LBL-27631

Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

Physics Division

Presented at the XVIII-th International Conference
on Differential Geometric Methods in Theoretical
Physics "Physics and Geometry,"
Lake Tahoe, CA, July 3–8, 1989, and to
be published in the Proceedings

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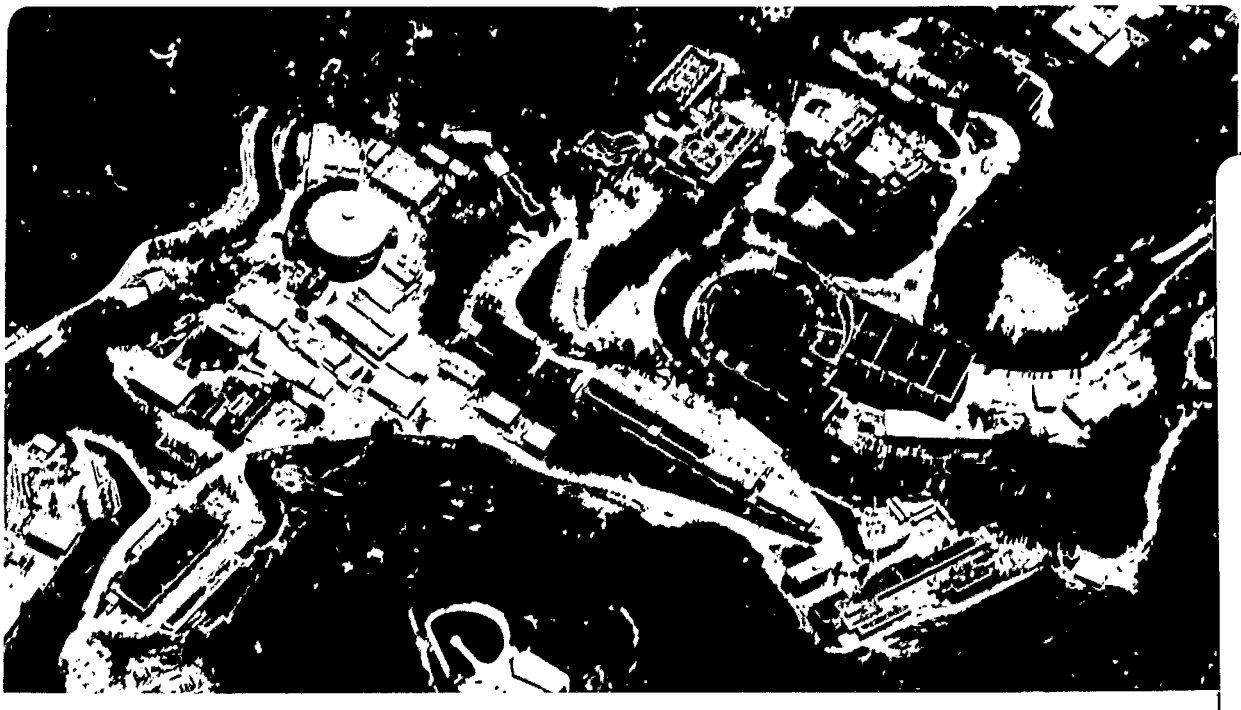
July 1989



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Flat Connection, Conformal Field Theory and Quantum Group*

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Abstract

General framework of linear first order differential equation for four-point conformal block is studied by using flat connection. Integrability and SL_2 invariance restrict possible form of flat connection. under a special ansatz classical Yang-Baxter equation appears as an integrability condition and the WZW model turns to be unique conformal field theory in that case. Monodromy property of conformal block can be easily determined by the flat connection.

* Talk given at the XVIII-th International Conference on Differential Geometric Methods in Theoretical Physics "Physics and Geometry" (July 3 - 8, Lake Tahoe). This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098.

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FLAT CONNECTION, CONFORMAL FIELD THEORY AND QUANTUM GROUP

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INTRODUCTION

Recently many people[1] are discussing general framework of rational conformal field theories (RCFT). There, one of the important concept is a connection matrix of conformal blocks. Once connection matrices are given for the conformal blocks, which are made from chiral vertex operators of a given chiral algebra by sandwiching them with SL_2 invariant vacuum, then we can construct the physical correlation functions invariant under monodromy transformations and determine operator product expansion coefficients in principle.

The connection matrices satisfy some polynomial equations[2] such as pentagon identity for example. Some people are trying to classify RCFT by solving the polynomial equations for connection matrices. The connection matrices, however, have only global information of conformal blocks. Moreover it is unclear how one can study dynamical or physical properties of the conformal blocks if playing only with the connection matrices.

On the other hand, we know that the conformal block satisfies a certain differential equation which is the result of null state structure of the representation.[3,4] The differential equation contains both global and local information of the conformal block.

In the present note we will study the differential equations for conformal blocks in general framework. Especially we would like to point out the importance of *flat connection*. It may give a new and more tractable tool for classification of RCFT.

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Indeed for a special case we can completely classify the flat connections and hence CFT. Another advantage of using flat connection in writing down a differential equation is that a monodromy property of conformal block can be easily determined. In some case we can see quantum group structure for connection matrices without giving their explicit forms.

DIFFERENTIAL EQUATIONS FOR CONFORMAL BLOCKS

It is known that four-point function is the first nontrivial multi-point function in CFT. Let us assume a four-point function ψ satisfy the following first order linear differential equation:

$$(\frac{\partial}{\partial z_i} - \omega_i)\,\psi(z_1, z_2, z_3, z_4) = 0 \qquad i = 1,2,3,4 \tag{1}$$

where ψ is a n-column vector of functions and ω_i's are $n \times n$ matrices of functions. In general, the number of fundamental solutions of (1), which are conformal blocks, is not greater than n.

Here one might wonder if differential equation for conformal block is always written in a first order form like (1). In the minimal model case, for instance, a correlation function (let us denote it by f) which contains primary field $\phi_{h_{(r,s)}}(z)$ is known to satisfy a differential equation of order rs with respect to z. Even in this case we can define a rs-column vector ψ consists of $f, \frac{d}{dz}f, \cdots$, and $(\frac{d}{dz})^{rs-1}f$ so that ψ satisfies first order equation of the form (1). With similar tricks we can see that for every known differential equation of four-point function we can rewrite it in first order form (1). Hence (1) is most general form.

Now let us consider requirements for ω_i. Among them two important things are (A) SL_2 invariance and (B) integrability.

(A) SL_2 *invariance*: Since we are considering CFT on a sphere, SL_2 invariance is one of necessary conditions. If we denote conformal dimension of each vertex operator by Δ_i ($i = 1,2,3,4$), then SL_2 invariance of ψ requires the following relations:

$$\sum_{i=1}^{4} \frac{\partial}{\partial z_i}\psi = 0 \tag{2a}$$

$$\sum_{i=1}^{4} (z_i \frac{\partial}{\partial z_i} + \Delta_i)\psi = 0 \tag{2b}$$

$$\sum_{i=1}^{4} (z_i{}^2 \frac{\partial}{\partial z_i} + 2z_i\Delta_i)\psi = 0 \tag{2c}$$

Using eq. (1), we can derive the conditions for ω_i:

$$\sum_{i=1}^{4} \omega_i\psi = 0 \tag{3a}$$

$$\sum_{i=1}^{4}(\Delta_i + z_i\omega_i)\psi = 0 \tag{3b}$$

$$\sum_{i=1}^{4}(2z_i\Delta_i + {z_i}^2\omega_i)\psi = 0 \tag{3c}$$

(B) *Integrability*: Eq. (1) consists of four equations. So they have to be consistent with each other. This is achieved if differential operators $\frac{\partial}{\partial z_i} - \omega_i$ $(i = 1,2,3,4)$ commute with each other. Thus we have another set of conditions:

$$\frac{\partial \omega_j}{\partial z_i} - \frac{\partial \omega_i}{\partial z_j} + [\omega_i, \omega_j] = 0 \tag{4}$$

If you regard ω_i as a connection, (4) tells ω_i has zero curvature. It is very natural that a flat connection governs conformal block since analytic continuation of conformal block function does not depend on small (infinitesimal) deformation of continuation path. We will see later how the flat connection actually determine the connection matrix.

Besides the above conditions (A) and (B), we need several physical requirements to obtain differential equations corresponding to some RCFT. Unitarity, boundary condition at infinity and gauge choice are some of examples. Classification of RCFT based on this direction is now under investigation. Here we describe a remarkable trial of classification using a certain ansatz.

Let us assume ω_i has following form:

$$\omega_i = \sum_{j\neq i} r_{ij}(z_{ij}) \tag{5}$$

where $z_{ij} = z_i - z_j$ and $r_{ij}(z_{ij})$ is a matrix of functions only depending on z_{ij} satisfying $r_{ij}(z_{ij}) = -r_{ji}(z_{ji})$. Then the integrability condition (4) becomes

$$[r_{ij}(z_{ij}), r_{ik}(z_{ik})] + [r_{ij}(z_{ij}), r_{jk}(z_{jk})] + [r_{ik}(z_{ik}), r_{jk}(z_{jk})] = 0 \tag{6}$$

This equation is nothing but *classical Yang-Baxter equation*[5] *(CYBE)!* Fortunately there exists a classification of its solutions by Belavin and Drinfel'd[6]; a) rational, b) trigonometric and c) elliptic solutions. It can be easily seen that among them only rational solution satisfies SL_2 invariance. The rational solution has a form

$$r_{ij}(z_{ij}) = \frac{\lambda}{2}\frac{1}{z_{ij}}\sum_a \rho_i(T^a) \otimes \rho_j(T^a) \tag{7}$$

where T^a's are the orthonormal basis of arbitrary simple Lie algebra and ρ_i is a certain representation. Normalization constant λ can be fixed by another requirement such as unitarity. Substituting this expression into eq. (1), we obtain Knizhnik-Zamolodchikov[4](KZ) equation.* This means the WZW model is a unique CFT

* Conversely, the fact that the integrability condition of KZ equation coincides with CYBE is first noted by Kohno[7].

under the ansatz (5). Although this ansatz does not include other interesting classes of CFT such as minimal model, the above discussion sufficiently exhibit powerfulness of this framework toward the classification of CFT.

MONODROMY PROPERTIES OF CONFORMAL BLOCKS

As mentioned before, an advantage of writing the differential equation in the form of eq. (1) is that it enables us to write down a formal result of analytic continuation of solution. Let γ be a path from a point $\{z_i\}$ to another point $\{z_i'\}$ in $C^4\backslash\Delta$ (we denote Δ as a subspace of C^4 in which any two coordinates z_i and z_j coincide). Then the analytic continuation $\psi_\gamma(z_i')$ of $\psi(z_i)$ along γ is expressed by a path-ordered integral:

$$\psi_\gamma(z_i') = P_\gamma \exp\Big(\int_{\{z_i\}}^{\{z_i'\}} \sum_j \omega_j dz_j\Big)\psi(z_i) \tag{8}$$

From this expression we can extract the property of connection matrix. Before doing so, let us consider gauge degrees of freedom of the differential equation.

Eq. (1) has a gauge invariance (we should rather say covariance) under the transformation

$$\begin{aligned} \psi &\to \psi' = \mathcal{U}\psi \\ \omega &\to \omega' = \mathcal{U}\omega\mathcal{U}^{-1} + (d\mathcal{U})\mathcal{U}^{-1} \end{aligned} \tag{9}$$

where $\mathcal{U}(z_1, z_2, z_3, z_4)$ is a gauge transformation matrix and $\omega = \sum_i \omega_i dz_i$. What is more important is a subclass of gauge transformation which leaves ω invariant. The condition that $\mathcal{U}$ leaves ω invariant is

$$d\mathcal{U} + [\mathcal{U}, \omega] = 0 \tag{10}$$

Then $\mathcal{U}\psi$ satisfies the same differential equation as ψ does, *i.e.* eq. (1), and can be expressed by a linear combination of fundamental solutions of (1). Defining a matrix $\Psi = (\psi^{(1)}, \psi^{(2)}, \cdots, \psi^{(n)})$ which consists of fundamental solutions $\psi^{(i)}$, we can summarize the action of $\mathcal{U}$ in the following form:

$$\mathcal{U}\Psi = \Psi U \tag{11}$$

where U is a constant matrix.‡ We call $\mathcal{U}$ that satisfies (10) *monodromy operator*§ and U *connection (monodromy) matrix.*

Now let us consider the braiding operation for which we take a path γ such that $z_i' = z_{p(i)}$ where $p(i)$ is a permutation. The analytic continuation operator along γ

$$T_\gamma = P_\gamma \exp\Big(\int_{\{z_i\}}^{\{z_{p(i)}\}} \omega\ \Big) \tag{12}$$

‡ We can easily see $d(\Psi^{-1}\mathcal{U}\Psi) = 0$ by using the relations (10), $d\Psi = \omega\Psi$, and $d\Psi^{-1} = -\Psi^{-1}\omega$.

§ Here the term *monodromy* is used somewhat extensively.

becomes a matrix of functions of z_i. Generally T_γ does not satisfy (10) unless $p(i) = i$. Instead,

$$dT_\gamma = \omega(z_{p(i)})\, T_\gamma - T_\gamma\, \omega(z_i) \tag{13}$$

is held. If we find a constant matrix $\mathcal{A}$ which satisfies $\mathcal{A}\omega(z_{p(i)})\mathcal{A}^{-1} = \omega(z_i)$, then $\mathcal{A}T_\gamma$ becomes a monodromy operator.

We take the WZW model for the illustration of above construction. In this case ω_i is given by eqs. (5) and (7), and ψ is a function of four vectors v_i of Lie algebra on which the representation $\rho_i(T^a)$ acts and four complex coordinates z_i. Taking an appropriate basis ψ becomes a column vector. Now a braiding operator, say $\mathcal{B}_{23}$, is constructed as follows: First we obtain an analytic continuation operator T_γ with an appropriate braiding path γ for z_2 and z_3

$$\psi_\gamma(v_1 \otimes v_2 \otimes v_3 \otimes v_4; z_1, z_3, z_2, z_4) = T_\gamma \psi(v_1 \otimes v_2 \otimes v_3 \otimes v_4; z_1, z_2, z_3, z_4) \tag{14}$$

where

$$T_\gamma = P_\gamma \exp\Big(\int_{\{z_1,z_2,z_3,z_4\}}^{\{z_1,z_3,z_2,z_4\}} \omega \Big) \tag{15}$$

Second we find a constant matrix which turns $\omega(z_1, z_3, z_2, z_4)$ back to $\omega(z_1, z_2, z_3, z_4)$; the answer is permutation operator $\mathcal{P}_{23}$ of $v_2 \otimes v_3$. Furthermore in this case we can explicitly perform the path-ordered integral and so the final result becomes

$$\mathcal{B}_{23} = \mathcal{P}_{23} e^{i\pi\lambda\Omega_{23}} \tag{16}$$

where $\Omega_{ij} = \frac{1}{2}\sum_a \rho_i(T^a) \otimes \rho_j(T^a)$.

We define a connection matrix $B_{i\,i+1}$ corresponding to braid operation $\mathcal{B}_{i\,i+1}$ according to the general story:

$$\mathcal{B}_{i\,i+1}\Psi = \Psi B_{i\,i+1} \tag{17}$$

Although the determination of B itself is generally a hard task, much of its important properties can be obtained from the properties of $\mathcal{B}$ *via* eq. (17). Indeed B and $\mathcal{B}$ have common properties such as eigenvalues and braid relation. The eigenvalues of $\mathcal{B}_{ij}$ are easily determined as $\epsilon_{ij}^k\, q^{c_k - c_i - c_j}$, where c_i is Casimir invariant evaluated on the representation ρ_i and $q = e^{i\pi\lambda}$. Index k runs over the representations which appear in the decomposition $\rho_i \otimes \rho_j = \oplus_k \rho_k$, and ϵ_{ij}^k is either $+1$ or -1 depending on whether ρ_k is constructed symmetrically or anti-symmetrically from ρ_i and ρ_j.

In this line of argument, Kohno[7] showed that the braiding matrices satisfy Hecke algebra if we choose SL_n for Lie algebra and n-dimensional fundamental representation for every ρ_i. Furthermore we can show[8,9] for the case of SU_2 that any connection matrices can be expressed by the $6j$-symbols[10] of *quantum* SU_2 and the normalization factors for the conformal blocks. They completely agree with the explicit evaluation of the connection matrices by Tsuchiya and Kanie.[11]

ACKNOWLEDGEMENTS

We would like to thank P. Valtancoli for the collaboration from which this work was inspired. We have also benefited from discussions with T. Ozawa. This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U. S. Department of Energy under Contract DE-AC 03-76SF00098.

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