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Dispersion Relations and Finite Size Effects in High-Energy Electroweak Interactions[1]

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ABSTRACT

Dispersion relations between the forward elastic scattering amplitude and the total inclusive cross section are shown to hold for B+L violating instanton-induced interactions in the electroweak sector of the Standard Model and in the two dimensional Abelian Higgs model. The result is due to instanton-antiinstanton interactions, which are evaluated in the framework of the dilute instanton approximation. Within that framework finite instanton-size corrections are considered, and it is shown that their effect is not sufficient to suppress multiple gauge vector or Higgs scalar production below the threshold $E_0 \approx M_H/\alpha_W$. These processes may be sensitive to deviations from the dilute gas approximation.

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1 Introduction

Interest in non-perturbative baryon (B) and lepton (L) number violating processes in the electroweak sector of the Standard Model has grown again after years, due to two different (and related) observations. The first one [1] is that at temperatures of the order of M_W/α_W, the cross sections of baryon number violating amplitudes could become strong. Various evidence for [2] and against [3] this possibility has been brought up in the literature in the past years. Anyway, a second, more recently observed phenomenon has been pointed out. In [4,5] it was shown that even in zero-temperature, high-energy scattering of weakly interacting particles, some baryon number violating cross sections become enormous at energies M_W/α_W[2], so large indeed that they eventually violate unitarity bounds. Put in another way, weak interactions, according to this analysis, become strong at energies around ten TeV.

The relevant process is, in the three family Standard Model, the inelastic scattering of two fermions into ten fermions plus an arbitrary number n of Higgs particles and/or gauge vectors. The largest cross sections are those for which $n \approx 1/\alpha_W$ [4,5,6]. The cross sections of these B+L violating processes are zero to all orders in perturbation theory. It is known long since that non-perturbative effects give rise to a nonzero amplitude [7], but that effect was neglected in the past, since its cross section is $O(\exp(-4\pi/\alpha_W))$.

The crucial observation of refs. [4,5,6] is that B+L violating interactions are completely pointlike for any number of emitted Higgs and gauge bosons in the zeroth-order instanton calculation. This fact makes up for the exponential suppression in $1/\alpha_W$ at energies $O(M_W/\alpha_W)$, due to the phase space growth. Violation of unitarity in pointlike interactions is obvious, so obvious indeed to make us remember that the approximations used in [4,5] break down when the number of external particles in a scattering amplitude becomes $O(1/\alpha_W)$. That fact was clearly stated in [4,5]. Moreover, the results of [4,5] were criticized in manifold ways in refs. [8]. In order to overcome these problems, the authors of [6,9,10] took into account some of the effects that could give rise to a form factor for the B+L violating processes.

[2] In all papers on the subject it is assumed that the Higgs and the W masses be of the same order of magnitude, in order to avoid the complication of dealing with two scales.

In particular, they considered the effects of the modification of the instanton background due to the presence of many external particles. The claim of [6,9,10] is that these corrections are unable to prevent the rapid growth of B+L violating amplitudes at large energies (i.e. at $E \gg M_W/\alpha_W$). This conclusion means, evidently, that electroweak interactions become strong at large energy.

In view of the relevance of that conclusion, it becomes crucial to evaluate possible corrections to that result, and try to verify if the approximation scheme used respects all basic phyiscal requirements. The most important, and more delicate requirement is that imposed by dispersion relations between forward elastic scattering (FES) amplitudes and total cross sections. The fact that the lowest order instanton calculation respects dispersion relations below the critical threshold $E = M_W/\alpha_W$ is by no means trivial. Moreover, inclusive calculations, like those necessary to evaluate FES amplitudes, may offer some advantage ove exclusive calculations.

The first one is that phase space approximations are overcome at once with all resummation problems typical of exclusive cross section calculations. More importantly, in the calculation of FES amplitudes the source of potential classical corrections becomes clearer. In this paper, for example, it will be shown that our approximation of large separation between instantons is adequate only for the vector boson emission, and even that only at sufficiently low energies.

Interesting considerations on the evaluation of inclusive cross sections and classical corrections to leading order results were made in ref [11]. Also in [11], the role of instanton-antiinstanton configurations was correctly pointed out.

The paper is organized as follows. In Section 2, we evaluate, as a startup, fermion number violating contributions to the FES of two fermions in the two-dimensional Abelian Higgs model, in the limit of large Higgs mass. Already in that simple example some interesting features emerge. For example, one may see that finite size corrections cannot modify the unitarity-violating growth of the total cross section. That growth may only get changed, at the classical level, by the breakdown of the dilute instanton approximation itself. Section 3 describes in some detail how to single out the relevant B+L violating contributions to the FES amplitude of two fermions in the lowest order instanton calculation of a four dimensional

SU(2) gauge theory coupled to fermions. This is the usual approximation of the Standard Model, corresponding to setting $\sin\theta_W = 0$, used in the literature on the subject [4,5,6]. The problem of disentangling the perturbative contribution to the total cross section from the non-perturbative one is in general quite complex. Within the approximation we use, however, the task is not so difficult. The reason is that widely separated instanton-antiinstanton configurations are the only ones wich contribute to FES amplitudes with intermediate states containing the desired number of fermions to lowest order in α_W. Always in Section 3 we show how to single out multiple Higgs particle emission from the total B+L violating cross section. All this machinery is used in Section 4 to evaluate the relevant quantities. There in particular it is found that the total B+L violating cross section grows, if finite instanton-size corrections are neglected, as $\exp(\text{const}(E/E_0)^{4/3})$, while the inclusive cross section for Higgs particle emission grows, as $\exp(\text{const}E^2/E_0^2)$ (cfr. [4,6]). From these behaviors one concludes that at energies below the critical threshold E_0, gauge vector production dominates over Higgs particle production. This conclusion does not change if finite instanton-size corrections, also discussed in Section 4, are taken into account. These corrections are indeed not sufficient to suppress gauge vector production at energies below threshold. The fate of multiple Higgs production is similar. For that process indeed, one can show that the exponential growth found at lowest order remains to all orders in the expansion parameter $\xi = \rho^2/a^2$, with ρ=instanton size and a=instanton-antiinstanton distance. In this case, a problem comes from the fact that, quite surprisingly, the relevant value of ξ is $O(1)$ for *all* energies. A brief discussion of the implications of these results is given at the end of the section.

2 The Abelian Higgs Model

Before turning to the case of four-dimensional electroweak interactions, it is useful to study the simpler case of the two-dimensional Abelian Higgs model in the limit of large Higgs mass. Besides having a natural scale for the instanton (i.e. vortex) configurations, this model linearizes in the $M_H \to \infty$ limit. Vortex-antivortex configurations are therefore exactly evaluable and

the (classical) theory is completely under control. The field content of the model is: a complex scalar ϕ, an abelian gauge vector A_μ, and a charged Dirac fermion ψ. The action in Euclidean space reads

$$\begin{aligned} S &= \int d^2x \frac{1}{4} F_{\mu\nu}^2 + |D_\mu \phi|^2 + \lambda(\phi^*\phi - \langle\phi\rangle^*\langle\phi\rangle)^2 + i\bar{\psi} D_\mu \gamma^\mu \psi, \\ D_\mu &= \partial_\mu + ieA_\mu. \end{aligned} \tag{2.1}$$

Since $\langle\phi\rangle$ plays the role of perturbation expansion parameter, we assume $|\langle\phi\rangle| \gg 1$, as in ref. [6], in order to deal with a weakly coupled theory. This model possesses a vortex solution, whose form is known exactly in the $\lambda \to \infty$ limit and reads, in polar coordinates:

$$\phi = e^{i\theta}\langle\phi\rangle, \quad A_r = 0, \quad A_\theta = \frac{M_W}{e} r K_1(M_W r) - \frac{1}{e}\theta, \; r > M_H^{-1}. \tag{2.2}$$

Here $M_W^2 = 2e^2|\langle\phi\rangle|^2$ is the gauge vector mass, and K_1 is a modified Bessel function. Action 2.1 evaluated on the background 2.2 remains finite in the infinite Higgs mass limit, once the value of fields in the region $r < M_H^{-1}$ is taken into account. Its value is $S_{vortex} = C|\langle\phi\rangle|^2$, with C a coupling constants independent number of order one.

In this theory non-perturbative effects give rise to the fermion number violating processes [6]:

$$f + f \to n \text{ Higgs} + m \text{ Gauge Vectors}. \tag{2.3}$$

To check dispersion relations, one must find a way of computing the FES amplitude $f + f \to f + f$, and extract its relevant part. In our case this means that one has to insert a projector $\mathcal{P}$ over intermediate states containing no fermions.

We proceed as follows. Consider the Green function

$$G(x_1, x_2, x_3, x_4) = \langle \bar{\psi}(x_1)\bar{\psi}(x_2)\mathcal{P}\psi(x_3)\psi(x_4)\rangle. \tag{2.4}$$

From it, one extracts a scattering amplitude by the usual LSZ procedure, i.e. by Fourier transforming 2.4 and amputating the external fermion propagators. The result is

$$\begin{aligned} &A(p_1, p_2, p_3, p_4) = (2\pi)^2\delta^2(p_1 + p_2 + p_3 + p_4) \\ &\langle \bar{\psi}' \overleftarrow{\not{p}}_1 u(p_1)\bar{\psi}' \overleftarrow{\not{p}}_2 u(p_2)\mathcal{P}\bar{u}(p_3)\not{p}_3\psi'(p_3)\bar{u}(p_4)\not{p}_4\psi'(p_4)\rangle, \end{aligned} \tag{2.5}$$

where $u(p_i)$ and $\bar{u}(p_i)$ are the appropriate external spinors, and the prime denotes the Fourier transform. Naturally $A(p_1, p_2, p_3, p_4)$ must be continued back to Minkowsky space momenta. This is done by evaluating it at imaginary p_i^0.

Calculating $G(x_1, x_2, x_3, x_4)$ is in general far from trivial. To lowest order and in the dilute gas approximation, however, one can argue as follows. The dominant contribution to the functional integral is the one given by substituting to the fermions their zero modes in the appropriate vortex background. Here we need four zero modes, two of charge $+e$ and two of charge $-e$. A smaller number of zero modes would indeed mean that in the Green function 2.4 there are internal fermion lines. On the other hand, the effect of $\mathcal{P}$ is, to lowest order in the semiclassical expansion, to put to zero all contributions from intermediate states containing fermions. A larger number of zero modes would make the functional integral vanish upon integration over the fermions. The dominant finite action configuration with 2+2 zero modes is the vortex-antivortex one. In the dilute vortex approximation the zero modes of positive charge are concentrated around the center of the vortex, while the negative charge ones are concentrated around the antivortex. Needless to say, the self-consistency of the dilute vortex approximation has to be checked at the end of the calculation.

To sum up, the Green function 2.4 reads

$$\begin{aligned} G(x_1, x_2, x_3, x_4) &= \text{const} \int d^2x d^2y \psi_0^1(x_1 - x)\psi_0^2(x_2 - x) \times \\ &\quad \bar{\psi}_0^3(x_3 - y)\bar{\psi}_0^4(x_4 - y) \exp(-S(A^{xy}, \phi^{xy})). \end{aligned} \tag{2.6}$$

Here, x and y are the usual collective coordinates of the classical vortex-antivortex configuration (A^{xy}, ϕ^{xy}). The vortex is centered about x and the antivortex about y. Let us denote by $\theta_x(z)$ the angle formed by the line passing through (x, z) and the x axis. Similarly, call $\theta_y(z)$ the angle between (y, z) and the x axis. The value of ϕ^{xy} is then

$$\phi^{xy}(z) = \langle\phi\rangle e^{i\theta_x(z) - i\theta_y(z)}, \tag{2.7}$$

while A_μ^{xy} is simply the sum of a vortex (A_μ^x) and an antivortex (A_μ^y) solution. That configuration solves exactly the equations of motion of 2.1 in the infinite Higgs mass limit. The dilution assumption enters when we

substitute the vortex (antivortex) zero modes $\psi_0^i(z-x)$ ($\bar{\psi}_0^i(z-y)$) into 2.6 instead of the exact fermionic background fields.

By performing the gauge transformation

$$\phi^{xy} \to \phi^{xy} e^{-i\theta_x(z)+i\theta_y(z)}, \quad A_\mu^{xy} \to A_\mu^{xy} + 1/e\partial_\mu\theta_x - 1/e\partial_\mu\theta_y, \tag{2.8}$$

one can easily evaluate $S(A^{xy}, \phi^{xy})$. This is done by dividing the space into three regions: two disks D_x, D_y of radius $\rho \approx M_H^{-1}$, centered about x and y respectively, and the outside region $R - D_x - D_y \equiv D$. In D the interference term in the bosonic action is

$$\int_D d^2z \frac{1}{2} F_{\mu\nu}^x F_{\mu\nu}^y + 2|\langle\phi\rangle|^2 A_\mu^x A_\mu^y, \quad F_{\mu\nu}^{x,y} \equiv \partial_\mu A_\nu^{x,y} - \partial_\nu A_\mu^{x,y}. \tag{2.9}$$

Since $A_\mu^x + A_\mu^y$ is a solution of the classical equations of motion, integral 2.9 reduces to

$$-\frac{1}{2}\int_{\partial D_x} dx^\rho \varepsilon^{\mu\nu} F_{\mu\nu}^x A_\rho^y - \frac{1}{2}\int_{\partial D_y} dx^\rho \varepsilon^{\mu\nu} F_{\mu\nu}^x A_\rho^y. \tag{2.10}$$

To evaluate the first integral, it is convenient to pass to polar coordinates centered about x, and notice that, in our gauge,

$$\begin{aligned}
&\int_{\partial D_x} dx^\rho \varepsilon^{\mu\nu} F_{\mu\nu}^x A_\rho^y = \int d\theta \frac{1}{\rho} F_{r\theta}^x A_\theta^y = \\
&-\frac{M_W^2}{e^2} K_0(M_W\rho) \int d\theta A_\theta^y = \pi \frac{M_W^2\rho^2}{e^2 a} K_0(M_W\rho) F_{\theta r}^y = \\
&\pi \frac{M_W^4\rho^2}{e^2} K_0(M_W\rho) K_0(M_W a).
\end{aligned} \tag{2.11}$$

Here $a \equiv |x-y|$. The second integral in 2.10 is evaluated in a similar way, and gives

$$\int_{\partial D_y} dx^\rho \varepsilon^{\mu\nu} F_{\mu\nu}^x A_\rho^y = 2\pi \frac{M_W^3\rho}{e^2} K_0(M_W a) K_1(M_W\rho). \tag{2.12}$$

Inside the disk D_x the fields are, approximately:

$$A_\theta = \frac{1}{2}(\rho^2 - r^2) M_W^2 K_0(M_W\rho) + M_W\rho K_1(M_W\rho), \quad A_r = 0, \quad \phi = 0. \tag{2.13}$$

These expressions are determined by matching the solutions of the equations of motion for A_μ inside and outside the disk. The interference term is

$$\int_{D_x} \frac{1}{2} F^x_{\mu\nu} F^y_{\mu\nu} + 2|\langle\phi\rangle|^2 A^x_\mu A^y_\mu = \int_{\partial D_x} dx^\rho \varepsilon^{\mu\nu} F^x_{\mu\nu} A^y_\rho =$$
$$\pi \frac{M_W^4 \rho^2}{e^2} K_0(M_W \rho) K_0(M_w a). \tag{2.14}$$

The contribution of the integral in D_y is the same.

We are studying the case $M_W \rho \approx M_W M_H^{-1} \ll 1$, so that we find

$$S(A^{xy}, \phi^{xy}) = 2C|\langle\phi\rangle|^2 - 2\pi|\langle\phi\rangle|^2 K_0(M_W a). \tag{2.15}$$

We have now all the elements to evaluate eq. 2.6, but it is simpler to compute directly the scattering amplitude $A(p_1, p_2, p_3, p_4)$. In fact, by Fourier-transforming, amputating the external fermionic legs, and putting all external momenta on-shell, we get

$$\begin{aligned} A(p_1, p_2, p_3, p_4) &= \text{const } (2\pi)^2 \delta^2(p_1 + p_2 + p_3 + p_4) \times \\ &\quad \bar{u}(p_1) F \bar{u}(p_2) F \bar{F} u(p_3) \bar{F} u(p_4) \mathcal{F} \\ \mathcal{F} &= \int d^2 a \exp(ia \cdot p_T - 2C|\langle\phi\rangle|^2 + 2\pi|\langle\phi\rangle|^2 K_0(M_W a)), \\ p_T &= p_1 + p_2. \end{aligned} \tag{2.16}$$

Amputated zero mode wave functions have been replaced, on the mass shell, by the momentum-independent constant spinors F and $\bar{F}$, following ref. [6].

The integration over angular variables in 2.16 is trivial, and yields

$$\begin{aligned} \mathcal{F} &= \pi \int da a J_0(aE) \exp(-2C|\langle\phi\rangle|^2 + 2\pi|\langle\phi\rangle|^2 K_0(M_W a)) \\ E &= p_1^0 + p_2^0. \end{aligned} \tag{2.17}$$

As pointed out before, the function $\mathcal{F}(E)$ must be evaluated at imaginary values of E.

By using the well known asymptotic expansions for J_0 and K_0:

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos(x - \pi/4), \quad K_0(x) \approx \sqrt{\frac{\pi}{2x}} \exp(-x), \quad x \gg 1, \tag{2.18}$$

and letting $E \to iE$, one finds

$$\begin{aligned}
\mathcal{F} &= \sqrt{\frac{\pi}{2}} M_W^{-2} \mathcal{E}^{-1/2} \int dx x^{1/2} \exp\left(\mathcal{E}x + \sqrt{2}\pi^{3/2}|\langle\phi\rangle|^2 x^{-1/2} e^{-x} + \right. \\
&\quad \left. -2C|\langle\phi\rangle|^2\right), \\
x &\equiv aM_W, \quad \mathcal{E} \equiv E/M_W. \qquad (2.19)
\end{aligned}$$

Evaluation of integral 2.19 can be done in the saddle point approximation. The stationary point is given by the solutions of the equation

$$\mathcal{E} - \sqrt{2}\pi^{3/2}|\langle\phi\rangle|^2 x^{-1/2} e^{-x} - \frac{\sqrt{2}}{2}\pi^{3/2}|\langle\phi\rangle|^2 x^{-3/2} e^{-x} = 0. \qquad (2.20)$$

Self-consistency of approximation 2.18 requires $\mathcal{E}x \gg 1$ and $x \gg 1$. With these constraints we find an approximate solution of 2.20 of the form

$$x \approx \log \frac{\sqrt{2}\pi^{3/2}|\langle\phi\rangle|^2}{\mathcal{E}} \left(\log \frac{\sqrt{2}\pi^{3/2}|\langle\phi\rangle|^2}{\mathcal{E}}\right)^{-1/2}, \quad 1 \ll \mathcal{E} \ll \sqrt{2}\pi^{3/2}\langle\phi\rangle. \qquad (2.21)$$

From this equation and the previous considerations we get

$$\begin{aligned}
\mathcal{F} &= \pi i M_W^{-2} \mathcal{E}^{-1} \left[\log \frac{\sqrt{2}\pi^{3/2}|\langle\phi\rangle|^2}{\mathcal{E}} \left(\log \frac{\sqrt{2}\pi^{3/2}|\langle\phi\rangle|^2}{\mathcal{E}}\right)^{-1/2}\right]^{1/2} \times \\
&\quad \exp\left(\mathcal{E} \log e \frac{\sqrt{2}\pi^{3/2}|\langle\phi\rangle|^2}{\mathcal{E}} \left(\log \frac{\sqrt{2}\pi^{3/2}|\langle\phi\rangle|^2}{\mathcal{E}}\right)^{-1/2} - 2C|\langle\phi\rangle|^2\right). \qquad (2.22)
\end{aligned}$$

Notice that $\mathcal{F}$ is imaginary, so that it gives a contribution to the inclusive cross section $f + f \to$ anything. Equation 2.22 coincides with formula 29 of ref. [6], up to the replacement $M_W \to M_H$. This difference is probably due to the fact that here we are also considering gauge vectors production, instead of only Higgs particles production, like in [6]. Equation 2.22 tell us that, stretching the limits of validity of our approximations, the inclusive cross section $f + f \to$ anything becomes $O(1)$ at energies $E \approx O(M_W|\langle\phi\rangle|^2)$.

It is also possible to evaluate $\mathcal{F}$ for energies $E \gg M_W|\langle\phi\rangle|^2$, with the caveat that the dilute vortex approximation is in this case quite questionable.

Resorting again to the saddle point calculation we find:

$$\begin{aligned}\mathcal{F} &= i\sqrt{2}\pi^2|\langle\phi\rangle|^2 M_W^{-2}\mathcal{E}^{-2}\exp(2\pi|\langle\phi\rangle|^2\log e\mathcal{E}/2\pi|\langle\phi\rangle|^2 - 2C|\langle\phi\rangle|^2),\\ x &= 2\pi|\langle\phi\rangle|^2/\mathcal{E}, \quad M_H \gg \mathcal{E} \gg 2\pi|\langle\phi\rangle|^2. \end{aligned} \tag{2.23}$$

Notice that in this case the growth of the total cross section turns out to be $\mathcal{E}^{\pi|\langle\phi\rangle|^2-1}$. This result seems to indicate that, in two dimensions, form factors due to classical corrections are not sufficient to stop the Abelian Higgs model from becoming a strongly interacting theory at $E \approx M_W|\langle\phi\rangle|^2$. Corrections to equation 2.23, which in principle can easily change this result, are to be expected when deviations to the dilute vortex approximation are taken into account.

3 SU(2) Gauge Interactions in Four Dimensions

We pass now to the four dimensional SU(2) electroweak interactions. As usual [4,5,6] we put $\sin\theta_W = 0$. The action in Euclidean space reads

$$\begin{aligned} S &= \int d^4x \frac{1}{2}\mathrm{tr}F_{\mu\nu}F_{\mu\nu} + D_\mu\phi^\dagger D_\mu\phi + \frac{\lambda}{4}(\phi^\dagger\phi - \langle\phi\rangle^\dagger\langle\phi\rangle)^2 + S_F,\\ D_\mu &= \partial_\mu - \frac{i}{2}gA_\mu^a\sigma^a, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \end{aligned} \tag{3.1}$$

Here S_F is the fermionic action, and σ^a are the usual Pauli matrices. In terms of the constants of lagrangian 3.1, the physical Higgs mass is $M_H = (\lambda/2\langle\phi\rangle^\dagger\langle\phi\rangle)^{1/2}$ and the gauge vector mass is $M_W = \sqrt{\kappa}M_H$, with $\kappa = g^2/\lambda$. In the following we will take κ of order one, to avoid complications due to the presence of two scales.

In the present case, the relevant B+L violating processes are [5]

$$q + q \to (3n_f - 2)\, q + n_f\, l + n\, W + m\, H, \tag{3.2}$$

where the q's denote quark fields, l corresponds to leptons, W to gauge vectors and H to physical Higgs particles. The number of families is n_f. In this process a fermion for each family of quarks and leptons is present

either in the initial or in the final state. Process 3.2 evidently violates B+L, while it conserves B−L.

In order to single out the relevant part of the FES $q + q \to q + q$, we follow the procedure of Section 2, up to some modifications. What we are looking for is, indeed, the lowest order contribution to a process with ten fermions in the intermediate state. The relevant Green function, from which the corresponding scattering amplitude is evaluated, is therefore

$$G(x_1, x_2, x_3, x_4) = \langle q(x_1)q(x_2)\mathcal{P}\bar{q}(x_3)\bar{q}(x_4)\rangle, \tag{3.3}$$

where $\mathcal{P}$ projects on intermediate states with the correct number of fermions. To lowest order in the semiclassical approximation, one can easily tell apart two different contributions to intermediate states with fermions. One is coming from fermionic zero modes around a classical finite-action configuration. The other one is the perturbative contribution coming from the fermionic propagator. This second contribution is suppressed, relatively to the first one, at least by a factor g^2. Thus, even in this case, the Green function 3.3 is easily written down explicitly to lowest order in g^2. In order to find this explicit representation, we must first recall that, in the Standard Model, the instanton solution is found by introducing an operator constraint [12] which, in our case, gives rise to the following form of the classical action [12]

$$S^I(\rho) = 2\pi^2 \frac{M_H^2 \rho^2}{\lambda} + \frac{1}{\lambda} O(v^4 \rho^4 \log v\rho). \tag{3.4}$$

Here ρ is the instanton size, i.e. another collective coordinate, besides the instanton's location. By v we mean $(\langle\phi\rangle^\dagger\langle\phi\rangle)^{1/2}$. This result is easily proven by substituting into action 3.1 the explicit form of the constrained instanton solution, which, in the singular gauge, reads [12,5]

$$\begin{aligned}
A_\mu &= x_\nu \Lambda_{\mu\nu} \mathcal{A}(x), \\
\mathcal{A}(x) &= \rho^2 M_W^2 \frac{K_2(M_W x)}{x^2}, \quad x \gg M_H^{-1}, \\
\mathcal{A}(x) &= 2\rho^2/x^2(x^2+\rho^2), \quad x < M_H^{-1}; \\
\phi(x) &= M_H \left(1 - \frac{1}{2}\rho^2 M_H^2 \frac{K_1(M_H x)}{x}\right) h, \quad x \gg M_H^{-1}, \\
\phi(x) &= M_H x (x^2+\rho^2)^{-1/2} h, \quad x < M_H^{-1}.
\end{aligned} \tag{3.5}$$

In this equation, h is a constant isospinor and $\Lambda_{\mu\nu}$ is defined, according to the conventions of ref. [5], as $\Lambda_{\mu\nu} = 1/2\bar{\eta}_{a\mu\nu}U^\dagger\sigma^a U$. The SU(2) matrix U gives the instanton orientation in isospin space. The antiinstanton solution is identical to 3.5 with the substitution $\bar{\eta}_{a\mu\nu} \to \eta_{a\mu\nu}$.

Next, we have to find the action of an instanton-antiinstanton configuration, which, as we will see later in this section, is the relevant one to our computations. We are interested in the dominant terms for large instanton separation a, i.e. for $a \gg \rho$. We call this approximation dilute instanton, or dilute instanton gas, with a slight abuse of terms.

Because of equation 3.5, one is supposed to consider two different cases, namely: when the distance a of the instanton pair is much larger than M_H^{-1}, and when $a \ll M_H^{-1}$. This happens because the instanton solutions behave in a qualitatively different way in the two regions. Actually we will always deal with the second situation. A heuristic motivation for this choice is that the interesting energy region for baryon and lepton number violating processes is $E \approx M_H/\lambda$ [4,5,6]. This fact, at least naively, implies that the relevant lenght scale is $a \approx M_H^{-1}\lambda \ll M_H^{-1}$.

In the SU(2) case actually, we need not to redo all calculations to figure out the form of the instanton-antiinstanton interaction term. All relevant calculations were indeed already done in refs. [13] and [14]. In this paper, we will simply sketch the main points of the derivation.

The first thing to notice is that, in the dilute instanton approximation, one may use the following argument [13], which we describe only for the gauge contribution to the classical action. When a is much larger than both the instanton and antiinstanton sizes ρ and ρ', we can split the integration over the four-dimensional space in two parts: inside and outside a ball B_x of radius R (ρ, $\rho' \ll R \ll M_H^{-1}$) centered about the instanton. Since the value of the antiinstanton field is small inside B_x, the value of the action is:

$$\begin{aligned} S^{I\bar{I}}(a,\rho,\rho') &= S^I(\rho) + S^I(\rho') + \delta S_{B_x} + \delta S_{R^4-B_x}, \\ \delta S_{B_x} &= 2\int_{B_x} d^4x \mathrm{tr} F^I_{\mu\nu} D^I_\mu A^{\bar{I}}_\nu + O((A^{\bar{I}})^2). \end{aligned} \tag{3.6}$$

Here $F^I_{\mu\nu}$ and D^I_μ denote the field strenght and covariant derivative in the instanton background. $A^{\bar{I}}_\mu$ is the antiinstanton solution. $\delta S_{R^4-B_x}$ is obtained by replacing the instanton with the antiinstanton. Using the equations of

motion for A^I_μ one finds [13]

$$\begin{aligned}
\delta S_{B_x} &= 2\int_{\partial B_x} d^3x \frac{x^\mu}{x} F^I_{\mu\nu} A^I_\nu \\
&= -\frac{32\pi^2}{g^2}[4(u_I u_{\bar{I}})^2 - 1]\rho^2\rho'^2 a^{-4} + \frac{1}{g^2} o(\rho^2\rho'^2 a^{-4}), \\
\delta S_{R^4-B_x} &= \delta S_{B_x}.
\end{aligned} \tag{3.7}$$

In equation 3.7, u_I is a unit SU(2) vector giving the instanton orientation in isospin space, $u_{\bar{I}}$ is the same object for the antiinstanton. The scalar field contribution to the classical action is found in a similar way [14] and reads:

$$\begin{aligned}
\delta S^{I\bar{I}}_{scalar}(\rho,\rho',a) &= -\frac{4\pi^2}{\lambda} M_H^2 \rho^2 \rho'^2 a^{-2}[2(u_I u_{\bar{I}})^2 - 1] + O(M_H^2 \rho^4 \rho'^2 a^{-4}) + \\
& O(M_H^2 \rho^2 \rho'^4 a^{-4}).
\end{aligned} \tag{3.8}$$

Note that the contribution of the scalar potential to the action is $O(M_H^4\rho^4)$, and is not present in eq. 3.8 because is of the same order of terms which were already discarded in eq. 3.4. We are not going to integrate over instanton orientations, rather, we notice that, in the saddle point approximation we are working in, the relevant configurations are those of minimal action. This means to take $u_I u_{\bar{I}} = 1$ in eqs. 3.7, 3.8. The total $I\bar{I}$ classical action reads then:

$$\begin{aligned}
S^{I\bar{I}}(\rho,\rho',a) &= \frac{16\pi^2}{g^2} + \frac{2\pi^2}{\lambda} M_H^2 \rho^2 + \frac{2\pi^2}{\lambda} M_H^2 \rho'^2 - \frac{96\pi^2}{g^2}\rho^2\rho'^2 a^{-4} + \\
& -\frac{4\pi^2}{\lambda} M_H^2 \rho^2 \rho'^2 a^{-2} + o(\rho^2\rho'^2 a^{-4}, M_H^2\rho^2).
\end{aligned} \tag{3.9}$$

To find the FES amplitude for the process $q + q \to q + q$, first of all, we ought to find an explicit expression for eq. 3.3. To lowest order in g^2 this is done as follows.

The fermionic action is not exactly zero in the $I\bar{I}$ sector, even when evaluated on instanton zero modes, due to the presence of the interference term

$$S^{I\bar{I}}_{fermionic} = \int d^4x \bar{\psi}^0_i(x - x_I)(i\partial\!\!\!/ + m^i_f)\psi^0_i(x - x_I). \tag{3.10}$$

Here, ψ_i^0 ($\bar{\psi}_i^0$) denote the fermionic zero modes in the instanton (antiinstanton) background. The index i is the fermion label and the m_f^i are the fermion masses. The center of the instanton is at x_I, and that of the antiinstanton at $x_{\bar{I}}$.

Due to this term, $I\bar{I}$ configurations give nonvanishing contributions to 3.3. These contributions are indeed exactly those coming from B+L violating processes. The corresponding FES amplitude is again computed by the LSZ procedure from 3.3. The Green function 3.3 itself reads, once we write explicitly the spinor indices α_i:

$$G(x_1,x_2,x_3,x_4)_{\alpha_1\alpha_2\alpha_3\alpha_4} = \int d\rho\rho^{-5}f(\rho\mu)\int d\rho'\rho'^{-5}f(\rho'\mu)\int d^4x_I d^4x_{\bar{I}}$$
$$\psi^1_{0\alpha_1}(x_1-x_I)\psi^2_{0\alpha_2}(x_2-x_I)\bar{\psi}^1_{0\alpha_3}(x_3-x_{\bar{I}})\bar{\psi}^2_{0\alpha_4}(x_4-x_{\bar{I}})\times$$
$$\prod_{i=3}^{4n_f}\int d^4x\bar{\psi}_i^0(x-x_I)(i\partial\!\!\!/+m_f^i)\psi_i^0(x-x_{\bar{I}})\exp(-S^{I\bar{I}}(\rho,\rho',a)). \qquad (3.11)$$

In this equation, we have explicitly written down the integration measure for all collective coordinates. This measure includes, in $f(\rho\mu)$ and $f(\rho'\mu)$, terms coming from the lagrangian constraint that is to be imposed in order to find stationary points in the action 3.1 [12,5]. The function $f(\rho\mu)$ also accounts for one-loop renormalization group effects. For this reason, it depends on the renormalization point μ. Its explicit form is

$$f(\rho\mu) = \mathrm{const}(\rho\mu)^\alpha. \qquad (3.12)$$

For SU(2) $\alpha = (43-8n_f)/6$ [7,5]. In order to get the FES amplitude $A(p_1,p_2,p_3,p_4)$ from $G(x_1,x_2,x_3,x_4)$ one Fourier-transforms 3.11 and amputates the external fermion legs. The amputated, on shell values of all fermionic zero modes in an instanton background were given in ref. [5], and read

$$\begin{aligned}[\psi_A(p)]_{amp} &= -2\pi i\rho m_f U^\dagger\chi,\\ [\psi_B(p)]_{amp} &= 2\pi i\rho p_\mu\bar{\sigma}^\mu U^\dagger\chi. \end{aligned} \qquad (3.13)$$

Here ψ_A (ψ_B) are in the (1/2,0) ((0,1/2)) representation of SO(4) $\approx$ SU(2)$\otimes$ SU(2). The χ are constant isospinors defined as

$$\chi = \begin{pmatrix}1\\0\end{pmatrix}, \quad \chi = \begin{pmatrix}0\\1\end{pmatrix}, \qquad (3.14)$$

for 'up' and 'down' fermions respectively [5]. The Fourier transforms $\psi'^{0i}(p)$ of the zero modes themselves read [6]

$$\psi'^{0i}_{\alpha}(p) = \frac{1}{p^2 + (m^i_f)^2} F^i_{\alpha}(p). \tag{3.15}$$

The $F^i_{\alpha}(p)$ reduce to 3.13 on the mass shell. By substituting equations 3.13 and 3.15 into the Fourier transform of 3.11, and after amputating all external legs, one gets a very simple form for the imaginary part of the FES amplitude $A(p_1, p_2, p_3, p_4)$. This quantity is the one of interest for checking dispersion relations, since it is related to the inclusive cross section of processes 3.2 by means of the optical theorem. One finds

$$\begin{aligned}
\mathrm{Im}A(p_1, p_2, p_3, p_4) &= (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4) h_{\alpha_1\alpha_2\alpha_3\alpha_4}(p_1, p_2, p_3, p_4) \\
&\quad \prod_{i=3}^{4n_f} \int d^4p_i 2\pi\theta(p^0_i)\delta(p^2_i + (m^i_f)^2)\varepsilon_i \mathrm{Im}\mathcal{I}(p_T), \\
\varepsilon_i &= \langle \chi^{\dagger}_i U p^{\mu}_i \sigma^{\mu} U^{\dagger} \chi_i \rangle_U, \\
\mathcal{I}(p_T) &= \int d^4a \int d\rho \rho^{-5} f(\rho\mu) \int d\rho' \rho'^{-5} f(\rho'\mu) \rho^{4n_f} \rho'^{4n_f} \\
&\quad \exp(-S^{II}(\rho, \rho', a) + ia \cdot p_T).
\end{aligned} \tag{3.16}$$

Here $\langle\rangle_U$ denotes the average over instanton orientations in isospin space. We called $h_{\alpha_1\alpha_2\alpha_3\alpha_4}$ a SO(4) invariant polynomial depending only on the momenta of the four external fermions, p_T is the total four-momentum available to gauge vector or Higgs production, i.e.

$$p_T = p_1 + p_2 - \sum_{i=3}^{4n_f} p_i. \tag{3.17}$$

In order to extract from 3.16 the cross section corresponding to given values of the intermediate fermion momenta, i.e. to calculate all cross sections for the processes 3.2 summed over n and m, one substitutes in 3.16 the appropriate kinematical invariants to the integrated phase space

$$\Phi = \prod_{i=3}^{4n_f} \int d^4p_i 2\pi\theta(p^0_i)\delta(p^2_i + (m^i_f)^2). \tag{3.18}$$

The actual computation of 3.16 is the subject of next section, here we only make a last observation. If one substitutes expression 3.9 to $S^{I\bar{I}}$ in 3.16, one may expand $\exp(-S^{I\bar{I}})$ as

$$\exp(-S^{I\bar{I}}) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{96\pi^2}{g^2} \rho^2 \rho'^2 a^{-4} \right)^n \exp\left(-\frac{16\pi^2}{g^2} - \frac{2\pi^2}{\lambda} M_H^2 \rho^2 + \right.$$
$$\left. -\frac{2\pi^2}{\lambda} M_H^2 \rho'^2 + \frac{4\pi^2}{\lambda} M_H^2 \rho^2 \rho'^2 a^{-2} \right). \tag{3.19}$$

On the other hand,

$$\frac{96\pi^2}{g^2} \rho^2 \rho'^2 a^{-4} \approx \int d^4x (\partial_\mu A_\nu^I - \partial_\nu A_\mu^I) A_\nu^I. \tag{3.20}$$

In other words, expansion 3.19 is an expansion in powers of the classical gauge fields A_μ^I, $A_\mu^{\bar{I}}$. By means of an argument similar to that used to figure out the inclusive cross section of processes 3.2, one may convince oneself that

$$\left[\int d^4x (\partial_\mu A_\nu^I - \partial_\nu A_\mu^I) A_\nu^I \right]^n \tag{3.21}$$

gives rise to a FES amplitude with n intermediate gauge vectors. This fact implies that, if one keeps only the $n = 0$ term in 3.19, i.e. if one considers in equation 3.16 only the Higgs field contribution to the classical action, one would find, via dispersion relations, the inclusive cross section of the process

$$q + q \to (3n_f - 2)q + n_f\, l + m\, H, \tag{3.22}$$

summed over m. For related consideration see also ref. [11].

4 Energy Dependence of Inclusive Cross Sections

We have now all the tools necessary to evaluate B+L violating contributions to inclusive cross sections, and check thereby whether the instanton calculations of refs. [4,5,6] satisfy dispersion relations. What we do in this section indeed, is to relate eq. 3.16 to the corresponding total cross section.

This is done as usual via the optical theorem, by equating initial and final momenta $p_1 = p_3$, $p_2 = p_4$, multiplying equation 3.16 by the appropriate external spinors $u_\alpha(p)$ and $\bar{u}_\alpha(p)$ and the usual kinematical factors. The result is

$$\begin{aligned}\sigma_T &= \frac{(2\pi)^4}{\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}}\bar{u}_{\alpha_1}\bar{u}_{\alpha_2}u_{\alpha_3}u_{\alpha_4}h_{\alpha_1\alpha_2\alpha_3\alpha_4}\times \\ &\prod_{i=3}^{4n_f}\int d^4p_i 2\pi\theta(p_i^0)\delta(p_i^2+(m_f^i)^2)\varepsilon_i \mathrm{Im}\mathcal{I}(p_T). \end{aligned} \tag{4.1}$$

In this equation, the only nontrivial thing to be calculated is $\mathcal{I}$.

Recalling equations 3.12, 3.13, 3.16, and defining

$$\mathcal{I}(p_T) = \mu^{2\alpha}\mathcal{J}(p_T), \tag{4.2}$$

one is left with the following integral

$$\begin{aligned}\mathcal{J}(p_T) &= \int d^4a d\rho d\rho'(\rho\rho')^c \exp\left(-\frac{16\pi^2}{g^2} - \frac{2\pi^2}{\lambda}M_H^2\rho^2 - \frac{2\pi^2}{\lambda}M_H^2\rho'^2 + \right. \\ &\left. \frac{96\pi^2}{g^2}\rho^2\rho'^2a^{-4} + \frac{4\pi^2}{\lambda}M_H^2\rho^2\rho'^2a^{-2} + ia\cdot p_T\right), \\ c &= 4n_f + \alpha - 5. \end{aligned} \tag{4.3}$$

Integrating over angular variables one finds that $\mathcal{J}$ reduces to

$$\begin{aligned}\mathcal{J} &= \frac{4\pi^2}{E}\int da d\rho d\rho' a^2(\rho\rho')^c J_1(Ea)\exp\left(-\frac{16\pi^2}{g^2} - \frac{2\pi^2}{\lambda}M_H^2\rho^2 + \right. \\ &\left. -\frac{2\pi^2}{\lambda}M_H^2\rho'^2 + \frac{96\pi^2}{g^2}\rho^2\rho'^2a^{-4} + \frac{4\pi^2}{\lambda}M_H^2\rho^2\rho'^2a^{-2}\right), \\ E &= |p_T|. \end{aligned} \tag{4.4}$$

Consider now the region $Ea \gg 1$, where the Bessel function in 4.4 can be approximated by

$$J_1(aE) \approx \sqrt{\frac{2}{\pi a E}}\cos(aE - 3/4\pi). \tag{4.5}$$

Since we are interested in the analytic continuation of integral 4.3 to Minkowsky space momenta, we make the substitution $E \to iE$ everywhere. Integral 4.4 reduces then, disregarding exponentially small terms, to

$$\begin{aligned}\mathcal{J} &= \frac{2(2\pi)^{3/2}}{E^{3/2}} \int da d\rho d\rho' a^{3/2} (\rho\rho')^c \exp\left(Ea - \frac{16\pi^2}{g^2} - \frac{2\pi^2}{\lambda} M_H^2 \rho^2 + \right. \\ &\left. - \frac{2\pi^2}{\lambda} M_H^2 \rho'^2 + \frac{96\pi^2}{g^2} \rho^2 \rho'^2 a^{-4} + \frac{4\pi^2}{\lambda} M_H^2 \rho^2 \rho'^2 a^{-2} \right). \end{aligned} \tag{4.6}$$

In order to be consistent with the approximations we made, integral 4.6 is to be evaluated in the saddle point approximation. Notice that, since we have assumed that $a \ll M_H^{-1}$, then

$$\frac{96\pi^2}{g^2} \rho^2 \rho'^2 a^{-4} \gg \frac{4\pi^2}{\lambda} M_H^2 \rho^2 \rho'^2 a^{-2}. \tag{4.7}$$

In computing the saddle point, we can therefore disregard the last term in the exponential of 4.6.

It is convenient to define new variables:

$$\xi = \rho^2 M_H^{-2} a^{-4}, \quad \xi' = \rho'^2 M_H^{-2} a^{-4}, \quad x = M_H a, \tag{4.8}$$

and set $\kappa = 1$.
The stationary point is at $\xi = \xi' = 1/48$, $x = (6g^2 E/\pi^2 M_H)^{1/3}$, i.e. at

$$a = \frac{1}{M_M} \left(\frac{6g^2 E}{\pi^2 M_H} \right)^{1/3}, \quad \frac{\rho}{a} = \frac{\rho'}{a} = \frac{\sqrt{3}}{12} \left(\frac{6g^2 E}{\pi^2 M_H} \right)^{1/3}. \tag{4.9}$$

Self consistency of the approximation requires:

$$\frac{6g^2 E}{\pi^2 M_H} \ll 1, \quad \frac{E}{M_H} \gg \left(\frac{\pi^2}{6g^2} \right)^{1/4}. \tag{4.10}$$

These conditions can be verified simultaneously for sufficiently small g^2. The first of conditions 4.10 ensures that $\rho \ll a \ll M_H^{-1}$, as well as that the expansion parameters in the saddle point approximation be all small. The second one comes from the requirement $Ea \gg 1$.

The matrix of quadratic fluctuations about the point 4.9 has a single negative eigenvalue. The gaussian integral about 4.9 is therefore imaginary and it gives

$$\begin{aligned}
\mathcal{J} &= i\frac{\sqrt{3}}{4}\left(\frac{1}{48}\right)^{c-1}\left(\frac{g^2}{\pi}\right)^3 M_H^{-2c-6}\left(\frac{E}{E_0}\right)^{4/3c-4/3}\times \\
&\quad \exp\left(-\frac{16\pi^2}{g^2}+\frac{\pi^2}{8g^2}\left(\frac{E}{E_0}\right)^{4/3}\right), \\
E_0 &= \frac{\pi^2 M_H}{6g^2}. \qquad (4.11)
\end{aligned}$$

Substituting equation 4.11 into 4.1 one finds a cross section which, in spite of the exponential suppression $\exp(16\pi^2/g^2)$ typical of instanton effects, becomes of $O(1)$ at energies $E \approx E_0$. This is consistent with the results of ref. [4,6], where an exponential growth of inclusive cross sections with energy was also found.

Before commenting about the reliability of this result, we must notice that, for energies below the critical value E_0, the exponential growth that we obtain is even larger than that of [4,6]. In [4,6] indeed, the growth was $\exp(\mathrm{const}(E/E_0)^2)$, whereas we find $\exp(\mathrm{const}(E/E_0)^{4/3})$. This should come as no surprise, since by formula 4.11 we are evaluating inclusive cross sections for processes involving the production of an arbitrary number of gauge vectors and Higgs scalars. In ref. [4,6] instead, only the contribution of multiple Higgs production was considered. Equation 4.11 suggests therefore that gauge vector production is the dominant B+L violating process for energies below E_0.

The inclusive cross section for processes 3.22 may also be computed by means of observation 3.19. The main step here is to compute the integral

$$\begin{aligned}
\mathcal{J}' &= 2(2\pi)^{3/2}E^{-3/2}\int da d\rho d\rho' a^{3/2}(\rho\rho')^c \exp\left(Ea-\frac{16\pi^2}{g^2}-\frac{2\pi^2}{\lambda}M_H^2\rho^2\right. \\
&\quad \left.-\frac{2\pi^2}{\lambda}M_H^2\rho'^2+\frac{4\pi^2}{\lambda}M_H^2\rho^2\rho'^2a^{-2}\right). \qquad (4.12)
\end{aligned}$$

The only difference between this integral and $\mathcal{J}$ is that the gauge field contribution to the interference term in the $I\bar{I}$ action has been dropped.

In order to evaluate $\mathcal{J}'$, one uses again the saddle point. In this case, it is convenient to go to variables $\xi = \rho^2/a^2$, $\xi' = \rho'^2/a^2$. The saddle point is found at $a = \lambda E/2\pi^2 M_H^2$, $\xi = \xi' = 1/2$. Notice that we started from the assumption ρ/a, $\rho'/a \ll 1$. As we see, this assuption works very marginally in this case. Nevertheless, it is still interesting to go on and see what form has the resulting $\mathcal{J}'$. The other self consistency condition is

$$\sqrt{\frac{2\pi^2}{\lambda}} \ll E \ll \frac{2\pi^2}{\lambda}. \tag{4.13}$$

These bounds are easily satisfied for $\lambda \ll 1$.

Saddle point approximation yields

$$\begin{aligned} \mathcal{J}' &= i\sqrt{2}\left(\frac{1}{2}\right)^{c+2}\left(\frac{\lambda}{\pi}\right)^3 M_H^{-2c-6}\left(\frac{E}{E_c}\right)^{2c-3/2}\exp\left(\frac{\pi^2}{\lambda}\left(\frac{E}{E_c}\right)^2 - \frac{16\pi^2}{g^2}\right), \\ E_c &= \frac{2\pi^2 M_H}{\lambda}. \end{aligned} \tag{4.14}$$

The integral turns out again to be imaginary, because the quadratic fluctuations about the saddle point have a single negative eigenvalue. The contribution to the total B+L violating cross section has thus the form $\exp(\mathrm{const}E^2/E_c^2)$. Since we set $g^2 \approx \lambda$, $\mathcal{J}'$ becomes of order one at $E \approx 4E_c$.

Already now we may draw a nontrivial conclusion, namely that dispersion relations between FES amplitudes and total cross sections are satisfied in the semiclassical calculation scheme of [4,5,6].

To go further, one must pay some attention to the validity of the approximations used here. The subject of classical corrections to the leading order results 4.11, 4.14 is of great importance, as emphasized in ref. [11].

One fact that may look suspicious is that the integrals 4.11, 4.14 turn out to be imaginary, and contribute by that to inclusive cross sections, because one is expanding about unstable stationary points[3]. Actually this objection is not so strong. The integral over collective coordinates indeed, has the exact form (in the dilute instanton approximation)

$$\int_0^\infty d\rho \int_0^\infty d\rho' \int_0^\infty daa^2(\rho\rho')^c J_1(Ea)\exp(-S^{I\bar{I}}(\rho,\rho',a)). \tag{4.15}$$

[3]Interchanging analytic continuation in E and integration over collective coordinates, while potentially dangerous in general, is acceptable in the saddle point approximation.

where

$$\begin{aligned}\lim_{\rho\to\infty} S^{I\bar{I}}(\rho,\rho',a) &= +\infty, \quad \lim_{\rho'\to\infty} S^{I\bar{I}}(\rho,\rho',a) = +\infty \\ \lim_{a\to\infty} S^{I\bar{I}}(\rho,\rho',a) &= S^{I}(\rho) + S^{\bar{I}}(\rho'). \end{aligned} \tag{4.16}$$

Equations 4.15, 4.16 ensure that the saddle point (better the steepest descent) method is appliable to the integration over ρ, ρ' [15]. The integral over a can be done by the same method if one decomposes the Bessel function $J_1(Ea)$ in terms of the Hankel's functions $H_1^{(1)}(Ea)$ and $H_1^{(2)}(Ea)$ and writes

$$\begin{aligned}\int_0^\infty daa^2 J_1(Ea)\exp(-\tilde{S}(a)) &= \int_{C_+} daa^2 H_1^{(1)}(Ea)\exp(-\tilde{S}(a)) + \\ &\quad i\int_{C_-} daa^2 H_1^{(2)}(Ea)\exp(-\tilde{S}(a)). \end{aligned} \tag{4.17}$$

In this equation $\exp(-\tilde{S}(a))$ is the result of the steepest descent integration over ρ and ρ'. The contours of integration C_+ and C_- are given in Figure 1. Equation 4.17 holds under the quite natural assumption that no singularities lie between the curves C_+, C_- and the half-line $a > 0$.

Another point which may be raised against equations 4.11 and 4.14, is that higher order terms in ξ, ξ' may change the exponential growth in energy. In order to deal with this problem one may examine corrections introduced by considering the improved form of $S^{I\bar{I}}$ given in ref. [14], where finite instanton-size corrections to equation 3.9 were considered.

Let us consider at first corrections to gauge vector production. The improved $I\bar{I}$ action of [14] is

$$S^{I\bar{I}}_{gauge} = -\frac{96\pi^2\rho^2\rho'^2}{g^2(a^2+\rho^2+\rho'^2)^2} + \frac{2\pi^2}{\lambda}M_H^2\rho^2 + \frac{2\pi^2}{\lambda}M_H^2\rho'^2 + \frac{16\pi^2}{g^2}. \tag{4.18}$$

The stationary point of $Ea - S^{I\bar{I}}$ is easily found, by going to variables $\xi = \rho^2/a^2$, $\xi' = \rho'^2/a^2$, to be at

$$\xi = \xi' = \frac{(E/E_0')^{2/3}}{48 - 2(E/E_0')^{2/3}}, \quad a = \frac{1}{M_H}(E/E_0')^{1/3} - \frac{1}{24M_H}(E/E_0'). \tag{4.19}$$

Here $E_0' = \sqrt{\kappa}E_0$, with E_0 defined as in equation 4.11. The value of $Ea - S^{I\bar{I}}$ at this point is

$$Ea - S^{I\bar{I}} = \frac{\pi^2}{8g^2}(E/E_0')^{4/3} - \frac{\pi^2}{288g^2}(E/E_0')^2 - \frac{16\pi^2}{g^2}. \tag{4.20}$$

The exponential $\exp(Ea - S^{I\bar{I}})$ as a function of E/E_0' reaches its maximum value

$$\exp(Ea - S^{I\bar{I}})\,|_{max} = \exp(+8\pi^2/g^2) \tag{4.21}$$

at $E = 48\sqrt{6}E_0'$. In other words, finite size corrections to the classical instanton action are not sufficient, by themselves, to stop the exponential growth of the inclusive cross section for gauge vector production. Substituting equation 4.20 into 4.1 one still get a cross section that reaches the unitarity limit for energies $O(M_W/\alpha_W)$.

Finite size corrections to Higgs boson production are dealt as follows. By going to the usual variables $\xi = \rho^2/a^2$, $\xi' = \rho'^2/a^2$, one may rewrite the scalar contribution to the classical $I\bar{I}$ action as:

$$Ea - S^{I\bar{I}}_{scalar} = Ea + \frac{2\pi^2}{\lambda}M_H^2 a^2[\phi(\xi,\xi') - \xi - \xi'] - \frac{16\pi^2}{g^2}. \tag{4.22}$$

The explicit form of $\phi(\xi,\xi')$, given in ref. [11], is

$$\phi(\xi,\xi') = \xi\xi'\left[\frac{1}{\sqrt{1+\xi}(1+\sqrt{1+\xi'})} + \frac{1}{\sqrt{1+\xi'}(1+\sqrt{1+\xi})}\right]. \tag{4.23}$$

It can be readily verified that

$$\xi + \xi' - \phi(\xi,\xi') \geq \frac{1}{4}\xi + \frac{1}{4}\xi'. \tag{4.24}$$

From the small ξ, ξ' expansion of $\phi(\xi,\xi')$ we know that the point $\xi = \xi' = 0$ is never a stationary point of the action 4.22. Therefore, at the stationary point for ξ and ξ' one finds:

$$\begin{aligned} Ea - S^{I\bar{I}}_{scalar}\,|_{stat} &= Ea - \frac{2\pi^2}{\lambda}M_H^2 a^2\tilde{\phi}(1,1) - \frac{16\pi^2}{g^2}, \\ \tilde{\phi}(1,1) &> 0. \end{aligned} \tag{4.25}$$

Here $\tilde{\phi}$ is the Legendre transform of ϕ. The saddle point in a gives:

$$\exp(Ea - S^{I\bar{I}}_{scalar}) = \exp\left(\frac{E^2\lambda}{8\pi^2 M_H^2 \tilde{\phi}(1,1)} - \frac{16\pi^2}{g^2}\right). \tag{4.26}$$

Even in this case, classical finite size corrections can not stop the growth of the inclusive cross section for B+L violating Higgs production.

Summing up all the previous results, we may draw the following conclusions.

Semiclassical instanton calculations, when properly done, preserve dispersion relations.

An exponential growth in inclusive cross sections, which eventually saturates the unitarity bounds, exists both in the two dimensional Abelian Higgs model and in spontaneously broken SU(2) gauge theories coupled to fermions. Classical corrections to the instanton action, which are present because instantons have nonzero size, do not change, by themselves, these results.

Finite size corrections, naturally, do not exhaust all possible contributions to equations 4.11, 4.14. For one thing, indeed, we never touched the problem of quantum corrections to our results. Moreover, we never left the framework of dilute instanton gas approximation. The inadequacy of that approximation is evident when the relevant values of ρ/a or ρ'/a versus the c.m. energy E are considered. For example, one notices immediately that the values of the square instanton radii over the square instanton distance in equation 4.19 become of order one for $E \approx E_0'$. The case of Higgs particle emission is even more dramatic. There indeed, the function $\phi(\xi, \xi')$ in equations 4.22, 4.23 does not depend on E, λ or g^2. Thus, the relevant values of ξ, ξ', i.e. the values at the saddle point, are always $O(1)$. This fact by itself should make us doubt about the actual validity of the dilute instanton approximation.

On the other hand, other sources of indeterminacy in our findings, like the fact that corrections $O(M_H^4 \rho^4 \log M_H \rho)$ were not considered, seem to play a minor role, according to the analysis of ref. [9,10]. The authors of [9,10], indeed, claim that that kind of corrections does not have the strenght of modifying an exponential growth in the inclusive cross section for multiple Higgs emission, in the case of a SU(2) supersymmetric theory. Since this conclusion does not seem to depend heavily on the particular

form of the theory, it seems possible to extend it to the Standard Model as well.

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Figure Captions

Figure 1 - The contours of integration C_+ and C_-, the crossed circle indicates the stationary point of $S(a)$.

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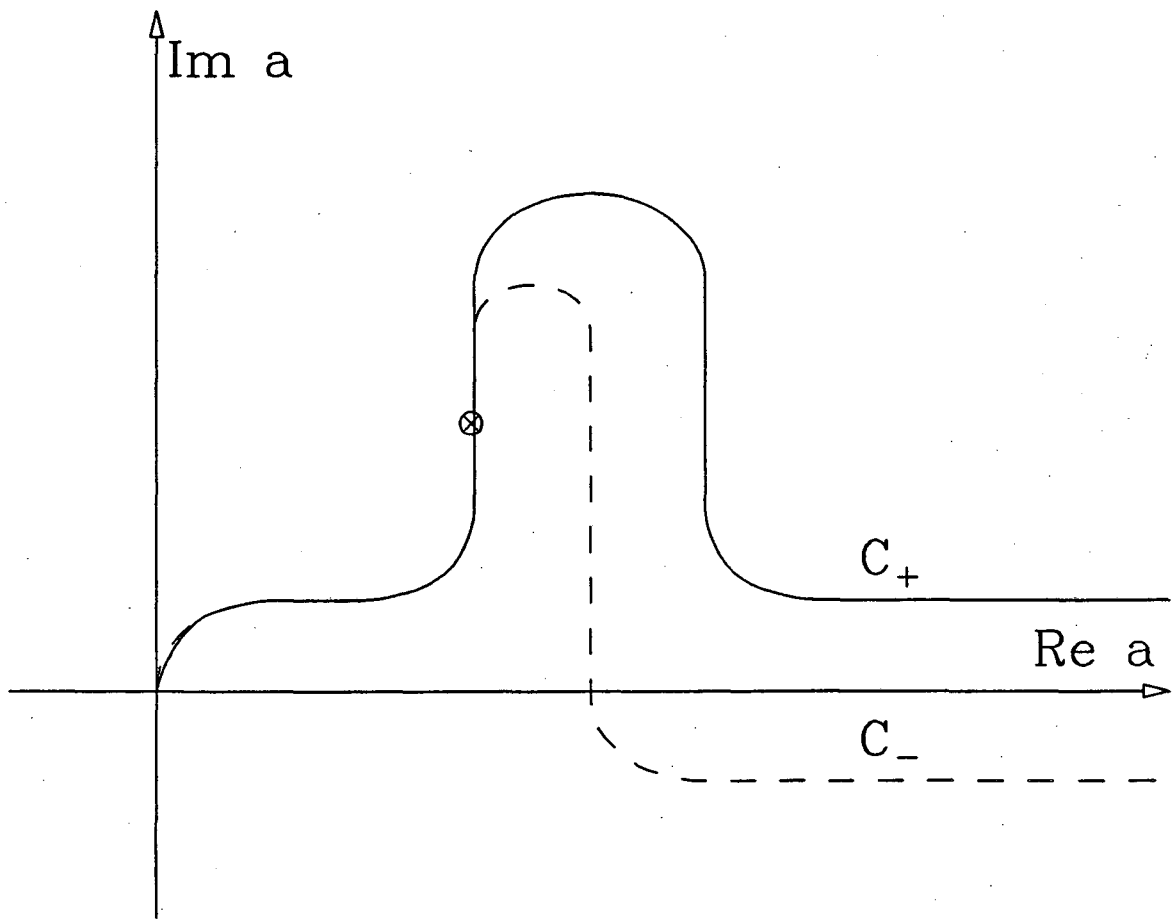

Figure 1