of the Landau diagram along which run *none* of the photon loops that constitute S are also contracted to points.

The Landau diagram that corresponds to any individual sector S can be shown to contract to a point by using the arguments developed earlier: the argument involving V_R and V_L shows that no photon line of nonzero length can occur in the spider diagram, and then the step-by-step consideration of the photon loops i, in the order of increasing i, shows that each of these loops must contract in turn to a point.

We thus conclude that, apart from the three α_s's corresponding to the three pole denominators, all of the α_j's must vanish. The three α_s's escape only in case the $i < g$ part is separable or empty.

The implication of this result is that for every point in the compact domain of integration in r_i space (away from the conventionally placed limiting surface $r_1 = \delta$) one can distort all of the Ω_i contours away from all of the singularities other than the three poles, and away also from these poles if their locations depend upon the Ω's.

8.6 Infrared finiteness in nonseparable diagrams.

The derivation in section 6 of the infrared finiteness of the integrations over the photon momenta was explicitly restricted to the case of separable diagrams. However, the argument applies essentially unchanged to the general case.

The restriction to separable diagrams fixed the directions that the photon loops flowed along the half-line h under consideration: each photon loop i incident upon h flowed *away* from the $*$ line that lies on one end of h. This entails that for any line segment j lying in h the associated function f_j contains a term $2p_s k_i$ if and only if the following condition is satisfied: exactly *one* end of the photon loop i that carries momentum k_i is incident upon the half-line h between the segment j and the $*$ segment that lies on one end of h.

This key property of f_j follows in general, however, directly from the formula

$$\begin{aligned} f_j &= \sigma_{js}(\Sigma_j^2 - \Sigma_s^2) \\ &= \sigma_{js}(\Sigma_j + \Sigma_s)(\Sigma_j - \Sigma_s) \\ &= \sigma_{js}(2p_s(K_j - K_s) + K_j^2 - K_s^2) \end{aligned}$$

where $\sum_j = p_s + K_j$, and $\sum_s = p_s + K_s$. The difference $K_j - K_s$ consists, apart from signs, of the sum of the k_i associated with the the photon loops i that are incident upon h precisely *once* in the interval between the $*$ segment and segment j. This entails the key property that was obtained in section 6 from the separability assumption, which is therefore not needed: the argument in section 6 covers also the nonseparable case.

8.7 QC Photons.

The arguments of the preceding subsections pertain to QQ photons, which are the photons having quantum, or Q-type, vertices on both ends. These photons propagate in accordance with the usual Feynman rule. This rule entails, as previously explained, that positive energy always flows from right to left along each QQ line segment of a Landau diagram. This fact played an important role in the preceding arguments.

In equation (2.36) of ref. 7 the propagation of QC photons is governed by the retarded propagator. This modification of the usual Feynman rule was a consequence of shifting the classical-part operator $U(L(x))$ to the right of the quantum-photon operator F. This shift breaks the original Feynman-rule QC contribution into two terms. The first arises from a (retarded) QC-line contribution to the c-number part of the operator $\tilde{F}^D_{opr}$; the second arises from the contraction of a creation operator (associated with a Q-type interaction) in $\tilde{F}^D_{opr}$ with an annihilation operator in $U(L(x))$. The sum of these two contributions brings the net contribution back to the form given in equation (2.25) of ref. 7, in which the Feynmam propagator of the QC photon is used in the evaluation of the c-number function $F^D(x)$. We shall use this original Feynman-propagator form. That allows us to include QC lines in the arguments used above to prove the possibility of distorting the Ω contours away from all Ω-dependent singularities.

9. Contributions of the Meromorphic Terms to the Singularity on the Triangle-Diagram Surface $\varphi = 0$.

In this section we describe the contributions to the singularity on the triangle-diagram singularity surface $\varphi = 0$ arising from the meromorphic parts of the three generalized propagators.

The arguments of sections 6, 7, and 8 show that in the typical pole-residue term (5.9) we can distort the contours in the Ω_j variables so as to keep the residue factors analytic, even in the limit when some or all of the r_j's become zero. In that argument we considered separately an individual half-line, but the argument is 'local': it carries over to the full set of six half-lines, with all the $|k_i|$ ordered. Thus for each fixed value of the set of variables $(r_i, ..., r_n;\ \Omega_1, ..., \Omega_n)$ the integration over the remaining variable of integration p gives essentially a triangle-diagram function: it gives a function with the same log φ-type singularity that arises from the simple Feynman triangle diagram itself, with, however, the location of this singularity in the space of the external variables (q_i, q_2, q_3) shifted by an amount (K_1, K_2, K_3), where the three vectors K_s are related to the photon momenta flowing along the three star lines of the original graph. Specifically, if we re-draw the photon loops so that they pass through *no* star line of the original graph (or equivalently through no star line of the Landau diagram), but pass, instead, *out* of the diagram at a vertex v_1, v_2 or v_3, if necessary, and then define the net momentum flowing out of vertex v_s to be

$$q_s = q_s(k) + K_s, \tag{9.1}$$

where K_s is the net momentum flowing out of vertex v_s along the newly directed photon loops, then, for fixed k, the function in (q_1, q_2, q_3) space will have a normal log φ triangle-diagram singularity along the surface $\varphi(q_1(k), q_2(k), q_3(k)) = 0$. For example, the original singular point at the point $\hat{q}$ in (q_1, q_2, q_3) space will be shifted to the point $(q_1, q_2, q_3) = (\hat{q}_1, \hat{q}_2, \hat{q}_3) + (K_1, K_2, K_3)$. This shift in the external variables q's shifts the momentum flowing along the three star lines to the values they would have if the photon moments k_i were all zero: it shifts the kinematics back to the one where no photons are present.

It is intuitively clear that the smearing of the location of this log φ singularity caused by the integration of the variables k_i will generally produce a

weakening of the log singularity at $\varphi(q) = 0$. For, in general, only the endpoint $r_1 = 0$ of the r_1 integration will contribute to the singularity at $\varphi(q) = 0$, and there is no divergence at $r_1 = 0$, by power counting, and hence no contribution from this set of measure zero in the domain of integration. The only exception arises from the set of separable diagrams. For these diagrams the K_s are all zero, and hence the integrations produce no smearing, and thus no weakening, of the log φ singularity.

To convert this qualitative argument to quantitative form we begin by separating the set of photon lines into two subsets that enter differently into the calculations. Let a *bridge* line in a diagram D' that corresponds to a term in the pole-residue decomposition (7.1) be a photon line j that 'bridges' over a star line: any closed loop in D' that contains the photon line segment j, and is completed by charged-particle segments that lie on the triangle D, passes along at least one star line. Let i be the smallest j such that photon line j is a bridge line. (Here we are using the ordering of the full set $(1, 2, \ldots, n)$ of photon labels that was specified in (6.1), not the ordering used in (7.1)). Thus each $k_j = \rho_j \Omega_j = r_1 ... r_j \Omega_j$ that appears in a star-line denominator, and hence in (9.1), contains a factor $\rho_i = r_1 ... r_i$. Let the set of variables $(k_1, ..., k_{i-1})$ be denoted by k_a, and let the set of variables $(k_i, ..., k_n)$ be denoted by k_b. And let r_a and r_b, and Ω_a and Ω_b be defined analogously. Then the function represented by D' can be written in the form

$$F(q) = \prod_{j=i}^{n} \int_{\Omega_j \widetilde{\Omega}_j = 1} d\Omega_j \int_0^1 r_j^{e_j} dr_j \; G(q, \Omega_b, r_b) \tag{9.2}$$

where

$$G(q, \Omega_b, r_b) = \int d^4p \prod_{j=1}^{i-1} \int_{\Omega_j \widetilde{\Omega}_j = 1} d\Omega_j \int_0^1 r_j^{e_j} dr_j$$

$$\prod_{s=1}^{3} \frac{1}{p_s(q, \Omega_b, r_b)^2 - m^2 = +i0} \; R(q, \Omega_b, r_b, \Omega_a, r_a). \tag{9.3}$$

Here R is the product of the three residue factors.

These intuitive conclusions are confirmed in Appendix G, where it is shown that in the nonseparable case the singularity on the surface $\varphi(q) = 0$ is contained in a finite sum of terms of the form $A_m \varphi \; (\log \varphi)^m$, where m is a positive integer that is no greater than the number of photons in the diagram, and A_m is analytic.

10. Operator Formalism.

We have dealt so far mainly with the meromorphic contributions. In order to treat the nonmeromorphic remainder it is convenient to decompose the operator $\widehat{C}_i$ into its "meromorphic and "nonmeromorphic" parts, $\widehat{C}_i^M$ and $\widehat{C}_i^N$.

The operator $\widehat{C}_i$ is defined in (2.5):

$$i\widehat{C}_i F(\tilde{p}) = \int_0^1 d\lambda_i \frac{\partial}{\partial p^{\mu_i}} F(p) \tag{10.1}$$

where

$$p = \tilde{p} + \lambda_i k_i. \tag{10.2}$$

Suppose

$$F(p) = A(p)B(p), \tag{10.3}$$

where $A(p)$ is analytic and $B(p)$ is $(p^2 - m^2)^{-1}$. An integration by parts gives

$$\begin{aligned} i\widehat{C}_i AB &= \int_0^1 d\lambda_i \Big[(\partial_{\mu_i} A)B + A(\partial_{\mu_i} B)\Big] \\ &= \int_0^1 d\lambda_i \Big[(\partial_{\mu_i} A) - (\partial A/\partial\lambda_i) \int^{\lambda_i} \partial_{\mu_i} \\ &\quad + A\left(\delta(\lambda_i - 1) - \delta(\lambda_i)\right) \int^{\lambda_i} \partial_{\mu_i}\Big] B, \end{aligned} \tag{10.4}$$

where the difference of delta functions, $(\delta(\lambda_i - 1) - \delta(\lambda_i))$ indicates that one is to take the difference of the integrand at the two end points.

The indefinite integral, computed by the methods used to compute (3.3), (3.6), and (3.7), is

$$\begin{aligned} \int^{\lambda_i} \partial_{\mu_i} B &\equiv \int d\lambda_i \frac{\partial}{\partial p^{\mu_i}} B \\ &= \frac{2p_{\mu_i}}{2pk_i} B - \frac{4(p_{\mu_i} k_i^2 - k_{i\mu_i} pk_i)}{d} \left[\int^{\lambda_i} B + \frac{1}{pk_i}\right]. \end{aligned} \tag{10.5}$$

Because the factor in front of the square bracket in (10.5) is independent of λ_i one can use a second integration by parts (in reverse) to obtain

$$
\begin{aligned}
i\widehat{C}_i AB \;=\; & \\
\int_0^1 d\lambda_i \Bigg[& (\partial_{\mu_i} A) - \left(\frac{\partial}{\partial\lambda_i} A\right) \frac{2p_{\mu_i}}{2pk_i} \\
& + A\left(\delta(\lambda_i - 1) - \delta(\lambda_i)\right) \frac{2p_{\mu_i}}{2pk_i} \\
& - \frac{4\left(p_{\mu_i} k_i^2 - k_{i\mu_i} pk_i\right)}{d} A \\
& + \frac{4(p_{\mu_i} k_i^2 - k_{i\mu_i} pk_i)}{d} \frac{k_i^2 (p^2 - m^2) A}{(pk_i)^2} \Bigg] B,
\end{aligned}
\tag{10.6}
$$

where the final term comes from the $1/pk_i$ term in the square bracket in (10.5) and has no singularity at $(p^2 - m^2) \equiv B^{-1} = 0$ for $pk_i \neq 0$.

Since all of the λ_i dependence in A is in $p = \bar{p} + \lambda_i k_i$ we may write

$$
\partial A / \partial \lambda_i = (\partial_{\mu_i} A) k^{\mu_i}. \tag{10.7}
$$

Hence the first two terms on the right side of (10.6) cancel, and one is left with

$$
\widehat{C}_i = \widehat{C}_i^M + \widehat{C}_i^N + \widehat{C}_i^R, \tag{10.8}
$$

where

$$
i\widehat{C}_i^M AB = \int_0^1 d\lambda_i \frac{2p_{\mu i}}{2pk_i} (\delta(\lambda_i - 1) - \delta(\lambda_i)) AB \tag{10.8a}
$$

$$
i\widehat{C}_i^N AB = -\frac{4(p_{\mu_i} k_i^2 - k_{\mu_i} pk_i)}{d} \int_0^1 d\lambda_i AB \tag{10.8b}
$$

$$
\begin{aligned}
i\widehat{C}_i^R AB \;=\; & \frac{4(p_{\mu_i} k_i^2 - k_{\mu i} pk_i)}{d} \int_0^1 d\lambda_i \Bigg[\left(\frac{\partial}{\partial\lambda_i} A\right) \\
& - A(\delta(\lambda_i - 1) - \delta(\lambda_i)) \Bigg] \frac{1}{pk_i} \\
= \; & \frac{4(p_{\mu_i} k_i^2 - k_{i\mu_i} pk_i)}{d} \int_0^1 d\lambda_i \frac{k_i^2 (p^2 - m^2)}{(pk_i)^2} AB.
\end{aligned}
\tag{10.8c}
$$

Notice that the contribution $\widehat{C}_i^R$ cancels the pole at $d = 0$ of the contribution $\widehat{C}_i^N$.

To efficiently manipulate these operators their commutation relations are needed. Recall from section 2 that the operators $\widehat{C}_i$ commute among themselves, as do the $\widehat{D}_i$:

$$[\widehat{C}_i, \widehat{C}_j] = 0 \tag{10.9a}$$

and

$$[\widehat{D}_i, \widehat{D}_j] = 0. \tag{10.9b}$$

The operators $\widehat{C}_i$ and $\widehat{D}_j$, properly interpreted, also commute:

$$[\widehat{C}_i, \widehat{D}_j] = 0. \tag{10.9c}$$

To verify (10.9c) note first that $\widehat{D}_j$ acts on generalized propagators (See (2.9)), and, by linearity, on linear superpositions of such propagators. However, Eq. (2.3) shows that the action on such an operand of the operator $(-\partial/\partial p_{\mu_i})$ in $\widehat{C}_i$ is the same as a $\widehat{D}_i$ with $k_i = 0$. Moreover, the replacement $p \to p + \lambda_i k_i$ commutes with $\widehat{D}_j$. Thus (10.9c) is confirmed, provided we stipulate that the integrations over the variables λ_i shall be reserved until the end, *after* the actions of all operators $\widehat{D}_i$ and differentiations. In fact, we see from (10.8) that the various partial operators $\widehat{C}_i^M$, $\widehat{C}_j^N$, and $\widehat{C}_k^R$ all commute: if we reserve the λ integrations until the end then each of the operations is implemented by multiplying the integrand by a corresponding factor, and those operations commute.

11. Nonmeromorphic Contributions

The D-coupling part of a Q-type coupling is meromorphic. Thus each of the $\widehat{C}$- and $\widehat{Q}$-type couplings can be expressed as by means of (10.8) as sum of of its meromorphic, nonmeromorphic, and residual parts. Then the full function can be expanded as a sum of terms in which each coupling is either $\widehat{C}$-type or $\widehat{Q}$-type, and is either meromorphic, nonmeromorphic, or residual. If any factor is residual then the term has no singularity at $(p^2-m^2)=0$, and is not pertinent to the question of the singularity structure on $\varphi=0$. Thus these residual terms can be ignored.

We have considered previously the terms in which every coupling is meromorphic. Here we examine the remainder. Thus terms not having least one nonmeromorphic coupling $\widehat{C}_i^N$ or $\widehat{Q}_i^N$ are not pertinent: they can also be ignored.

All couplings of the form $\widehat{Q}_i^M$ can be shifted to the right of all others, and this product of factors $\widehat{Q}_i^M$ can then be re-expressed in terms of the couplings $\widetilde{Q}_i^M$. That is, the terms corresponding to the different orderings of the insertions of the *meromorphic* couplings Q_i^M into the charged-particle lines can be recovered by using (2.9), (2.15), and (5.8). The various couplings $\widehat{C}_i^N$ are then represented, apart from the factor standing outside the integral in (10.8*b*), simply by an integration from zero to one on the associated variable λ_i.

In this paper we are interested in contributions such that every photon has a Q-type coupling on at least one end. In sections 6 and 7 the variables ρ_i's corresponding to photons i having a $\widetilde{Q}_i^M$-type coupling on (at least) one end were expressed in terms of the variables r_j, and it was shown that the contributions from all of the $\widetilde{Q}_i^M$-type couplings lead to an r_j dependence that is of order at least one in each r_j. The $\widehat{C}_i^M$-type couplings do not upset this result. Thus the general form of the expression that represents any term in the pole-residue expansion of the product of meromorphic couplings $\widetilde{Q}_i^M$ and $\widehat{C}_i^M$ is

$$\prod_j \int_{\Omega_j \widetilde{\Omega}_j=1} d\Omega_j \prod_i \int_0^1 r_i^{e_i} dr_i \; AB, \tag{11.1}$$

where the e_i are nonnegative integers, and A and B have the forms specified in section 10, provided the Ω contours are distorted in the way described in section 8. (For convenience, the scale has been defined so that the upper limit δ of the

integration over r_1 is unity.)

For these meromorphic couplings the integrations over the variables λ_i have been eliminated by the factors $\delta(\lambda_i - 1)$ and $\delta(\lambda_i)$. But for any coupling $\widehat{C}_i^N$ there will be, in addition to the integration from zero to one on the variable r_i, also an integration from zero to one on the variable λ_i: it comes from (10.8*b*).

The calculations in Appendices *I* and *J* show that in the nonmeromorphic case the singularities on the triangle diagram singularity surface $\varphi = 0$ are no stronger than $\varphi(\log\varphi)^{n+1}$, where n is the number of photons in the diagram. Even if the log factors from the diagrams of different order in e^2 should combine to give a factor like $\varphi^{-(1/137)}$ this, when combined with the form $\varphi(\log\varphi)^{n+1}$, would not produce a singularity as strong as the $\log\varphi$ singularity that arises from the separable diagrams.

12. Comparison to Other Works

Block and Nordsieck[17] recognized already in 1937 that the bulk of the very soft photon contribution to a scattering cross-section was correctly predicted by classical electromagnetic theory, that the process therefore involved arbitrarily large numbers of photons, and that this circumstance rendered perturbation theory inapplicable. They calculated the probability for the scattering of a charged particle by a local interaction V by taking the absolute value squared of the matrix element of V between charged-particle states "clothed" with a cloud of bremsstrahlung soft photons. This cloud has components with arbitrarily large numbers of photons.

The basic ideas of Block and Nordsieck were cast into a more modern form by Yennie, Frautschi, and Suura[1]. The sources of the dominant contributions to soft-photon processes were identified as the incoming and outgoing charged-particle legs of a Feynman diagram. Due to the softness of the photons incident upon these legs, the Feynman propagators corresponding to the various segments of these external legs all remain close to the mass shell, and this leads to apparent infrared divergences. However, by summing over all of the different orders in which the photons can couple into the external lines the net effects were converted into exponential forms, and the exponentials for the real and virtual photon processes were shown to nearly cancel, provided cross sections were calculated. This argument, though simple in principle, was nonrigorous, awkward, and arduous in practice.

A major conceptual advance was made by V. Chung,[3] who recognized that not only cross sections but also the scattering amplitudes (i.e., the S-matrix) could be made well defined if one used final states constructed not from the ordinary photon vacuum, but rather from the *coherent states* associated with the classical fields derived from classical electromagnetic theory. Chung's arguments rested, however, on the nonrigorous results of Yennie, Frautschi, and Suura.

Kibble[5] formulated the ideas of Chung in a more rigorous way, and examined the structure of the Green's functions of field theory, as contrasted to the structure of the S matrix. In field theories that deal only with massive particles one can extract the S-matrix from the Green's function by means of reduction formulas. These formulas exploit the pole character of the single-

particle propagator—and also the factorization property of the residue of that pole—to identify the S matrix as a factor in the residue of a multiple-pole in the Green's function. However, Kibble showed that in QED this method encountered difficulties, arising from the fact that the singularity corresponding to a charged-particle leg appeared to be not a pole, but rather a gauge dependent power of $(p^2 - m^2)^{-1}$. Moreover, in computing the "S-matrix" from the Green's function the result depended on the order in which the limits are taken to the mass shell of the propagators of the various incoming and outgoing charged particles. These peculiarities were further studied by Zwanziger.[6]

The present work began as an effort to resolve these problems, by making use of the fact that if the charged-particle propagator is defined in a way that is closely related to physical properties then this propagator must have a factorizable pole singularity. For this is the unique form of singularity that corresponds to the physically observed fact that a stable particle propagates, without dissipation, from source to sink (at least in simple idealized cases) along straight lines in spacetime, and carries from source to sink only the information carried in its momentum and spin state.[11]

To make the problem better defined mathematically and more closely related to physics the situation under consideration was changed from the one considered by Kibble. The first change was to consider a process in which charge was strictly conserved. The action of an operator that creates charge at a point x is not physical, because it upsets charge conservation. Even letting charges move in from infinity can create problems, because there are then contributions to currents from line integrals that extend to infinity, and the simple form of gauge invariance, $\partial^\mu J_\mu = 0$, fails to hold if ill-defined contributions from infinity are ignored. [See ref. 7 Eq. (1.7), and ref. 5(b) Eqs. (3.20) and (4.5)].

The second change was to invoke the essential idea of Block and Nordsieck — namely the incorporation of the classical aspect of the problem into the description *before* the application of the perturbation procedure. This incorporation is here done at the level of the *in* $\rightarrow$ *out* transition operator (rather that at the level of "in" and "out" states) and is done in coordinate space, which is where the connection between quantum and classical electrodynamics is most easily formulated. In the usual momentum space this connection is not so simple because for an n-photon system the Fourier transform leads to $3n$ dimensional

configuration space, rather than 3 dimensional physical space. The key next step was the recognition that the "classical part" of the transition operator can be generated by an appropriate "classical part" of the quantum-mechanical interaction term. The integration of this classical part of the interaction leads to a *unitary* transformation that incorporates already into the scattering operator the cancellation between the real and virtual process that was identified by Yennie, Frautschi, and Suura at the level of the transition probability. The unitarity transformation generates a coherent state that yields, automatically, the detailed combinatoric factors derived by Chung's lengthy calculations.

The formulas for the residual "quantum" part of the computation are similar to those obtained by Grammer and Yennie.[2] However, our residual part is what is left after the *full* classical part is removed, not merely the first approximation to it, obtained by shifting all soft bremsstrahlung photons to the origin.

It is this shift of the classically prescribed part from the "in" and "out" states to the transition operator itself, and the associated shift of the source points of the soft bremsstrahlung photons to the spacetime points where the deflections of the charged particles occur, that is the essential charge inaugurated by the work pursued here.[7] Without this shift the classical part does not accurately represent the dominant contribution in the large-distance limits.

After reference 7 was submitted several works related to ours have appeared. Morchio and Strocchi[18] have continued their work within the general arena of Wightman-type field theories, endeavoring to provide a foundation for a Haag-Ruelle scattering theory that would apply to QED. That their work is somehow related to ours is suggested by a conjecture they have made,[19] which brings in some formulas similar to some of ours. However, their approach has still some way to go before it could address the detailed questions about the triangle-diagram function addressed and answered in the present work.

d'Emilio and Mintchev[20] have initiated an approach that is more closely connected to the one pursued here. They have considered charged-field operators that are nonlocal in that each one has an extra phase factor that is generated by an infinite line integral along a ray that starts at the field point x. Their formula applied to the case of a product of three current operators located at the three vertices (x_1, x_2, x_3) of our closed triangular loop *could* be made to yield precisely the phase that appears in Eq. (1.7) of ref. 7. However, that would

involve making the direction of the ray associated with each field operator $\psi(x)$ depend upon the argument of the *other* field operator in the coordinate-space Green's function $\langle T\psi(x)\bar{\psi}(x')\rangle_0$ in which it appears.

d'Emilio and Mintchev do not follow that tack. Instead, they keep the direction of the ray associated with each field operator $\psi(x)$ fixed, *then* go to momentum space, and then find, for some simple cases (charged-particle propagation and vertex correction), that reasonable results are obtained only if the directions of the rays associated with the charged-particle operators that create or destroy the electrons are set equal to the momenta of the particles that are created or destroyed.

Of course, charged-particle propagators generally occur under integral signs, whereas the directions of the rays are treated as constants. If these "constant" directions are allowed to depend upon the momentum p then the inverse Fourier transform would, of course, not yield the original coordinate-space Feynman function.

The "intuitive" reason given why the one particular choice of the directions of the rays gives reasonable answers relies on the idea of "the classical currents responsible for the emission of soft photons". But classical-current arguments ought to be formulated in coordinate space.

Such a formulation (i.e., a coordinate-space formulation) would suggest letting the direction of the ray that occurs in the d'Emilio-Mintchev formula be the direction of the line between the two arguments x and x' of the coordinate-space charged-particle propagator. Then, due to a partial cancellation, the two infinite line integrals would collapse to a single finite line integral running between the two points x and x'. Then, in the case of our triangular closed loop, the phase factors associated with the lines on the three sides of the triangle would combine to give just the phase factors appearing in (1.7) of reference 7.

This coordinate-space procedure, which would seem to be the physically reasonable way to proceed, would bring the d'Emilio-Mintchev formulation to the first stage of the work pursued here and in reference 7. Whether their alternative momentum-space formulation would give results comparable to those obtained here is a question that remains as yet unanswered. Also, it is not clear how their formulas are connected to the standard Feynman rules: Are

their formulas derivable from the standard rules by a change of the order of summation (as ours are), or will their theory, when spelled out in detail, be basically inequivalent to the standard QED as defined by the Feynman rules?

Many of the problems dealt with in the present work, such as those that arise from the singular character of the singularity surface $k^2 = 0$ of the photon propagator at $k = 0$, and the subtle problems that arise when many photon momenta simultaneously tend to zero at different rates, should not be changed much by a shift from our basically coordinate-space approach to other approaches. These are basic technical problems that will inevitably arise in practical calculations, and they have to be dealt with adequately.

The introduction of the line integral of A_μ along the path of the charged particle seems to play a key role in the adequate treatment of the long distance, classical, and infrared properties of quantum electrodynamics. Thus one would expect that some similar procedure would be necessary in the case of other gauge theories of massless particles. However, we have made no effort to extend the methods developed here to those other interesting cases.

Appendix A. The $(\log \varphi)^2$ singularity.

It was shown in reference 17 that the function corresponding to the diagram of Fig. 2, but with ordinary Feynman vertices γ_ν for the couplings, is well defined (for photon mass equal to zero) but has a leading singularity of the form $(\log \varphi)^2$ on the Landau triangle diagram surface $\varphi = 0$. This $(\log \varphi)^2$ form "explains", in a certain sense, the ill-defined character of the vertex function associated with the small triangle diagram in Fig. 5d: if that vertex function did not diverge then the contribution to the singularity of $\varphi = 0$ from this diagram would necessarily be of the form $(\log \varphi)$, which it is not. Thus the divergence of this vertex function is a signal that the singularity on $\varphi = 0$ is stronger than $\log \varphi$.

This $(\log \varphi)^2$ singularity, if not mitigated in some way, would produce a violation of the geometric fall-off properties required by the classical concept of stable particles. However, it has been known since the time of Block and Nordsieck that the logarithm divergence (in k_{min}, or photon-mass cut off) associated with the small triangle in Fig. 5d *is* in fact mitigated by the effects of the radiated photons, and that one can obtain the finite physically correct results by means of perturbative corrections to a zeroth-order non-perturbative result that incorporates with sufficient precision the classically describable aspects of the electromagnetic field. The program initiated in reference 7 was precisely to treat these classically describable aspects with sufficient precision to allow physically and mathematically correct results to be obtained in the macroscopic domain. By a judicious splitting of the photon field into classical and quantum parts, the full effect of all of the classical photons was combined into a well-defined unitary operator U, leaving over a residual quantum-photon operator F. The compensating effects of the real and virtual photons are incorporated into the unitary operator U. In the present work it is shown that the residual operator F has, in the c-number sector, on the triangle diagram surface $\varphi = 0$, the normal $\log \varphi$ character that generates, in conjunction with U, the physically mandated asymptotic fall-off properties in coordinate space that are associated with the process that is represented by this diagram.

Appendix B. Proof of the triviality of the contribution from the factor $\delta(\Omega_j\tilde{\Omega}_j - 1)$ to the Landau loop equations.

In discussing the singularities of the meromorphic parts in §8 we made full use of the fact that the row in the Landau matrix corresponding to $\Omega_j\tilde{\Omega}_j - 1$ reduces to zero under the closed loop conditions for Ω_j-column, the r_j-column and the r_{j+1}-column. We give here a proof of this fact.

In view of the definition of the integral, the functions f_i other than the various $\Omega_j^2, \Omega_j\tilde{\Omega}_j - 1$ and r_j have the following form (B.1) or (B.2), where ϵ_m, and ϵ'_t are each either 0 or +1 or −1:

$$f_i = (p_\ell + \sum \epsilon_m r_1 ... r_m \Omega_m)^2 - m^2 \tag{B.1}$$

$$\begin{aligned} f_i &= 2(p_\ell + \sum \epsilon_m r_1 ... r_m \Omega_m)(\Omega_s + \sum \epsilon'_t r_{s+1} ... r_t \Omega_t) \\ &\quad + r_1 ... r_s (\Omega_s + \sum \epsilon'_t r_{s+1} ... r_t \Omega_t)^2. \end{aligned} \tag{B.2}$$

Let H_j denote the first-order differential operator given by $\Omega_j \frac{\partial}{\partial \Omega_j} - r_j \frac{\partial}{\partial r_j} + r_{j+1} \frac{\partial}{\partial r_{j+1}}$. Then the following equations hold:

$$H_j(p_\ell + \sum \epsilon_m r_1 ... r_m \Omega_m) = 0 \quad \text{for any } j \text{ and } \ell. \tag{B.3}$$

$$\begin{aligned} &H_j(\Omega_s + \sum \epsilon'_t r_{s+1} ... r_t \Omega_t) \\ &= \begin{cases} \Omega_s + \sum \epsilon'_t r_{s+1} ... r_t \Omega_t & \text{if } j = s \\ 0 & \text{if } j \neq s \end{cases}, \end{aligned} \tag{B.4}$$

$$\begin{aligned} &H_j(r_1 ... r_s (\Omega_s + \sum \epsilon'_t r_{s+1} ... r_t \Omega_t)^2) \\ &= \begin{cases} r_1 ... r_s (\Omega_s + \sum \epsilon'_t r_{s+1} ... r_t \Omega_t)^2 & \text{if } j = s \\ 0 & \text{if } j \neq s \end{cases} \end{aligned} \tag{B.5}$$

Hence H_j annihilates each f_i of the form (B.1) and each f_i of the form (B.2) with $s \neq j$, and it reproduces each f_i of the form (B.2) with $s = j$.

Since the Ω_j-column etc. in the Landau matrix is given by $\partial f_i / \partial \Omega_j$ etc., this property of the operator H_j entails, under the Ω_j, r_j, and r_{j+1} closed-loop conditions, that

$$0 = \Omega_j (\sum_i \alpha_i \frac{\partial f_i}{\partial \Omega_j}) - r_j (\sum_i \alpha_i \frac{\partial f_i}{\partial r_j}) + r_{j+1} (\sum_i \alpha_i \frac{\partial f_i}{\partial r_{j+1}})$$

$$= \sum_i \alpha_i H_j f_i$$

$$= \sum_{i \in I(j)} \alpha_i f_i + 2a_j \Omega_j^2 + 2\beta_j \Omega_j \widetilde{\Omega}_j - \gamma_j r_j + \gamma_{j+1} r_{j+1},$$

where $I(j)$ denotes the set of indices i such that f_i is of the form (B.2) with $s = j$, and a_j, β_j and γ_j denote the Landau parameters associated with Ω_j^2, $\Omega_j \widetilde{\Omega}_j - 1$, and r_j, respectively. It follows from (8.4a) that all terms except for $\beta_j \Omega_j \widetilde{\Omega}_j = \beta_j$ on the right-hand side of (B.6) vanish. This entails the required fact, namely that the row corresponding to $\Omega_j \widetilde{\Omega}_j - 1$ must have coefficient $\beta_j = 0$ and hence give no net contribution to the Landau loop equations.

Appendix C. The Landau diagram corresponding to a term in the pole-residue decomposition.

To confirm the geometric representation of the Landau equations described in connection with Eq.(8.6) recall first that the pole-residue denominators corresponding to non $*$ charged lines are

$$f_j = \sigma_{js}(\Sigma_j^2 - \Sigma_s^2) + i0, \tag{C.1}$$

where the sign σ_{js} is defined below (8.3). For each side $s\epsilon\{1,2,3\}$ one may verify immediately that the contribution from the side s of the triangle of *direct* lines V_s is just the contribution to the p loop equation arising from the charged-particle line segments that lie on side s of the original graph.

For the photon loop ℓ there is first a contribution $\alpha_\ell k_\ell$, and then the contributions corresponding to charge-particle line segments along which the loop flows. There are contributions of this latter kind only from segments corresponding to those (one or two) sides s of the triangle along which the loop runs, and we can consider separately the contributions from each of those sides s.

There are three cases:

Case 1. The photon loop ℓ in the Feynman graph runs along the segment $j\epsilon J(s)$ but does *not* run along the $*$ segment lying on side s. In this case the contribution to the ℓ loop equation proportional to α_j is

$$\begin{aligned} \alpha_j \frac{1}{2}\frac{\partial f_j}{\partial k_\ell} &= \sigma_{js}\alpha_j \frac{1}{2}\frac{\partial}{\partial k_\ell}(\Sigma_j^2 - \Sigma_s^2) \\ &= \beta_{js}\Sigma_j. \end{aligned} \tag{C.2}$$

Case (2a). The loop ℓ of the Feynman graph flows along the $*$ segment of side s, but does *not* flow along the non $*$ segment j lying on side s. Then the contribution to the ℓ loop equation proportional to α_j is

$$\begin{aligned} \alpha_j \frac{1}{2}\frac{\partial f_j}{\partial k_\ell} &= \sigma_{js}\alpha_j \frac{1}{2}\frac{\partial}{\partial k_\ell}(\Sigma_j^2 - \Sigma_s^2) \\ &= \beta_{js}(-\Sigma_s). \end{aligned} \tag{C.3}$$

Case (2b). The loop ℓ of the Feynman graph flows along the $*$ segment of side s of the graph and also along the non $*$ segment $j\epsilon J(s)$. Then the

contribution to the ℓ loop equation proportional to α_j is

$$\begin{aligned}\alpha_j \frac{1}{2}\frac{\partial f_j}{\partial k_\ell} &= \beta_{js}\alpha_j \frac{1}{2}\frac{\partial}{\partial k_\ell}(\Sigma_j^2 - \Sigma_s^2) \\ &= \beta_{js}(\Sigma_j - \Sigma_s). \qquad (C.4)\end{aligned}$$

Notice that, according to (C.2), (C.3), and (C.4),there is, for each $j \epsilon J(s)$, a contribution $\beta_{js}\Sigma_j$ to the photon loop equation ℓ if and only if the loop ℓ in the graph passes along the segment j. There is also, for each $j \epsilon J(s)$, a contribution $-\beta_{js}\Sigma_s$ if and only if this loop passes along the star line s in the graph. There is also a contribution $\alpha_s\Sigma_s$ if and only if this loop passes along the star line s of the graph. These results are summarized by the rules (8.6).

Appendix D. The contribution from a one-photon nonseparable meromorphic part.

As is shown in §8 for the general case, we can distort the Ω-contour so that $Im\Omega^2 > 0$ at $\Omega^2 = 0$, and $Imp_j\Omega > 0$ $(j = 1,2)$ at $p_j\Omega = 0$. Then each denominator of the integrand of F_b of Fig. 3 except for three pole-factors, i.e., $p^2 - m^2, (p_1 + r\Omega)^2 - m^2$ and $p_2^2 - m^2$, is different from 0. The r-integration $\int_0^\delta rdr/[(p_1 + r\Omega)^2 - m^2]$ can be explicitly performed, and when $p_1\Omega \neq 0$ its dominant singularity along $p_1^2 = m^2$ is

$$-\frac{(p_1^2 - m^2)\log(p_1^2 - m^2)}{4(p_1\Omega)^2}$$

Combining this singularity, instead of the ordinary pole $1/(p_1^2 - m^2)$, with the other two poles, i.e., $1/(p^2 - m^2)$ and $1/(p_2^2 - m^2)$, we perform the p-integration and find a singularity $A(q,\Omega)\varphi(q)^2 \log\varphi(q)$, with A being analytic. Performing the Ω-integration along the compact distorted contour, the dominant singularity of F_b is $\varphi^2 \log\varphi$.

Appendix E. The contribution from a pair of non-meromorphic parts arising from one photon.

The contribution of I in (4.6) to the amplitude is

$$F = \int_{|\Omega|=1} \frac{d^4\Omega}{\Omega^2 + i0} \int_0^\delta \frac{dr}{r} \int d^4p_3 \frac{1}{p_3^2 - m^2 + i0} \log \frac{(q_1 + p_3 + r\Omega)^2 - m^2 + i0}{(q_1 + p_3)^2 - m^2 + i0}$$
$$\times \log \frac{(p_3 - q_3 - r\Omega)^2 - m^2 + i0}{(p_3 - q_3)^2 - m^2 + i0} \qquad (E.1)$$

(in the notations given in Fig. 4 in §8.1.) Here the Ω-contour is deformed so that $Im\Omega^2 > 0$ and $Imp_j\Omega > 0$ $(j = 1,2)$ (*cf.* §8). Performing the p_3-integration we find

$$\int_{|\Omega|=1} \frac{d^4\Omega}{\Omega^2 + i0} \int_0^\delta \frac{dr}{r} \left(G(q_1 + r\Omega, q_3 + r\Omega)\right.$$
$$\left. - G(q_1 + r\Omega, q_3) - G(q_1, q_3 + r\Omega) + G(q_1, q_2)\right). \quad (E.2)$$

where

$$G(q_1, q_2) = \varphi(q_1, q_2)^2 \log(\varphi(q_1, q_2) + i0).$$

Since

$$(\partial/\partial r)\varphi(q_1+r\Omega, q_3+r\Omega) = (\partial\varphi/\partial q_1+\partial\varphi/\partial q_3)\cdot\Omega = (\alpha_1\rho_1+\alpha_2\rho_2)\cdot\Omega = -\alpha_3 p_s\cdot\Omega \neq 0$$

holds by the Landau equation, we can find non-vanishing functions $a(q_1, q_3, r, \Omega)$ and $b(q_1, q_3)$ for which

$$\varphi(q_1 + r\Omega, q_3 + r\Omega) = a(q_1, q_3, r, \Omega)(r - b(q_1, q_3)\varphi(q_1 + r\Omega, q_3))$$

holds (Cf. App. H). Similar decompositions hold also for $\varphi(q_1 + r\Omega, q_2)$ and $\varphi(q_1, q_2 + r\Omega)$. Hence application of the results in Appendix I to $\int_0^\delta \frac{dr}{r}(G(q_1 + r\Omega, q_3 + r\Omega) - G(q_1, q_3))$ etc. entails that the r-integration in (E.2) produces a singularity of the form $\varphi(q_1, q_3)^2(\log(\varphi(q_1, q_3) + i0)^2$ near $\varphi = 0$. Since the Ω-integration (along a suitably detoured path) is over the compact set, F itself behaves as $\varphi^2(\log(\varphi + i0))^2$.

Appendix F. The contribution from a coupling of a non-meromorphic part with either a meromorphic part or a C-part.

If a meromorphic part is coupled with a non-meromorphic part, the RHS of (E.1) is replaced by an integral of the following form:

$$\int_{|\Omega|=1} \frac{d^4\Omega}{\Omega^2 + i0} \int_0^\delta dr \int d^4p_3 \frac{1}{p_3^2 - m^2 + i0} \log \frac{(q_1 + p_3 + r\Omega)^2 - m^2 + i0}{(q_1 + p_3)^2 - m^2 + i0}$$
$$\frac{1}{(p_3 - q_3)^2 - m^2 + i0} \; \frac{1}{2(p_3 - q_3)\Omega + r\Omega^2 + i0}. \qquad (F.1)$$

By deforming the Ω-contour as in §8 so that $Im\Omega^2 > 0$ and $Imp_j\Omega > 0 (j = 1, 2)$ [with $p_1 = q_1 + p_3, p_2 = p_3 - q_3$], we find the singularity of this integral near $\varphi(q_1, q_3) = 0$ is $\varphi \log(\varphi + i0)$, as there is no potentially divergent factor $1/r$.

If the meromorphic part is replaced by a C-term, then the dominant singularity is given by an integral similar to (F.1) but with the replacement of the residue factor $1/(2(p_3 - q_3)\Omega + r\Omega^2)$ by $1/r(p_3 - q_3)\Omega$. Hence a potentially divergent factor $1/r$ arises. But this problem is circumvented by combining the singularity originating from $\log((q_1 + p_3 + r\Omega)^2 - m^2 + i0)$ and that from $\log((q_1 + p_3)^2 - m^2 + i0)$; the results in Appendix I show, with a reasoning similar to (but simpler than) that in Appendix E, that the resulting singularity is $\varphi(\log(\varphi + i0))^2$.

Appendix G. Weakness of the singularity in the general non-separable meromorphic case.

To confirm the weakness of the singularity in the non-separable meromorphic case we first need to verify

$$\frac{\partial}{\partial \rho_i}\varphi(q-\Delta)|_{\varphi=0} \neq 0, \tag{G.1}$$

where $\rho_i = r_1 \cdots r_i$, with i being the index labelling the first *bridge* line; i.e., i is the smallest j such that the photon line j has a meromorphic coupling on both ends, and completes to a closed loop — constructed according to the rules specified below Eq.(8.2) — that flows along at least one $*$-segment. The k_i-dependent vector Δ is chosen so that at $\varphi = 0$ the pole factor associated with each $*$-segment can be evaluated at the critical point $p_s(q-\Delta)$ ($s = 1,2,3$), defined below Fig. 4, with $q = (q_1, q_2, q_3)$ the set of external variables defined there.

The vector Δ is constructed in the following way. Introduce for each bridge line i an open flow line $L(k_i)$ that passes along this photon line i, but along no other photon line, and along no $*$-segment. Instead, the flow line $L(k_i)$ *enters* the diagram at one of the three vertices v_i and *leaves* at another. Specifically, let e be an end-point of the photon line i, and let s be the side of the triangle on which e lies. This point e separates s into two connected components, s^0 and s^*, where s^* is the part of s that contains the $*$-segment. Run $L(k_i)$ along the component s^0. At the end-point of s^0 that coincides with a vertex v_i of the triangle diagram, run $L(k_i)$ out along the external line q_j ($j = 1,2$ or 3). Do the same for the other end-point of the line i. Include on $L(k_i)$ also the segment i itself. This produces a continuous flow line. Orient it so that it agrees with the orientation of the line i. This oriented line is the flow line $L(k_i)$. Then for each external line q_j along which $L(k_i)$ runs add to the vector q_j either $+k_i$ or $-k_i$ according to whether the orientation of $L(k_i)$ is the same as, or opposite to, the orientation of the external line q_j along which $L(k_i)$ runs. Sum up the contributions from all of the bridge lines. This shift in $q = (q_1, q_2, q_3)$ is the vector Δ.

The function of interest has the form

$$F(q) = \prod_{j=1}^{n} \int_{\Omega_j \widetilde{\Omega}_j = 1} d\Omega_j \prod_{j=i+1}^{n} \int_0^1 r_j^{e_j} dr_j \prod_{j=1}^{i} \int_0^1 r_j^{e_j} dr_j$$
$$\times A(q, \Omega, r) \log \varphi(q - \Delta), \tag{G.2}$$

where

$$D \log \varphi(q - \Delta) + E = \int_{p \approx \widehat{p}} d^4 p \prod_{s=1}^{3} \frac{1}{p_s^2 - m^2 + i0}, \tag{G.3}$$

and A, D, and E are holomorphic.

Here

$$p_1 = p + q_1 + \sum_{m=i}^{n} \epsilon_{1m} k_m,$$
$$p_2 = p - q_3 + \sum_{m=i}^{n} \epsilon_{2m} k_m, \tag{G.4}$$

and

$$p_3 = p + \sum_{m=i}^{n} \epsilon_{3m} k_m,$$

with each ϵ_{sm} either zero or one.

We are interested in the singularity of this function at the point $\widehat{q}$ on $\varphi(q) = 0$. This singularity comes from the p-space point $\widehat{p} = p(\widehat{q})$, and we can consider the p-space domain of integration to be some small neighborhood of $\widehat{p}$. Similarily, the domain in (r, Ω) is confined to a region R in which the following conditions hold:

$$(\widehat{p} + \widehat{q}_1) \cdot \sum_{m=1}^{n} \epsilon_{1m} k_m / \rho_i \approx i\epsilon_1$$
$$(\widehat{p} - \widehat{q}_3) \cdot \sum_{m=1}^{n} \epsilon_{2m} k_m / \rho_i \approx i\epsilon_2$$
$$\widehat{p} \cdot \sum_{m=1}^{n} \epsilon_{3m} k_m / \rho_i \approx i\epsilon_3. \tag{G.5}$$

That is, the real parts of the three denominators in $(G.3)$ are close to zero, and the imaginary parts are positive: $\epsilon_s \geq 0 (s = 1, 2, 3); \sum \epsilon_s > 0$. It was shown in section 8 that the contours can be distorted in a way such that (G.5) holds in a neighborhood of the points contributing to the singularity at $\widehat{q}$.

Note that all of the k_m that contribute to $(G.5)$ belong to bridge lines, and hence have a factor ρ_i. Thus none of the $r_j (j \leq i)$ enter into $(G.5)$. Hence the region R is independent of the variables $r_j (j \leq i)$.

The quantity $\Delta = (\Delta_1, \Delta_2, \Delta_3)$ is added to $q = (q_1, q_2, q_3)$, and it satisfies, in analogy to $\sum q_i = 0$, the condition $\sum \Delta_i = 0$. This trivector Δ is a sum of terms, one for each bridge line. For each bridge line j the corresponding term in Δ is proportional to k_j. If line j bridges over (only) the star line on side $s = 1$ then the contribution to Δ is $(-k_j, k_j, 0)$. If line j bridges over (only) the star line on side $s = 2$ then the contribution to Δ is $(0, -k_j, k_j)$. If the line j bridge (only) over the star line on side $s = 3$ then the contribution to Δ is $(k_j, 0, -k_j)$.

The gradient of $\varphi(q)$ is also a trivector. The condition $\sum q_i = 0$ in q space means that the gradient (which is in the dual space) is defined modulo translations: $\Delta_i \to \Delta_i + X$, all i. Thus one can take $\nabla\varphi$ to have a null second component. Then at the point $\hat{q}$ of interest the gradient has the form[15]

$$\nabla\varphi = (\alpha_1 \hat{p}_1, 0, -\alpha_2 \hat{p}_2), \tag{G.6}$$

provided the sign and normalization of φ are appropriately defined. Hence the quantity on the left-hand side of (G.1) is, at $q - \Delta = \hat{q}$,

$$\begin{aligned} \frac{\partial\varphi(q-\Delta)}{\partial\rho_i} &= -\nabla\varphi \cdot \frac{\partial\Delta}{\partial\rho_i} \\ &= \sum_{s=1}^{3} \sum_{m=i}^{n} \alpha_s \hat{p}_s \epsilon_{sm} k_m / \rho_i \end{aligned} \tag{G.7}$$

which, according to (G.5), is nonzero, as claimed in (G.1). Use has been made here of the Landau equation $\sum \alpha_s \hat{p}_s = 0$.

Using (G.1) we now employ the result in Appendix H to normalize the defining function φ of the Landau surface so that we may apply Proposition I.3 of Appendix I to the integral F in question. It follows from Lemma H.1 in Appendix H that the following normalization holds on a neighborhood of the point in question:

$$\varphi(q - \Delta) = B(q, \rho_i, k'/\rho_i)\ (\rho_i - \varphi(q)/C(q, k'/\rho_i)), \tag{G.8}$$

where B and C are different from 0 at any point in question, and k' denotes the totality of the bridge lines k_j. Note that each bridge k_j contains a factor ρ_i and

that k'/ρ_i is independent of ρ_i. Let us now apply Proposition I.3 in Appendix I to the following integral I:

$$I = \int_0^\delta r_1^{e_1} dr_1 \int_0^1 r_2^{e_2} dr_2 \cdots \int_0^1 r_i^{e_1} dr_i \log(\rho_i - \varphi/C). \qquad (G.9)$$

Then we find [modulo a function analytic at $\varphi = 0$]

$$I = E(q, k'/\rho_i)\ (\varphi(q)/C(q, l'/\rho_i))^N \left(\sum_{j=0}^{i} a_j(q, k'/\rho_i) \right.$$
$$\left. \times (\log(\varphi(q)/C(q, k'/\rho_i)))^j \right) \quad (G.10)$$

with $N \geq 1$, and E and a_j being holomorphic in their arguments, and, in particular, in the r_j's ($j > i$).

The function A in (G.2) is holomorphic. This factor has no important effect on the result: it can be incorporated by using Remark I.3(ii) of Appendix I.

Appendix H. A normalization of the function defining a Landau surface.

The purpose of this section is to prove the following lemma, which is an adaptation of the implicit function theorem (or the Weierstrass preparation theorem in the theory of holomorphic functions of several variables) to the Landau surface shifted by a vector Δ determined by photons that bridge star lines. (cf. §11.).

Lemma H.1. Let $\varphi(q)$ denote a defining function of the Landau surface for the triangle diagram and let $\hat{q}$ be a point on the surface. Let i be the smallest j such that j identifies a bridge photon line. (A *bridge* photon line is a photon line that has meromorphic couplings on both ends and that completes — via the rules defined below (8.2) — to a closed photon loop that passes along at least one star line.) Then on a sufficiently small neighborhood of q_0 and for sufficiently small $\rho_i = r_1 \cdots r_i$ there exist non-vanishing holomorphic functions $B(q, \rho_i, k'/\rho_i)$ and $C(q, k'/\rho_i)$ such that

$$\varphi(q - \Delta) = B(q, \rho_i, k'/\rho_i)(\rho_i - \varphi(q)/C(q, k'/\rho_i)) \qquad (H.1)$$

holds, where k' denotes the collection of bridge lines.

Proof. Since i is the first bridge photon line, any bridge photon line k_ℓ has the form $k_\ell = \rho_i r_{i+1} \cdots r_\ell \Omega_\ell$. Hence k'/ρ_i is actually independent of ρ_i. It follows from (G.1) of Appendix G that $\partial\varphi(q-\Delta)/\partial\rho_i|_{\rho_i=0} \neq 0$ holds. Hence the Weierstrass preparation theorem guarantees the local and unique existence of a non-vanishing holomorphic function $B(q, \rho_i, k'/\rho_i)$, and a holomorphic function $R(q, k'/\rho_i)$, which vanishes for $q_i = q_0$, for which the following holds:

$$\varphi(q - \Delta) = B(q, \rho_i, k'/\rho_i)(\rho_i - R(q, k'/\rho_i)) \qquad (H.2)$$

Setting $\rho_i = 0$ in (H.2) we find

$$\varphi(q) = B(q, 0, k'/\rho_i)(-R(q, k'/\rho_i),$$

that is,

$$R(q, k'/\rho_i) = \varphi(q)/(-B(q, 0, k'/\rho_i)).$$

Hence by choosing $C(q, k'/\rho_i) = -B(q, 0, k'/\rho_i)$ we obtain (H.1).

Appendix I. Some auxiliary integrals.

The purpose of this appendix is to find an explicit form of the singularities of several integrals that we encounter in dealing with infrared problems. The simplest example is this sort is the following integral $I(t)$:

$$I(t) = \int_0^\delta [\log(r+t+i0) - \log(t+i0)]dr/r, \quad (\delta > 0).$$

In spite of the divergence factor $1/r$, $I(t)$ is well defined as a (hyper) function of t. In order to see this, it suffices to decompose $I(t)$ as

$$\int_0^{t/2} (\log(r+t) - \log t)dr/r + \int_{t/2}^\delta (\log(r+t) - \log t)dr/r$$

with $Im\ t > 0$: the well-definedness of the second integral is clear, while the fact that

$$\log(r+t) - \log t = \log(1 - \frac{r}{t}) \sim \frac{r}{t}$$

holds in the domain of integration of the first integral entails its well-definedness. Furthermore $I(t)$ (thus seen to be well-defined) satisfies the following ordinary differential equation:

$$t\frac{d}{dt}I(t) = \log(t+i0) - \log(t+\delta+i0). \tag{I.1}$$

Hence $(t\frac{d}{dt})^2$ is holomorphic near $t=0$. Then it follows from the general theory of ordinary differential equations that $I(t)$ has the form

$$C_2(\log(t+i0))^2 + C_1(\log(t+i0)) + h(t), \tag{I.2}$$

where C_1 and C_2 are constants and $h(t)$ is holomorphic near $t=0$. Furthermore, by substituting (I.2) into (I.1) and comparing the coefficients of singular terms at $t=0$, we find $C_2 = 1/2$.

This computation can be generalized as follows:

Proposition I.1. Let $J(\alpha, j; t)$ ($\alpha \neq 0, 1, 2, \cdots; j \geq 1$) denote the following integral:

$$\int_0^{\delta=\rho_0} \frac{d\rho_1}{\rho_1} \int_0^{\rho_1} \frac{d\rho_2}{\rho_2} \cdots \int_0^{\rho_{j-2}} \frac{d\rho_{j-1}}{\rho_{j-1}} \int_0^{\rho_{j-1}} (t+\rho_j+i0)^\alpha d\rho_j.$$

Then the singularity of $J(\alpha, j; t)$ near $t = 0$ is of the following form with some constants C_ℓ ($\ell = 0, \cdots, j-1$):

$$\begin{cases} (t+i0)^{\alpha+1}(\sum_{\ell=0}^{j-1} C_\ell(\log(t+i0))^\ell), & \text{if } \alpha \neq -1 \\ \sum_{\ell=0}^{j-1} C_\ell(\log(t+i0))^{\ell+1}, & \text{if } \alpha = -1 \end{cases}. \tag{I.3}$$

Remark I.1. If α is a non-negative integer, the integral $J(\alpha, j; t)$ is not singular at $t = 0$.

Proof of Proposition I. 1. The well-definedness can be verified by the same method as was used for the above example $I(t)$. To find its singularity structure, we again make use of an ordinary differential equation as follows:

$$(t\frac{d}{dt} - (\alpha+1))J(\alpha, j; t) = \left(t\frac{d}{dt} - (\alpha+1)\right)\int_0^\delta \frac{d\rho_1}{\rho_1}\int_0^{\rho_1}\frac{d\rho_2}{\rho_2}\cdots$$

$$\int_0^{\rho_{j-2}}\frac{d\rho_{j-1}}{\rho_{j-1}}((t+\rho_{j-1}+i0)^{\alpha+1} - (t+i0)^{\alpha+1})/(\alpha+1)$$

$$= \int_0^\delta \frac{d\rho_1}{\rho_1}\int_0^{\rho_1}\frac{d\rho_2}{\rho_2}\cdots\int_0^{\rho_{j-2}}\frac{d\rho_{j-1}}{\rho_{j-1}}(t(t+\rho_{j-1}+i0)^\alpha - (t+\rho_{j-1}+i0)^{\alpha+1})$$

$$= -\int_0^\delta \frac{d\rho_1}{\rho_1}\int_0^{\rho_1}\frac{d\rho_2}{\rho_2}\cdots\int_0^{\rho_{j-2}}(t+\rho_{j-1}+i0)^\alpha d\rho_{j-1} = -J(\alpha, j-1; t)(j \geq 2).$$

Repeating this computation, we finally obtain

$$(t\frac{d}{dt} - (\alpha+1))^{j-1}J(\alpha, j; t) = (-1)^{j-1}\int_0^\delta (t+\rho_1)^\alpha d\rho_1,$$

and hence we find $(t\frac{d}{dt} - (\alpha+1))^j J(\alpha, j; t)$ is holomorphic near $t = 0$. Again, by using the general theory of ordinary differential equations, we obtain the required formula (I.3).

Remark I.2. Although we do not need the exact values of C_ℓ's, we note that C_{j-1} in (I.3) is simply given by $(-1)^{j-2}/(j-1)!(\alpha+1)$ if $\alpha \neq -1$. In order to find this value it suffices to insert (I.3) into the recurrence relation $(td/dt - (\alpha+1))J(\alpha, j; t) = -J(\alpha, j-1; t)$ and use the trivial relation

$$\left(t\frac{d}{dt} - (\alpha+1)\right)J(\alpha, 2; t) = -\int_0^\delta (t+\rho_1+i0)^\alpha d\rho_1$$

$$= ((t+i0)^{\alpha+1}/(\alpha+1)) - ((t+\delta+i0)^{\alpha+1}/(\alpha+1))$$

as the starting point of the induction. Similarly C_{j-1} for $\alpha = -1$ is equal to $(-1)^j/j!$. The coefficient of the most singular term can be similarly computed explicitly for the integrals to be dealt with in subsequent propositions.

The following modification of Proposition I.1 is often effective in actual computations.

Proposition I.2. (i) Let $K(\alpha, j; t)$ $(\alpha \neq 0, 1, \cdots; j \geq 1)$ denote the following integral:

$$\int_0^\delta \frac{d\rho_1}{\rho_1} \int_0^{\rho_1} \frac{d\rho_2}{\rho_2} \cdots \int_0^{\rho_{j-2}} \frac{d\rho_{j-1}}{\rho_{j-1}} \int_0^{\rho_{j-1}} \rho_j (t + \rho_j + i0)^\alpha d\rho_j.$$

Then its singularity near $t = 0$ is of the following form for some constants $C_\ell (\ell = 0, \cdots, j-1)$:

$$\begin{cases} (t+i0)^{\alpha+2} \left(\sum_{\ell=0}^{j-1} C_\ell (\log(t+i0))^\ell \right) & \text{if } \alpha \neq -1, -2 \\ t^{\alpha+2} \left(\sum_{\ell=0}^{j-1} C_\ell (\log(t+i0))^{\ell+1} \right) & \text{if } \alpha = -1 \text{ or } -2 \end{cases}. \tag{I.4}$$

(ii) Let n be a non-negative integer and let $I(n, j; t)(j \geq 1)$ denote the following integral:

$$\int_0^\delta \frac{d\rho_1}{\rho_1} \int_0^{\rho_1} \frac{d\rho_2}{\rho_2} \cdots \int_0^{\rho_{j-2}} \frac{d\rho_{j-1}}{\rho_{j-1}} \int_0^{\rho_{j-1}} (t + \rho_j)^n \log(t + \rho_j + i0) d\rho_j.$$

Then its singularity near $t = 0$ is of the following form with some constants $C_\ell (\ell = 0, \cdots, j-1)$:

$$t^{n+1} \left(\sum_{\ell=0}^{j-1} C_\ell (\log(t+i0))^{\ell+1} \right). \tag{I.5}$$

(iii) Let $\widetilde{I}(n, j; t)(n$; a non-negative integer, and $j \geq 1)$ denote the following integral:

$$\int_0^\delta \frac{d\rho_1}{\rho_1} \int_0^{\rho_1} \frac{d\rho_2}{\rho_2} \cdots \int_0^{\rho_{j-2}} \frac{d\rho_{j-1}}{\rho_{j-1}} \int_0^{\rho_{j-1}} \rho_j (t + \rho_j)^n \log(t + \rho_j + i0) d\rho_j.$$

Then its singularity near $t = 0$ is of the following form with some constants $C_\ell (\ell = 0, \cdots, j-1)$:

$$t^{n+2} \left(\sum_{\ell=0}^{j-1} C_\ell (\log(t+i0))^{\ell+1} \right). \tag{I.6}$$

Proof. Since $\rho_j(t+\rho_j)^\alpha = (t+\rho_j)^{\alpha+1} - t(t+\rho_j)^\alpha$, (i) and (iii) follow respectively from Proposition I.1 and from (ii) above. Hence it remains to prove (ii). Since

$$\frac{d^{n+1}}{dt^{n+1}}((t+\rho_j)^n \log(t+\rho_j+i0)) = \frac{n!}{t+\rho_j+i0} + P_n,$$

where P_n is a polynomial of $(t+\rho_j)$, Remark I.1, entails that

$$\frac{d^{n+1}}{dt^{n+1}} I(n,j;t) + n! J(-1,j;t) + h(t) \tag{I.7}$$

holds with a holomorphic function $h(t)$. On the other hand, near $t = 0$ a straightforward computation shows

$$\int^t dt\ t^n (\log(t+i0))^m = \frac{t^{n+1}}{n+1} \sum_{r=0}^{m} \frac{(-1)^r m!(\log(t+i0)^{m-r}}{(m-r)!(n+1)^r} \tag{I.8}$$

holds for non-negative integers n, and $m \neg 1$. Combining (I.7) and Proposition I.1, we use (I.8) repeatedly to find (I.5). We also note that Remark I.2 entails $C_{j-1} = (-1)^j/j!(n+1)$ in this case.

The following proposition is a key result of this Appendix.

Proposition I.3. Let $I(t)$ denote the following integral (I.9), where e_j $(j = 1, 2, \cdots, n)$ is a non-negative integer:

$$\int_0^\delta r_1^{e_1} dr_1 \int_0^1 r_2^{e_2} dr_2 \cdots \int_0^1 r_n^{e_n} dr_n \log(t + r_1 \cdots r_n + i0). \tag{I.9}$$

Then its singularity near $t = 0$ is a sum of finitely many terms of the form

$$Ct^N (\log(t+i0))^m \tag{I.10}$$

with a constant C and positive integers $N(\geq min\ \ e_j + 1)$ and $m(\leq n)$.

<u>Proof.</u> First of all, let us re-scale the parameter r_1 and the variable t as follows:

$$r_1 = \delta r_1', \quad t = \delta s$$

Then I becomes

$$\delta^{-e_1-1} \int_0^1 (r_1')^{e_1} dr_1' \int_0^1 r_2^{e_2} dr_2 \cdots \int_0^1 r_n^{e_n} dr_n (\log(s + r_1' r_2 \cdots r_n) + \log \delta). \tag{I.11}$$

The contribution from $\log \delta$ in (I.11) is a finite constant. Thus we may assume from the first that $\delta = 1$. Then the roles of $r_j's$ in (I.9) are uniform, and hence we may re-number the index j so that

$$e_1 \geq e_2 \geq \cdots \geq e_n. \tag{I.12}$$

Let us introduce new variables σ_j by

$$\sigma_1 = r_1, \quad \sigma_2 = r_1 r_2, \cdots, \quad \sigma_n = r_1 \cdots r_n. \tag{I.13}$$

Because of the re-numbering done above, the $\sigma_j (j \leq n-1)$ may be different from the ρ_j introduced in Appendix G. The integral I (with $\delta = 1$) can be now expressed as

$$\int_0^1 \frac{\sigma_1^{d_1}}{\sigma_1} d\sigma_1 \int_0^{\sigma_1} \frac{\sigma_2^{d_2}}{\sigma_2} d\sigma_2 \cdots \int_0^{\sigma_{n-2}} \frac{\sigma_{n-1}^{d_{n-1}}}{\sigma_{n-1}} d\sigma_{n-1} \int_0^{\sigma_{n-1}} \sigma_n^{e_n} d\sigma_n \log(t+\sigma_n+i0), \tag{I.14}$$

where $d_j = e_j - e_{j+1}$. The number d_j is nonnegative by (I.12), and this non-negativity makes our reasoning much simpler: that is why we re-numbered the index j.

The first integration in (I.14), i.e. $\int_0^{\sigma_{n-1}} \sigma_n^{e_n} d\sigma_n \log(t+\sigma_n+i0)$, can be done in a straightforward manner: using the identity $\sigma^e = \sum_{j=0}^{e} c_j t^j (t+\sigma)^{e-j}$, where c_j is some constant, we find it is a sum of terms of the form

$$\begin{aligned} &Ct^j\{(t\ +\sigma_{n-1})^{e_n-j+1} \log(t+\sigma_{n-1}+i0) \\ &\qquad - t^{e_n-j+1} \log(t+i0)\} \end{aligned} \tag{I.15}$$

and polynomials of the form

$$C' t^j \{(t+\sigma_{n-1})^{e_n-j+1} - t^{e_n-j+1}\},$$

where C and C' are some constants.

If $d_{n-1} \geq 1$, the same computation can be done for the second integration (I.14). In this case we do not need to combine the first term and the second term in (I.15). That is, we perform the integration of these terms separately. If d_{n-1} is equal to 0, we first define an integral J by

$$\int_0^1 \frac{\sigma^{d_1}}{\sigma_1} d\sigma_1 \int_0^{\sigma_1} \frac{\sigma_2^{d_2}}{\sigma_2} d\sigma_2 \cdots \int_0^{\sigma_{n-2}} \frac{d\sigma_{n-1}}{\sigma_{n-1}} \{(t+\sigma_{n-1})^\alpha \log(t+\sigma_{n-1}+i0) - t^\alpha \log(t+i0)\},$$

where $\alpha = e_n - j + 1$ is a positive integer. Then we have

$$\begin{aligned} &(t\frac{d}{dt}\ - \alpha)J(t) \\ &\quad = \int_0^1 \frac{\sigma_1^{d_1}}{\sigma_1} d\sigma_1 \int_0^{\sigma_1} \frac{\sigma_2^{d_2}}{\sigma_2} d\sigma_2 \cdots \int_0^{\sigma_{n-2}} d\sigma_{n-1} \{ -\alpha(t+\sigma_{n-1})^{\alpha-1} \log(t+\sigma_{n-1}+i0) \\ &\quad + ((t+\sigma_{n-1})^{\alpha-1} - t^{\alpha-1})/\sigma_{n-1}\} \end{aligned}$$

Since the contribution from $((t+\sigma_{n-1})^{\alpha-1}-t^{\alpha-1})/\sigma_{n-1}$ is finite and analytic in t (actually a polynomial), the main contribution to $J(t)$ is from $-\alpha(t+\sigma_{n-1})^{\alpha-1}\log(t+\sigma_{n-1}+i0)$. But this is the same integral discussed at the first step. Repeating this procedure we finally find that I has the form $\sum_{j=0}^{e_n} t^j I_j$, where I_j satisfies the following equation:

$$\prod_{\ell=1}^{n(j)}(t\frac{d}{dt}-\alpha_\ell(j))I_j(t)=\sum_k(C_k\log(t+i0)+C'_k)t^k+A, \qquad (I.17)$$

where A is analytic at $t=0$. Here $n(j)\le n-1$, $\alpha_\ell(j)$ is an integer $\ge e_n-j+1, k$ ranges over a finite subset of integers $\ge e_n-j+1$, and C_k and C'_k are constants. As a solution of the equation (I.17), $I_j(t)$ [modulo a function analytic at $t=0$] is a sum of terms of the form

$$Ct^N(\log(t+i0))^m$$

with a constant C and integers $N\ge e_n-j+1$ and $m\le n$. Thus $I(t)$ consists of terms of the required form (I.10). Note that $\min e_j=e_n$ by the re-numbering of the index j.

Remark I.3. (i) If $\log\,(t+r_1...r_n+i0)$ in (I.9) is replaced by $(t+r_1...r_n+i0)^\alpha$ (α: non-integer), the resulting integral is a finite sum of terms of the form

$$Ct^{\alpha+e}(\log(t+i0))^m \qquad (I.10')$$

with an integer $e\ge\min e_j+1$ and a non-negative integer $m\le n-1$. If $\alpha=-1$, then the condition on e is the same as above but the condition on m is replaced by $m\le n$.

(ii) Let $a(r')$ be an analytic function of $r'=(r'_1...,r'_n)$, in a closed "cube" $C=[\epsilon,1]\times...\times[\epsilon,1](\epsilon>0)$. Then the following integral $F(a)$ has the same singularity as $I(t)$, or a weaker one :

$$F(a)=\int_\epsilon^1 dr'_1...\int_\epsilon^1 dr'_{n'}a(r')\int_0^\epsilon r_1^{e_1}dr_1...\int_0^\epsilon r_n^{e_n}dr_n\log(t+r'_1...r'_{n'}...r_n+i0).$$

In fact, for r' in C we find

$$\begin{aligned}&\int_0^\epsilon r_1^{e_1}dr_1...\int_0^\epsilon r_n^{e_n}dr_n\log(t+r'_1...r'_m r_1...r_n+i0)\\ =&\int_0^\epsilon r_1^{e_1}dr_1...\int_0^\epsilon r_n^{e_n}dr_n\{\log(\frac{t}{r'_1...r'_m}+r_1...r_n+i0)\\ &+\log(r'_1...r'_m)\}.\end{aligned}$$

Since the contribution from $\log(r'_1 ... r'_m)$ to $F(a)$ is an analytic function, it suffices to consider the contribution from $\log(\frac{t}{r'_1 ... r'_{n'}} + r_1 ... r_n + i0)$. Proposition I.3 then tells us that it is a sum of terms of the form

$$\int_\epsilon^1 dr'_1 ... \int_\epsilon^1 dr'_{n'} a(r') \frac{t^N}{(r'_1 ... r'_{n'})^N} (\log(t + i0) - \log(r'_i ... r'_{n'}))^m.$$

Hence the singularity of $F(a)$ near $t = 0$ is a sum of terms of the form (I.10). Note that the effect of changing the upper end-point of the integral in (I.9) to ϵ is absorbed by the harmless change of scaling in r variables and t variable (as was employed at the beginning of the proof of Proposition I.3), on the condition that ϵ is a fixed positive constant.

To generalize Proposition I.3 to the form needed in Appendix J we prepare the following Lemma:

Lemma I.1. Let $L_m(t; a)(n = 1, 2, ...; a$ a strictly positive constant), denote the following integral:

$$\int_0^a \frac{(\log w)^m}{t + w + i0} dw.$$

Then the singularity structure of L_m near $t = 0$ is as follows:

$$L_m = \sum_{j=1}^{m+1} C_j (\log(t + i0))^j + h(t) \qquad (I.18)$$

where $C_j (j = 1, ..., m + 1)$ are some a-dependent constants and $h(t)$ is an a-dependent holomorphic function near $t = 0$.

Proof. Since $(\log w)^m \theta(w) \theta(a - w) (a > 0)$ is well-defined as a product of locally summable functions, the convolution-type integral $L_m(t; a)$ is well-defined, and it is a boundary value of a holomorphic function on $\{\mathrm{Im}\, t > 0\}$ near $t = 0$. To find out its explicit form (I.18), we first apply an integration by parts:

$$\begin{aligned} L_m &= \lim_{\kappa \downarrow 0} \Big\{ - \int_\kappa^a \frac{m(\log(w + i0))^{m-1}}{w + i0} \log(t + w + i0) dw \\ &\quad + (\log a)^m \log(a + t + i0) - (\log(\kappa + i0))^m \log(t + \kappa + i0) \Big\} \\ &= \lim_{\kappa \downarrow 0} \Big\{ - \int_\kappa^a \frac{m(\log(w + i0))^{m-1}}{w + i0} \log\left(\frac{t + w + i0}{t + i0}\right) dw \\ &\quad - \left(\int_\kappa^a \frac{m(\log(w + i0))^{m-1}}{w + i0} dw \right) \log(t + i0) - (\log(\kappa + i0))^m \log(t + \kappa + i0) \Big\} \end{aligned}$$

$$+ (\log a)^m \log(a+t+i0)$$

$$= \lim_{\kappa \downarrow 0} \left\{ -\int_\kappa^a \frac{m(\log(w+i0))^{m-1}}{w+i0} \log\left(\frac{t+w+i0}{t+i0}\right) dw \right.$$

$$\left. - (\log(\kappa+i0))^m \log \frac{t+\kappa+i0}{t+i0} \right\} + (\log a)^m \log \frac{t+a+i0}{t+i0} \qquad (I.19)$$

Let us note that, if $\operatorname{Im} t > 0$,

$$(\log(\kappa+i0))^m \log \frac{t+\kappa+i0}{t+i0}$$

$$= (\log(\kappa+i0))^m \left(\frac{\kappa+i0}{t+i0} - \frac{1}{2}\left(\frac{\kappa+i0}{t+i0}\right)^2 + \ldots \right) \longrightarrow 0$$

as $\kappa \downarrow 0$. Hence we obtain

$$L_m = M_m - (\log a)^m \log(t+i0) + (\log a)^m \log(t+a+i0), \qquad (I.20)$$

where

$$M_m = \lim_{\kappa \downarrow 0} \left(-\int_\kappa^a \frac{m(\log(w+i0))^{m-1}}{w+i0} \log \frac{t+w+i0}{t+i0} dw \right). \qquad (I.21)$$

Since $(\log\,(w+i0))^{m-1}$ is locally summable, the reasoning used to verify the well-definedness of the integral $\int_0^\delta \frac{dr}{r} \log \frac{t+r+i0}{t+i0}$ (cf. the beginning of this appendix) is applicable also to M_m. To find out the explicit form of M_m, let us first note

$$M_1 = C_2(\log(t+i0))^2 + C_1(\log(t+i0)) + h(t)$$

holds near $t=0$ for some constants C_1, C_2 and some holomorphic function $h(t)$. (Cf. (I.2)). Thus we can verify (I.18) for $n=1$. For $n>1$, we use mathematical induction: Let us suppose (I.18) is verified for $1 \le m \le m_0$. Since

$$t\frac{d}{dt} M_{m_0+1} = \int_0^a \frac{(m_0+1)(\log(w+i0))^{m_0}}{t+w+i0} dw = (m_0+1) L_{m_0}, \qquad (I.22)$$

using the induction hypothesis, we find that $(t\frac{d}{dt})^{m_0+1} L_{m_0}$ is holomorphic near $t=0$. This means that $(t\frac{d}{dt})^{m_0+2} M_{m_0+1}$ is holomorphic near $t=0$. Otherwise stated,

$$M_{m_0+1} = \sum_{j=1}^{m_0+2} \tilde{C}_j (\log(t+i0))^j + \tilde{h}(t)$$

holds near $t=0$ for some constants $\widetilde{C}_j(j=1,...,m_0+2)$ and some holomorphic function $\widetilde{h}(t)$. Therefore (I.20) implies that (I.18) is true for $m=m_0+1$. Thus the induction proceeds.

Proposition I.4. (i) Let $K_{n,m}(t)(n,m=1,2,3,...)$ denote the following integral (with $\delta>0$) :

$$\int_0^\delta (\log r_0)^m dr_0 \int_0^1 ... \int_0^1 \prod_{j=1}^n dr_j (t+r_0r_1...r_n+i0)^{-1}. \qquad (I.23)$$

Then the singularity structure of $K_{n,m}(t)$ near $t=0$ is as follows:

$$K_{n,m}(t) = \sum_{j=1}^{n+m+1} C_j(\log(t+i0))^j + h(t), \qquad (I.24)$$

where $C_j(j=1,...,n+m+1)$ are some constants and $h(t)$ is some holomorphic function near $t=0$.

(ii) Let $J_{n,m}(t)(n,m=1,2,3,...)$ denote the following integral:

$$\int_0^\delta (\log r_0)^m dr_0 \int_0^1 ... \int_0^1 \prod_{j=1}^n dr_j \log(t+r_0r_1...r_n+i0). \qquad (I.25)$$

Then the singularity structure of $J_{n,m}(t)$ near $t=0$ is as follows:

$$J_{n,m}(t) = \sum_{j=1}^{n+m+1} C_j t(\log(t+i0))^j + h(t), \qquad (I.26)$$

where $C_j(j=1,...,n+m+1)$ are some constants and $h(t)$ is some holomorphic function near $t=0$.

Proof. (i) Let $\rho_j(j=0,1,...,n)$ denote $\prod_{i=0}^j r_i$. Then $K_{n,m}$ assumes the following form:

$$\int_0^\delta \frac{(\log\rho_0)^m}{\rho_0} d\rho_0 \int_0^{\rho_0} \frac{d\rho_1}{\rho_1} ... \int_0^{\rho_{n-2}} \frac{d\rho_{n-1}}{\rho_{n-1}} \int_0^{\rho_{n-1}} d\rho_n (t+\rho_n+i0)^{-1}.$$

Hence we find

$$(-t\frac{d}{dt})^n K_{n,m}(t) = \int_0^\delta \frac{(\log\rho_0)^m}{t+\rho_0+i0} d\rho_0.$$

Therefore Lemma I.1 shows that $(-t\frac{d}{dt})^{n+m+1}K_{n,m}(t)$ is holomorphic near $t=0$. This entails (I.24).

(ii) Since $\frac{d}{dt}J_{n,m} = K_{n,m}$, the result (i) entails

$$\frac{d}{dt}J_n = \sum_{j=1}^{n+m+1} C_j(\log(t+i0))^j + h(t) \qquad (I.27)$$

holds for some constants C_j and a holomorphic function $h(t)$. Hence, by integrating both sides of (I.27), we find (I.26). Here we have used a formula

$$\int^t (\log t)^N dt = t\left(\sum_{\ell=0}^{N}(-1)^\ell \frac{N!}{(N-\ell)!}(\log t)^{N-\ell}\right).$$

The following generalization of Proposition I.4 is used in Appendix J.

Proposition I.5. Let $L_{n,m}(t)(n,m = 1,2,3,...)$ denote the following integral (with $\delta_0 > 0$), where $e_j(j = 0,1,...,n)$ is a non-negative integer:

$$\int_0^{\delta_0}(\log r_0)^m r_0^{e_0} dr_0 \int_0^1 ... \int_0^1 \prod_{j=1}^{n} r_j^{e_j} dr_j \log(t + r_0 r_1 r_2 ... r_n + i0).$$

Then the singular part of $L_{n,m}(t)$ near $t = 0$ is a finite sum of terms of the following form:

$$C_{N,p}\, t^N (\log(t+i0))^p, \qquad (I.28)$$

where $C_{N,p}$ is a constant, N is a non-negative integer ($\geq \min_{0\leq j\leq n} e_j + 1$) and p is a positive integer ($\leq n+m+1$).

Proof. Making use of the scaling transformation of r_0 and t as in the proof of Proposition I.3, we may assume without loss of generality that $\delta_0 = 1$. Furthermore, as the role of variables $r_j(j = 1,...,n)$ is uniform, we may assume, by re-labelling of the variables $r_j(j = 1,...,n)$, that $e_1 \geq e_2 \geq ... \geq e_n$. If $e_0 \geq e_1$ then the method used in the proof of Proposition I.3, supplemented by Lemma I.1, establishes the required result. However, this condition cannot be expected to hold in general, and hence we must generalize. Introducing the new variables $\sigma_j = r_0 r_1 .. r_j$ $(j = 0,1,...,n)$ we find

$$\begin{aligned} L_{n,m}(t) &= \int_0^1 (\log\sigma_0)^m \sigma_0^{d_0-1} d\sigma_0 \int_0^{\sigma_0} \sigma_1^{d_1-1} d\sigma_1 ... \int_0^{\sigma_{n-2}} \sigma_{n-1}^{d_{n-1}-1} d\sigma_{n-1} \\ &\times \int_0^{\sigma_{n-1}} \sigma_n^{e_n} \log(t+\sigma_n+i0) d\sigma_n, \end{aligned}$$

where $d_j = e_j - e_{j+1}$. As noted above, the proof is finished if $d_0 \geq 0$. Let us consider the case $d_0 < 0$. We then use mathematical induction on m. When $m = 1$, we use the following:

$$\begin{aligned}
&\frac{d}{d\sigma_0}\left((\log\sigma_0)\sigma_0^{d_0}F(\sigma_0,t)\right) \\
&\quad = d_0(\log\sigma_0)\sigma_0^{d_0-1}F(\sigma_0,t) + \sigma_0^{d_0-1}F(\sigma_0,t) \\
&\quad + (\log\sigma_0)\sigma_0^{d_0}\frac{\partial F(\sigma_0,t)}{\partial\sigma_0}. \qquad (I.29)
\end{aligned}$$

Choosing

$$\int_0^{\sigma_0}\sigma_1^{d_1-1}d\sigma_1 \ldots \int_0^{\sigma_{n-2}}\sigma_{n-1}^{d_{n-1}-1}d\sigma_{n-1}\int_0^{\sigma_{n-1}}\sigma_n^{e_n}\log(t+\sigma_n+i0)d\sigma_n$$

as $F(\sigma_0,t)$, we obtain

$$\begin{aligned}
d_0L_{n,1} &= (\log\sigma_0)\sigma_0^{d_0}F(\sigma_0,t)|_{\sigma_0=1} - \lim_{\sigma_0\downarrow 0}\left((\log\sigma_0)\sigma_0^{d_0}F(\sigma_0,t)\right) \\
&\quad - \int_0^1\sigma_0^{d_0-1}F(\sigma_0,t)d\sigma_0 - \int_0^1(\log\sigma_0)\sigma_0^{e_0-e_1}\sigma_0^{e_1-e_2-1} \\
&\quad \times \int_0^{\sigma_0}\sigma_2^{d_2-1}d\sigma_2 \ldots \int_0^{\sigma_{n-1}}\sigma_n^{e_n}\log(t+\sigma_n+i0)d\sigma_n. \qquad (I.30)
\end{aligned}$$

For notational convenience, let $A_j (j = 1,2,3,4)$ denote the j-th term in RHS of (I.30). Since $F(1,t)$ is a well-defined integral (cf. Proposition I.3), A_1 vanishes because of the trivial fact $\log 1 = 0$. To confirm that A_2 also vanishes, we note that

$$\left|\frac{1}{t+\sigma_n}\right| \leq C_\epsilon$$

holds if $\operatorname{Im} t \geq \epsilon > 0$ and σ_n is real. Then we find, for $\sigma_0 \geq 0$,

$$\begin{aligned}
|F(\sigma_0,t)| &\leq C_\epsilon\int_0^{\sigma_0}\sigma_1^{d_1-1}d\sigma_1 \ldots \int_0^{\sigma_{n-1}}\sigma_n^{e_n}d\sigma_n \\
&= \frac{C_\epsilon\sigma_0^{e_1+1}}{\prod_{j=1}^n(e_j+1)}.
\end{aligned}$$

Therefore

$$|A_2| \leq C_\epsilon\lim_{\sigma_0\downarrow 0}\frac{(\log\sigma_0)\sigma_0^{e_0+1}}{\prod_{j=1}^n(e_j+1)} = 0.$$

Since ϵ is an arbitrary positive number, this means that A_2 vanishes.

The term A_3 has the same structure as the integral discussed in Proposition I.3, and hence its singular part is a sum of terms of the form (I.28). Note that we can re-label all variables including r_0 if we go back to r-variables from σ-variables in the integral A_3; the factor $\log \sigma_0$ has disappeared in A_3.

Finally let us study A_4. As it has the form $L_{n-1,1}$, we can apply the above procedure to it. Repeating this procedure , we eventually end up with one of the following two integrals (i) or (ii), together with terms of the form $(I.28)$:

(i) $L_{n',1}(n' < n)$ with $d_0 \geq 0$

(ii) $\int_0^1 (\log \sigma_0)\sigma_0^{c_0+1} \log(t + \sigma_0 + i0) d\sigma_0$.

By using Lemma I.1 together with the method of the proof of Proposition I.3, we can verify that the singular part of either of them is a sum of terms of the form (I.28).

Thus the proof is finished if $m = 1$. Let us consider next the case $m \geq 2$. We then use

$$\begin{aligned}
&\frac{d}{d\sigma_0}\left((\log \sigma_0)^m \sigma_0^{d_0} F(\sigma_0, t)\right) \\
&\quad = d_0 (\log \sigma_0)^m \sigma_0^{d_0-1} F(\sigma_0, t) + m(\log \sigma_0)^{m-1} \sigma_0^{d_0-1} F(\sigma_0, t) \\
&\quad + (\log \sigma_0)^m \sigma_0^{d_0} \frac{\partial F(\sigma_0, t)}{\partial \sigma_0}. \qquad (I.31)
\end{aligned}$$

As before, we concentrate our attention on the case $d_0 < 0$. Choosing as $F(\sigma_0, t)$ the same integral as was used when $m = 1$, we find that the same reasoning as before applies to the contribution from the LHS of $(I.31)$ and the third term on the RHS of $(I.31)$. When integrated over [0, 1] (with respect to σ_0), the second term on the RHS of $(I.31)$ turns out to be $mL_{n,m-1}$. Thus the induction proceeds, completing the proof.

Appendix J. Computation for the Nonmeromorphic Case

The computation in the nonmeromorphic case is similar to the computation for the meromorphic case described in Appendices G and H. Let the special index i be now the smallest integer such that photon line i is either a bridge line or a photon line with a nonmeromorphic coupling on at least one end.

If line i is a bridge line (and hence, by definition, has a meromorphic coupling on each end, and bridges across a $*$-segment) then the argument used for the meromorphic case continues to work. This is because the condition (G.1) of appendix G continues to hold, and each variable λ_j associated with a nonmeromorphic coupling acts just like a variable $r_j (j > i)$ of appendices G and H.

If, on the other hand, the index i labels a line with a nonmeromorphic coupling on at least one end then (G.1) may fail, because in this case the variable k_i may enter into Δ only in the form $\lambda_i k_i$ (or $\lambda'_i k_i$). For example, if the photon line i runs between two different sides, s and s', and has a nonmeromorphic coupling on both ends then, according to (10.8b), the vector k_i enters into Δ only in the combinations $\lambda_i k_i$ or $\lambda'_i k_i$, where λ_i and λ'_i are the variables associated with the two different nonmeromorphic couplings of line i. Hence the derivative on Δ occurring in (G.7) will introduce a factor λ_i or λ'_i into each k_i-dependent contribution to (G.7). Since λ_i and λ'_i vanish in the domain of integration, and all other contributions have factors $r_j (j > i)$, which can vanish, the property (G.1) can fail.

Similarly, if only one end of line i is coupled nonmeromophically, say into the side s, but the closed loop i does not pass through the star line for either of the other two sides $s' \neq s$, then again (G.1) can fail, for essentially the same reason.

These failures of (G.1) cannot be avoided by simply using $\rho'_i = \lambda_i \rho_i$ (or $\lambda'_i \rho_i$) in place of ρ_i, because the condition in (H.1) on k'/ρ_i fails if ρ_i is replaced by ρ'_i.

In this appendix the "self-energy" photons that couple nonmeromorphically on both ends onto the same side s will be ignored: they are treated in Appendix K.

To deal with the new cases we introduce the set of variables $x_j (j \epsilon J)$ to

represent both the $r_j(j > i)$, and also the occurring variables $\lambda_j(j \geq i)$ and $\lambda'_j(j \geq i)$. This set $x_j(j\epsilon J)$ replaces the set $r_j(j > i)$ that occurs in the arguments of Appendices G and H.

Using the evaluation (G.6) for the constant gradient vector $\nabla\varphi$ we define a new variable

$$\begin{aligned} \rho &= -\nabla\varphi \cdot \Delta \\ &= \sum_{s=1}^{3} \sum_{m=i}^{n} \alpha_s \widehat{p}_s \epsilon'_{sm} k_m, \end{aligned} \tag{J.6}$$

where the reasoning leading to (G.7) has been used. However, ϵ'_{sm} can be 0, 1, λ_m, or λ'_m, with the latter two possibilities coming from the possible nonmeromorphic couplings.

In the case under consideration the photon line i has a nonmeromorphic coupling on one or both ends. If this line i has a nonmeromorphic coupling on only one end, and the ϵ'_{si} associated with the other end is 1, then (G.1) again holds, and the method used in the meromorphic case again works. In the remaining, cases (namely those for which $\epsilon'_{si} \neq 1$ for all s) the function $r_0 = \rho/\rho_i$ has a term $\lambda_i p_s \Omega_i$ (or $\lambda'_i p_s \Omega_i$) and no other dependence on λ_i (or λ'_i). Hence the variable r_0 may be introduced as a new variable, replacing λ_i (or λ'_i), provided the associated coefficient $p_s \cdot \Omega_i$ is nonzero.

The arguments of section 8, slightly extended to include the λ_j, show that $\widehat{p}_s \cdot \Omega_i$ can be taken to be nonzero near points in the integration domain that lead to a singularity of the integral at $\widehat{q}$. Hence the transformation to the new set of variables (with r_0 replacing λ_i or λ'_i) is a holomorphic transformation: all analyticity properties are maintained.

The derivative at $(q - \Delta) = \widehat{q}$ of $\varphi(q - \Delta)$ with respect to ρ is

$$\begin{aligned} \frac{\partial\varphi(q-\Delta)}{\partial\rho} &= \nabla\varphi \cdot \frac{\partial(-\Delta)}{\partial\rho} \\ &= \frac{\partial(-\nabla\varphi \cdot \Delta)}{\partial\rho} \\ &= \frac{\partial\rho}{\partial\rho} = 1. \end{aligned} \tag{J.7}$$

Thus (G.1) is now satisfied (with $\rho = r_0 r_1 ... r_i$ in place of $\rho_i = r_1 r_2 ... r_i$), and we can use the method of Appendices G and H.

The function $F(q)$ of (G.2) now takes the form

$$F(q) = \prod_{j=1}^{n} \int_{\Omega_j \widetilde{\Omega}_j = 1} d\Omega_j \prod_{j=1}^{i} \int_0^1 r_j^{e_j} dr_j \int dr_0 G(q, r, r_0), \tag{J.8}$$

where

$$\begin{aligned} G(q, r, r_0) &= \prod_{j \in J} \int_0^1 x_j^{e_j} dx_j \delta(r_0 + \rho_j^{-1} \nabla\varphi \cdot \Delta) \\ &\quad \times A(q, \Omega, r, x) \log\varphi(q - \Delta), \end{aligned} \tag{J.9}$$

and $\log\varphi(q - \Delta)$ is defined in (G.3) and (G.4), but with the ϵ'_{sm} in place of the ϵ_{sm}. Notice that the $\int dr_0$ can be cancelled by the δ function to give the generalization of (G.2) engendered by the action of the nonmeromorphic-part operators of (10.8b).

The expression for G given in (J.9) is well defined only for real ∇ (i.e., only for real $k_j (j \geq i)$). A more general definition is this: (1), leave the $\int dr_0$ and δ function out of (J.8) and (J.9); (2), change the variable λ_i (or λ'_i) to r_0; (3), replace the $\int d\lambda_i$ (or $\int d\lambda'_i$) by $\int dr_0$; (4), identify G as the integrand of this integral over dr_0.

Near the point $\widehat{q}$ one can write

$$\begin{aligned} \varphi(q - \Delta) &\cong \varphi(q) - \nabla\varphi \cdot \Delta \\ &= \varphi(q) + \rho. \end{aligned} \tag{J.10}$$

Insertion of (J.10) into (J.8) and (J.9) gives

$$\begin{aligned} F(q) &= \prod_{j=1}^{n} \int_{\Omega_j \widetilde{\Omega}_j = 1} d\Omega_j \prod_{j=1}^{i} \int_0^1 r_j^{e_j} dr_j \int dr_0 \log(\varphi(q) + \rho) \\ &\quad \times f(r_0, r, \Omega, q), \end{aligned} \tag{J.11}$$

where

$$f(r_0, r, \Omega, q) = \prod_{j \in J} \int_0^1 x_j^{e_j} dx_j \delta(r_0 + \nabla\varphi \cdot \widetilde{\Delta}) A(q, \Omega, r, x), \tag{J.12}$$

and $\widetilde{\Delta} \equiv \Delta / \rho_i$.

Equation (J.11) exhibits the smearing of the $\log\varphi(q)$ singularity. If $f(r_0, r, \Omega, q)$ were to have a δ function singularity at $r_0 = 0$ then the expression (J.12) would

yield a singularity of the form $\log \varphi(q)$. But if f has only a milder singularity at $r_0 = 0$ then $F(q)$ will have a weaker singularity at $\varphi(q) = 0$.

Let us examine, then, the form of $f(r_0, r, \Omega, q)$. Let the particular x_j that is λ_i be called simply λ. Then $\nabla\varphi \cdot \tilde{\Delta}$ will be $(a\lambda + P)$, where P is a sum of terms each of which is a coefficient of the form $p_s(q)\Omega_j$ times a product $r_{i+1}r_{i+2}...r_j$, or $r_{i+1}r_{i+2}...r_j\lambda_j$, or $r_{i+1}r_{i+2}...r_j\lambda'_j$. Eventually the coefficients $p_s(q)\Omega_j$ will be shifted to nonzero complex numbers. But we shall evaluate the integrals first at points where each $p_s(q)\Omega_j = 1$, $a = A = 1$, and each $e_j = 0$.

Consider first, then, for $0 < r_0 < 1$, the simple example

$$f(r_0) = \int_0^1 d\lambda \int_0^1 dx_1 \int_0^1 dx_2\ \delta(r_0 - \lambda - x_1x_2). \tag{J.13}$$

Using the δ function to eliminate the $\int d\lambda$ we obtain (with Θ the Heaviside function)

$$\begin{aligned} f(r_0) &= \int_0^1 dx_1 \int_0^1 dx_2\ \Theta(r_0 - x_1x_2) \\ &= \int_0^1 dx_1 \int_0^{r_0/x_1} dx_2\ \Theta(1 - \frac{r_0}{x_1}) \\ &+ \int_0^1 dx_1 \int_0^1 dx_2\ \Theta(\frac{r_0}{x_1} - 1) \\ &= r_0 \int_{r_0}^1 \frac{dx_1}{x_1} + \int_0^{r_0} dx_1 \\ &= r_0(-\log r_0 + 1). \end{aligned} \tag{J.14}$$

Thus in this case the singularity of the function $f(r_0)$ is much weaker than $\delta(r_0)$: $f(r_0)$ is bounded and tends to zero as r_0 tends to zero.

The general form of $f(r_0)$ is

$$f(r_0) = \prod_{j \in J} \int_0^1 dx_j\ \Theta(r_0 - P), \tag{J.15}$$

where P is as defined above. One sees immediately that $f(r_0)$ is bounded, and tends to zero with r_0.

To begin the study of the general form of $f(r_0)$ let us consider a case slightly more complicated than (J.14):

$$f(r_0) = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3\ \Theta(r_0 - x_1x_2x_3)$$

$$= \int_0^1 dx_1 \int_0^1 dh\ \Theta(r_0 - x_1 h)$$
$$\times \int_0^1 dx_2 \int_0^1 dx_3\ \delta(h - x_2 x_3)$$
$$= \int_0^1 dx_1 \int_0^1 dh\ \Theta(r_0 - x_1 h) H(h), \qquad (J.16)$$

where (for $0 < h < 1$)

$$H(h) = \int_0^1 dx_2 \int_0^1 dx_3\ \delta(h - x_2 x_3)$$
$$= \int_0^1 \frac{dx_2}{x_2}\ \Theta(1 - \frac{h}{x_2})$$
$$= \int_h^1 \frac{dx_2}{x_2} = (-\log h). \qquad (J.17)$$

Notice that the last line of (J.16) has the same form as the first line of (J.14), but with a different function H. Substituting the function $H(h)$ from (J.17) into (J.16) one obtains, for $0 < r_0 < 1$,

$$f(r_0) = \int_0^1 dx_1 \int_0^1 dh\ \Theta(\frac{r_0}{x_1} - h)(-\log h)$$
$$= \int_{r_0}^1 dx_1 \int_0^{r_0/x_1} dh(-\log h)$$
$$+ \int_0^{r_0} dx_1 \int_0^1 dh(-\log h)$$
$$= \sum_{n,m} C_{nm} (r_0)^n (\log r_0)^m, \qquad (J.18)$$

where only a finite number of the constant coefficients C_{nm} are nonzero.

A function of one variable x having, in $0 < x < 1$, the form $\sum C_{nm} x^n (\log x)^m$, and bounded in $0 \leq x \leq 1$, with some finite number of nonzero coefficients C_{nm}, will be said to have form F. Thus the functions $f(r_0)$ specified in $(J.14)$ and $(J.16)$ both have form F.

In fact, the general function $f(r_0)$ of the form specified in $(J.15)$ has form F. To see this note first that if we replace the factor $H(h) = (-\log h)$ in (J.16) by any function $H(h)$ of form F then $f(r_0)$ has form F:

$$f(r_0) = \int_{r_0}^1 dx_1 \int_0^{r_0/x_1} dh\ H(h)$$

$$+\int_0^{r_0} dx_1 \int_0^1 dh\ H(h)$$
$$= r_0 \int_{r_0}^1 \frac{dx'}{(x')^2} \int_0^{x'} dh\ H(h)$$
$$+ r_0 \int_0^1 dh\ H(h) \qquad (J.19)$$

Then $(I.8)$ gives the result that if $H(h)$ has form F then $f(r_0)$ has form F. (Note that every term in $\int_0^{x'} dh H(h)$ has a factor x', and hence the denominator $(x')^2$ is reduced to x'.) So our problem is to show that $f(r_0)$ can be reduced to the form (J.16) with $H(h)$ having the form F.

To show this let $I(g)$ be some function of form F and consider the integral operator H_h defined by

$$H_h[I(g)] = \int_0^1 dx \int_0^1 dg\ \delta(h - xg) I(g). \qquad (J.20)$$

Then, for $0 < h < 1$,

$$H_h I = \int_h^1 \frac{dx}{x}\ I(\frac{h}{x})$$
$$= \int_h^1 \frac{dx'}{x'}\ I(x'), \qquad (J.21)$$

where $x' = h/x$. Then $(I.8)$ and Lemma $I.1$ entail that if I has form F, so does $H_h I$.

Repeated application of this result shows that if $P = x_1 x_2 ... x_p$ then $f(r_0)$ has form F. One first combines $x_{p-1}x_p$ into h_p, then combines $x_{p-2}h_p$ into h_{p-1}, etc.. At each stage the functions I and H have form F, and hence one finally gets (J.16) with $H(h)$ having form F, as required.

The general form of P is not just a single product $r_{i+1}...r_j$: it is a sum of such terms with different values of j, some of which can be multiplied by λ_j or λ'_j. However, these other terms can be brought into the required form by a generalization of the operator technique used above.

Let us again consider first a simple case:

$$f(r_0) = \int_0^1 dx \int_0^1 dg \int_0^1 dt\ \Theta(r_0 - xt - xg) I(g), \qquad (J.22)$$

where $I(g)$ has form F, and t could be a λ_j. Then

$$f(r_0) = \int_0^1 dx \int_0^2 dh\ \Theta(r_0 - xh)H(h), \tag{J.23}$$

where

$$H(h) = \int_0^1 dg \int_0^1 dt\ \delta(h - t - g)I(g). \tag{J.24}$$

Thus

$$\begin{aligned} f(r_0) &= \int_0^1 dx \int_0^1 dh\ \Theta(r_0 - xh)H(h) \\ &+ \int_0^1 dx \int_1^2 dh\ \Theta(r_0 - xh)H(h). \end{aligned} \tag{J.25}$$

For $0 < h < 1$ the function $H(h)$ is

$$\begin{aligned} H(h) &= \int_0^1 dg \int_0^1 dt\ \delta(h - t - g)I(g) \\ &= \int_0^1 dg\ I(g)\Theta(h - g)\Theta(1 - (h - g)) \\ &= \int_0^h dg\ I(g), \end{aligned} \tag{J.26}$$

which has form F. Thus the first term in (J.25) gives a contribution $f_1(r_0)$ to $f(r_0)$ that has form F. The second term is, for $0 < r_0 < 1$,

$$\begin{aligned} f_2(r_0) &= \int_0^1 dx \int_1^2 dh\ \Theta(r_0 - xh)H(h) \\ &= \int_0^1 dx \int_1^2 dh\ \Theta(\frac{r_0}{x} - h) \int_{h-1}^1 dg\ I(g) \\ &= \int_0^1 dx \int_1^{r_0/x} dh\ \Theta(\frac{r_0}{x} - 1)\ \Theta(2 - \frac{r_0}{x}) \int_{h-1}^1 dg\ I(g) \\ &+ \int_0^1 dx \int_1^2 dh\ \Theta(\frac{r_0}{x} - 2) \int_{h-1}^1 dg\ I(g) \\ &= \int_{r_0/2}^{r_0} dx \int_1^{r_0/x} dh \int_{h-1}^1 dg\ I(g) \\ &+ \int_0^{r_0/2} dx \int_1^2 dh\ \int_{h-1}^1 dg\ I(g) \\ &= r_0 \int_1^2 \frac{dx'}{(x')^2} \int_1^{x'} dh \int_{h-1}^1 dg\ I(g) + (r_0/2) \times \text{const.} \\ &= r_0 \times \text{const.}, \end{aligned} \tag{J.27}$$

which is also of form F.

The two important points are: (1), that the integral operator that reduces a sum $t+g$ to a single h, just like the operator that reduces a product tg to h, preserves form F; and (2), the extra part $h > 1$ does not disrupt the argument: it adds only a term $r_0 \times$ const.

By taking combinations of these two kinds of operators, and a third kind with t fixed at unity, rather than being integrated over, one can reduce any one of the possible functions $(r_0 - P)$ to $(r_0 - x_1 h)$ combined with an $H(h)$ of form F. Thus all functions $f(r_0)$ of the kind $(J.15)$ will be of the form F , provided we make the simple assignments $1 = a = A = p_s \Omega_j = e_j + 1$. The remaining task is to show that essentially the same result follows even when we do not make these simple assignments. One other problem also needs to be addressed: we have computed the integrals on the variables r_j under the assumption that the variables Ω_j are held fixed, whereas the distortions in the variables Ω_j can depend on the r_j.

By following through the arguments just given, but with the e_j now allowed to be nonnegative integers, one finds that the conclusions are not disrupted: the positive power n of r_0 in $f(r_0)$ can be increased, and the positive power m of $\log r_0$ can be decreased, but changes in the opposite direction do not occur. Hence the singularities are at most weakened.

To deal simultaneously with the problems of the dependence of $A(q, \Omega, r, x)$ upon (r, x), and the dependence of the distortion in Ω upon (r, x), we introduce a sufficiently small number $\epsilon = 1/N > 0$, and divide the domain of integration $0 < x < 1$ in each of the variables x_j and r_j into a sum of N intervals of length ϵ, such that (1), the distortion of the set of Ω variables can be held fixed over each separate product interval, and (2), for any *subset* σ of the set of variables r_j and x_j, and for the corresponding space S formed by the product over σ of the corresponding set of *leading* intervals $0 < (r_j, x_j) < \epsilon$, the dependence of A on these variables can be represented by a power series that converges within S for each point in the space formed by the product over the complementary set of variables of the nonleading intervals $\epsilon < x < 1$. (See Remark I.3(ii).) The variables can then be re-scaled so that the original integration domains run from 0 to N, and the leading intervals (formerly from 0 to ϵ) now run from from 0 to 1. The earlier arguments can then be applied to the re-scaled problem, with

the concept 'form F' replaced by 'form F': a function of one variable x is said to have form F' if and only if it is bounded in the interval $0 \leq x \leq 1$, and in $0 < x < 1$ can be written in the form

$$\sum_m A_m(x)(\log x)^m,$$

where the sum is over a finite set of integers m, and each $A_m(x)$ is analytic on $0 < x < 1$. The contributions from the integrations over the nonleading domains $1 < x < N$ do not disrupt the arguments, and formula $(I.8)$ shows that extra factors $n+1$ are introduced into the denominators at each integration, so that convergence at the level of the integral is, if anything, improved over the original convergence at the level of the integrand. This takes care of these two problems.

The final step is to remove the assumption that the coefficients of the various terms of P are unity: these coefficients are actually the quantities $p_s\Omega_j$.

There is no problem in allowing these coefficients to be strictly-positive Ω-dependent functions: the constants C_{nm}, or the functions A_m, then simply become analytic functions of the variables $p_s\Omega_j$ over these strictly-positive domains. In fact, these coefficients can be continued into the complex domain without affecting the character of the singularity at $r_0 = 0$ provided we keep each coefficient away from the cut along the negative real axis in that variable, and keep the point C in the space of the collection of these coefficients away from all points where $a\lambda + P(C, x_j, r_j) = 0$ for some point in the product of the open domains of integration $0 < \lambda < 1$, and (for all j) $0 < x_j < 1$ and $0 < r_j < 1$. Here $a = p_s\Omega_i$.

The points in the domain of integration over the variables r_j, x_j, Ω_j that contribute to the singularity at $\varphi = 0$ are points where each of the three star-line factors is evaluated at, or very close to, the associated pole. The arguments in section 8 show that in this region the first of the variables $p_s\Omega_j$, namely $p_s\Omega_i = a$ can be shifted into upper-half plane $\mathrm{Im}\, a > 0$, and the collection of contours C can be distorted so that $a\lambda + P$ is shifted into the upper-half-plane provided $0 < \lambda < 1$ and, for all j, $0 < r_j < 1$ and $0 < x_j < 1$. This is exactly the condition that is needed to justify the extension of the results obtained above for positive real coefficients to the complex points of interest.

The dependence of the distortion of the contour on λ needs to be described. When one introduces the nonmeromorphic couplings, and hence the $\int d\lambda$, into

the formula, the Landau matrix acquires a new column, the $d\lambda$ column. However, the parameter λ enters in an almost trivial way: the pole residues associated with the side s of the triangle into which the vertex associated with λ is coupled are changed from $p_s(\Omega_j + ...)$ into $(p_s + \lambda_i k_i)(\Omega_j + ...)$, and the pole denominator $(p_s)^2 - m^2$ is changed to $(p_s + \lambda_i k_i)^2 - m^2$. The new set of Landau equations can be satisfied at each of the two end points $\lambda_i = 1$ and $\lambda_i = 0$. These two solutions correspond to diagrams in which the vertex associated with λ_i is placed at one end or the other of the side s of the triangle. Both solutions to the triangle-diagram equations exist, and, because of the null contributions in all $d\Omega_j$ columns, the two solutions yield two different ways of distorting the Ω_j contours, Δ_1 and Δ_0, the first corresponding to $\lambda_i = 1$, the second corresponding to $\lambda_i = 0$. An allowed distortion that gives these two cases and smoothly interpolates to the intermediate values of λ_i is $\lambda_i\Delta_1 + (1 - \lambda_i)\Delta_0$. It keeps the imaginary part of the pole denominator strictly positive (near the zero of the real part) for all values of λ_i in the domain $0 \leq \lambda_i \leq 1$. This distortion, or some approximation to it, can be used in the argument given above.

For the remaining integrations on the dr_j $(j \leq i)$ one uses Propositions I.4(ii) and I.5, and Remark I.3(ii). This gives for the singular part of $F(q)$ at $\widehat{q}$ a function of form F′, in some appropriately scaled variable $\varphi(q)$, multiplied by an analytic function of q.

Appendix K. Computation for Self-Energy Case.

Contributions from photon lines j coupled nonmeromorphically on both ends into the same side s of the triangle were excluded from the discussion in Appendix J. For these values of j one can, in order to exclude double counting, impose the condition $\lambda_j \geq \lambda'_j$, where λ_j is the λ-parameter associated with the nonmeromorphic coupling on the tail of photon line j, and λ'_j is the λ-parameter associated with the nonmeromorphic coupling on the head of line j. Momentum k_j flows along line j from its tail to its head, according to our conventions.

The formulas of section 2 refer to momentum k_j flowing out of the charged-particle line at the tail of the photon line j. The coupling at the head can be treated like the coupling at the tail, but with a reversal of the sign of k_j. Then the effect of the two couplings into the same side s is to replace p_s by $p_s + (\lambda_j - \lambda'_j)k_j$, and to integrate on λ_j from zero to one and on λ'_j from zero to λ_j. The condition $\mathrm{Im}\, p_s\Omega_j > 0$ then retains its usual from. The reduction of the domain of integration does not upset the arguments of Appendix J.

To bring this case into accord with Appendix J we use the following transformations:

$$
\begin{aligned}
&\int_0^1 d\lambda \int_0^\lambda d\lambda' f((\lambda - \lambda')k) \\
&= \int_0^1 \lambda d\lambda \int_0^1 d\lambda'' f(\lambda(1-\lambda'')k) \\
&= \int_0^1 \lambda d\lambda \int_0^1 d\lambda''' f(\lambda\lambda''' k) \\
&= \int_0^1 dh f(hk) \int_0^1 \lambda d\lambda \int_0^1 d\lambda''' \delta(h - \lambda\lambda''') \\
&= \int_0^1 dh f(hk) \int_0^1 d\lambda \int_0^1 d\lambda''' \delta(\frac{h}{\lambda} - \lambda''') \\
&= \int_0^1 dh f(hk) \int_h^1 d\lambda \\
&= \int_0^1 dh f(hk)(1-h).
\end{aligned}
$$

The variable h plays the role played by λ in appendix J.

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