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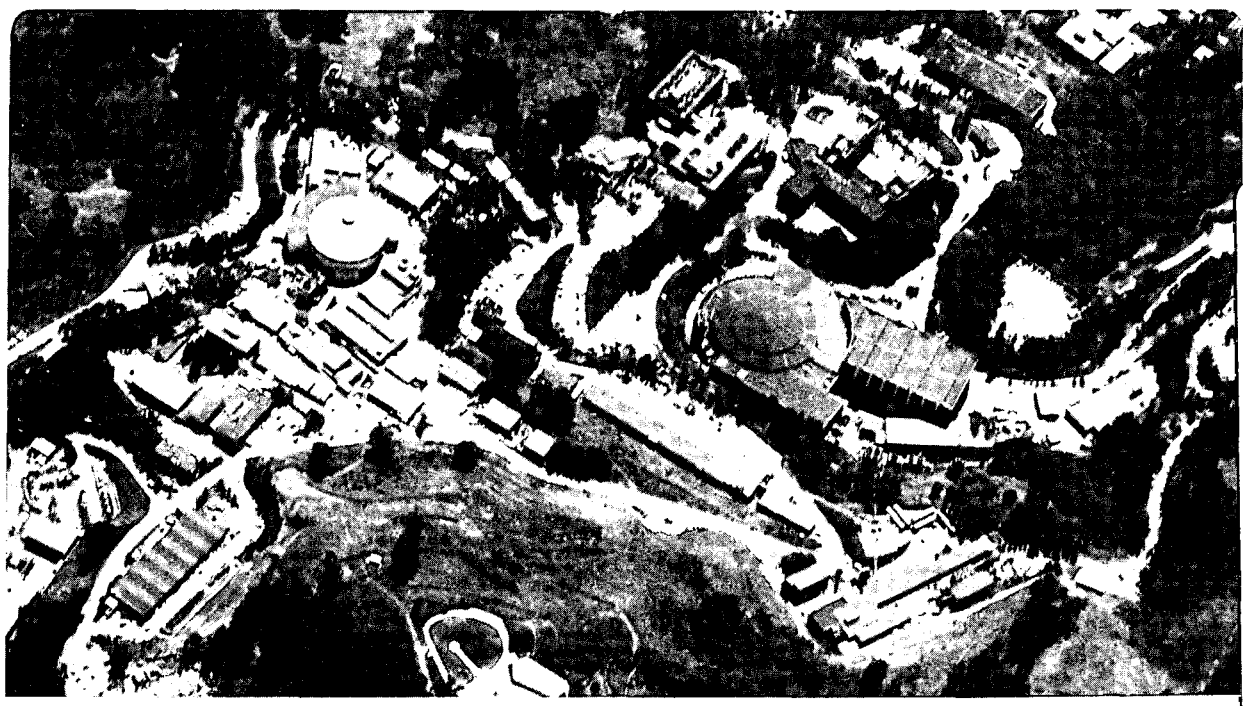

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CAPILLARY SURFACES IN MICROGRAVITY *

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CAPILLARY SURFACES IN MICROGRAVITY

Paul Concus and Robert Finn

We present here a selection of older and newer results on capillary surfaces, which in our view complement each other and form elements of a new and developing theory. Although much of the material applies in general gravity fields, we have tried to emphasize the particular interest related to a microgravity setting, and the applications peculiar to microgravity. We have made no attempt at completeness. Our attention is directed naturally to matters on which we feel a special competence because we discovered them ourselves; beyond that, we have included work by others that has stimulated our interest and to which we could respond. Even in this limited context, much has been omitted in the interest of exhibiting clear lines of conceptual development. We have included almost no proofs. For readers seeking more complete understanding of the field or detailed information on particular points, we trust we have included enough references to provide a basis for a fruitful literature search. For those with a more casual interest as well as (and especially) for new initiates to the wealth of exciting and challenging problems that have appeared in the last decades and continue to appear, we hope to have provided a useful overview of significant features of the current state of the art.

For the general notion of capillary surface and associated variational characterization, we refer the reader to [1, Chapter 1]. We restrict ourselves here to particular configurations.

1. The capillary tube

We consider a semi-infinite cylindrical tube of homogeneous material and general section, closed at one end by a base Ω, in a uniform gravity field g either zero or directed toward the base. We attempt to cover Ω by a volume V of fluid, making contact angle γ with the walls, and to characterize the resultant free surface S (see Fig. 1). We at first

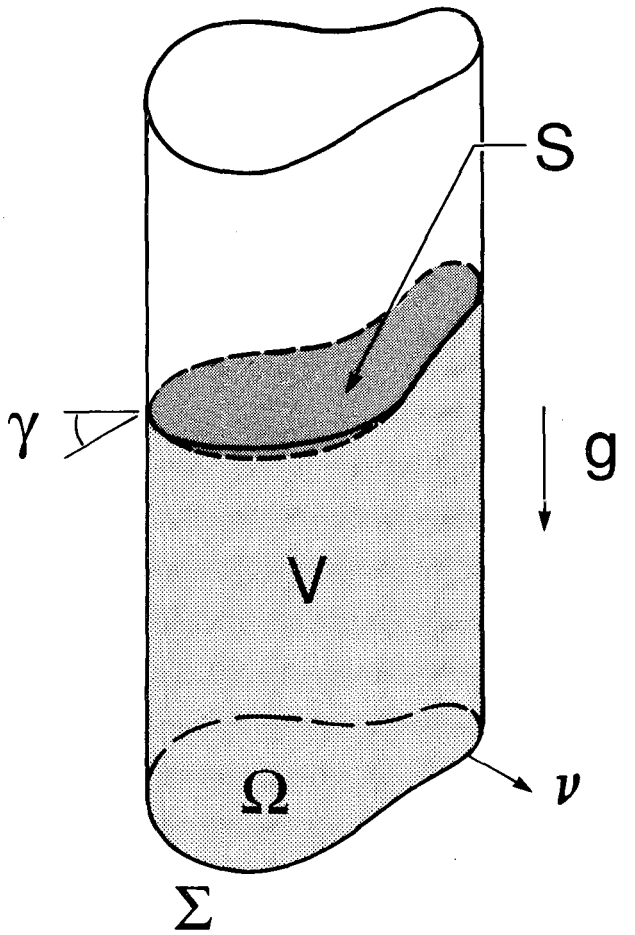

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Figure 1. Partly filled cylindrical tube with base Ω.

seek the solution as a "graph" $z = u(x, y)$ over Ω and are led by the principle of virtual work to look for a stationary "point" for the energy functional

$$E[u;\gamma] \equiv \int_\Omega \sqrt{1+|\nabla \mathrm{u}|^2}dxdy - \beta \oint_\Sigma uds + \frac{1}{2}\kappa \int_\Omega u^2 dxdy \tag{1}$$

under the constraint

$$V = \int_\Omega udxdy. \tag{2}$$

Here Σ is the boundary of Ω; $\beta = \cos\gamma$ is the relative adhesion coefficient (wetting energy density) of the fluid on the walls, constant for homogeneous materials; $\kappa = \rho g/\sigma$, where ρ is the density change across S and σ the surface tension. The indicated direction for the gravity field corresponds to $g > 0$. The coordinates are chosen so that $z = 0$ corresponds to the base Ω. For details of the derivation, see, e.g., [1, Chapter 1].

The Euler-Lagrange equation for (1) becomes

$$\operatorname{div} Tu = \kappa u + \lambda, \quad Tu = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \tag{3}$$

in Ω, with

$$\nu \cdot Tu = \beta \tag{4}$$

on Σ; ν is the exterior unit normal on Σ, and λ is a Lagrange multiplier corresponding to the constraint (2).

To determine the constant λ, we integrate (3) over Ω, obtaining by the divergence theorem and (2)

$$\lambda|\Omega| = -\kappa V + \oint_\Sigma \nu \cdot Tuds,$$

which by (4) yields

$$\lambda = \frac{-\kappa V + \beta|\Sigma|}{|\Omega|}. \tag{5}$$

The symbols $|\Sigma|, |\Omega|, \ldots$ denote the length or area of the indicated geometric quantity. Here we note that if V is replaced by $V + C$, where C is a constant, then addition of $C/|\Omega|$ to u yields again a solution of (3),(4) under the (new) constraint (2); that is, once a solution for some V is known, then solutions for any V are obtained by rigid vertical translation. It is necessary to show that by choosing V large enough we will have $u > 0$, so

that Ω is covered by S. Clearly this can be achieved (by the indicated family) if and only if u is bounded below in Ω. We shall see that such a bound cannot always be achieved, even for relatively simple geometries, and thus that the problem is in many interesting cases mathematically ill-posed.

We note that the left side of (3) can be identified as twice the mean curvature H of the surface S. Also, (4) states that S meets the cylinder walls Z over Σ in the constant angle γ. Thus the problem posed by (3),(4),(5) can be interpreted geometrically: we are to find a surface of prescribed mean curvature $H(z)$, which bounds a prescribed volume V over the cylinder base Ω, and which meets Z in a prescribed angle γ.

1.1. The circular tube. Here we seek solutions with rotational symmetry about an axis. Even in this case, the question of existence of a solution with prescribed data is not a triviality; the first proof in the literature appears in Johnson and Perko [2], over 150 years after the initial estimates of Young and of Laplace for the center and meniscus heights of the (presumed) solution surface.

In 1806, Laplace presented his celebrated estimate for the height u_0 of the surface on the axis of a narrow tube of radius a. The estimate is equivalent to approximation of the surface by a spherical cap. In the volume-constrained case considered here we find, in terms of nondimensional variables $\mathcal{U}_0 = \frac{1}{a}u_0$, $\mathcal{V} = \frac{1}{a^3}V$, and contact angle γ,

$$\mathcal{U}_0 \approx \frac{1}{\pi}\mathcal{V} - \frac{1}{\cos\gamma} + \frac{2}{3}\frac{1-\sin^3\gamma}{\cos^3\gamma} \equiv \mathcal{L}[\mathcal{V};\gamma]. \tag{6}$$

Laplace offered no proof for the asymptotic validity of (6), and no indication of how small a must be for prescribed accuracy. The first formal proof of asymptotic correctness of (6) was given by Siegel [3]. In [4] the explicit bounds in terms of Bond number $B = \kappa a^2$,

$$\mathcal{L}[\mathcal{V};\gamma] < \mathcal{U}_0 \quad \text{for all} \quad B, \quad 0 \le \gamma < \frac{\pi}{2}, \tag{7}$$

$$\mathcal{U}_0 < \mathcal{L}[\mathcal{V};\gamma] + \frac{\cos\gamma(1+2\sin\gamma)}{6(1+\sin\gamma)^4}B, \quad B < 6, \quad 0 \le \gamma < \frac{\pi}{2}, \tag{8}$$

are established. Both inequalities reverse if $\frac{\pi}{2} < \gamma \leq \pi$, with $\mathcal{V}$ replaced by $-\mathcal{V}$. It should be noted that these bounds are strict, not merely asymptotic. It can be shown that (7),(8) are best possible, in the sense that

$$\mathcal{U}_0 = \mathcal{L}[\mathcal{V};\gamma] + \frac{\cos\gamma(1+2\sin\gamma)}{6(1+\sin\gamma)^4}B + O(B^2) \tag{9}$$

uniformly in γ, as $B \to 0$.

Similar bounds are established in [5] and [6] for the height $\mathcal{U}_1$ on the contact line and for the "meniscus height" $\mathcal{U}_1 - \mathcal{U}_0$. Siegel [7] has given bounds that hold throughout the trajectory $0 \leq r \leq a$, although they are somewhat less exact at the end points.

1.2. General sections. If the system (3),(4),(5) is to reflect reality, one should expect solutions to exist when containers have reasonable sections, other than that of the disk. This topic has been studied mathematically in considerable generality (see [1] for references). For simplicity, we restrict attention here to sections Ω whose boundaries Σ consist of a finite number n of smooth curves that meet in well-defined corners $P_1, \ldots, P_n$, with interior half-angles $\alpha_1, \ldots, \alpha_n$. We note that the condition (4) cannot be prescribed at the P_j, since ν is not defined at these points. Nevertheless, if $g > 0$ we obtain the general result that *there is exactly one surface $z = u(x,y)$ over Ω, such that* (3) *holds strictly in Ω, the given volume V is achieved, and* (4) *holds strictly on Σ except at the $\{P_j\}$.* This result was first proved by M. Emmer [8] under some restrictions. For greater generality, see [9] and [1, Chapter 7].

Observe that uniqueness holds without growth conditions at the $\{P_j\}$. In this respect, the behavior of solutions of (3) differs strikingly from that typically encountered for solutions of linear problems, for which failure to prescribe boundary conditions at even a single point can lead to nonuniqueness. The nonlinearity in the present problem constrains the solution near the exceptional boundary points, even though no data are prescribed there.

The uniqueness follows from a general comparison principle, which is central to the material that follows. We define $Nu \equiv \operatorname{div} Tu - \kappa u$.

CP: *Suppose $Nu \geq Nv$ in Ω, $\nu \cdot Tv \geq \nu \cdot Tu$ on $\Sigma \backslash \{P_j\}$. There follows*

i) *if $\kappa > 0$ then $v \geq u$ in Ω; equality holds at any point if and only if $v \equiv u$ in Ω,*

ii) *if* $\kappa = 0$ *then* $v(x,y) \equiv u(x,y)+$ const. *in* Ω.

This statement is adequate for the applications to be made here. More general statements (and results) can be found in [10]. The crucial point is that the result holds without growth hypotheses at the exceptional points $\{P_j\}$.

Application I: Suppose Ω contains a corner of interior angle 2α, as indicated in Fig. 2. Let $v(x,y)$ denote a lower hemisphere, whose equatorial circle lies over Γ. Since div Tv is twice the mean curvature of the surface $v(x,y)$, we have

$$\text{div } Tv = \frac{2}{\delta} \tag{10}$$

in the disk B_δ bounded by Γ; also v meets the vertical (planar) walls Z over Σ^* in the constant angle $\gamma_0 = \frac{\pi}{2} - \alpha$. Let $u(x,y)$ satisfy

$$\text{div } Tu = \kappa u, \quad \kappa > 0, \tag{11}$$

in Ω, with boundary angle γ such that $\gamma_0 \leq \gamma \leq \pi/2$. Then $\nu \cdot Tv \geq \nu \cdot Tu$ on $\Sigma^* \cap B_\delta$, while $\nu \cdot Tv = 1$ on $\Gamma \cap \Omega$. We observe that $|\nu \cdot Tu| < 1$ on $\Gamma \cap \Omega$ in virtue of the finiteness of $|\nabla u|$, and we choose $\Sigma = \Sigma^* \cup (\Gamma \cap \Omega)$, $\{P_i\}$ to be the three intersection points $\Gamma \cap \Sigma^*$. We adjust v by an additive constant, so that its minimum is $v_0 = \frac{2}{\kappa\delta}$. We then have by (11)

$$\text{div } Tv = \frac{2}{\delta} = \kappa v_0 \leq \kappa v \tag{12}$$

and thus $Nv \leq Nu$ in $\Omega \cap B_\delta$. From **CP** we find immediately $u < v$ in $\Omega \cap B_\delta$. Similarly, $u > -v$. But $v \leq v_0 + \delta$ and *there follows*

$$|u| < \frac{2}{\kappa\delta} + \delta \tag{13}$$

throughout $\Omega \cap B_\delta$. *This inequality holds for any configuration for which* γ *lies in the interval* $\left|\frac{\pi}{2} - \gamma\right| \leq \alpha$, *including the end points.* But outside that interval a very different behavior prevails. *In fact* [11, 12], *if* $\left|\frac{\pi}{2} - \gamma\right| > \alpha$ *there exist positive constants* C, ϵ *such that*

$$\left| u - \frac{\cos\theta - \sqrt{k^2 - \sin^2\theta}}{k\kappa r} \right| < Cr^\epsilon, \quad k = \frac{\sin\alpha}{\cos\gamma}, \tag{14}$$

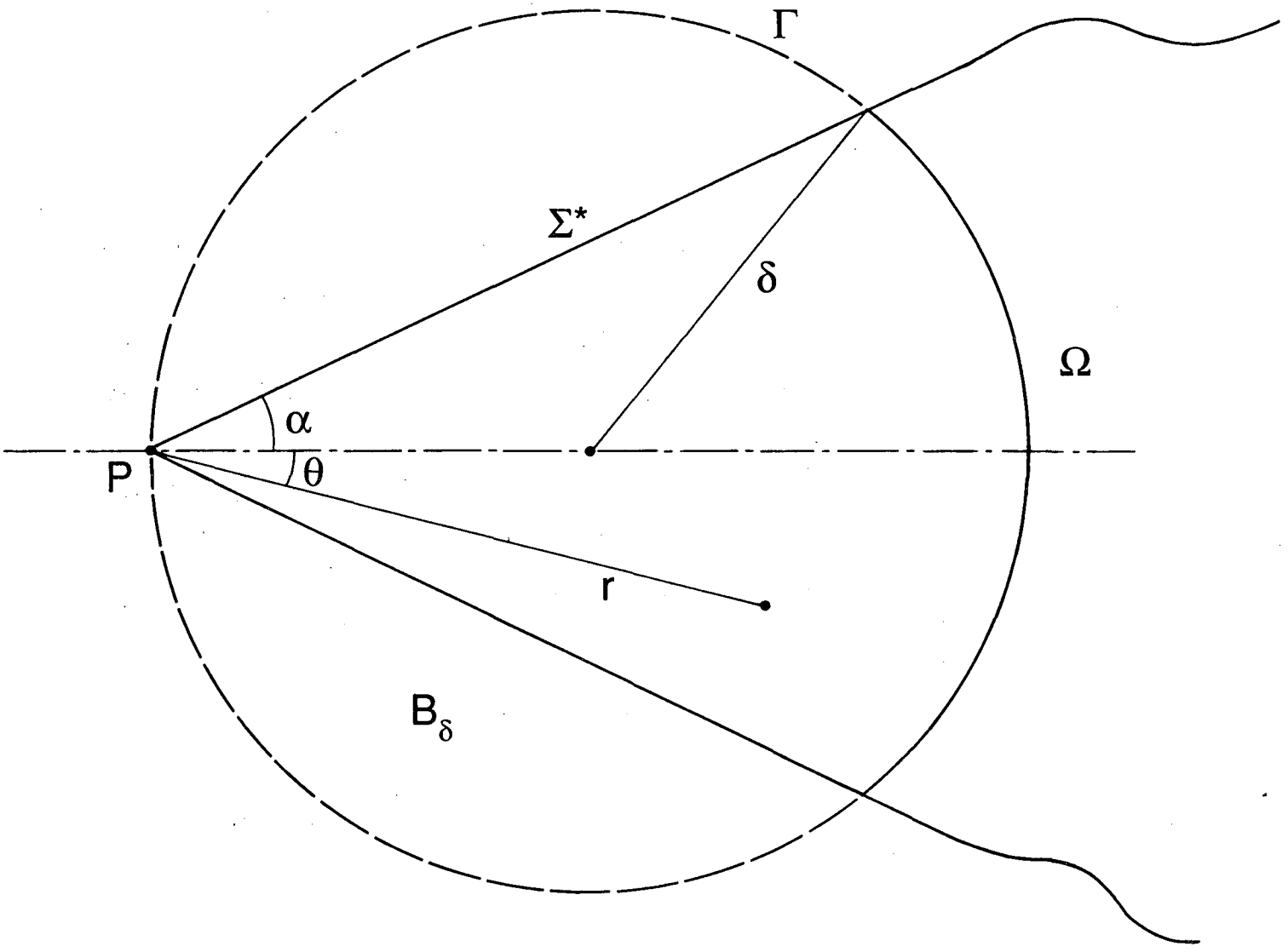


Figure 2. Domain Ω containing a corner. The disk B_δ.

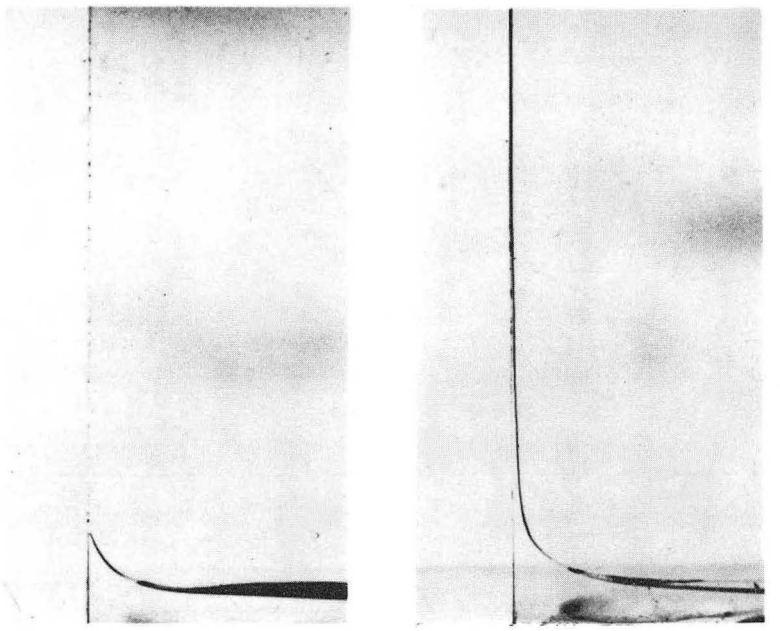
near P, see Fig. 2. Thus, in this case, $|u| \to \infty$ at P, and we see that *there is a discontinuous change in behavior as* $\left|\frac{\pi}{2} - \gamma\right|$ *passes through* α.

The discontinuity can be used effectively for accurate measurement of contact angle, at least when $\left|\frac{\pi}{2} - \gamma\right|$ is not too large. Fig. 3 shows results of a "kitchen sink" experiment carried out by Tim Coburn in the Medical School of Stanford University, using distilled water between plates of acrylic plastic. A change (effected by hand) of about 2° or 3° in the half-angle between the plates, yielded the change in observed peak height at P, from slightly under the bound (13) when $\left|\frac{\pi}{2} - \gamma\right| \leq \alpha$, to over ten times that bound when $\left|\frac{\pi}{2} - \gamma\right| > \alpha$, thus leading to a preliminary estimate of between 78° and 81° for the contact angle γ_{wa} between water and acrylic plastic. Coburn's procedure was repeated recently under more controlled conditions by Mark Weislogel at NASA Lewis Research Center. For his materials he obtained accurately repeatable results of $\gamma_{wa} = 80° \pm 2°$. By comparison, he found the reproducibility of other, standard methods to be more strongly affected by hysteresis, with uncertainty several times as large.

The accuracy of the above method decreases rapidly as $\left|\frac{\pi}{2} - \gamma\right|$ increases, since the critical opening angle then becomes large and the height changes become restricted to a small neighborhood of the vertex, where observations are difficult. This point is addressed in the considerations that follow.

Application II: The above results are for $g > 0$. If $g = 0$, the corner effect becomes still more striking. In fact, *if* $g = 0$ *and* Ω *contains a corner at which* $\left|\frac{\pi}{2} - \gamma\right| > \alpha$, *then there is no solution of* (3),(4) *over* Ω *for any* λ. This can be proved very simply by applying the divergence theorem to (3) over a domain cut off by a segment Γ at the corner, using the boundary condition (4) on Σ^*, and the bound $|Tu| < 1$ on Γ. If $\left|\frac{\pi}{2} - \gamma\right| > \alpha$, a contradiction is immediately obtained by letting $\Gamma \to P$, see [13].

What does such a result mean physically? To get a feeling for what happens, observe that when $\left|\frac{\pi}{2} - \gamma\right| > \alpha$ and the boundary segments at the corner are long enough, a circular arc Γ of radius $1/\lambda$ can be positioned in the corner to meet Σ^* in angle γ at both intersection points (see Fig. 4). The vertical (cylindrical) surface determined by Γ defines, in a limiting sense, a solution of (3) that fills the corner, and leaves the remainder of the



(a) (b)

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Figure 3. Discontinuous dependence on data: $g > 0$. (a) $\alpha \approx 12°$, (b) $\alpha \approx 9°$.

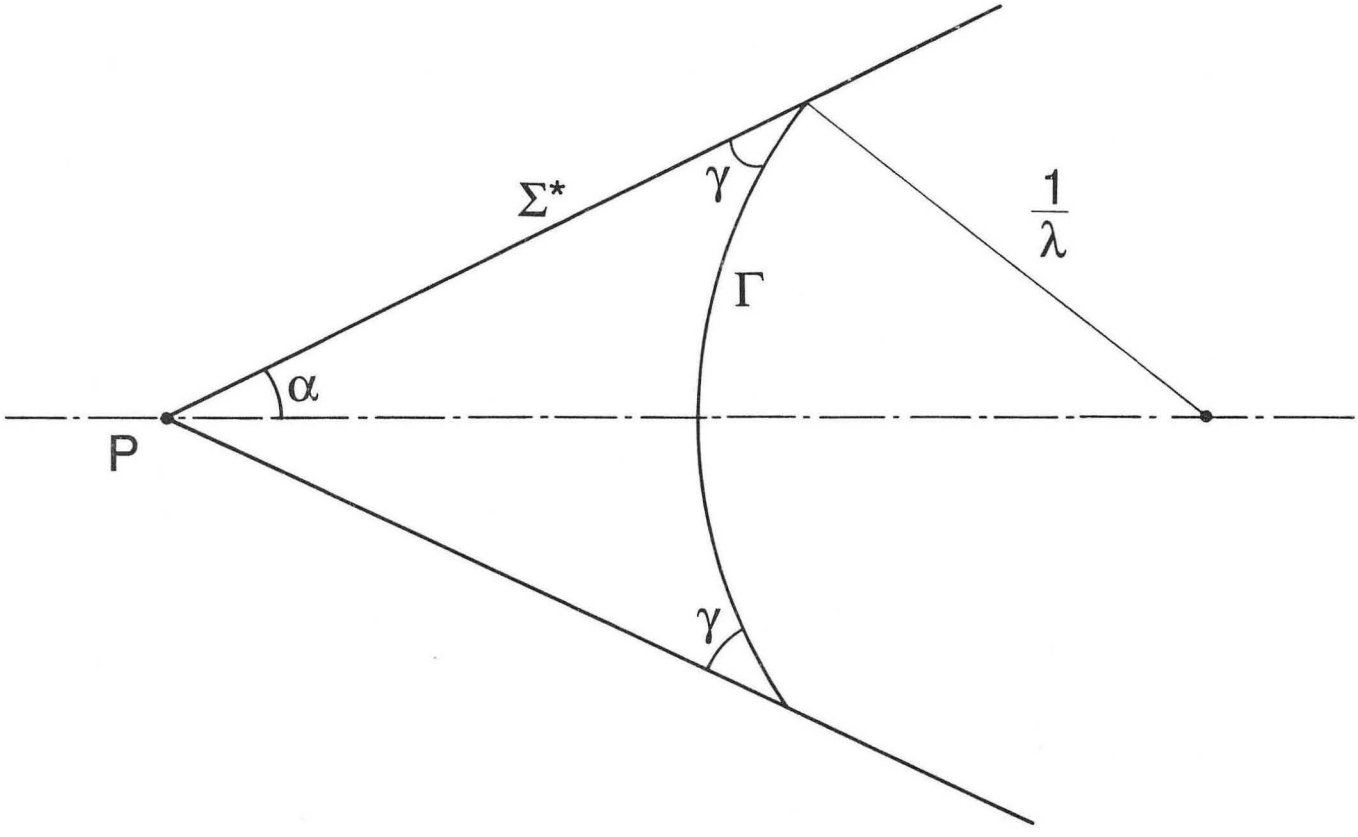


Figure 4. Corner domain with circular arc Γ.

base uncovered. (No such configuration can be found when $\left|\frac{\pi}{2}-\gamma\right| \leq \alpha$.) It is reasonable to expect that actual solutions attempt to adopt that shape, to the extent permitted by the remaining geometry of the container. This view is supported by experiments made by W. Masica at the NASA Lewis Research Center Zero Gravity Facility, using a cylindrical container with hexagonal cross section. In this case, if $\left|\frac{\pi}{2}-\gamma\right| \leq \alpha$, a solution surface exists and can even be given explicitly as the spherical cap

$$u(x,y;\delta) = u_0 \mp \sqrt{\frac{\delta^2}{\cos^2\gamma} - r^2}, \tag{15}$$

where δ is the radius of the inscribed circle, and u_0 is determined by the constraint

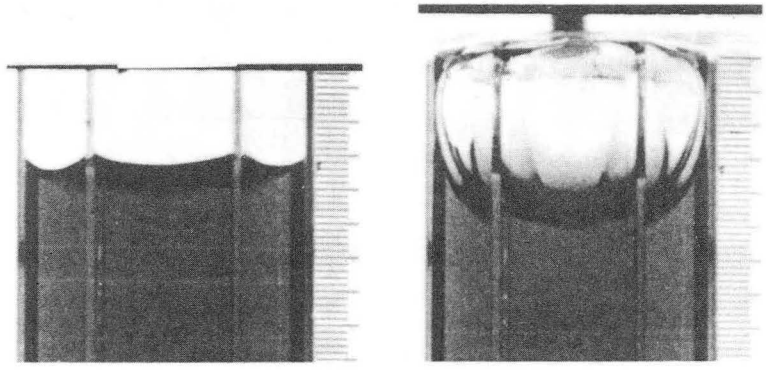
$$\int_\Omega u\,dx\,dy = V. \tag{16}$$

Fig. 5 shows the results of drop tower experiments, using identical acrylic plastic containers and two different liquids. In Fig. 5a the liquid is a 20% ethanol in water solution, for which $\gamma \approx 48°$, and there holds $\left|\frac{\pi}{2}-\gamma\right| < \alpha$; here the expected spherical cap is obtained. In Fig. 5b the liquid is a 30% ethanol in water solution, for which $\gamma \approx 25°$, and there holds $\left|\frac{\pi}{2}-\gamma\right| > \alpha$; the fluid now attempts to fill the corners, and climbs to the top of the container, meeting the top and the upper boundary walls in the prescribed angle. The fluid height over the remainder of the base is significantly lowered.

This behavior is the key to the following observations [14]. Consider the domain of Fig. 6, bounded by two line segments and a circular arc of radius δ. We suppose $g > 0$ and $\left|\frac{\pi}{2}-\gamma\right| \leq \alpha$. It can then be shown [9] that a solution of (3) exists, with data γ on Σ. We note that a hemisphere of radius $\delta/\cos\gamma$ and center at O meets the bounding cylinder Z in the constant angle γ (except at the vertex, where the angle is undefined). Applying **CP** in ways similar to that used above, we are led to the inequality

$$\frac{V}{|\Omega|} - \frac{\delta}{\cos\gamma}\left(1 - \frac{\sqrt{k^2-1}}{k}\right) < u(x,y;g) < \frac{V}{|\Omega|} + \frac{\delta}{\cos\gamma}\left(1 - \frac{\sqrt{k^2-1}}{k}\right) \tag{17}$$

throughout Ω when $0 \leq \gamma < \pi/2$, and the same inequality with $u \to -u$, $\gamma \to (\pi-\gamma)$ when $\frac{\pi}{2} < \gamma \leq \pi$. To fix the ideas, consider the case $0 \leq \gamma < \pi/2$. If V is prescribed in advance, the formal solution $u(x,y;g)$ might be partly below the base Ω, which is physically



(a) (b)

Figure 5. Discontinuous dependence on data: $g = 0$. (a) $\alpha+\gamma \geq \pi/2$, (b) $\alpha+\gamma < \pi/2$.

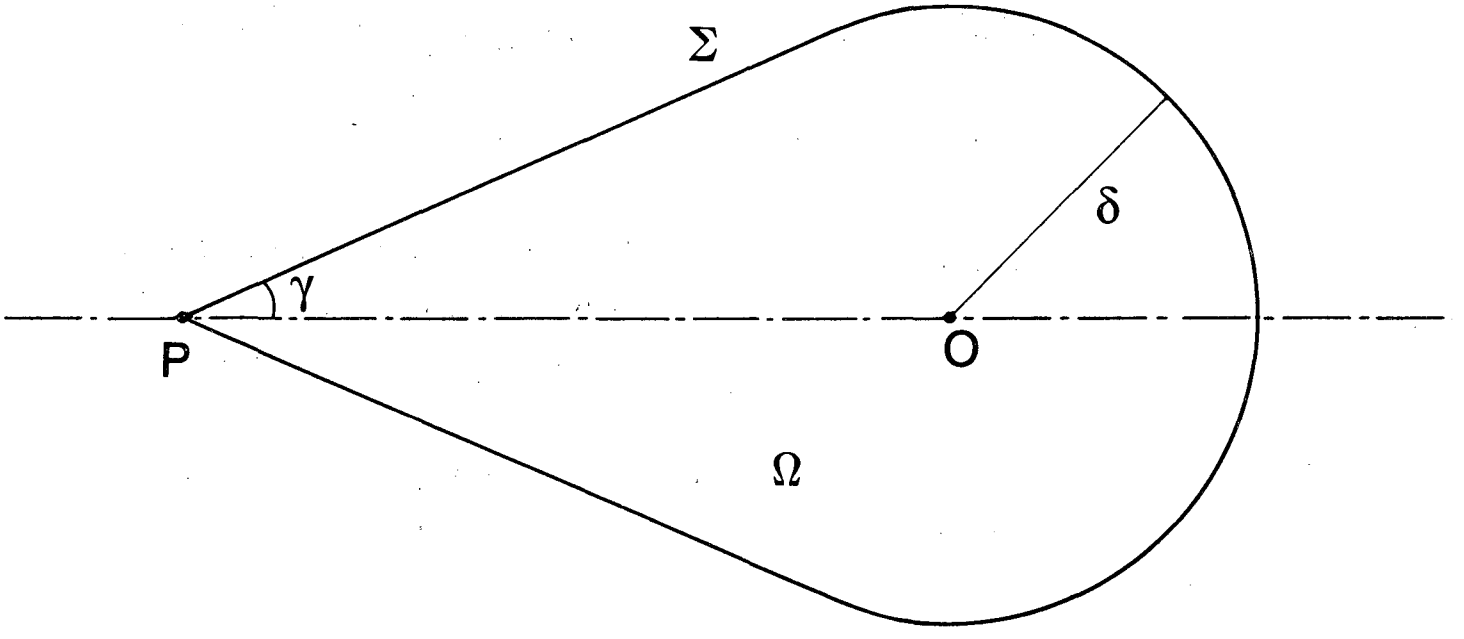


Figure 6. "Ice-cream cone" domain.

unacceptable. However, we see from (17) that by choosing V large enough so that the left side of (17) is positive, the entire surface can be made to lie strictly above Ω. We note further that (17) is independent of g and continues to hold as stated as $g \searrow 0$, the surfaces converging in fact to the spherical cap indicated above.

Now we suppose $\left|\frac{\pi}{2} - \gamma\right| > \alpha$. In this case the fluid tries to fill out the corner (as indicated above), as $g \searrow 0$; thus, if the cylindrical container is sufficiently high then the bottom Ω must become uncovered, except for a small neighborhood of P. In particular, $u(0,0;g)$ will violate (17) for small enough g. Thus, we have found a procedure for determination of contact angle, which does not require measurements in the corner. This procedure can be expected to yield great accuracy for values γ reasonably distant from 0 and π, perhaps in the range $\left|\frac{\pi}{2} - \gamma\right| < 70°$.

Application III: As we have seen, when $g = 0$, surfaces $u(x,y)$ satisfying (3),(4) need not always exist. Explicit geometric criteria have been given only under restrictive conditions, cf., Theorem 8 in [15] and [16]. However, the following useful (indirect) criterion appears in [15], see also [1, Chapters 6,7].

Let $R_\gamma = \frac{|\Omega|}{|\Sigma|\cos\gamma}$. A solution of (3),(4) in Ω exists if and only if, for every strict subarc Γ (in Ω) of a semicircle of radius $|R_\gamma|$ that meets Σ with angle γ at both intersection points, as indicated in Fig. 7, there holds

$$\Phi[\Omega^*] \equiv |\Gamma| - |\Sigma^*|\cos\gamma + \frac{1}{R_\gamma}|\Omega^*| > 0. \tag{18}$$

In many cases of interest there can be found (up to inessential displacements) only a finite number of curves Γ with the required properties, and thus the matter can be settled by examining a finite number of cases. Of particular interest are those situations in which (18) holds vacuously, in the sense that there is no Γ satisfying the conditions. That is the case, for example, regardless of γ, in the configuration of Fig. 8, bounded by two parallel lines and semicircles, and we are thus assured of existence, regardless of h, for that configuration.

This situation changes dramatically if we consider instead the configuration of Fig. 9, bounded by two non-parallel lines and smoothly-joined circular arcs. We observe from (18)

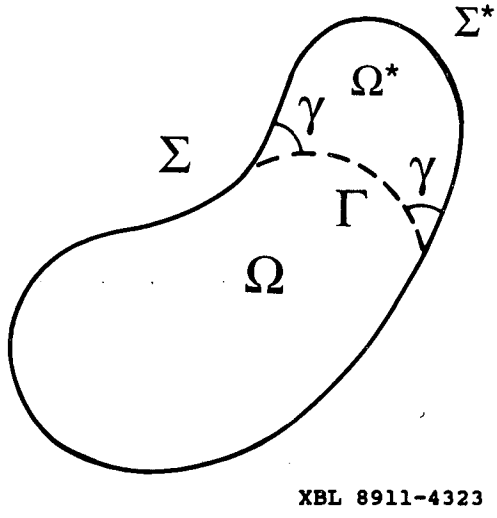


Figure 7. Domain partitioned by circular arc Γ of radius R_γ.

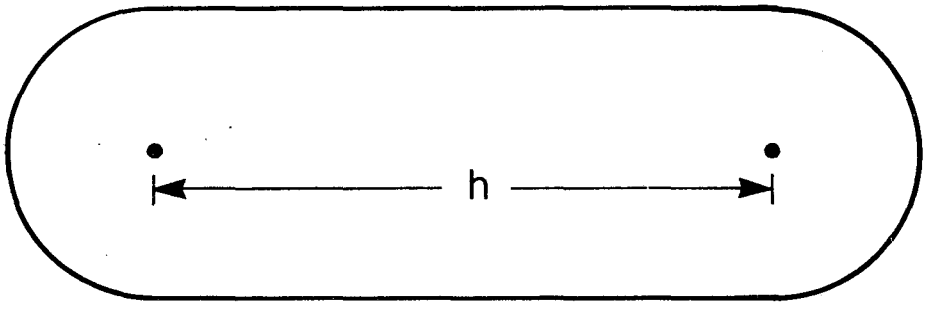


Figure 8. Domain with parallel sides and semicircular ends.

that to any domain Ω there corresponds a critical value $\gamma_0(\Omega)$ of γ (and the supplementary critical value $\pi - \gamma_0$), such that a solution exists if $\left|\frac{\pi}{2} - \gamma\right| < \left|\frac{\pi}{2} - \gamma_0\right|$ and fails to exist if $\left|\frac{\pi}{2} - \gamma\right| > \left|\frac{\pi}{2} - \gamma_0\right|$. For the Ω of Fig. 8 we have $\gamma_0 = 0$ (or π); nevertheless if we let $\alpha \to 0$ in Fig. 9 we find $\lim_{\alpha \to 0} \gamma_0(\Omega) = \pi/2$. That is, *when the sides are parallel, a solution exists for any γ; but if they are non-parallel, then the closer they become to being parallel the more solutions are excluded.* This nonuniformity in behavior is illustrated by the curves in Fig. 10, relating the critical γ_0 in $[0, \frac{\pi}{2}]$ to ρ for differing values of α. For the regions below and to the left of the curves, solutions do not exist. As $\alpha \to 0$ the curves tend to the entire upper and right hand boundary segments; but $\alpha = 0$ yields only the indicated single point (1,0). (The depicted curves join piecewise linearly the tabular values calculated for increment 0.025 in ρ.)

Application IV: The domains Ω^* that appear when $\Phi = 0$ for some Γ as above can be interpreted physically. As remarked above, solution surfaces over Ω exist when $\left|\frac{\pi}{2} - \gamma\right| < \left|\frac{\pi}{2} - \gamma_0\right|$. Letting $\gamma \to \gamma_0$ from within the range of solvability, we obtain a sequence $u^{(n)}$ of surfaces satisfying (3) with $\kappa = 0$; *this sequence can be normalized to converge throughout Ω^* to $\pm\infty$ (according as $\gamma_0 \lessgtr \frac{\pi}{2}$) and throughout $\Omega \backslash \bar{\Omega}^*$ to a solution* $u^0(x, y)$. Thus, we can consider $u^0(x, y)$ to determine a "generalized solution" over Ω, that is identically positive infinite over Ω^*. Such "solutions" were first introduced in a formal mathematical way by Miranda [17] and are basic for the general existence theory [1, Chapters 6,7].

Alternatively, we could have normalized the $u^{(n)}$ to converge to a solution in Ω^* and to $\mp\infty$ in $\Omega \backslash \Omega^*$. These properties can be exploited in conjunction with particular geometries (chiefly modifications of that in Fig. 11) to obtain procedures for accurate measurement of small contact angles. For example, in Fig. 11 the fluid height becomes infinite in the shaded region when $\left|\frac{\pi}{2} - \gamma\right| \geq \left|\frac{\pi}{2} - \gamma_0\right|$. This work is currently in progress; preliminary results are encouraging.

Application V: What happens if we fix γ and let $g \searrow 0$ (or equivalently Bond number $B \searrow 0$)? This question is of considerable importance for Application II. It is known that

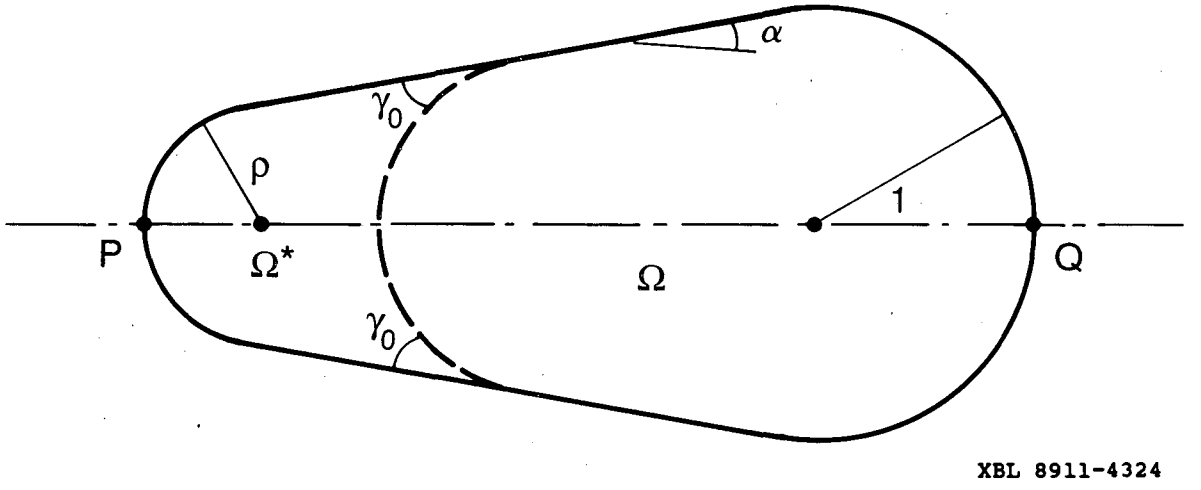


Figure 9. Domain with non-parallel sides and circular-arc ends.

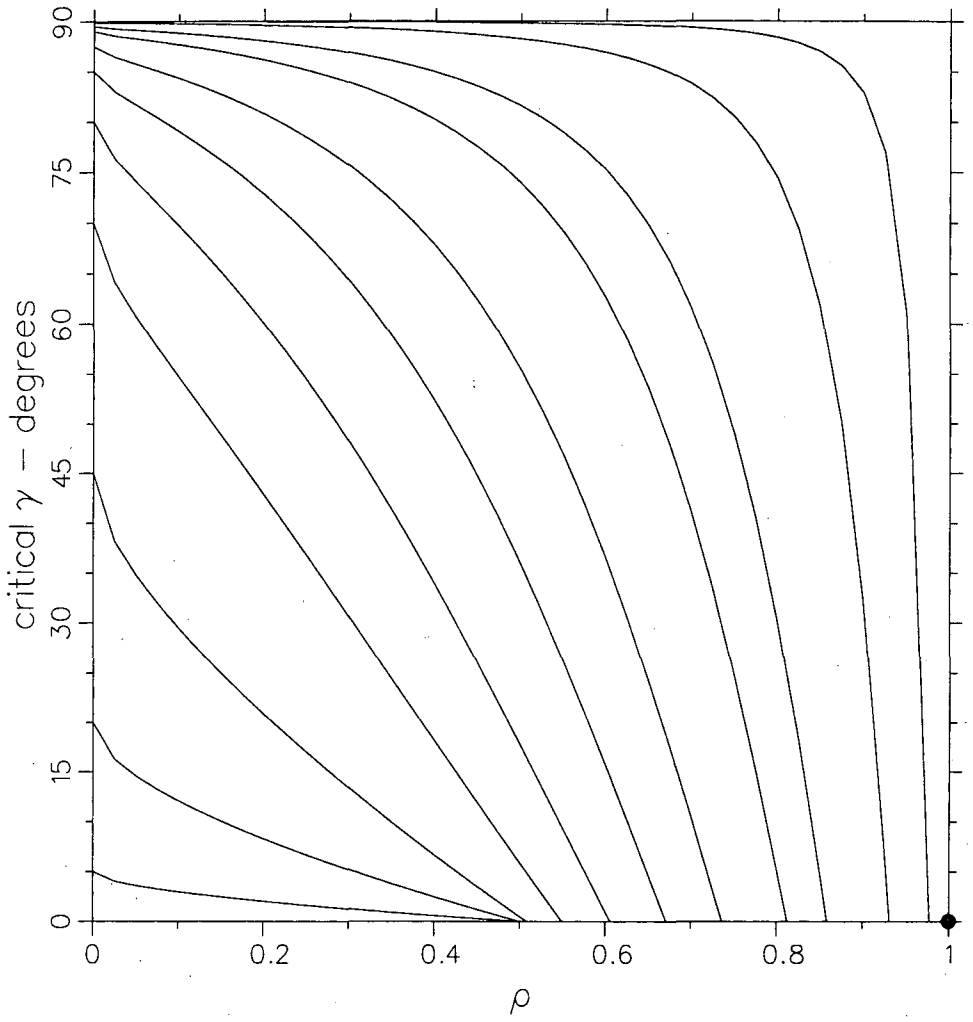


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Figure 10. Critical γ vs. ρ, for α = 85°, 70°, 45°, 20°, 10°, 5°, 2.5°, 1°, .5°, .1°, .01°, from lower left to upper right. For $\alpha = 0°$ the curves degenerate to the single point $\rho = 1, \gamma_0 = 0°$.

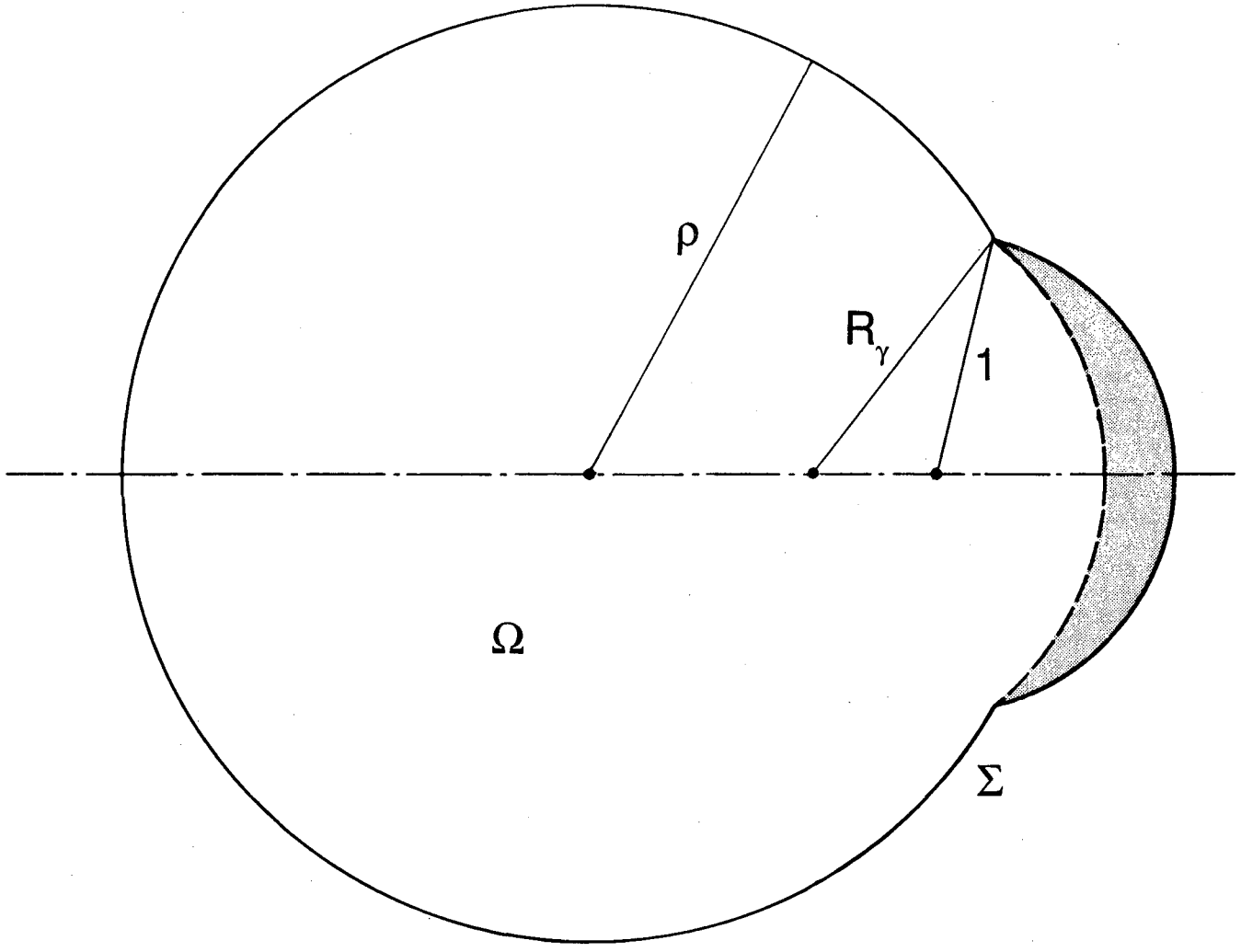

Figure 11. Two-circle domain.

for reasonably smooth Ω a solution $u^B \equiv u(x,y;B)$ always exists when $B > 0$. Siegel [18] proved that *whenever a solution* $u^0 = u(x,y;0)$ *exists, then (under some restrictions) there exists a constant* C *such that* $|u^B - u^0| < CB$ *uniformly in* Ω. Tam [19] later removed the restrictions, and also to some extent the requirement that u^0 exist. He showed that *whenever* Ω^* *is uniquely determined there exist functions* $C_1(B), C_2(B)$ *such that*

$$\lim_{B\to 0}\left[u^B + C_1(B)\right] = \begin{cases} \pm\infty \text{ in } \bar{\Omega}^*, \text{ according as } \gamma_0 \lessgtr \frac{\pi}{2} \\ \text{solution in } \Omega\backslash\bar{\Omega}^* \end{cases}$$

$$\lim_{B\to 0}\left[u^B + C_2(B)\right] = \begin{cases} \text{solution in } \Omega^* \\ \mp\infty \text{ in } \Omega\backslash\Omega^*, \text{ according as } \gamma_0 \lessgtr \frac{\pi}{2}\,. \end{cases}$$

This is the sense in which transition to zero gravity must be interpreted in general (practical) situations.

2. Uniqueness and non-uniqueness

Surfaces $u(x,y)$ that satisfy (3),(4) in capillary tubes are uniquely determined when $g \geq 0$, and provide an absolute minimum for the mechanical energy. This follows easily from **CP** above. Miranda [20] showed that any energy minimizing surface that covers Ω must in fact have the form $u(x,y)$ (i.e., the surface cannot bend over itself if it minimizes). More recently, Vogel [21] showed that there can be no stationary (equilibrium) configuration distinct from the minimizer. Thus, *capillary surfaces in cylindrical tubes are strongly stable configurations that cover the base simply.*

A drop of liquid of volume V resting on a horizontal plane (Fig. 12) is also unique and energy minimizing [1, Chapter 3, Note 3]. Now imagine the plane continuously deformed, through a family of convex surfaces, into the cylinder (Fig. 13). Does the liquid surface remain unique? In general, certainly not! Consider the configuration of Fig. 14, in which a conical base meets the right-circular-cylindrical wall in a 45° angle, and suppose $\gamma = 45°$. If the cone is filled from the bottom until just below the juncture point, a horizontal surface at that height provides a solution, with the volume V of the cone. Now observe that any volume $V^+ > V$ yields a capillary surface meeting the cylinder walls. This surface cannot be horizontal in view of the 45° contact angle. If fluid is removed until the contact line is just above the juncture point, a congruent surface is obtained with volume $V^- < V$. By

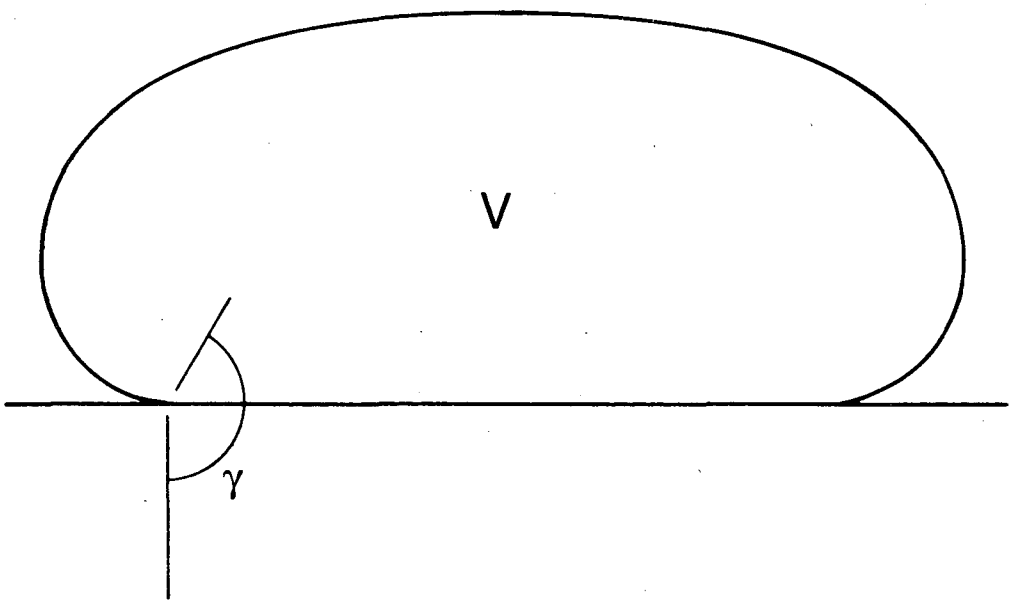

Figure 12. Sessile drop on a horizontal plane.

considerations of continuity, there must be an intermediate configuration distinct from the horizontal one, and with the same volume.

This discussion can be modified so as to yield an entire continuum of equilibrium interfaces, all of which bound the same volume and have identical mechanical energy [22,23]. In this continuum, the horizontal surface has been shown to be unstable [23,24] and thus it is unlikely to be observed physically. Using these considerations, *one can design an axially symmetric container, as close to a circular cylinder as desired, which when half filled with liquid admits a continuum of symmetric solutions, but for which no interface of minimizing energy can be symmetric* [23]. In Fig. 15 the radial section is shown of such a container admitting a continuum of symmetric solutions for $\gamma = 60°$ and zero gravity, along with some of the solution surfaces. The procedure works in any gravity field, however the size of the container becomes small and the effects can be difficult to observe, except under microgravity conditions.

3. Stability

If $g > 0$ then the free surface in a capillary tube yields a global energy minimum and hence is stable. Under reasonable conditions, this will also be so when $g = 0$, but in some senses there can be exceptions. For example, in the configuration of Fig. 6 when $\left|\frac{\pi}{2} - \gamma\right| = \alpha$ the free surface is clearly unstable under small perturbations of contact angle. For a detailed discussion of the instability see Langbein [25].

Similarly, a liquid drop on a horizontal surface (sessile drop) is stable for any $g \geq 0$. As was shown above, if the support surface is curved, stability cannot in general be expected. For a horizontal support surface Π with $g < 0$ (or equivalently with $g > 0$ and the drop contacting Π from below) one obtains the "pendent drop", for which stability criteria have a long history (see, e.g. [1, Section 4.15]); we indicate here recent results of Wente [26]. To describe them, we normalize the equation to the form

$$\operatorname{div} Tu = -u \tag{19}$$

in terms of nondimensional variables ($\sqrt{|\kappa|}\, u \to u$, $\sqrt{|\kappa|}\, r \to r$). It can be shown [27] that for any prescribed $u_0 < 0$, there exist global axisymmetric solutions for which $u(0) = u_0$;

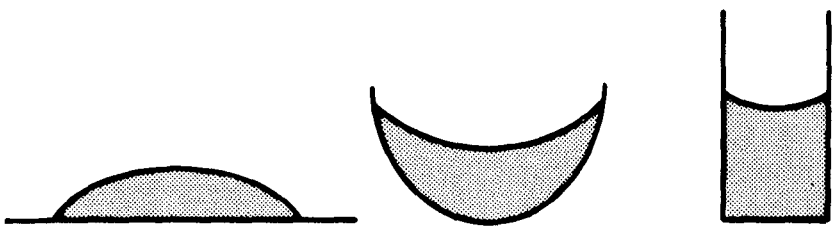

Figure 13. Fluid on a plane, in an "intermediate" container, and in a cylinder.

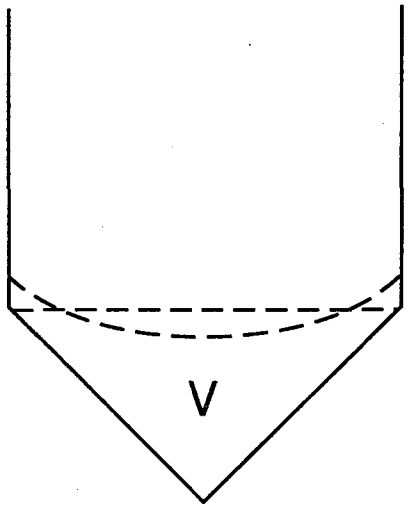

Figure 14. Nonuniqueness.

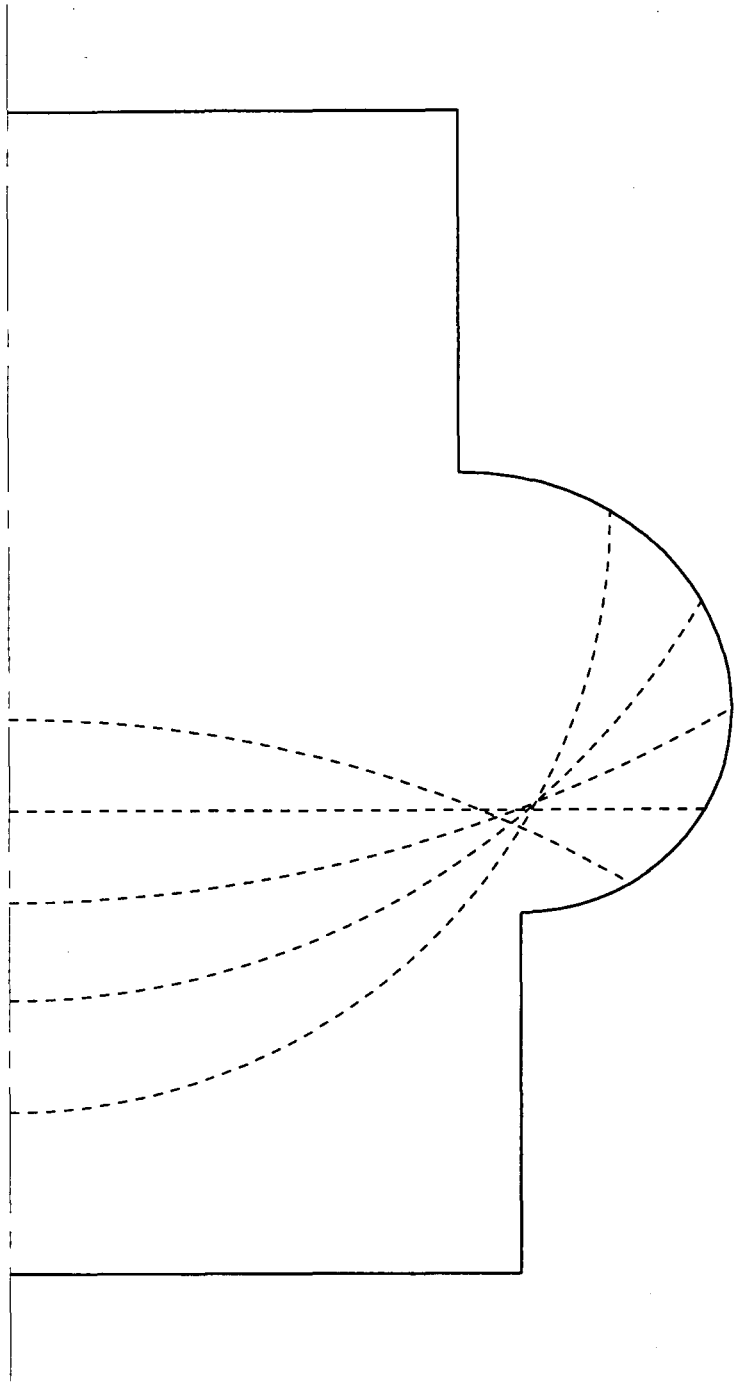

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Figure 15. Radial section of a container (solid curve) with selected free surfaces (dashed curves) from the continuum having the same energy, contact angle, and enclosing the same liquid volume with the container. $\gamma = 60°$, $g = 0$.

these solutions are bounded in volume, up to the horizontal contact plane with positive u, and (for large $|u_0|$) exhibit a succession of "bubble-like" shapes, see Fig. 16. Wente shows [26]:

a) *Suppose* $0 < \gamma < \pi$. *Then for any sufficiently small volume* V, *there exist stable drops that are convex and resemble spherical caps. These drops are generated by profile curves whose tip is at* u_0, *where* $|u_0|$ *is large. As* V *increases, so does* u_0, *until an inflection point is reached. This drop is stable. With further increase of* V, u_0 *decreases; the drop profile contains an inflection but continues in some interval to remain stable. Instability occurs prior to occurrence of a second inflection* (see Fig. 16).

b) *If* $\gamma = 0$ *all profile curves contain an inflection. The curves are generated by starting with the solution* $u \equiv 0$, *then letting* u_0 *decrease and considering the portion of the curve up to the first maximum. The limit of stability is reached prior to the appearance of a vertical point, i.e., before* $|u_0| \approx 2.5678$ (cf., [27]).

c) *For any angle of contact, the drop height increases monotonically with volume throughout the range of stability.*

Wente also establishes stability criteria for two other configurations, in which the fluid drop hangs from a fixed circular aperture, under conditions either of constant pressure (the nearly empty medicine dropper) or of constant volume (the filled medicine dropper). In the latter case he shows that *stable configurations can occur in which both a bulge and a neck appear.* Fig. 17 shows a verification of this behavior in the latter author's kitchen sink, with a drop of water (colored with soluble ink) suspended in ricinoleic acid, which has a density of about 0.95, so that a near neutrally buoyant (small Bond number) condition arises.

If $g = 0$, then all sessile (or pendent) drops on a horizontal homogeneous support plane are spherical caps. That is also the case for the "medicine dropper" problem with circular orifice. If the orifice is not circular, one is led to a generalization of the classical Plateau problem for minimal surfaces: *among all surfaces through a given closed curve* C *and bounding with a given surface through* C *a prescribed volume* V, *to find one that minimizes area.* The solution (soap-bubble) is a surface of constant mean curvature H.

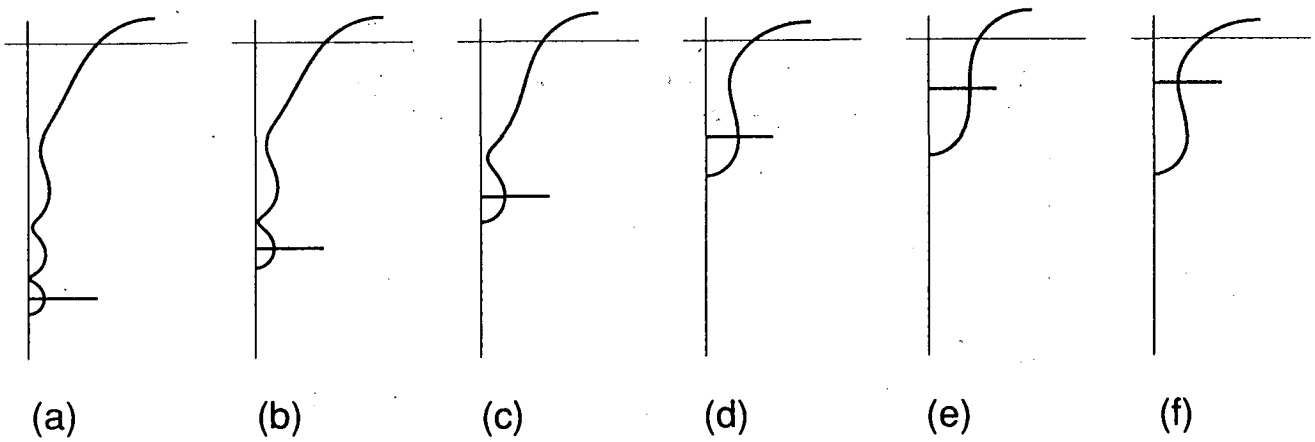

Figure 16. Pendent drop formation with increasing volume, $\gamma = 90°$. The line segments indicate the plane of support. Configurations (a)–(e) are always stable; (f) will be stable if the increase of volume from (e) is sufficiently small.

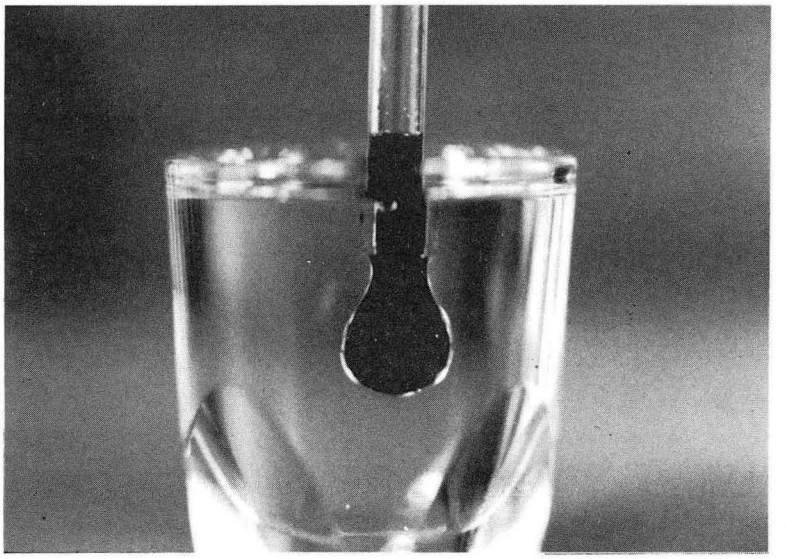

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Figure 17. Stable pendent drop with neck and bulge.

An existence proof was given by Wente [29]. Wente also showed that for sufficiently small Bond number a pendent drop through the orifice and supported by $\mathcal{C}$ must exist [30].

Another stability problem that has attracted much attention is that of a liquid column (liquid bridge) of volume V joining two parallel planes of distance h in the absence of gravity. The problem can be traced back to Rayleigh [31], who studied the linearized stability of infinite liquid columns whose sections are circular and found energy-decreasing perturbations of period equal to the circumference. The column joining parallel planes, with contact angle $\pi/2$, was studied independently by Athanassenas [32] and by Vogel [33]. These authors obtained, by different procedures, the result that *any stable configuration is a circular cylinder; if* $V > \frac{1}{\pi}h^3$ *the cylinder is stable, if* $V < \frac{1}{\pi}h^3$ *it is unstable.* We note that the onset of instability appears at half the cylinder length at which it occurs for the classical Rayleigh instability. That is because of the differing boundary condition, which allows a larger choice of perturbations in the present case.

Vogel continued his study with a series of papers [34, 35, 36] investigating stability criteria for contact angle $\gamma \neq \pi/2$, and also for differing angles γ_1, γ_2 on the two planes. He found a remarkable diversity in the kinds of behavior that can occur. We summarize here some of the results.

The drop is necessarily rotationally symmetric and described by its meridional distance $f(x)$ *from an axis, with* $x \in [0, h]$. $f(x)$ *must satisfy the equation*

$$ff'' - (1 + f'^2) = 2H(1 + f'^2)^{3/2} f, \tag{20}$$

for some constant H*, and the boundary conditions*

$$f'(0) = -\cot\gamma_1 \quad , \quad f'(h) = \cot\gamma_2. \tag{21}$$

The following two conditions suffice for stability:

1) *The Sturm-Liouville problem*

$$L(\psi) \equiv -\left(\frac{f\psi'}{(1 + f'^2)^{3/2}}\right)' - \frac{\psi}{f(1 + f'^2)^{1/2}} = \lambda\psi \tag{22}$$

$$\psi'(0) = \psi'(h) = 0$$

has exactly one negative eigenvalue;

2) *There is a smoothly parametrized family* $f(x;\epsilon)$ *with* $f(x;0) \equiv f(x)$, *such that* $\gamma_1'(\epsilon) = \gamma_2'(\epsilon) \equiv 0$, *and such that*

$$H'(0)V'(0) > 0.$$

Here $H(\epsilon), V(\epsilon)$ are the mean curvature and volume of the drops in the family. Physically this means that volume increases with increasing pressure in the drop.

If $L(\psi)$ has two or more negative eigenvalues or if $H'(0)V'(0) < 0$ then the drop is unstable.

For sufficiently large V, *a stable drop exists whose profile is uniformly close to an arc of a circle.*

If $\gamma_1 + \gamma_2 \neq \pi$ *then the family of all profile curves without inflections can be parametrized by* H, *with condition* 1) *above holding throughout the family.*

Every convex drop is stable.

There exist stable drops with inflection points in the profile curve (and which are thus not convex).

If $|f''(x)| > a_0 > 0$ for a solution of (20, 21) then the first two eigenvalues λ_0, λ_1 of (22) satisfy $\lambda_0 < 0 < \lambda_1$. Thus, the only way an instability can develop is for condition 2) to fail. If that occurs, Vogel notes that for given γ_1, γ_2 each profile determines a pair of values H, V; he defines the point (H, V) corresponding to a profile on which $V'(H)$ changes sign to be a point of instability of type 2.

Another way for a family of profiles to become unstable is for condition 1) to fail; if λ_1 passes through zero while condition 2) continues to hold, then we refer to a point of instability of type 1.

In what follows, we consider the case $\gamma_1 = \gamma_2$. Let Π be the plane parallel to the two given ones and midway between them. Vogel shows [35] that *at a type* 1 *point the instability manifests itself through a perturbation that is asymmetric with respect to* Π; *at a type* 2 *point, symmetric perturbations give rise to the instability.*

Vogel found numerical evidence to support the view that there is an angle $\gamma_0 \approx 31.14°$, such that if $\gamma_1 = \gamma_2 > \gamma_0$ the family of inflectionless profiles will become type 1 unstable,

corresponding to a bifurcation that occurs when an inflection appears at the boundary (he proves the existence of the bifurcations), while if $\gamma_1 = \gamma_2 < \gamma_0$ the family of inflectionless profiles will exhibit a type 2 instability. For $\gamma_1 = \gamma_2$ no stable configurations with interior inflections were found. If $\gamma_1 \neq \gamma_2$, no type 1 instability was observed.

In [34] and in [36], Vogel presents a systematic numerical investigation of stability criteria and modes of breakdown, for varying combinations of angles γ_1, γ_2. In Figs. 18 and 19 (taken from [35]), the two modes of breakdown are exhibited in two specific cases. Fig. 18 corresponds to $\gamma_1 = \gamma_2 = 60°$, and we see that, with decreasing H, $V'(H) > 0$ until a bifurcation occurs. In Fig. 19, $\gamma_1 = \gamma_2 = 20°$; in this case $V'(H)$ changes sign prior to the bifurcation.

We remark that since no breakdown by bifurcation was observed when $\gamma_1 \neq \gamma_2$, the former of the above two cases is presumably an isolated event; a slight change in γ_1 or γ_2 should lead to a splitting into two distinct branches, with type 2 instability replacing the bifurcation.

4. Convexity

When is a capillary surface convex? For a given constant contact angle $\gamma \neq 0, \pi$ in a capillary tube, it does *not* suffice that the base domain Ω be convex. To see that, consider Ω as indicated in Fig. 9, with $0 < \gamma_0 < \pi/2$, where γ_0 is the critical angle. Take a sequence $\gamma \searrow \gamma_0$. Then the corresponding zero gravity solutions $u(x, y; \gamma)$ of (3),(4) exist and can be normalized so that $u \to u^0(x, y)$ (a solution of (3) with $\kappa = 0$) uniformly in $\Omega \backslash \bar{\Omega}^*$, $u \to \infty$ uniformly in Ω^*. The solutions are symmetric with respect to the line PQ, and thus $\frac{\partial u}{\partial x}|_P = -\tan\gamma \to -\tan\gamma_0$, $\frac{\partial u}{\partial x}|_Q = \tan\gamma \to \tan\gamma_0$. So if γ is close enough to γ_0 that $u(P; \gamma) - u(Q; \gamma) > |P - Q| \tan\gamma$, an inflection must appear in the curve lying directly above PQ in the surface, and thus convexity will fail. A similar construction can be effected also when $g > 0$ (Korevaar [37]).

Chen and Huang [38] proved that *if $g = 0$ and $\gamma = 0$ or π then convexity of Ω implies convexity of the solution surface.* Under some restrictions, Korevaar [37] found by a different method the same result when $g > 0$.

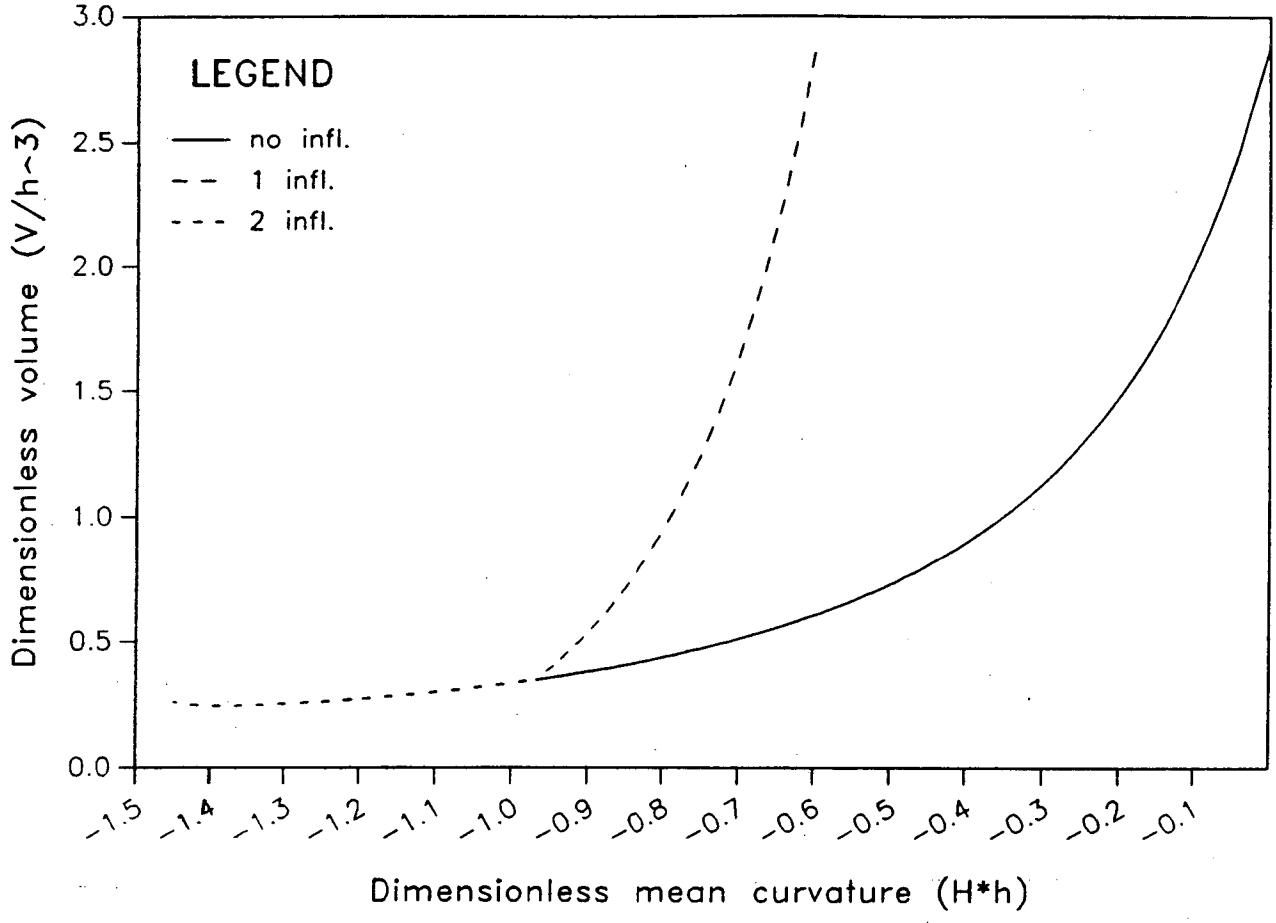

Figure 18. V vs. H for $\gamma_1 = \gamma_2 = 60°$.

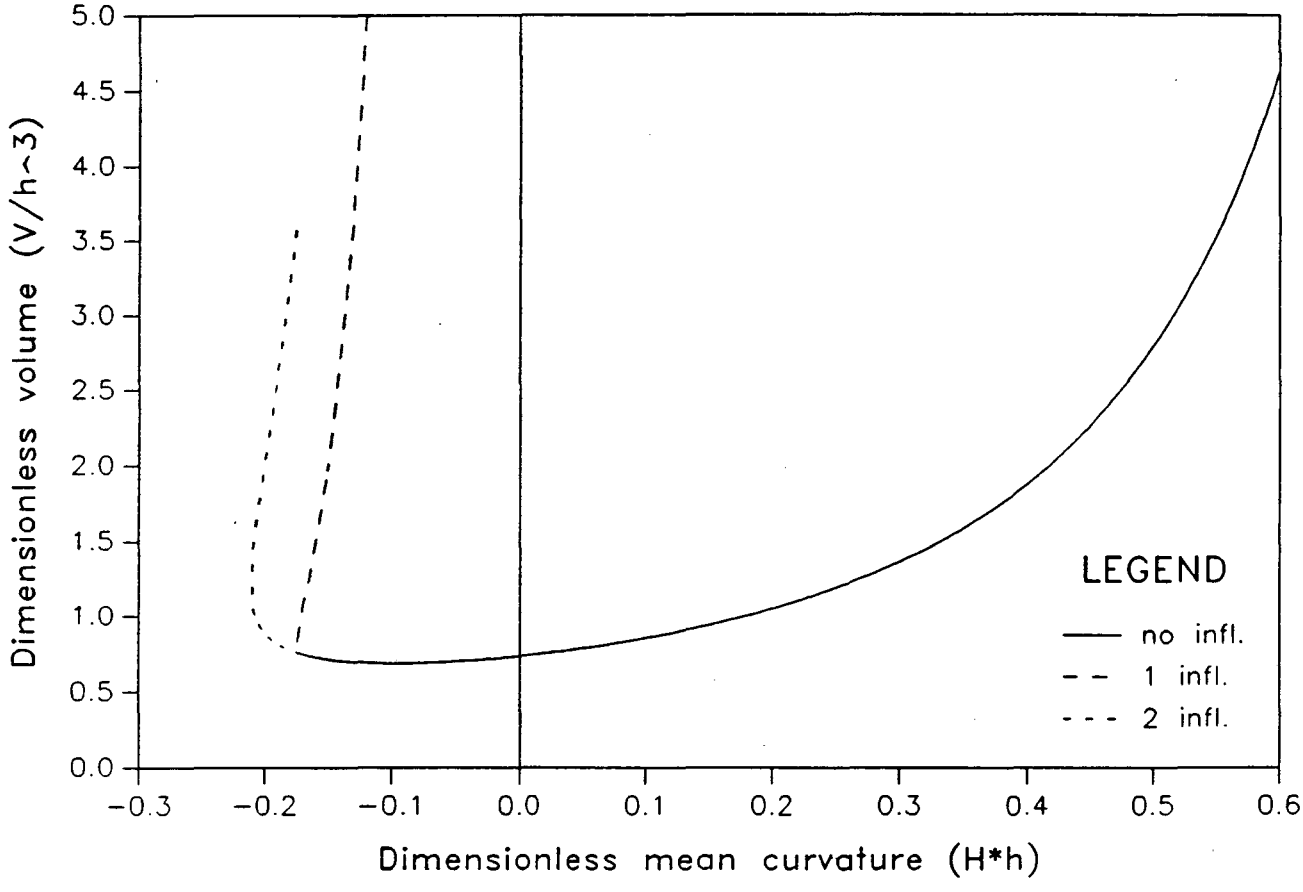

Figure 19. V vs. H for $\gamma_1 = \gamma_2 = 20°$.

Although the surface itself need not be convex when Ω is convex, Chen [39] proved that *there can be only one minimal point if* $0 < \gamma < \pi/2$. Chen's proof (and result) were improved in some respects by Siegel [40]. Huang [41] proved that *if* $g = 0$ *and* Ω *is convex, then the (unique) minimal point must have distance at least* $|\Omega|/|\Sigma|$ *from the boundary* Σ *of* Ω. *If in addition* Ω *admits a line of symmetry, then the level curves of the solution surface are convex.*

In [10] it is shown by example that a drop resting on a horizontal plane (with variable contact angle so that it need not be symmetric) can wet a convex region and nevertheless not itself be convex. We have been informed that A.N. Wang has now found an example for which also the level curves are not all convex.

We close by mentioning a striking result of Vogel [21] on sessile drops with variable contact angle: *if* $0 < \gamma < \pi/2$, *then the entire surface projects simply onto the support plane* Π. *For any distribution of* γ, *let* z_0 *be the maximum height of the drop above* Π. *Then the portion of the drop with height exceeding* $\frac{1}{2}z_0$ *projects simply onto* Π.

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