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Theory of X-Ray Holography Including Phase and Amplitude Contrast and Finite Transfer Functions

M.R. Howells

January 1989

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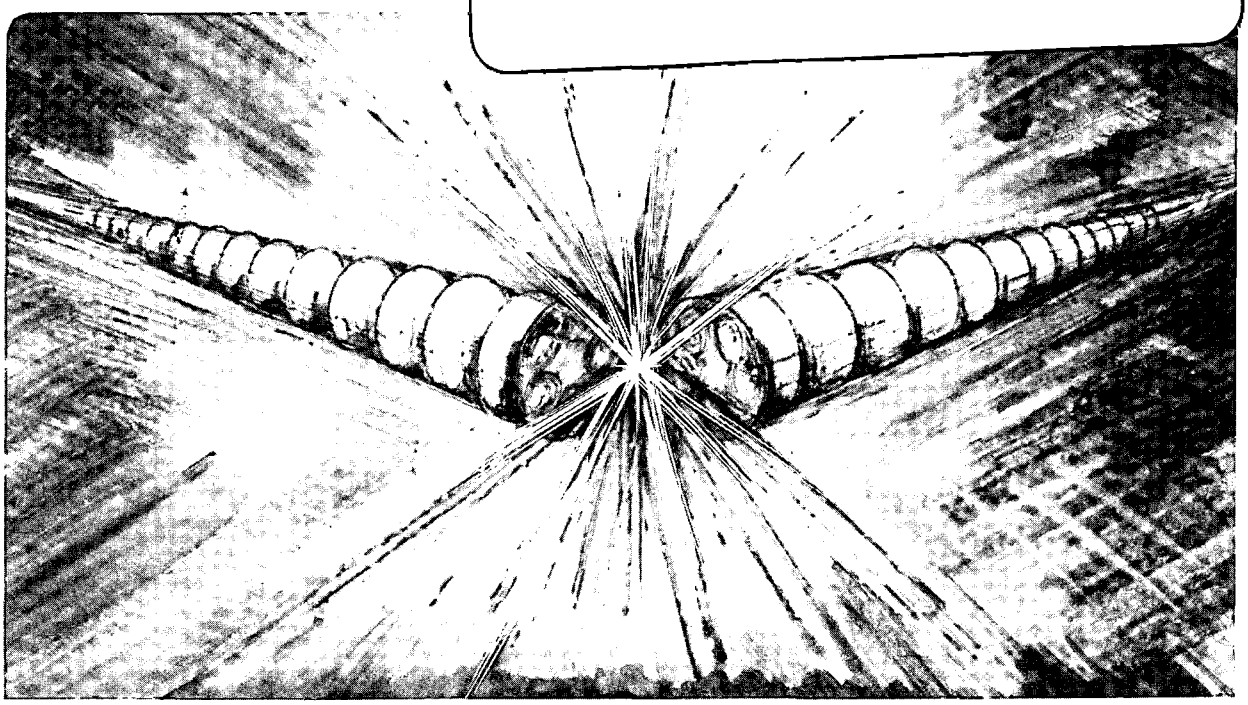

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THEORY OF X-RAY HOLOGRAPHY

INCLUDING PHASE AND AMPLITUDE CONTRAST

AND FINITE TRANSFER FUNCTIONS

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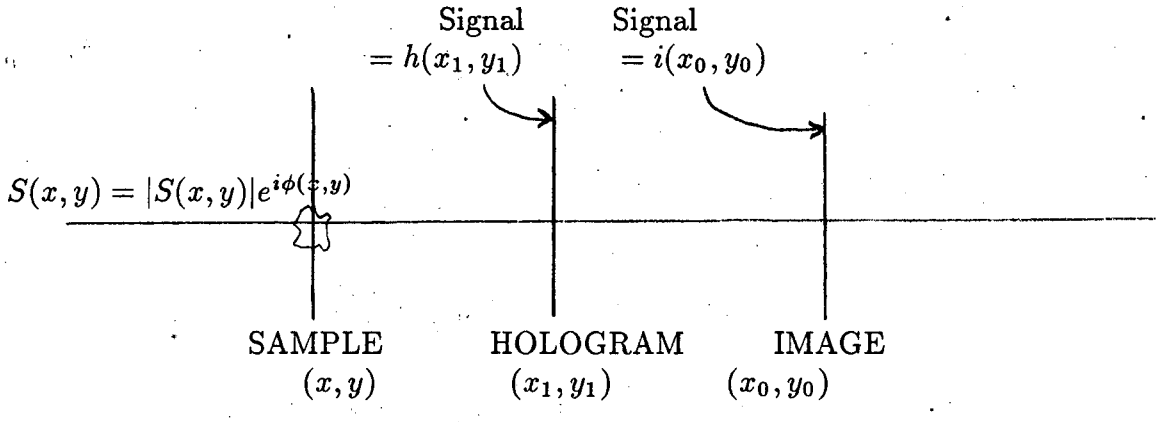
Introduction

We present the theory in broadly the same way as for visible holography, specializing to the Gabor, in-line case with plane-wave illumination, but allowing phase and amplitude contrast and a fairly general transfer function. Much of this theory is treated by a number of earlier authors [1,2,3] but none with the particular choice of issues that we wish to address. These are those mentioned plus numerical reconstruction from a digitized hologram and of course application to the soft x-ray spectral range. For simplicity and clarity, we consider first an amplitude and phase object consisting of a single, on-axis pixel and an ideal, perfect transfer function. The notation is explained in Figure 1 and is similar to that of Goodman [4].



Abbreviations: $r_i^2 = x_i^2 + y_i^2$ signal $\rightarrow$ lower case

$\mathbf{x}_i = x_i\mathbf{i} + y_i\mathbf{j}$ transform $\rightarrow$ upper case

Fig.1

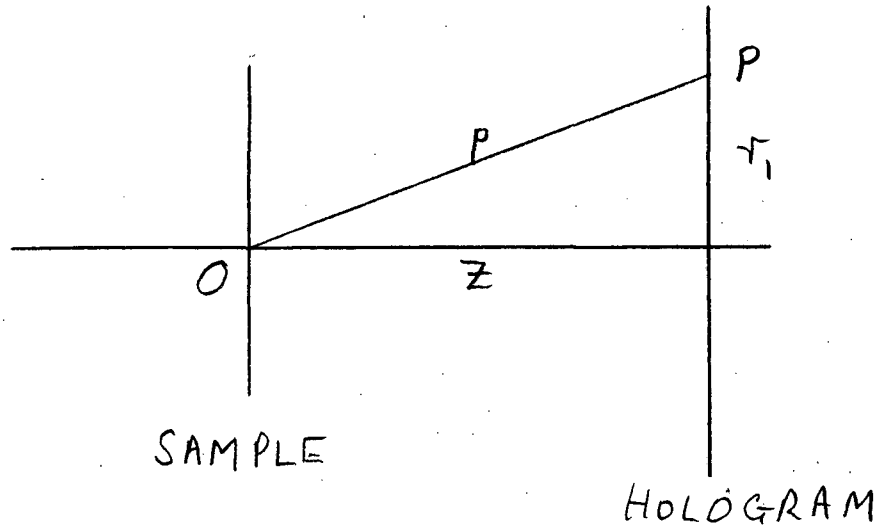


Fig. 2

Hologram Formation When The Sample Has Only One Pixel

A reference wave $\mathbf{r}$ is incident from the left and a small fraction of it is scattered by the sample. The approximately spherical scattered wave and the approximately plane reference wave are superposed in the hologram plane. From Fig. 2 the intensity of the hologram is given by

$$I_H = (\mathbf{h} + \mathbf{r})(\mathbf{h}^* + \mathbf{r}^*) \tag{1}$$

where $\mathbf{h} = he^{i\phi_h}$, $\mathbf{r} = re^{i\phi_r}$. Therefore

$$I_H = h^2 + r^2 + ar\left[e^{i(\phi_h - \phi_r)} + e^{-i(\phi_h - \phi_r)}\right] = h^2 + r^2 + 2arcos(\phi_h - \phi_r) \tag{2}$$

If we assume (temporarily) that the reference wave and scattered wave are in phase in the sample plane then using the notation of Fig. 2,

$$\Delta\phi = \phi_h - \phi_r = \frac{2\pi}{\lambda}(p - z) = k\left[\sqrt{z^2 + r_1^2} - z\right] \simeq k\left[z\left(1 + \frac{1}{2}\frac{r_1^2}{z^2} + \cdots\right) - z\right] \simeq \frac{kr_1^2}{2z} \tag{3}$$

Therefore

$$I_H = r^2\left\{1 + \left(\frac{h}{r}\right)^2 + 2(\frac{h}{r})\cos\left(\frac{kr_1^2}{2z}\right)\right\} \tag{4}$$

Now for x-ray holography, the sample is always a weak scatterer: $h/r \ll 1$. (The value of h/r is calculated in the next section.) This means that we can always neglect the "intermodulation" term $(h/r)^2$ and all the other terms of this order and higher that might be generated by system non-linearities. Equation (4) now represents a system of cosine fringes with center positions the same as the zones of a Fresnel Zone plate. Such an arrangement is known as a Gabor Plate. The fringe visibility can be determined from Eq.(4) as follows:

$$I_{\max} = r^2 + 2hr : \qquad (\cos\ \text{term} = +1)$$

$$I_{\min} = r^2 - 2hr : \qquad (\cos \text{ term} = -1)$$

$$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{2h}{r} \tag{5}$$

where we assume the scattered wave is uniform over the small angle of collection and V and h are related to the strength of the pixel in the sample plane that scattered the wave.

Coherent X-ray Scattering by a Single Pixel

To understand the physics of how a single pixel at $\mathbf{x}$ in the sample plane scatters x-rays to form a Gabor plate of contrast $V(\mathbf{x})$ in the hologram, we start with the differential cross-section for Thompson scattering of unpolarized x-rays by a single electron [5]

$$\frac{d\sigma}{d\Omega} = R_0^2 \, \frac{1 + \cos\theta}{2} \tag{6}$$

where σ is the cross-section, Ω the solid angle, R_0 the classical electron radius and θ the angle of the scattered x-ray to the axis. We are interested in scattering into a small aperture of half-angle θ, so $d\Omega \simeq \pi\theta^2$ and we will consider a pixel with a coherent scattering strength equivalent to F electrons. Equation (6) then gives

$$d\sigma_F = \pi R_0^2 \theta^2 |F|^2 \ .$$

We can now calculate the intensity ratio (photons scattered into the aperture/reference photons in the same aperture), K, of the interfering beams

$$K = \frac{\frac{d\sigma_F}{\Delta} I_0 \Delta}{I_0 \pi z^2 \theta^2} = \frac{R_0^2 |F|^2}{z^2} \ , \tag{6a}$$

where I_0 is the incoming flux and Δ the geometrical area presented to the incoming beam by the pixel.

Now we know that $V = 2h/r$, which means that it depends on the ratio of the amplitudes of the interfering beams or $\sqrt{K}$. Thus

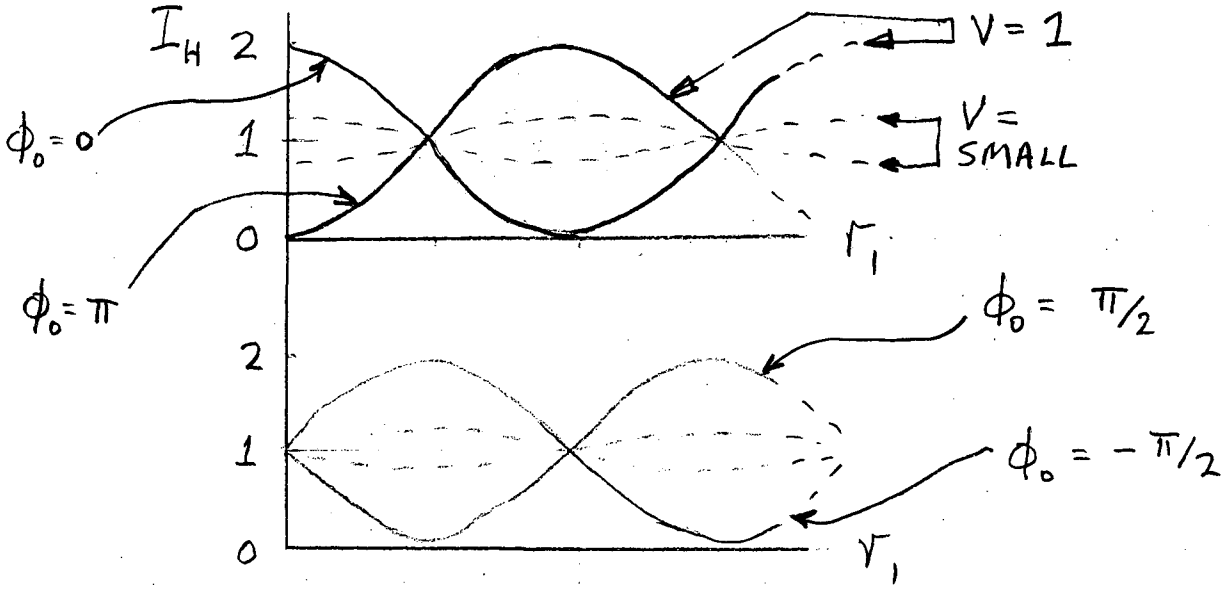
$$V = \frac{2R_0|F|}{z} \tag{6b}$$

F can be found in a given case as $F = F_1 + iF_2$ where F_1 and F_2 are the Henke scattering factors f_1 and f_2 summed over the atoms in the pixel [6].

Normalization of the Diffraction Pattern

So far we have not represented the sample properly. We now consider the axial pixel with complex transparency

$$S(\mathbf{x})\delta(\mathbf{x}) \equiv s_0 e^{\phi_0} \ . \tag{7}$$



$$I_H = 1 + V\cos\left(\frac{kr_1^2}{\lambda z} + \phi_0\right) \quad (\text{eq } 10)$$

Fig. 3

We define ϕ_0 to include the phase change $-\pi/2$ that takes place on scattering, so ϕ_0 equals the actual phase delay in the pixel minus $\pi/2$ (delays are considered positive). For an amplitude object we would have $\phi_0 = -\pi/2$. To get the relation between s, V and h we use the Fresnel-Kirchoff equation in the Fresnel approximation [4], Eq.(A1), from which, using Eq.(7) and the definition of $\delta(\mathbf{x})$

$$\mathbf{h}(0) = \frac{r}{i\lambda z} s_0 e^{i\phi_0} \exp\left[\frac{i\pi}{\lambda z}|\mathbf{x}_1|^2\right]\Bigg|_{x_1=0} \Delta \tag{7a}$$

or

$$h(0) \equiv |\mathbf{h}| = \frac{r s_0 \Delta}{\lambda z} \tag{8}$$

which is constant over the hologram in our approximation. Thus

$$V = \frac{2h}{r} = \frac{2 s_0 \Delta}{\lambda z} \tag{9}$$

From Eqs. (7a) and (4) we now see how to incorporate ϕ_0, and we finally get for I_H

$$I_H(\mathbf{x}_1) = r^2 \left\{1 + V\cos\left(\frac{kr_1^2}{\lambda z} + \phi_0\right)\right\} \tag{10}$$

The hologram is thus a phase-shifted Gabor Plate of contrast V. The effect of the phase shift is given in Fig.(3), where use has been made of

$$\cos\left(\frac{kr_1^2}{\lambda z} \pm \phi_0\right) = Re\left[\exp i\left(\frac{kr_1^2}{\lambda z} \pm \phi_0\right)\right] = \cos\left(\frac{kr_1^2}{\lambda z}\right)\cos\phi_0 \pm \sin\left(\frac{kr_1^2}{\lambda z}\right)\sin\phi_0 \,.$$

Image Reconstruction

Turning now to the reconstruction, we consider only the case of illumination with the original plane reference beam, because that is what we simulate in a numerical reconstruction. Assuming for the moment that the record-develop-readout sequence is perfectly linear, we can take a hologram record equal to I_H. Using Eq.(1A) we then have

$$i(\mathbf{x}_1) = \frac{r}{i\lambda x}\int_{holog} I_H(\mathbf{x}_1) e^{\frac{i\pi}{\lambda z}[(x_0-x_1)^2+(y_0-y_1)^2]}\, dx_1 dy_1 \tag{11}$$

Substituting for $I_H(\mathbf{x}_1)$ from Eq.(10)

$$I_H(\mathbf{x}_1) = r^2\left\{1+\frac{V}{2}\left[e^{\frac{ikr_1^2}{2z}}e^{i\phi_0}+e^{\frac{-ikr_1^2}{2z}}e^{-i\phi_0}\right]\right\} \tag{12}$$

and expanding the square in the exponent of Eq.(11)

$$\begin{aligned} i(\mathbf{x}_0) = \frac{r^3}{i\lambda z}e^{\frac{i\pi}{\lambda z}(x_0^2+y_0^2)}\Bigg\{&\int_{-\infty}^{+\infty} 1\cdot e^{\frac{i\pi}{\lambda z}(x_1^2+y_1^2)}e^{\frac{-2\pi i}{\lambda z}(x_0x_1+y_0y_1)}\, dx_1\, dy_1 \\ &+\frac{V}{2}\int_{-\infty}^{+\infty}\left[e^{\frac{i\pi r_1^2}{\lambda z}}e^{i\phi_0}+e^{\frac{-i\pi r_1^2}{\lambda z}}e^{-i\phi_0}\right]e^{\frac{i\pi}{\lambda z}(x_1^2+y_1^2)}e^{\frac{-2\pi i}{\lambda z}(x_0x_1+y_0y_1)}\, dx_1\, dy_1\Bigg\} \end{aligned} \tag{13}$$

The first integral is the Fourier transform of a propagator [Eq.(A3b)]. Some explanation of the algebra of propagators is given in the Appendix. Inserting its value (which cancels the factor in front) and compressing the second term, we get (remembering $x_1^2+y_1^2=r_1^2$)

$$\begin{aligned} \frac{i(\mathbf{x}_0)}{r^3} = 1+\frac{e^{\frac{i\pi r_0^2}{\lambda z}}}{i\lambda z}\Bigg\{&\frac{Ve^{i\phi_0}}{2}\int_{-\infty}^{+\infty} e^{\frac{2\pi i r_1^2}{\lambda z}}e^{\frac{-2\pi i}{\lambda z}(x_0x_1+y_0y_1)}\, dx_1 dy_1 \\ &+\frac{Ve^{-i\phi_0}}{2}\int_{-\infty}^{+\infty} e^{\frac{-2\pi i}{\lambda z}(x_0x_1+y_0y_1)}\, dx_1 dy_1\Bigg\} \end{aligned} \tag{14}$$

The first term is another Fourier transform of a propagator, and the second is the definition of a δ-function [Eq.(A2)]. Evaluating these,

$$\frac{i(\mathbf{x}_0)}{r^3} = 1+\frac{e^{\frac{i\pi r_0^2}{\lambda z}}}{i\lambda z}\left\{\frac{Ve^{i\phi_0}}{2}\frac{i\lambda z}{2}e^{\frac{-i\pi r_0^2}{2\lambda z}}+\frac{Ve^{-i\phi_0}}{2}\lambda^2 z^2\delta(x_0)\delta(y_0)\right\} \tag{15}$$

$$= 1+\frac{Ve^{i\phi_0}}{4}e^{\frac{i\pi r_0^2}{2\lambda z}}-i\lambda z\frac{Ve^{-i\phi_0}}{2}\delta(\mathbf{x}_0) \tag{16}$$

These three terms are respectively, the zero-order term, the unwanted, out-of-focus term from the virtual image and the real image of the object. The unwanted term combined with the zero-order term will form another Gabor plate, this time at distance $2z$ from the sample as indicated by the replacement $z \to 2z$ in the propagator. The image term is

a perfect image because we have considered an infinite hologram with an ideal transfer function. We will gradually introduce more realistic assumptions as we proceed.

For the present case we have defined our single pixel to be on-axis (always possible with a single pixel), so specializing Eq.(16) to $\mathbf{x}_0 = 0$

$$i(0,0) = r^3 \left[1 - \frac{iVe^{i\phi_0}}{4} + \frac{Ve^{-i\phi_0}\lambda z \delta(\mathbf{x}_0)}{2}\right] \tag{17}$$

If we had an amplitude object, this would be (setting $\phi_0 = -\pi/2$)

$$i(0,0) = r^3 \left[1 - \frac{iV}{4} + \frac{V\lambda z \delta(\mathbf{x}_0)}{2}\right] \tag{18}$$

Examination of Eq.(18) shows that the image signal and unwanted twin-image contribution can be distinguished because one is real and the other imaginary. In Eq.(17) it is possible to determine both V, which gives the amplitude of the object, and ϕ_0, which is its phase, from a knowledge of the complex number $i(0,0)$. Note that obtaining both the amplitude and the phase of the image is one of the advantages of numerical reconstruction compared to use of a laser beam. Thus, for a single pixel, the twin-image signal can always be distinguished. Unfortunately, when there is a large number of pixels, the separation of the twin-image contribution is more difficult but, as we shall see, not impossible. We continue with consideration of the single-pixel case a little further because it allows us a clearer treatment of the effect of the non-ideal transfer function of the overall system.

Non-Ideal Frequency Transfer Functions

First, suppose that the spatial frequency response of the system is ideal except for the use of finite optical aperture of radius a. We can allow for this condition in our treatment by replacing the last integral in Eq.(14) with

$$I_1 = \int_{-\infty}^{+\infty} \operatorname{circ}\left(\frac{r_1}{a}\right) e^{\frac{-2\pi i}{\lambda z}(x_1 x_0 + y_1 y_0)}\, dx_1 dy_1 \tag{19}$$

where

$$\operatorname{circ}\left(\frac{r_1}{a}\right) = \begin{cases} 1, & \text{if } r_1 \le a; \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

Equation (19) is a two-dimensional Fourier transform of a circularly symmetric function which we can represent as a function of a single frequency variable $\rho = \frac{r_0}{\lambda z}$, by means of the Hankel transform [4].

$$I_1 = 2\pi \int_0^{\infty} \operatorname{circ}\left(\frac{r_1}{a}\right) J_0(2\pi r_1 \rho) r_1\, dr_1\,. \tag{21}$$

Changing the variable to $s = r_1/a$, we obtain the standard form [7]

$$I_1 = 2\pi a^2 \int_0^1 J_0(2\pi s a \rho) s\, ds \tag{22}$$

This gives us finally

$$I_1 = \pi a^2 \left[\frac{2J_1(2\pi a\rho)}{2\pi a\rho}\right], \tag{23}$$

where the function in the square bracket equals unity for $\rho = 0$. This normalization is confirmed by examination of Eq.(19) with $x_0 = y_0 = r_0 = 0$.

If we now replace the previous value of the integral I_1, namely $\lambda^2 z^2 \delta(\mathbf{x}_0)$, with Eq.(23), we obtain a new form for Eq.(16), *viz.*

$$\frac{i(\mathbf{x}_0)}{r^3} = 1 + \frac{ve^{i\phi_0}}{4} e^{\frac{i\pi r_0^2}{2\lambda z}} - \frac{ve^{-i\phi_0}}{2} \frac{e^{\frac{i\pi r_0^2}{\lambda z}}}{i\lambda z} \pi a^2 \left[\frac{2J_1(2\pi a\rho)}{2\pi a\rho}\right] \tag{24}$$

Examination of the above treatment shows us how to deal with more general frequency transfer functions that could more adequately account for real-world limitations to the recording, readout and processing of x-ray holographic fringe patterns. Provided we know the modulus of the frequency transfer function, say $T(\rho)$, that represents our methods for dealing with fringes of frequency ρ, we substitute $T(\rho)$ for $\left[\frac{2J_1(2\pi a\rho)}{2\pi a\rho}\right]$. $T(\rho)$ must be normalized to unity for $\rho = 0$. πa^2 is replaced by the integrated signal over the whole aperture divided by the signal per unit area at $\rho = 0$. A practical way to find $T(\rho)$ is to examine the available data, namely the hologram. It can be shown that the power spectrum of an ideally recorded hologram of a single pixel is

$$W(\rho) = \delta(\rho) + 4r^2 s_0^2 \cos^2[\pi\rho^2\lambda z - \phi_0] \tag{25}$$

For a superposition of a large number of such recordings with different values of ϕ_0 and s_0, we can argue that the average power spectrum will be constant. In this case the function $T(\rho)$, which is to represent non-ideal effects, is simply the square root of the hologram power spectrum suitably normalized to unity "near" $\rho = 0$ to avoid the $\delta(\rho)$.

General Samples with Many Pixels

We now consider the theory of Gabor holography for a general sample with many pixels giving both phase and amplitude contrast. We will use the algebra of optical propagators (functions like $\exp[i\pi|\mathbf{x}|^2/\lambda z]$) which is given in detail in the Appendix. Using the same notation as before, we get for the hologram intensity pattern

$$I_H = (\mathbf{h} + \mathbf{r})(\mathbf{h}^* + \mathbf{r}^*) = |\mathbf{r}|^2 \left[1 + \frac{|\mathbf{h}|^2}{|\mathbf{r}|^2} + \frac{\mathbf{h}}{\mathbf{r}} + \frac{\mathbf{h}^*}{\mathbf{r}^*}\right]_0$$

Neglecting the intermodulation term and considering the reference wave to be real and constant, we form a reconstruction by illuminating the hologram with the original reference wave. For the wave just behind the hologram, we obtain

$$rI_H = r^2[r + \mathbf{h} + \mathbf{h}^*] \tag{26}$$

where

$$\mathbf{h}(\mathbf{x}_1) = \frac{r}{i\lambda z} s(\mathbf{x}) \; * \; e^{\frac{i\pi|\mathbf{x}|^2}{\lambda z}} \Big|_{x_1} \tag{27}$$

and

$$\mathbf{h}^*(\mathbf{x}_1) = \frac{-r}{i\lambda z} \; s^*(\mathbf{x}) \; * \; e^{\frac{-i\pi|\mathbf{x}|^2}{\lambda z}} \Big|_{x_1} \tag{28}$$

which are from Eq.(A1). The superscript * indicates a complex conjugate; otherwise the $*$ symbol indicates convolution.

To get the complex amplitude at the real image, we must propagate the signal rI_H in Eq.(26) to the real image plane at $+z$. Therefore,

$$i(\mathbf{x}_0) = r^2[r + \mathbf{h}(\mathbf{x}_1) + \mathbf{h}^*(\mathbf{x}_1)] \; * \; \frac{1}{i\lambda z} e^{\frac{i\pi|\mathbf{x}|^2}{\lambda z}} \Big|_{x_0} \tag{29}$$

The first term of this equation is a convolution of a propagator with a constant. Since the algebra of propagators is to hand, we regard this as a convolution of two propagators with $z_1 = \infty$ and $z_2 = z$ [see Eq. (A.4b)]. The second and third terms are double convolutions involving the unknown function $s(\mathbf{x})$ and two propagators. Since convolution is commutative, we may consider the propagator parts first to obtain

$$i(\mathbf{x}_0) = r^3 \left\{ 1 - \frac{s^*(\mathbf{x})}{2} * e^{\frac{i\pi|\mathbf{x}|^2}{\lambda z}} \Big|_{x_0} - i\lambda z s(\mathbf{x}) * \delta(\mathbf{x}) \Big|_{\mathbf{x}_0} \right\} . \tag{30}$$

Here, the z-values used to convolve the two propagators by means of Eq.(A4b) are: for the unity term, $z_1 = \infty \;\; z_2 = z$; for the second term, $z_1 = z \;\; z_2 = z$; and for the final term, $z_1 = -z \;\; z_2 = z$. A further complication, explained in the appendix, gives rise to the delta function in the image term. Using the sifting property of the latter we finally obtain

$$i(\mathbf{x}_0) = r^3 \left\{ 1 - \frac{s(\mathbf{x})}{2} * e^{\frac{i\pi|\mathbf{x}|^2}{2\lambda z}} \Big|_{x_0} - i\lambda z s(\mathbf{x}_0) \right\} . \tag{31}$$

The out-of-focus, twin-mage term has now become a functional that involves the superposition of signals from **all** points of the object. Since the object is unknown, we cannot, immediately, distinguish the twin-image signal from the zero-order term (unity) and the real image (third term).

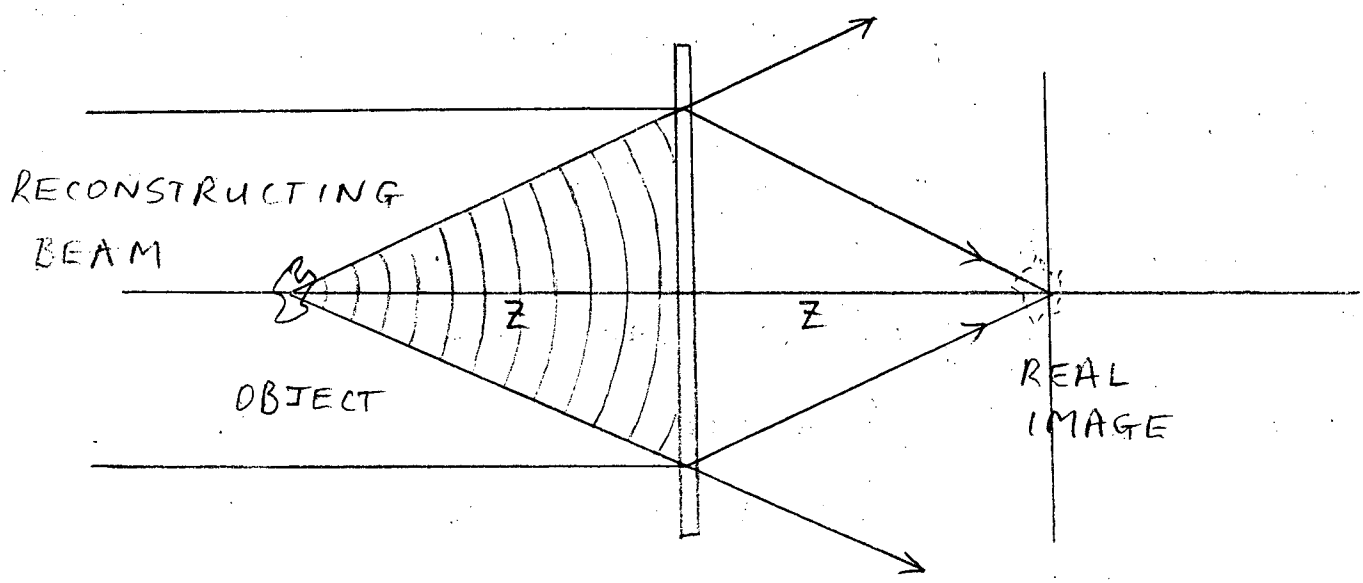
Expression (31) has been derived for the case of a hologram of infinite extent and ideal transfer function. We can include the effects of a finite aperture by making the replacement

$$\lambda^2 z^2 \delta(\mathbf{x}) \rightarrow \pi a^2 \left[\frac{2J_1(2\pi a\rho')}{2\pi a\rho'} \right] \tag{32}$$

where $\rho'(= r/\lambda z)$ is the dummy variable in the last convolution in Eq.(30). We are assuming here that the effect on the twin image term of diffraction at the aperture is negligible. The consequence of more general transfer function effects can be included as before.

Twin Image Suppression by Iterative Procedures

From the diagram one can see why the image signal is a second hologram at distance $2z$. Provided the hologram is much larger than the object, we can anticipate that the energy of the twin-image signal that arrives in or near the area of the real image will be small compared to the energy diffracted into the real image. If this is so, then an iterative approach to the signal-processing problem posed by Eq.(31) suggests itself. On the above assumption, the result $i(\mathbf{x}_0)$ in Eq.(31) minus the zero-order term should provide a reasonable approximation to $s(\mathbf{x})$. Therefore, we can use it instead of $s(\mathbf{x})$ in the second term of (31), and thereby obtain an approximation to the twin-image term and an improved approximation to the desired real-image term. This procedure can be repeated to give an iterative algorithm for analyzing the data. Such an algorithm has the advantage that it only requires a calculation (convolution with a propagator) that must have been coded already to get the first approximation.



APPENDIX

Fresnel-Kirchoff Equation:

Using our notation the diffracted signal, ψ, due to a plane wave of amplitude r incident on a complex transparency $t(\mathbf{x})$ is given by [4]

$$\psi(\mathbf{x}_1) = \frac{r}{i\lambda z}\int_{\text{transp.}} t(\mathbf{x})\exp\left[\frac{i\pi}{\lambda z}|\mathbf{x}-\mathbf{x}_1|^2\right] d\mathbf{x} \equiv \frac{r}{i\lambda z} t(\mathbf{x}) * \exp\left(\frac{i\pi|\mathbf{x}|^2}{\lambda z}\right) \qquad (A1)$$

This definition is based on the Fresnel approximation.

Dirac Delta Function:

If we take the defining property of the δ-function to be

$$\int_{-\infty}^{+\infty} f(x)\delta(x-x_0)\,dx = f(x_0)\,,$$

then the statement

$$\delta(x) = \frac{1}{\lambda z}\int_{-\infty}^{+\infty} e^{-\frac{2\pi i x x'}{\lambda z}}\,dx' \qquad (A2)$$

should be understood to mean

$$I_2 = \frac{1}{\lambda z}\int_{-\infty}^{+\infty}\int e^{-\frac{2\pi i x x'}{\lambda z}} f(x)\,dx_1 dx = f(0)\,.$$

In fact, changing the order of integration, this gives

$$I_2 = \int_{-\infty}^{+\infty} F(x_1/\lambda z)\,d(x_1/\lambda z) = f(0)\,,$$

where F is the Fourier transform of f. The final equality follows because the integral is the same as the inverse Fourier transform of F at $x = 0$. This shows that Eq.(A2) is equivalent to the defining equation of the δ-function.

Propagator Algebra:

We would like to do various operations with the function

$$\psi(\mathbf{x}:d) = e^{\frac{i\pi|\mathbf{x}|^2}{\lambda z}}$$

which represents the amplitude distribution in the $\mathbf{x}$-plane due to a unit spherical wave originating from a point at distance z. Such functions are variously known as propagators, Fresnel functions and Vander Lugh functions, and are the impulse response function

(Green's function) of the scalar diffraction problem in the Fresnel approximation. An elaborate algebra for them is provided in the engineering literature [10,11]. Consider the one-dimensional Fourier transform

$$\mathcal{F}\left[e^{\frac{i\pi x^2}{\lambda z}}\right] = \int_{-\infty}^{+\infty} e^{\frac{i\pi x^2}{\lambda z}} e^{\frac{-2\pi i x x_1}{\lambda z}}\, dx\ .$$

Completing the square,

$$= e^{\frac{-i\pi x_1^2}{\lambda z}} \int_{-\infty}^{+\infty} e^{\frac{i\pi}{\lambda z}(x-x_1)^2}\, d(x-x_1)$$

This integral can be evaluated by use of the standard form

$$\int_{-\infty}^{+\infty} e^{\frac{-x^2}{2\sigma^2}}\, dx = \sigma\sqrt{2\pi}$$

where $-1/2\sigma^2 \equiv i\pi/\lambda z$, so that in one dimension we get

$$\mathcal{F}\left[e^{\frac{i\pi x^2}{\lambda z}}\right] = \sqrt{i\lambda z}\, e^{\frac{-i\pi x_1^2}{\lambda z}} \equiv \sqrt{i\lambda z}\, e^{-i\pi X_1^2 \lambda z} \qquad (A3a)$$

In two dimensions this leads to

$$\mathcal{F}\left[e^{\frac{i\pi|\mathbf{x}|^2}{\lambda z}}\right] = i\lambda z e^{\frac{-i\pi|\mathbf{x}|^2}{\lambda z}} \equiv i\lambda z e^{-i\pi|\mathbf{X}_1|^2\lambda z} \qquad (A3b)$$

where $X_1 = x_1/\lambda z$ is the true frequency-plane variable. These results allow us to find the convolution of two propagators:

$$C_1 = \exp[i\pi x^2/\lambda z_1] \ * \ \exp[i\pi x^2/\lambda z_2].$$

By the convolution theorem and Eq.(A3a)

$$\mathcal{F}(C_1) = \sqrt{i\lambda z_1}\, e^{-i\pi x_1^2 \lambda z_1} \sqrt{i\lambda z_2}\, e^{-i\pi x_1^2 \lambda z_2}\ ,$$

$$\mathcal{F}(C_1) = i\lambda\sqrt{z_1 z_2}\, e^{-i\pi x_1^2 \lambda(z_1+z_2)}$$

Now, using Eq.(A3a) again,

$$C_1 = \mathcal{F}^{-1}\mathcal{F}(C_1) = \sqrt{i\lambda\frac{z_1 z_2}{z_1+z_2}}\, e^{\frac{i\pi x^2}{\lambda(z_1+z_2)}} \qquad (A4a)$$

In two dimensions we have

$$C_2 = e^{\frac{i\pi|\mathbf{x}|^2}{\lambda z_1}} \ * \ e^{\frac{i\pi|\mathbf{x}|^2}{\lambda z_2}} = i\lambda\frac{z_1 z_2}{z_1+z_2} e^{\frac{i\pi|\mathbf{x}|^2}{\lambda(z_1+z_2)}} \qquad (A4b)$$

Formulae (4a and 4b) fail if $z_1 = -z_2$. This corresponds to a converging (diverging) spherical wave of radius z being propagated forward (backward) by z [Eq.(A1)]. In other words, we are computing the signal at a focus and we expect a singularity.

If $z_1 = z$, $z_2 = -z$, Eq.(A4b) becomes

$$C_2 = \lambda^2 z^2 \lim_{d\to 0} \left[\frac{1}{i\lambda d} e^{\frac{i\pi|\mathbf{x}|^2}{\lambda d}} \right] .$$

Now it is said that

$$\lim_{d\to 0} \left[\frac{1}{i\lambda d} e^{\frac{i\pi|\mathbf{x}|^2}{\lambda d}} \right] = \delta(\mathbf{x}) . \tag{A5}$$

For this to be so, we must have

$$\lim_{d\to 0} \frac{1}{i\lambda d} \int_{-\infty}^{+\infty} e^{\frac{i\pi|\mathbf{x}|^2}{\lambda d}} f(\mathbf{x})\, d\mathbf{x} = f(0) .$$

To prove this, consider the one-dimensional limit

$$L = \lim_{d\to 0} \frac{1}{\sqrt{i\lambda d}} \int_{-\infty}^{+\infty} e^{\frac{i\pi x^2}{\lambda d}} f(x)\, dx .$$

For this limit, the contribution to the integral is negligible for all values of x except ones lying near the stationary point $x = 0$ of the argument of the propagator. We may thus use the Principle of Stationary Phase, which in general form states the following [12]:

$$\int_{\text{Stat. pt.}}^{\infty} f(x) e^{ikg(x)}\, dx = \sqrt{\frac{-\pi}{2ig''(x_0)}} f(x_0) e^{ikg(x_0)} / \sqrt{k}$$

where $k = 2\pi/\lambda$ and x_0 is a stationary point of $g(x)$. In our case, $g(x) \equiv x^2/2d$ and $x_0 = 0$. Making these substitutions and doubling the result to account for both sides of the stationary point, we find $L = f(0)$. This shows that

$$\lim_{d\to 0} \left[\frac{1}{\sqrt{i\lambda d}} e^{\frac{i\pi x^2}{\lambda d}} \right] = \delta(x) .$$

Combining this with a similar result for $\delta(y)$, the desired two-dimensional result, Eq.(A5), follows.

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