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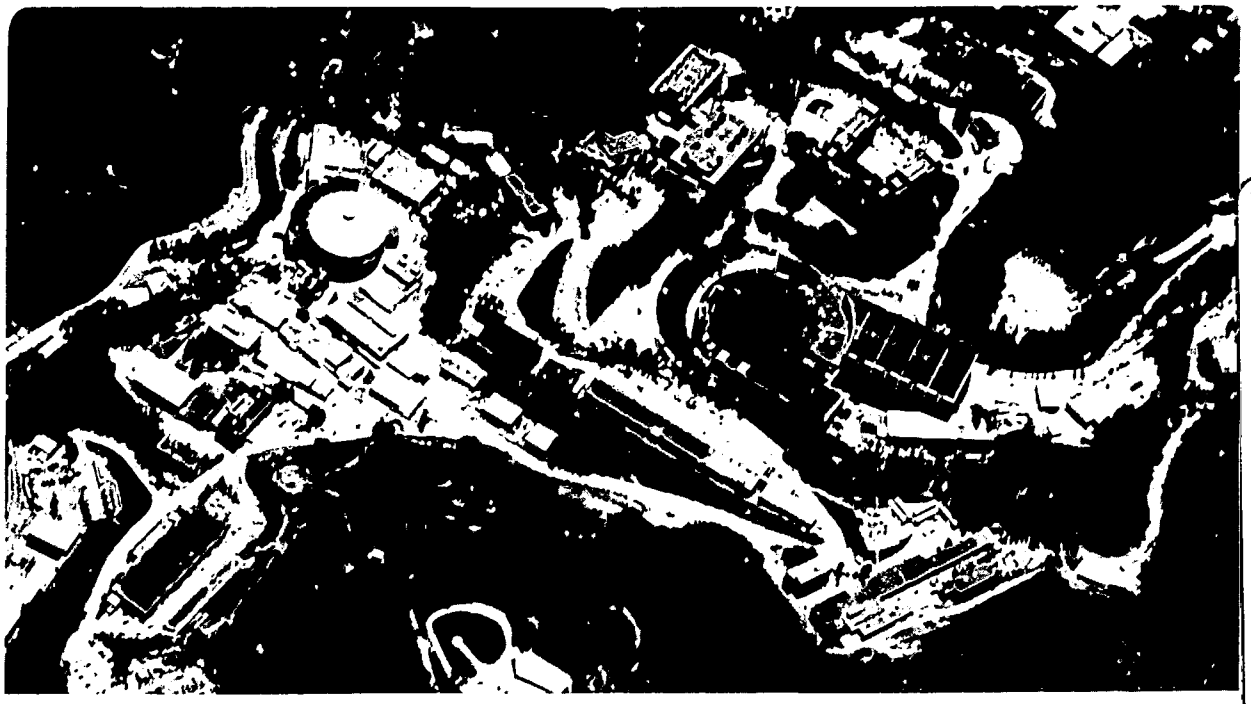

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A BRIEF OVERVIEW OF VORTEX METHODS

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A BRIEF OVERVIEW OF VORTEX METHODS

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Abstract

We review some recent work in vortex methods. We discuss the derivation of the equations of motion, inviscid and viscous flow, application of boundary conditions, some aspects of convergence theory, and fast summation methods for the vortex interaction. Vortex methods applied to vortex sheets, vortex pairing and stability, shear layers and mixing, internal and external flows, reactive flows, three-dimensional flows and turbulence modeling are then reviewed. Finally, we discuss the results of a numerical convergence study, and provide results of a vortex simulation of flow over a backward-facing step.

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A Brief Overview of Vortex Methods

In the past 10-15 years, vortex methods have been applied to a variety of fluid flow problems. The essential aspect of these methods is the focus on the *vorticity* of a flow, which is the local "twist" of the fluid. In an incompressible fluid, the time evolution of vorticity dictates the essential physics of the unfolding flow.

A. Velocity vs. Vorticity.

Traditionally, flow is described by the Navier-Stokes equations, which are written in terms of the fluid velocity at any point, and express Newton's law that force equals mass times acceleration. Since the vorticity ξ is defined to be the curl of the velocity ($\xi = \nabla \times \mathbf{u}$), the vorticity may be derived from knowledge of the fluid velocity. For our purposes, the reverse observation will be equally important: the velocity may be determined from the vorticity by means of a Biot-Savart integral.

Equations of motion in terms of velocity are especially convenient when boundary conditions are introduced. Imagine flow in a vessel or around an object. Suppose we divide the velocity $\mathbf{u}$ into a component u_n normal to the wall and a component u_τ tangential to the wall. Then the statement $u_n = 0$ on solid walls means fluid cannot pass through the obstacle, and $u_\tau = 0$ enforces that fluid in direct contact with the wall must itself not slip along the wall. The boundary conditions for the vorticity are not as straightforward.

Given this, why have vortex methods, which approximate the solution to the equations of motion written in terms of vorticity evolution, proved so valuable? The answer is because fluid vorticity often lies in concentrated regions of the flow. The situation is analogous to the gravitational field induced by the planets. The mass of the solar system is concentrated in relatively few places, and it is easier to focus on the positions and strengths of these masses at any time rather than the evolving gravitational field at any point in space. Thus, a vorticity formulation reduces the

mathematical description to its essential components. Furthermore, the physical meaning of the tangential boundary conditions becomes apparent; by requiring that solid walls "grab" onto the flow, a large shearing vorticity must be imparted to the flow. Thus, the boundary conditions both confine the flow ($u_n = 0$) and add vorticity ($u_\tau = 0$).

In two special cases, a vorticity formulation becomes particularly simple. If the flow is *inviscid*, that is, one in which vorticity does not diffuse due to viscosity or "friction" in the flow, patches of vorticity are self-advected and move without diffusing. In addition, in the absence of any vorticity ($\xi = 0$), the problem reduces to one of potential flow requiring the solution of Laplace's equation in the given domain.

B. Numerical Approximations

There are only a few explicit solutions to the equations of motion. Consequently, the goal of a numerical algorithm is to provide an efficient approximation to the solution of these equations, together with a systematic way of refining the approximation to increase the accuracy of the results. An approximation based on a vorticity formulation can focus computational resources on following the limited amount of vorticity in the system, rather than the velocity **u** which exists everywhere.

The first numerical calculation of a flow using an approximation to the vorticity equations is due to Rosenhead[191], who studied the evolution of a vortex sheet. Imagine one inviscid fluid on top of another one, each initially moving with constant (but different) speed. Since $\xi = \nabla \times \mathbf{u}$, the vorticity is initially zero everywhere except along the infinitely thin boundary between the two. This interface is unstable to small perturbations and rolls up into large spiral structures as the flow evolves. As an example, the flow above and below an airfoil meet as they exit the trailing edge, with the top flow moving faster than the bottom one. The *vortex sheet* that comes off the wing rolls up into a complicated pattern.

Since the velocity may be recovered from the vorticity, and the flow is assumed inviscid, all of the dynamics are contained in the evolution of this infinitely thin sheet of vorticity. (In a real

physical flow, the sheet has a finite thickness). Rosenhead approximated the motion of the vortex sheet by discretizing the infinite line of vorticity into a finite number of discrete point vortices. Each point vortex induced a velocity field of the form $1/r$, where r was distance to the vortex. By superposition, the evolution of a finite collection of point vortices initially placed along the line where $\xi \neq 0$ could be calculated. Thirty years before the advect of large-scale computing, Rosenhead computed the motion of this finite collection of points by hand, and produced a reasonable portrait of the rollup of a vortex sheet.

As clever as this work was, it contained two subtle complications. First, point vortices induce a singular velocity field; as $r \rightarrow 0$, their velocities becomes infinite. Second, vortex sheets develop a singularity as they evolve, even starting from smooth initial data. Thus, singular objects were being used to follow an object that develops a singularity. Attempts to improve on this calculation by using more discrete vortices revealed numerous instabilities and breakdowns, see, for example, the paper by Birkhoff[39]. The source of these instabilities was not evident. For the most part, vortex techniques, i.e., the systematic resolution of vorticity into a finite number of objects which would be more accurate with more elements, were considered unworkable.

C. Modern Vortex Methods

The introduction of modern vortex methods as a practical technique began with the calculations of flow past a cylinder by Chorin[57]. Several factors contributed to the success of these calculations. First, discrete vortex elements, called vortex "blobs", were used which induced a bounded velocity field with finite kinetic energy. Second, the diffusion of vorticity was simulated by means of a random walk imposed on the vortex trajectories. Third, the no-slip ($u_\tau = 0$) tangential velocity boundary condition was enforced by the creation of new vorticity; while numerous techniques had previously created single discrete point vortex elements at boundary separation points in response to a Kutta condition, see Clements and Maul[67], this represented the first approximation to the appli-

cation of the no-slip condition around the entire body. Finally, the method was applied to viscous flow past a cylinder, where integrated quantities such as the drag could be calculated and compared to physical experiments.

Since that time, vortex methods have become a practical tool for computing a wide variety of engineering flows. In recent years, they have been applied to external flows past cylinders, wedges and blunt objects, flow inside pistons, flame propagation and turbulent combustion, ship design, thermal bubbles, blood flow in the heart, flow past buildings, and flow in the mantle. They provide tools for studying what happens in more pathological problems, such as vortex sheets in two dimensions and vortex rings and filaments in three dimensions. In addition, they link together a variety of other techniques for approximating vorticity, such as point vortex methods, vortex-in-cell algorithms, and discrete Kutta-condition models. Finally, mathematical interest in proving convergence of this algorithm has provided the setting for significant theoretical insight into the underlying equations.

D. Outline

In this paper, we will review some past work in the evolution of vortex methods. No review can be exhaustive or all-inclusive, and undoubtably we have missed important work. However, we hope to survey the major themes. With few exceptions, we have focussed on reviewed articles. We apologize in advance for omissions, and hope that such work can be brought to our attention.

A collection of excellent review articles on a variety of topics related to vortex methods and vortex motion have appeared in recent years, and have served as guides for this review. An early review by Clements and Maul[67] discusses vortex shedding by discrete vortices for Kutta conditions. Leonard[142,143] has written two comprehensive reviews of work in vortex methods. Saffman and Baker[195] review some more theoretical aspects of vortex motion. Vortex shedding from bluff bodies in a variety of settings has been covered by Bearman[34] and Bearman and Gra-

ham[36], and vortex flows in aerodynamics by Smith[214]. Some more theoretical work in vortex filaments has been provided by Widnall[234], and vortex breakdown by Hall[112] and Leibovich[141]. A full discussion of the motion of few point vortices and vortex patches is given by Aref[9] and Aref and Kambe[13], and more general perspectives by Aref[10] and Zabusky[237]. A review of the field of computational fluid mechanics is provided by Rizzi and Engquist[190], and the field of particle simulations in electromagnetism, which is related to general particle methods, by Buneman et. al.[48]. An excellent overview of the importance of vorticity is provided by Saffman[194], and a clear exposition on the mathematical theory behind vorticity formulations and vortex motion was given by Majda[150]. Finally, a recent review by Majda[149] discusses the interplay between numerical and theoretical components of vortex methods and computational techniques.

The outline of this paper is as follows:

Outline

I. Introduction
A. Velocity vs. Vorticity
B. Numerical Methods
C. Modern Vortex Methods
D. Outline

II. Equations of Motion
A. Navier-Stokes Equations and the Vorticity Transport Equation
B. Lagrangian Perspective: Particle Trajectories
C. Modern Vortex Methods: Approximating the Lagrangian Particle Trajectories

III. Developments in Numerical Approximation to the Equations of Motion
A. Free Space Inviscid Flows: Smoothing and Accuracy
B. Viscous, Incompressible Flow
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IV. Physical Problems
A. Vortex Sheets
B. Vortex Pairing/Mixing/Shear Layers/Two Fluids
C. External and Internal Flows: Cylinders/Bluff Bodies/Chambers
D. Reactive Flows: Combustion/Flame Propagation
E. Three Dimensions: Filaments/Rings/Shear Layers
F. Turbulence Studies/Physics

V. Example: Modeling of Flow over a Backward-Facing Step
A. "Convergence" of a Random Method Applied to an Unstable Problem
B. Experimental and Numerical Results
C. Summary of Convergence Results

VI. Concluding Remarks

II. Equations of Motion

In this section, we discuss the formulation and approximations underlying vortex methods. This explanation benefits from considerable hindsight, and follows the discussion by Anderson and C.Greengard[6] and Majda[149].

A. The Navier-Stokes and Vorticity Transport Equations

The Navier-Stokes equations for incompressible, viscous flow in a region D with boundary ∂D are

$$\frac{D\mathbf{u}}{Dt} = -\nabla P + \frac{1}{R}\nabla^2\mathbf{u} \quad \text{in } D \tag{2.1}$$

$$\nabla\cdot\mathbf{u} = 0 \qquad \text{in } D \tag{2.2}$$

$$\mathbf{u} = 0 \qquad \text{on } \partial D$$

Here, $P = P(\mathbf{x},t)$ is the pressure, R is the Reynolds number, $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ is the three-dimensional velocity vector at $\mathbf{x} \in R^3$, and (D/Dt) is the material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$$

Equation (2.1) is an expression of Newton's law $\mathbf{F} = m\mathbf{a}$, while Eqn. (2.2) states that the flow is incompressible. The boundary conditions require that both the normal and tangential components of $\mathbf{u}$ vanish on solid walls.

We define the *vorticity* vector ξ as $\xi = \nabla\times\mathbf{u}$. The curl of Eqn. (2.1), together with a few vector identities, result in the **Vorticity Transport Equation**

$$\frac{D\xi}{Dt} = (\xi\cdot\nabla)\mathbf{u} + \frac{1}{R}\nabla^2\xi \tag{2.3}$$

Equation (2.3) states the material derivative $\frac{D\xi}{Dt}$ of vorticity depends on two terms: *vortex stretching* $(\xi\cdot\nabla)\mathbf{u}$ and *vorticity diffusion* $(1/R)\nabla^2\xi$. In the limit of infinite Reynolds number, the diffusion term disappears, and only the normal boundary condition can be imposed, see Chorin and

Marsden[63]. In the special case of two-dimensional flow, the vorticity vector ξ is normal to the velocity $\mathbf{u}$ and the term $(\xi\cdot\nabla)\mathbf{u}$ vanishes. These comments are summarized in the chart below.

	3–D	2–D
Viscous	$\frac{D\xi}{Dt} = (\xi\cdot\nabla)\mathbf{u} + \frac{1}{R}\nabla^2\xi$	$\frac{D\xi}{Dt} = \frac{1}{R}\nabla^2\xi$
	$\mathbf{u} = 0$ on ∂D	$\mathbf{u} = 0$ on ∂D
Inviscid	$\frac{D\xi}{Dt} = (\xi\cdot\nabla)\mathbf{u}$	$\frac{D\xi}{Dt} = 0$
	$u_N = 0$ on ∂D	$u_N = 0$ on ∂D

In order to "close" Eqn. (2.3), we must recover $\mathbf{u}$ from the vorticity. Given that

$$\nabla\cdot\mathbf{u} = 0 \qquad \xi = \nabla\times\mathbf{u} \tag{2.4}$$

there exists a vector function $\psi(\mathbf{x})$ such that $\mathbf{u} = \nabla\times\psi$ and

$$\nabla^2\psi = -\xi \tag{2.5}$$

In three dimensions, ψ is known as the velocity potential; in two dimensions, ψ is called the stream function, and both ξ and ψ are normal to the plane of motion. We see that ψ satisfies a Poisson equation with the vorticity distribution as the right-hand-side. This means that we may write ψ, and consequently the velocity $\mathbf{u}$, in terms of ξ by making use of the fundamental solution to the Laplace operator ∇^2. Recall that the solution to Eqn. (2.5) is given by

$$\psi(\mathbf{x},t) = \int L(\mathbf{x}-\mathbf{z})\,\xi(\mathbf{z})\,d\mathbf{z} \tag{2.6}$$

where

$$L(\mathbf{x}) = \begin{Bmatrix} \frac{-1}{2\pi}\log|\mathbf{x}| & \mathbf{x}\in R^2 \\ \frac{1}{4\pi}\frac{1}{|\mathbf{x}|} & \mathbf{x}\in R^3 \end{Bmatrix}$$

Since $\mathbf{u} = \nabla\times\boldsymbol{\psi}$, we have that

$$\mathbf{u}(\mathbf{x},t) = \int K(\mathbf{x}-\mathbf{z})\boldsymbol{\xi}(\mathbf{z})d\mathbf{z} \tag{2.7}$$

where the kernel K is defined by

$$K(\mathbf{x}) = \left\{ \begin{array}{ll} \dfrac{1}{2\pi}\dfrac{(-x_2,x_1)}{|\mathbf{x}|^2} & \mathbf{x}\in R^2 \\ \dfrac{1}{4\pi|\mathbf{x}|^3}\begin{Bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{Bmatrix} & \mathbf{x}\in R^3 \end{array} \right\} \tag{2.8}$$

where $\mathbf{x}=(x_1,\ldots,x_N)$ for N=2,3. Thus we have the following closed system for the evolution of $\boldsymbol{\xi}$, namely

$$\frac{D\boldsymbol{\xi}}{Dt} = (\boldsymbol{\xi}\cdot\nabla)\mathbf{u} + \frac{1}{R}\nabla^2\boldsymbol{\xi} \tag{2.9}$$

$$\mathbf{u}(\mathbf{x},t) = \int K(\mathbf{x}-\mathbf{z})\boldsymbol{\xi}(\mathbf{z})d\mathbf{z} \tag{2.10}$$

where K is given in Eqn. (2.8). We note that the kernel K is singular in both two and three dimensions.

B. Lagrangian Perspective

Equations (2.9),(2.10) are an *Eulerian* formulation of the vorticity equations. That is, if we are given $\boldsymbol{\xi}(\mathbf{x},0)$ at each point $\mathbf{x}$ in the domain at t=0, they describe the change of $\boldsymbol{\xi}$ at any point $\mathbf{x}$.

Suppose, on the other hand, we view the initial condition $\boldsymbol{\xi}(\mathbf{x},0)$ as describing the vorticity of the particle initially located at $\mathbf{x}$, and follow the motion of particles located at all possible starting points. This leads to a *Lagrangian* formulation for the particle trajectories, which we now formulate. Throughout this section, we shall assume that the flow is inviscid.

Let $\mathbf{X}(\boldsymbol{\alpha},t)$ give the position $\mathbf{X}$ at time t of that particle initially located at $\boldsymbol{\alpha}$. Then the particle trajectory is described by the initial value problem

$$\frac{d\mathbf{X}(\boldsymbol{\alpha},t)}{dt} = \mathbf{u}(\mathbf{X}(\boldsymbol{\alpha},t),t) \tag{2.11}$$

$$= \int K(\mathbf{X}(\alpha,t) - \mathbf{z})\,\xi(\mathbf{z})d\mathbf{z}$$

$$= \int K(\mathbf{X}(\alpha,t) - \mathbf{X}(\beta,t))\,\xi(\mathbf{X}(\beta,t),t)d\beta$$

$$\mathbf{X}(\alpha,0) = \alpha \tag{2.12}$$

where we have replaced the dummy variable $\mathbf{z}$ into the trajectory form by the change of variable $\mathbf{z} = \mathbf{X}(\beta,t)$ and used the incompressibility of the flow to imply that particle trajectories conserve volume and thus the Jacobian of the transformation is one.

Equations (2.11),(2.12) describe the evolution of a particle. The integrand depends on the positions of all other particles, as well as their vorticity. Thus, we must provide a recipe for computing the vorticity $\xi(\mathbf{X}(\beta,t),t)$. For three-dimensional flow, recall that

$$\frac{D\xi}{Dt} = (\xi\cdot\nabla)\mathbf{u}$$

Thus, one straightforward technique, suggested in Anderson and C.Greengard[6], is to add an additional equation for the rate of change of vorticity for the particle following the trajectory $\mathbf{X}(\alpha,t)$, namely

$$\frac{d\xi(\mathbf{X}(\alpha,t),t)}{dt} = \xi(\mathbf{x}(\alpha,t),t)\cdot\nabla\mathbf{u} \tag{2.13}$$

$$= \Big(\xi(\mathbf{X}(\alpha,t),t)\cdot\nabla\Big)\int K(\mathbf{X}(\alpha,t) - \mathbf{X}(\beta,t))\,\xi(\mathbf{X}(\beta,t),t)d\beta.$$

Alternatively, as suggested by Beale and Majda[32], one can use that

$$\xi(\mathbf{X}(\alpha,t),t) = [\nabla_\alpha\,\mathbf{X}(\alpha,t)]\cdot\xi(\mathbf{X}(\alpha,0),0) \tag{2.14}$$

, see Chorin and Marsden[63]. The difference is that that Eqn. (2.13) is an expression for the time derivative of the vorticity in term of the velocity, whereas Eqn. (2.14) directly describes the vorticity in terms of the spatial derivative of the particle trajectory.

In two-dimensional flow, the additional equation necessary to calculate the vorticity of the moving particles is trivial. Since $\frac{D\xi}{Dt} = 0$, we may directly write that

$$\xi(\mathbf{X}(\alpha,t),t) = \xi(\mathbf{X}(\alpha,0),0) \tag{2.16}$$

C. Modern Vortex Methods: Approximating the Lagrangian Particle Trajectories

The above equations, while mathematically correct, are difficult to approximate numerically. The difficulty occurs because the kernel is singular in both two and three space dimensions. Thus, if particles come close together, they can exert extremely large velocities on one another. The idea behind modern vortex methods is to smooth the kernel to limit this singular interaction. The first vortex blob calculations set the magnitude of the velocity inside a given cut-off size δ to be constant, thus eliminating the singularity. A large number of smoothed kernels have been constructed since then, providing vortex methods of various orders of accuracy. A fuller discussion of the techniques for choosing the smoothing function is discussed by the article by Hald in this book. For the rest of this section, we shall simply assume that the kernel K has been replaced by an appropriate smoothed kernel K_δ.

The basic idea behind vortex methods is to discretize in both space and time the initial value problem described by

$$\text{(PARTICLE UPDATE)}: \quad \frac{d\mathbf{X}(\boldsymbol{\alpha},t)}{dt} = \int K_\delta(\mathbf{X}(\boldsymbol{\alpha},t) - \mathbf{X}(\boldsymbol{\beta},t))\,\boldsymbol{\xi}(\mathbf{X}(\boldsymbol{\beta},t),t)\,d\boldsymbol{\beta} \tag{2.16}$$

$$\text{(VORTICITY UPDATE)}: \quad \left\{ \begin{array}{ll} \dfrac{d\boldsymbol{\xi}(\mathbf{X}(\boldsymbol{\alpha},t),t)}{dt} = \left(\boldsymbol{\xi}(\mathbf{X}(\boldsymbol{\alpha},t),t)\cdot\nabla\right)\int K_\delta(\mathbf{X}(\boldsymbol{\alpha},t) - \mathbf{X}(\boldsymbol{\beta},t))\,\boldsymbol{\xi}(\mathbf{X}(\boldsymbol{\beta},t),t)\,d\boldsymbol{\beta} & \\ \qquad\qquad \text{or} & \text{in } 3D \\ \qquad \boldsymbol{\xi}(\mathbf{X}(\boldsymbol{\alpha},t),t) = [\nabla_\alpha \mathbf{X}(\boldsymbol{\alpha},t)]\cdot\boldsymbol{\xi}(\mathbf{X}(\boldsymbol{\alpha},0)) & \\ & \\ \qquad \boldsymbol{\xi}(\mathbf{X}(\boldsymbol{\alpha},t),t) = \boldsymbol{\xi}(\mathbf{X}(\boldsymbol{\alpha},0),0) & \text{in } 2D \end{array} \right\}$$

$$\text{(INITIAL CONDITIONS)}: \quad \left\{ \begin{array}{ll} \mathbf{X}(\boldsymbol{\alpha},t) = \boldsymbol{\alpha} & \text{Initial positions of particles} \\ \boldsymbol{\xi}(\mathbf{X}(\boldsymbol{\alpha},0),0) & \text{Initial vorticity of particles} \end{array} \right\}$$

Given an initial vorticity distribution $\boldsymbol{\xi}(\mathbf{x},0)$, we begin by constructing a lattice Λ^h in R^N, N=2,3, with mesh size h. Let $\mathbf{j}h$ be the mesh points of Λ^h, where $\mathbf{j}$ is any N-tuple with integer coefficients. The motion of the set of particles originally located on the nodes of this lattice may be

approximated by a discrete approximation to

$$\text{(PARTICLE UPDATE)}: \quad \frac{d\mathbf{X}(\mathbf{j}h,t)}{dt} = \sum_{\mathbf{i}h \in \Lambda^h} K_\delta(\mathbf{X}(\mathbf{j}h,t) - \mathbf{X}(\mathbf{i}h,t))\,\xi(\mathbf{X}(\mathbf{i}h,t),t)h^N \tag{2.17}$$

$$\text{(VORTICITY UPDATE)}: \quad \left\{ \begin{array}{ll} \dfrac{d\xi(\mathbf{X}(\mathbf{j}h,t),t)}{dt} = \Big(\xi(\mathbf{x}(\mathbf{j}h,t),t)\cdot\nabla\Big) \displaystyle\sum_{\mathbf{i}h \in \Lambda^h} K_\delta\,(\mathbf{X}(\mathbf{j}h,t) - \mathbf{X}(\mathbf{i}h,t))\,\xi(\mathbf{X}(\mathbf{i}h,t),t)h^N & \\ \qquad\qquad \text{or} & \text{in } 3D \\ \xi(\mathbf{X}(\mathbf{j}h,t),t) = [\nabla^h_{\mathbf{j}h}\,\mathbf{X}(\mathbf{j}h,t)]\cdot\xi(\mathbf{X}(\mathbf{j}h,0),0) & \\ \xi(\mathbf{X}(\mathbf{j}h,t),t) = \xi(\mathbf{X}(\mathbf{j}h,0),0) & \text{in } 2D \end{array} \right\}$$

$$\text{(INITIAL CONDITIONS)}: \quad \left\{ \begin{array}{ll} \mathbf{X}(\mathbf{j}h,t) = \mathbf{j}h & \text{Initial positions of particles on grid} \\ \xi(\mathbf{X}(\mathbf{j}h,0),0) & \text{Initial vorticity of particles on grid} \end{array} \right\}$$

Here, the operator $\nabla^h_{\mathbf{j}h}$ refers to a finite difference operator on the mesh.

Equations (2.17) form a finite number of coupled differential equations for initial vorticity with compact support. Finally, the time derivative is approximated by a suitable finite difference operator to provide a complete algorithm for updating the positions of the particles in time. This completely specifies the vortex method for inviscid flow.

III. Developments in Numerical Approximations to the Equations of Motion

In this section, we discuss some of the advances made in numerical approximations based on Eqns. (2.17).

A. Free Space Inviscid Flows: Smoothing and Accuracy

We begin by discussing inviscid flow in free space. In the previous section, the equations of motion for viscous, incompressible flow in free space were approximated by a finite set of ordinary differential equations for the particle trajectories. The kernel K given in Eqn. (2.8) is singular, and thus particles that come close together exert large velocities on each other. The idea behind modern vortex methods is to smooth the kernel to limit this singular interaction. How does one construct this smoothed kernel K_δ? One possibility is as follows. Consider two-dimensional flow, and let $f(\mathbf{x})$ be a radially symmetric function such that $\int_{R^2} f(x)dx=1$. Define

$$f_\delta(\mathbf{x}) = \frac{1}{\delta^2} f(\frac{\mathbf{x}}{\delta}) \tag{3.1}$$

As $\delta \to 0$, $f_\delta(\mathbf{x})$ approaches the Dirac delta function. Thus, we define the kernel by the convolution

$$K_\delta(\mathbf{x}) = K * f_\delta(\mathbf{x}) = \int_{R^2} K(\mathbf{x}-\mathbf{x}')f_\delta(\mathbf{x}')\,d\mathbf{x}' \tag{3.2}$$

As $\delta \to 0$, $K_\delta \to K$.

The accuracy of the vortex method depends on several factors: (1) h, the size of the mesh size used in the discretization of the initial vorticity distribution, (2) δ, the amount of smoothing in Eqn. (3.1), (3) the choice of smoothing function f, and (4) the time step Δt used solving the ordinary differential equations for the particle trajectories. Since the introduction of smoothed vortex methods, theory has been developed studying accuracy as a function of these parameters and convergence.

Hald and Del Prete[111] constructed the first proof of the convergence of vortex methods in two dimensions. They considered several types of smoothed kernels, including the one used in

practice by Chorin, and established short time convergence. This was followed by the seminal paper in vortex theory, Hald's 1979 proof of long-time convergence of the vortex approximation, see Hald[109]. In this paper, Hald analyzed the relation between h, δ and the rate of convergence. His main result is as follows:

(Hald 1979): If $h = \delta^2$, then for a certain class of smoothing functions, the difference between the computed particle trajectories and the exact particle trajectories is of $O(h^2)$.

Thus, as one uses more vortex elements to resolve an initial vorticity approximation (that is, as $h \rightarrow 0$), linking the smoothing size to the number of vortices yields convergence of the position of the vortices to their exact trajectories.

On the basis of this work, Beale and Majda[32,33], showed that carefully designed vortex methods for smooth inviscid flow could provide results of any desired degree of accuracy in both two and three dimensions. Their three-dimensional vortex method (Eqn. 2.14) updates three-dimensional vortex stretching by computing derivatives along the particle paths. These papers were responsible for starting the major interest in the theory of vortex methods. The proofs were technical, and much of the work that followed has been aimed at simplifying the arguments. Cottet[72], using work on particle methods for the Vlasov-Poisson equation by Cottet and Raviart[73], gave a stronger and simpler consistency argument, followed by an even simpler consistency argument by Anderson and C.Greengard[6]. Anderson and Greengard also suggested a different three-dimensional stretching algorithm, namely the one given in Eqn. (2.13), in which the kernel is explicitly differentiated. In addition, they provide the compact form of the Beale-Majda three-dimensional scheme given in Eqn. (2.14), as well as a convergence proof which takes into account the time step error along in the integration along particle trajectories. Hald[110] then showed convergence for an extremely wide class of two-dimensional methods, requiring only Holder continuity in the vorticity field, as well as fourth order convergence for the classical Runge-Kutta techniques for the ordinary differential equations integration. Proofs of the three-dimensional method with explicit

differentiation also have been provided by Beale[29] and C.Greengard[102].

In tandem with these theoretical investigations were numerical convergence studies of the actual accuracy obtained in practice (see, for example, those performed by Sethian and Ghoniem [208]. Beale and Majda [30] showed how to construct explicit velocity kernels for high order accurate methods, and verified the increased accuracy on a test problem of the evolution of radially symmetric patch of vorticity. Perlman [174] made a careful, detailed study of the optimal choice of δ compared to h in practice for short and long time flow.

B. Viscous, Incompressible Flow

The extension of vortex methods to viscous flow requires treatment of the viscous diffusion term $(1/R)\nabla^2\xi$ in Eqn. (2.9). The Lagrangian formulation summarized in Eqn. (2.17) assumes an inviscid formulation and yields a set of ordinary differential equations for the particle trajectories. The addition of viscosity generates a set of *stochastic* differential equations, in which the diffusion term joins the right-hand-side of Eqn. (2.16), and represents a probabilistic perturbation away from the deterministic particle paths described by the inviscid case. The motion of a continuum of particles along trajectories with both a deterministic and probabilistic component constitutes an *exact* solution to the viscous equations of motion. No approximation is involved until the infinite system is reduced to a finite number of particles. A good introduction to the stochastic differential equation view of vortex methods was given by Chang[51].

Numerical approximation to the infinite set of stochastic ordinary differential equations for the vorticity evolution usually begins with "operator splitting". For simplicity, consider the vorticity transport equation (2.9) for two-dimensional flow, namely

$$\partial_t \xi = -(\mathbf{u}\cdot\nabla)\xi + (1/R)\nabla^2\xi \tag{3.3}$$

where we have decomposed the total derivative into its two components. Suppose we discretize the problem in time but not in space, i.e., we consider the evolution of a system of particles with discrete time step Δt. Let ξ^n be the approximate vorticity distribution at time $n\Delta t$. Suppose we

decompose the vorticity update $\partial_t \xi$ into two terms as follows. Let ξ_{adv}^{n+1} be the solution at time step $(n+1)\Delta t$ of the pure advection equation

$$\partial_t \xi = -(\mathbf{u}\cdot\nabla)\xi \tag{3.4}$$

with $\xi_{adv}^{n}=\xi^{n}$. The solution to this equation is approximated by integration along particle trajectories using Eqn. (2.16). Similarly, let ξ_{vis}^{n+1} be the approximate solution to the diffusion equation

$$\partial_t \xi = \frac{1}{R}\nabla^2\xi \tag{3.5}$$

with $\xi_{vis}^{n}=\xi^{n}$. If we let $\xi^{n+1} = \xi_{adv}^{n+1} + \xi_{vis}^{n+1}$ at the end of the time step, we should get a solution to Eqn. (3.3) as $\Delta t \rightarrow 0$. This technique is known as "operator splitting". Some analysis of the effect of operator splitting was provided by Beale and Majda[31].

How do we solve the vorticity diffusion equation (3.5)? Since the advection term has been formulated in a Lagrangian setting, it is desirable to provide a Lagrangian approach to the heat equation. An infinite collection of particles undergoing Brownian motion yields a solution to the heat equation, see Feller[86]. Thus, to accomplish both advection and diffusion, Chorin[57] updated the positions of the infinite system of particles by (1) advancing them by their induced velocity field and (2) adding an appropriately chosen random step.

It might seem, at first glance, that such a method is highly inaccurate, since a relatively large number of particles are needed to adequately approximate the diffusion equation. However, note that the random motion is being applied to approximate the evolution of *vorticity*, not velocity. The Biot-Savart integration acts to smooth the solution, and accurate solutions can therefore be obtained with a reasonable number of particles. A variety of numerical experiments have been performed to examine the error associated with a random walk solution to the diffusion equation in the context of vortex methods, see, for example, Ghonieim and Sherman[100], Milinazzo and Saffman[161], Sethian and Ghoniem[208]. Detailed calculations by Roberts[187] verify convergence of Chorin's original random walk algorithm applied to a model problem.

One can thing of other ways to approximate the diffusion equation (3.4). One possibility is a finite difference approximation. However, the introduction of a grid and the process of interpolating vorticity to the grid and back can result in considerable smoothing, known as "artificial viscosity". Thus, the major feature of a Lagrangian representation may be compromised, and the size of the Reynolds number can be limited by the grid resolution. Another proposed approach is core spreading, in which the core of each vortex element spreads as an exact solution to the diffusion equation. This technique was used by Kuwahara and Takami [136] in their early vortex work. However, C.Greengard [103] has shown that while the vorticity is corrected diffused in such techniques, it is advected incorrectly, and the limit of the computed solution will not be a solution of the Navier-Stokes equations.

Convergence of the random vortex method is measured in terms of expected values and variances. Due to the stochastic nature of the equations, probabilistic techniques come into play. As a test problem, Hald[108] studied an approximation based on a random walk to a particular reaction-diffusion equation, and showed that the expected value of the computed solution tended to the correct solution, and that the variance tended to zero as the number of particles was increased. Marchioro and Pulvirenti[153] gave a proof of weak convergence of the two-dimensional random vortex method. This was followed by Goodman[101] who provided a stronger proof of convergence. The state-of-the-art in the analysis of the random vortex method has been developed recently by Long[147]. He has succeeded in showing realistic convergence rates for the random vortex method in both two and three space dimensions.

C. Boundary Conditions

(1) Normal Boundary Conditions

For both inviscid and viscous flow, the addition of the normal boundary condition on solid walls is not difficult. Given a region D, suppose we require that $\mathbf{u}\cdot\mathbf{n}=0$ on the boundary ∂D, where $\mathbf{n}$ is the inward normal vector. Let $\mathbf{u}_{vor}(\mathbf{x},t)$ be the velocity field obtained from the distribution of vorticity. Suppose we find a potential flow $\mathbf{u}_{pot} = \nabla\phi$ such that $(\mathbf{u}_{pot}+\mathbf{u}_{vor})\cdot\mathbf{n} = 0$ on ∂D. Then

superposition of the vorticity flow $\mathbf{u}_{vor}$ with the potential flow $\mathbf{u}_{pot}$ yields a flow which satisfies the normal boundary condition by construction and has the same vorticity (since $\nabla\times\mathbf{u}_{pot} = \nabla\times\nabla\phi = 0$).

For a simple geometry, finding the potential function ϕ is fairly straightforward. Although this potential function must be found at every time step, it is usually a minor part of the calculation, compared to the vortex-vortex interaction. Some of the earliest calculations used the method of images, such as calculations of flow past a cylinder, see Cheer[52], Chorin[57]. Other possibilities are conformal mapping to the upper half plane followed by the method of images, see Ghoniem, Chorin and Oppenheim[94], and fast Poisson solvers, Sethian[206]. In the case of complex geometries, considerable work may be involved. Finally, appropriate inflow and outflow conditions may be difficult to pose, (see the discussion in Section IV.).

(2) Tangential Boundary Conditions

The addition of the tangential no-slip condition ($\mathbf{u}\cdot\boldsymbol{\tau}=0$) ($\boldsymbol{\tau}$ is the unit tangent to solid walls) on ∂D for viscous flow adds considerable complication. A thin transition zone, known as the *boundary layer*, must develop between the vanishing tangential velocity at the wall and the rapidly moving flow some distance away from the wall, see Batchelor[27] and Schlicting[203]. By way of explanation, let $\mathbf{u}=(u,v)$ and suppose that the x axis is tangential to the wall. Then $\partial u/\partial y$, which is the derivative of the tangential velocity in the direction normal to the wall, must be large. On the other hand, $\partial v/\partial x$, which is the derivative of the normal velocity along the wall, is small, since, by the addition of the potential flow, v is zero along the x-axis. Since vorticity is defined by $\xi = \partial v/\partial x - \partial u/\partial y$, the tangential boundary condition is a major source of vorticity in the flow. At large Reynolds numbers, particularly in the case of bluff bodies, streamlines break away near the sides of the body and enclose the fluid behind the body in an unsteady manner. This phenomena is known as "boundary layer separation", and leads to the downstream shedding of large physical vortices.

One way to approximate this effect is to release physical vortices into the flow from a predetermined separation point. A "Kutta condition" can be used to impose an *a priori* estimate of

the strength of the shed vortices. One such technique views the separation point as a source of an array of line vortices. For example, this model was used to study the periodic wake past a cylinder, see Abernathy and Kronauer[1], Gerrard[91], Sarpkaya[201]. An early, detailed set of calculations which created physical vortices for flow past bluff bodies is described in Clements[66], and a review of such techniques may be found in Clements and Maul[67]. Further discussion of boundary layer theory can be found in Batchelor[27] and Schlicting[203].

In Chorin[57], vorticity creation was extended to the entire boundary through the following technique. The length of boundary was broken into intervals, and within each interval a smoothed vortex "blob" was created at each time step. The strength of the blob was chosen to nullify the tangential velocity, and hence, by construction, satisfy the no-slip condition. There are two advantages to this technique. First, no a priori assumption about the location of the separation point is made, since the entire body acts as a line source of vorticity. Second, since vortex elements with smoothed velocity fields are used, more vortices may be used to resolve the vortex sheet along the boundary while avoiding the instabilities that plague point vortex approximations.

The technique of adding vortex blobs at the boundary in response to the no-slip condition is not very accurate. Part of this inaccuracy occurs because a small motion displacement in newly created vortex elements can cause significant variation in the tangential velocity at the next time step. Thus, large counterrotating vortices must be created at the next time step to enforce the no-slip condition. This can result in a large number of vortices near the boundary with one sign essentially balanced by a large number of vortices of opposite sign. Since the vortex-vortex interaction is expensive, this soon proves unworkable.

There are still further sources of inaccuracy if one tries to use vortex blobs to satisfy the no-slip boundary condition, as can be seen from the following argument, see Chorin[60]. The no-slip condition requires that we create a vortex sheet along the x-axis of strength U. Then we might imagine that the velocity at (x,y) right above the wall should be $+U$, and at $(x,-y)$ right below the wall should be $-U$. By the definition of the vorticity in the Navier-Stokes equation, we have that

$\xi(x,y)=\partial v/\partial x - \partial u/\partial y$. Note that $\xi(x,y) \neq \xi(x,-y)$, thus substantial inaccuracy results from using vortex blobs to approximate this boundary condition. In order to remedy this situation, an alternative was introduced in Chorin[61]. Close to smooth solid walls, the Navier-Stokes equations can be replaced by the Prandtl boundary layer equations, which are derived under the assumption that $\partial v/\partial x \ll \partial u/\partial y$, and that diffusion of vorticity occurs mostly in a direction normal to the wall. The vorticity in the boundary layer is then discretized by finite length vortex sheets. The jump in the tangential velocity across the sheet determines its strength. Just as for the vortex blobs, the velocity of each sheet can be constructed by summing the influence of all sheets located in a narrow neighborhood nearby, (see Chorin[61], Sethian[206], for the equations of motion). In Chorin[61], this algorithm was used to approximate flow past a semi-infinite flat plate. Cheer[52] coupled vortex sheets in the boundary to vortex blobs in the interior to study flow past a cylinder. Sheets became blobs when they moved more than a certain distance away from the wall, and vice versa; in both cases, conservation of circulation was maintained. Other applications using this hybrid algorithm of vortex sheets and vortex blobs include flow past a backward facing step by Ghoniem, Chorin and Oppenheim[94] and flow in a closed cavity by Sethian[206]. A hybrid algorithm which used vortex sheets in the boundary coupled to finite differences in the interior was studied by Shestakov[211]. An extension of the solid wall vortex sheet creation algorithm to three-dimensional flow is described in Chorin[60].

There is some controversy over the use of the Prandtl boundary layer equations near the wall coupled to the vortex blob Navier-Stokes solver in the interior. Two issues are often raised. First, the Prandtl boundary layer equations do not apply when separation occurs, thus, at least formally, they cannot be used in such situations. Second, the edge of the boundary layer is not well-defined, and the use of a fixed transition point from sheets to blobs may not match the size of the physical boundary layer. We believe that an appropriate viewpoint is to interpret the use of the Prandtl boundary layer equations and vortex sheet discretization as merely a numerical device to get vorticity off the boundary and into the main part of the flow. The size of this zone does not correspond to the

size of the physical boundary layer. Along these lines, interesting studies have been performed by Teng[224,225], in which vortices whose ellipticity varies with distance from the wall were used to approximate vorticity throughout the flow, thus providing a smooth transition from the wall to the interior.

Proving convergence of a random vortex method with creation of vorticity at the boundary is a difficult problem. Some theoretical work on this topic is provided by Benfatto and Pulvirenti[37]. Hald[107] studied a particular vorticity creation algorithm for the equations of free convection near a boundary discussed by Ghoniem and Sherman[100] in their survey paper. Hald showed that both the pointwise and least squares error of the computed solution tended to zero as the time step went to zero and the number of particles increased. For the random sheet method discussed above, numerical convergence tests have been performed by Puckett[180]. The convergence properties of the complete random vortex element method applied to a test problem of flow over a backward facing step have been studied in Sethian and Ghoniem[208].

D. Fast Summation Methods:

The approximation of the vorticity into N discrete vortex elements leads to an N-body problem, namely that each vortex interacts with each of the others. This is an $O(N^2)$ algorithm, which is expensive for large N. Several authors have investigated fast summation algorithms to lower the operation count.

A straightforward approach is to solve directly for the stream function by introducing a finite difference grid. In this technique, the discrete vorticity field is interpolated to neighboring grid points, and a fast Poisson solver is used to find the stream function at grid points. Finite difference approximations then yield the velocity field at the grid points, which is then interpolated back to the vortex locations. This "vortex-in-cell" technique has an operation count of $O(N \log N)$, and was used in some of the earliest vortex calculations, see Christianson[64] and Christianson and Zabusky[65]. It has a long history, and is a major component of many N-body calculations in a variety of areas,

see Eastwood and Hockney[85]. It is related to the particle-in-cell techniques invented by Harlow[113] and Harlow[114]. More recent applications of vortex-in-cell techniques to problems in fluid mechanics are described in Leonard[143], Baker[18], Tryggvason[226].

The process of interpolating the vorticity onto a mesh can introduce significant inaccuracy into the computation of vortex interaction. In the close interaction between neighboring vortices, the smearing of vorticity onto an intermediate mesh can introduce artificial viscosity which dominates the real physical viscosity in the system. As a compromise between the direct N-body method and the grid-based fast Poisson solution, various techniques have been developed to divide the computation into near and far field effects. One set of techniques uses fast solvers for the contributions of far away effects, and direct summation for the nearby elements. For a summary of such P^3M (particle-particle, particle-mesh) techniques, see Hockney and Eastwood[119]. One such algorithm in the context of vortex methods was given in Anderson[4] using in part ideas in Mayo[155]. Other work in this type of fast summation method may be found in Appel[8], Barnes and Hut[24], Boris[41]. Recently, a fast summation technique using a multi-pole expansion was developed by L. Greengard and Rokhlin for the far particles, see Carrier, L.Greengard and Rokhlin[50], L.Greengard and Rokhlin[104]. Further advances in rapid summation techniques have come from the use of computers with parallel architectures, significantly reducing the computational time, see Baden[17].

IV. Physical Problems

In this section, we provide references for the application of vortex methods to particular physical problems. Each of these applications has a vast literature, and are topics of ongoing interest as well as, in some cases, controversy. It is presumptuous, and probably well-nigh impossible, to presume that all the issues can be straightened out and presented in a concise, linear manner. Our goal is to summarize the topics of interest in each area, sketch some of the variations on vortex methods that have been used to attack the problems, and provide a few references.

A. Vortex Sheets

Many flows are characterized by sharp changes of velocity in narrow zones. For example, in parallel shear flow, the vorticity is concentrated in a narrow transition region between the two moving fluids. In the limit, the thickness of this shear layer goes to zero, and suggest the study of "vortex sheets". In this two-dimensional flow problem, the vorticity is zero everywhere excect along an infinitely thin line or curve. As an example, flow around the trailing edge of a wing forms a vortex sheet whose strength depends on the given wing design. The ensuing motion and roll-up of the vortex sheet affects both the drag on the wing and the flight of following aircraft.

The first numerical study of vortex sheets using vortex methods was due to Rosenhead[191], who discretized a vortex sheet into a small number ($N \approx 10$) of point vortices placed along an initial curve with periodic boundary conditions. The results obtained using this technique were physically reasonable, This was followed by a calculation by Westwater[232] of a finite vortex sheet carrying elliptically varying circulation, (the "elliptically loaded wing"). In both of these calculations, the positions of the point vortices were updated in time using Euler's method and the vortex sheet was reconstructed by interpolating through the point vortices. Later attempts to achieve better results using more points and more accurate time integration schemes failed. Contrary to expectation, the calculations produced irregular point vortex motion; see, for example, the calculations by Birkhoff[39]. As discussed earlier, this cast doubt on the validity of vortex methods. The difficulties

stemmed both from the use of singular point vortices and from the subtle nature of the vortex sheet evolution equation. In particular, a workable numerical algorithm must be able to handle potential singularities in an evolving vortex sheet.

Since these initial calculations, a variety of techniques have been tried to obtain a more accurate picture of the roll-up phenomenon and to understand the nature of the singularity that develops. The equations of motion for the evolving sheet are usually written in terms of a complex curve in two-dimensional flow, see Birkhoff[39]. The curve's motion is given by a nonlinear integrodifferential equation involving a Cauchy principal value integral. The various approaches may be broadly divided into two groups; those that attempt a higher order accurate discretization of the singular integral, and those that instead desingularize the integral before discretization.

A good starting point is the review article by Saffman and Baker[195]. Some other papers concerning numerical, asymptotic, and theoretical aspects of vortex sheets, are Anderson[5], Baker[18], Baker[19], Baker, Meiron and Orszag[20], Chorin and Bernard[62], DiPerna and Majda[79], DiPerna and Majda[80], Fink and Soh[88], Guiraud and Zeytounian[106], Higdon and Pozrikidis[117], Kaden[125], Krasny[133] Krasny[134], Krasny[135], Kuwahara and Takami[136], Mangler and Weber[152], Moore[165], Moore[166], Moore[167], Pullin[182], Pullin[183], Pullin and Phillips[185], Schwartz[204], Vandervooren[231].

B. Vortex Pairing/Mixing/Shear Layers/Two Fluids

The goal of many vortex calculations is to understand fundamental questions about mixing, shearing, and transport. As a practical example, consider two chemical solutions which must be adequately mixed before use. What is the most efficient means to accomplish this mixing? Some possibilities are agitators placed in the containers or aimed injection streams. The large vortex structures created by such mechanisms determine the mixing, and some configurations are more efficient than others. This leads to fundamental questions of vortex interactions. A wide array of theoretical, numerical, and experimental investigations have been undertaken, for example, Brown and Roshko[46] have provided detailed experimental analysis of large coherent structures in turbulent

shear layers.

One set of model problems analyze the stability of complicated shapes and distributions of vorticity, including the interaction of patches of constant vorticity. Of particular interest is the effect of mutually induced velocity on neighboring patches and the intertwining of bounding contours. Another model problem is the vortex street, which idealizes the wake shed behind a stationary cylinder placed in a uniform flow. Here, the stability and later coalescence of vorticity initially organized in two infinite parallel rows of opposite signed vortices is studied to analyze the roll-up of vortex structures and oscillating wake patterns.

To begin, some experimental studies of vortex pairing have been performed by Winant and Browand [236], and of vortex stability in a rotating fluid by Griffiths and Linden [105]. The interaction of a pair of vortices has been studied in several different settings. Saffman[193] studied the approach of such a pair to a plane surface, and Saffman and Szeto[199] studied steady equilibrium shapes for two equal uniform vortices rotating about each other. Pierrehumbert[175], using a relaxation method, found a family of steady pairs which can move through a fluid without change of shape; Saffman and Tanveer[200] enlarged the class of possible solutions. Dritschel[83] provided equilibrium shapes for two to eight corotating uniform vortices.

An early numerical point vortex study of vortex motion is due to Christiansen[64), who studied the merger of two corotating patches of vorticity. This was followed by a calculation by Christiansen and Zabusky[65] of the motion and merger of an asymmetric four vortex system. In order to study in more detail the motion of patches of constant vorticity in two dimensions, Zabusky, Hughes and Roberts[238] introduced the method of "contour dynamics". The idea behind this method is as follows. Rather than follow the evolution of vorticity, which is the underlying notion in the vortex methods described in Section III, suppose that (1) vorticity is distributed on a compact set and (2) vorticity may be subdivided into distinct regions of with constant magnitude within each patch. Let γ_i be the contour that bounds patch i. By using the Biot-Savart law, an evolution equation for the moving contours γ_i can be derived, see Zabusky, Hughes and Roberts[238]. The

goal of contour dynamics algorithms, which reduces the dimension of the problem by one, is to accurately follow the motion of the contours, capturing the interaction of the patches of vorticity. The contours are followed by a marker particle technique in which the contour is represented by points distributed along the curve, and appropriate finite difference approximations are used to compute the ensuing motion. Calculations using contour dynamics include the coherence of stretched vortices by Jacobs and Pullin[124], and Pullin and Jacobs[184], plasma cloud dynamics by Overman, Zabusky and Ossakow[172], interacting patches by Dritschel[82], and Overman and Zabusky[171], the Kelvin-Helmholtz instability by Pozrikidis and Higdon[179], and the dynamics of vorticity fronts by Stern and Pratt[220]. As a by-product, detailed technology has been developed on the frontier of marker particle technology for front tracking, see Dritschell[81], Zabusky and Overman[239]. However, in the application of this technology to problems in fluid mechanics, there are significant limitations. Bounding contours can only have meaning for inviscid, two-dimensional flow. Merger is impossible for such flow as a simple consequence of the reversibility of the flow map which forces each contour to stay closed. Any attempt to introduce viscosity to the bounding contours is an ad-hoc adjustment in the contours, rather than in the vorticity itself.

As mentioned earlier, vortex streets have been used as model problem to analyze vortex roll-up and the wake development. Baker, Saffman and Sheffield [22] studied the structure of a linear array of hollow vortices of finite cross-section, in which the vorticity of each vortex is concentrated just on the boundary. This was followed by a series of papers by Saffman and Schatzman[196], Saffman and Schatzman[197], Saffman and Schatzman[198]. which investigate questions of vortex street stability as a function of spacing parameters, vortex strengths, and core size. The stability of two infinite parallel rows of finite core vortices was also studied by Kida[128], whose work was revisited by Meiron, Saffman and Schatzman[157]. Calculations of roll-up into vortex streets using a large number of point vortices was performed by Aref and Siggia[14]. The stability of vortex arrays in three dimensions has been studied by Robinson and Saffman[189]. Other papers in this areas are Bromilow and Clements[44], Ferziger[87].

Shear flows are significantly more complicated mechanisms for mixing. Imagine two fluids, moving at different speeds and parallel to a splitter plate separating the two. When the edge of the plate is reached, shearing is introduced as the differentially moving fluids come in contact. It has been studied in the detail in the laboratory, see Batt[28], Bradshaw[43], Browand[45], Brown and Roshko[46], Freymuth[89], Freymuth, Banf and Palmer[90], Ho and Huang[118], Pui and Gartshorne[181], Winant and Browand[236]. In a series of papers, Corcos, Lin and Sherman, see Corcos and Lin[69], Corcos and Sherman[70,71], Lin and Corcos[144] studied various aspects of the mixing layer and unstable free shear layers using a variety of mathematical techniques. Analysis of the dynamics of stretched vortices is provided by Neu[169,170]. Some theoretical investigations may be found in Moore and Saffman[168], Pierrehumbert and Widnall[177]. Vortex methods applied to shear flows and the mixing layer have achieved considerable success; we refer the interested reader to numerical calculations in Acton[2], Aref and Siggia[14], Ghoniem and Ng[98], Heidarneijad and Ghoniem[115], Inoue[122], Inoue and Leonard[123], Meiburg[156], Meng[159], Meng and Thomson[160].

A variety of problems involve following the complicated interface that develops between two regions of flow, such as between fluids of differing density and/or viscosity. One example is the Rayleigh-Taylor instability, which occurs when a light fluid is pushing a heavier fluid. Rather than remain flat, the interface can develop long spikes which can reconnect or break away into bubbles. Some examples where this instability can occur are in the collapse of a massive star, the laser implosion of deuterium-tritium fusion targets, and the electromagnetic implosion of a metal liner. A simple example is the novelty-store toy in which fluids of differing densities are trapped between two glass plates. By upending the apparatus, the lighter fluid rises to the top and long spikes form in the interface. Bubbles can break off from the interface and merge with other bubbles. The interface between the two fluids becomes highly complex, breaking into numerous different parts with wildly varying shapes.

In their most complicated form, the equations of motion are the equations of full viscous, compressible flow plus interface effects. Some important factors controlling the growth of instability are (1) the density ratio, which governs the growth of small amplitude perturbations (2) surface tension, which stabilizes wavelengths shorter than a critical wavelength (3) the viscosity, which reduces growth rate and regularizes the flow (4) compressibility, which reduces growth rate, and (5) heterogeneity, which can excite instabilities of various wavelengths. For an overview of some work on the Rayleigh-Taylor instability, see Sharp[210].

Vortex methods have been applied to many of these problems. One technique is to reformulate the problem as a boundary integral, and discretize the boundary integral by a collection of vortex-like objects along the interface. The integral is desingularized to smooth the propagating interface. An early study of the Rayleigh-Taylor instability and the Saffman-Taylor instability may be found in Meng and Thomson[160]. Variations on vortex techniques and coupling to other numerical methods applied to Kelvin-Helmholtz and Rayleigh-Taylor problems may be found in Baker, Meiron and Orszag[20], Baker, Meiron and Orszag[21], Kerr[127], Pullin[183], Tryggvason[226]. Applications to Hele-Shaw flow may be found in [Tryggvason and Aref[227], Tryggvason and Aref[228]. The motion of a cylinder of fluid was studied in Rottman, Simpson and Stansby[192] using vortex sheets, vortex blobs and vortex-in cell methods, and inviscid flow about a rising cylinder was studied in Telste[223].

The extension of vortex methods to complicated flows with variable density is not straightforward. Begin with the Navier-Stokes equations (2.1), namely

$$\frac{D\vec{u}}{Dt} = -\frac{\nabla P}{\rho} \tag{4.1}$$

Here we have ignored viscosity and included the density term. Taking the curl of Eqn. (4.1) produces the vorticity transport equation. If the density is constant, the curl of the right-hand-side vanishes, since ρ may be factored out and we have the vector identity $\nabla\times\nabla P = 0$. In general, however, the *baroclinic* term $\nabla\times\nabla P = 0$ is not zero, and somewhat difficult to handle in the context of vortex methods. Under the assumption of small density variation, Anderson[5] used the Boussinesq

approximation to the baroclinic term on the right-hand-side and developed a vortex method for the transport of density gradients in the flow, similar to the advection of velocity gradients (that is, vorticity) by traditional vortex methods. Using this technique, he performed detailed calculations of a rising thermal. His calculations show the spiral arms of the rising thermals. This refinement approach between h and δ was later used in the study of vortex sheets by Krasny[134,135]. The extension of vortex methods to problems with large density variations (the "baroclinic" term) remains an open question. Such an extension would be particularly valuable in a variety of phenomena, such as density jumps across propagating flames. One approach has been suggested by Ghoniem and Givi[96].

C. External and Internal Flows: Cylinders/Bluff Bodies/Chambers

As discussed earlier, boundary conditions on solid walls produce a thin boundary layer where velocity gradients can be large. Traditional finite difference schemes often require prohibitively small meshes in order to accurately capture the mechanisms at the boundary. The grid-free approach of vortex methods offers an alternative in a variety of engineering applications.

There are three categories of flows involving solid bodies. First are external flows around bluff bodies. Some examples are flows past cylinder, airfoils, flat plates, and wedges. The second category contains inlet/outlet flows, in which confining walls trap much of the flow, and inlet/outlet conditions are required. Examples are flow over a backward-facing step and flow through choked nozzles. In the third category are internal flows in completely confined regions, such as pistons and driven cavities. In each of these areas, the fundamental questions are about the development of large eddies, wakes, and mixing.

In all cases, a potential flow must be found which, through superposition with the induced velocity field, satisfies the normal boundary condition on solid walls. In the case of flow past a cylinder, (or, by conformal mapping, an airfoil), this might be accomplished through the method of

images. In the case of flow in a rectangular region a fast Poisson solver can be used. However, given more complicated geometries, the solution of this potential problem can require significant attention. In the case of inlet/outlet flows, correct boundary conditions may not be obvious. Care must be taken at outlets to pose correct boundary conditions which are neither inconsistent with the flow dynamics nor unnecessarily restrict or influence flow upstream of the exit.

There is another source of difficulty. In response to the no-slip condition, vortices are created at the boundary which then enter the flow regime. In inlet/outlet flows, vortices can leave and no longer influence the region of interest. In external flows, one might assume that vortices far enough away from the solid body do not influence the flow and may be discarded with little effect. However, in internal flows, such as driven cavities, the number of vortices must increase. Eventually, this number can grow so large that computing the mutual vortex interaction is prohibitively expensive. This is one of the reasons why fast summation methods of the vortex interaction are of great importance, even for two-dimensional flows. Ideally, one would like to have a technique for combining vortices at some point to reduce the number. However, significant error is usually introduced when this is done.

As an introduction, we refer the interested reader to vortex calculations studying flow past cylinders, see Cheer[52], Chorin[57], Kimura and Tsutahara[129], Laitone[137], Smith and Stansby[216], Stansby and Dixon[219], Vanderveg and DeBoom[230], flow past airfoils, see Basu and Hancock[25], Basuki and Graham[26], Cheer[52], Katz[126], flow past blunt bodies, see Bearman and Fackrel[35], Boldman, Brinch and Goldstein[40], Clements[66], Smith and Stansby[215], Stansby[218], flow over a backward-facing step, see Ghoniem, Chorin and Oppenheim[94], Ghoniem and Gagnon[95], Ghoniem and Sethian[99], Sethian and Ghoniem[208], flow past a flat plate, see Chorin[61], Chorin[60], Inoue[121], Kiya and Arie[130], Kiya, Sasaki and Arie[131], Sarpkaya[202], flow past a bed, see Smith and Stansby[217], flow past buildings, see Summers, Hanson and Wilson[221], flow past immersible elastic boundaries, see Mendez[158], flow past in heart valves, see McCracken and Peskin[148], and flow in closed chambers, see Sethian[206], Shes-

takov[211], Sung, Latione and Patterson[222].

D. Reactive Flow: Combustion/Flame Propagation/Plasmas

In reactive flows that occur in turbulent combustion, one important problem is the interaction between the hydrodynamics and the propagating flame. Much of the efficiency of combustion comes from the large-scale mixing effects of the flow, which brings more reactants in contact with the flame, promoting shorter combustion times. The equations of turbulent combustion are complex, since the full Navier-Stokes equations must be coupled to complete reaction chemistry models of combustion. Fortunately, in many situations, the problem can be simplified. Since the chemistry time scale is fast relative to the hydrodynamic scale, the assumption of premixed turbulent combustion is often made. Under this model, the flame is taken as infinitely thin, pressure fluctuations are neglected, the Mach number is assumed small, and combustion is characterized by a single-step irreversible chemical reaction taking place at a constant rate. In this simplified model, the central issues are the feedback mechanisms between the propagating flame and the developing eddy structures.

Even with these assumptions, the underlying physical mechanisms remain formidable. The total velocity field is affected by a number of components. From a purely hydrodynamic side, the flow dynamics are controlled by the confinement geometry (which may be moving, as in a piston), initial vorticity distribution, and inlet/outlet conditions. The flame burns at a speed which may depend on the local curvature and pressure changes within the vessel. As the flame burns, it acts as a source of specific volume as reactants are converted to products, and an elliptic exothermic velocity field results which affects the entire flow dynamics. At the same time, vorticity is shed along the flame front, which adds a rotational velocity field which globally influences the flow.

Vortex methods have been used to isolate the components of this process to understand the controlling mechanisms and interactions. They have been coupled to flame propagation algorithms to study the effects of mixing and fluid transport on flame dynamics in a variety of combustion settings. In particular, the effects of such factors as Reynolds number, confinement geometries, inflow

and outflow conditions, exothermic heat release, vorticity shedding along flames, pressure changes in confined vessels, and curvature-dependent flame speeds have been addressed. We refer the reader to Chorin[55], Ghoniem, Chorin and Oppenheim[94], Ghoniem, Chen and Oppenheim[93], Ghoniem and Givi[96], Ghoniem and Knio[71], Majda and Sethian[151], Sethian[206], Sethian[207].

E. Three Dimensions: Filaments/Rings/Shear Layers

In three dimensions, vortex stretching occurs. Consequently, discrete vortex elements must be adaptively added to the calculation in order to maintain accuracy as filaments stretch and contort. This can create a large number of vortex elements, hence fast summation techniques to compute the vortex-vortex interactions are valuable.

Three popular model problems for study in three dimensions are vortex filaments, vortex rings, and shear layers. Vortex methods have been used to study the stability of an evolving vortex filaments, the breakdown of the solution, and the concentration of vorticity into smaller sets. Some papers are the numerical calculations in Chorin[53], Chorin[54], Chorin[60], Siggia[212], and the theoretical discussion in Tsai and Widnall[229], Widnall[233], Neu[169]. Further theoretical investigations into three-dimensional stability and calculations using vortex methods may be found in Moin, Leonard and Kim[162], Liu, Tavantzis and Ting[145], Pierrehumbert[176], Aref and Flinchem[12], Pumir and Siggia[186].

For vortex rings, vortex methods have been used to study the instabilities that develop on a single ring, and the merger of two rings which exhibit "leap-frogging" as each ring induces a velocity on the other. Some work on the theoretical, numerical and experimental issues can be found in Anderson and Greengard[7], Del Prete[75], Dhanak and DeBerbardinis[78], Knio and Ghoniem[132], Maxworthy[154], Leonard[142].

Finally, the analysis of three-dimensional shear layers using vortex methods may be found in Ashurst and Meiburg[16], Knio and Ghoniem[132], with some theoretical and experimental studies in Agui and Hesselink[3], Landman and Saffman[138], Lasheras and Choi[140], Robinson and Saffman[188]. Some theoretical issues are discussed in Corcos and Lin[69], Corcos and Sherman[70], Lin and Corcos[144]. and analysis of the dynamics of stretched vortices may be found in Neu[170]. Additional three-dimensional vorticity calculations may be found in Chorin[60], Couet, Buneman and Leonard[74].

F. Turbulence Studies/Physics

A number of interesting experiments have been conducted to numerically analyze some of the more theoretical issues in turbulence, such as folding, intermittency, possible fractalization of vorticity, and the cascade of energy through various scales. Studies have included estimation of Hausdorff dimensions for flows using filament methods, energy statistics of large numbers of point vortices, sub-grid lattice models to produce inertial range exponents and inertial range statistics, and chaotic motion of point vortices driven by unsteady agitators. As a starting point we refer the interested reader to Aref[11], Chorin[53], Chorin[54], Chorin[56], Chorin[58], Chorin[59], Pointin and Lundgren[178], Siggia and Aref[213].

Lastly, we mention some related work of interest. First, there are similarities between some of the techniques in vortex methods and those in large plasma simulations. We refer the interested reader to Barnes, Kamimura, Leboef and Tajima[23], Birdsall and Langdon[38], Brunel, Leboef, Tajima and Dawson[47], Cohen, Langdon and Friedman[68], Denavit[76], Hewett and Langdon[116], Langdon, Cohen and Friedman[139]. A few papers involving particle/grid methods in other areas are Loewenstein and Matthews[146], Brackbill and Ruppel[42], Dukowicz[84], Monaghan and Gingold[164]. Some general papers in the area of particle methods may be found in Monaghan[163], Eastwood and Hockney[85], Gingold and Monaghan[92], Hockney and Eastwood[119], Williamson[235]. Finally, we mention the connection between vortex methods and superfluid turbulence explored by Buttke[49]. This relationship is analyzed in detail in an article

elsewhere in this volume.

V. Measuring the Accuracy of a Random Vortex Simulation

A. "Convergence" of a Random Method Applied to an Unstable Flow

The difficulty associated with measuring the accuracy of the random vortex method lies in both the method itself and the physical problem under study. For example, imagine flow past a cylinder. Suppose we compare a very local, sensitive quantity, such as the velocity at a particular point in the flow, with an integrated quantity, such as the drag. At low enough Reynolds numbers, measurements of both the velocity and the drag one body diameter downstream are repeatable from one laboratory experiment to the next. In this regime, the flow can be viewed as deterministic on the smallest scales. On the other hand, at a much higher Reynolds number, an oscillatory wake develops. Pointwise, instantaneous probe measurements vary wildly between physical experiments, whereas the drag is suitably averaged so that it remains unchanged from one experiment to the next. One the smallest scales, this flow is "random", on the largest, it is predictable.

What quantities, structures, and flow attributes can one find on the middle scales to measure this "repeatability" issue? Suppose one had a numerical method that supposedly "solved" the equations of motion describing the flow. What would one expect the method to do? Certainly it should predict the largest scales. More detailed than that, it is as much a question about the flow regime as it is about the numerical algorithm.

It was with this in mind that we have analyzed the accuracy of random vortex methods. In Sethian and Ghoniem [208], we studied two-dimensional viscous flow over a backward-facing step. The assumption of two-dimensional flow ignores such three-dimensional effects as vortex stretching. This is a significant limitation. However, the issue of repeatability of the smallest scales remains, and convergence of the two-dimensional vortex method is itself a complex question.

We began with a study of laminar flow, where the solution is smooth and steady and we can expect convergence to pointwise, instantaneous values. Other methods are more economical in this regime, however the point of the study was to examine the convergence of the vortex method. We

then increased the Reynolds number, through the transition regime where unsteady flow occurs into, the turbulent regime. At each stage, we measured the effects of refining numerical parameters on the computed solution.

We chose to study flow over a backward facing step because of the wealth of experimental data available. The results of numerous laboratory experiments, see Armaly, Durst and Pereira[15], Denham and Patrick[77], Honji[120], Periaux[173], are described in Sethian and Ghoniem[208] and summarized as follows. Within the laminar regime, the flow is stable, and there are experimentally measured values for the location of the reattachment point and the recirculation zone size and shape. Thus, "converged" numerical quantities and experimental data can be compared. In the transitional and turbulent regime, laboratory experiments yield integrated quantities such as the averaged recirculation zone length and averaged velocity profiles which can be compared with numerical calculations. In addition, there is a periodic and regular pattern to the generation and development of coherent eddy structures which which should not change under refinement of numerical parameters.

B. Summary of Convergence Results

The results of extensive numerical experiments of this flow over a step, are described in detail in Sethian and Ghoniem [208]. Details of the code itself may be found in Ghoniem, Chorin and Oppenheim [94]. We may summarize the results our convergence study as follows.

In the laminar regime, where there is a steady, stable, recirculating profile, we found the following.

1) The variance around computed means decays in inverse proportion to the number of vortex elements.

2) The computed means converge pointwise to a time-independent profile.

3) The calculations accurately predict the experimentally measured relationship between Reynolds number and the size and length of the recirculation zone.

4) The accuracy primarily depends on the number of vortex elements. The time step, boundary layer resolution are of secondary importance.

In the transitional/turbulent regime, physical experiments indicate a periodic mechanism of eddy shedding downstream of the recirculation bubble, and coherent traveling eddy structures. Here, we found the following.

1) The calculations accurately predicted the qualitative large-scale mechanisms of the flow dynamics, which remained unchanged as the numerical parameters were refined.

2) The calculations produced numerically converged values for integrated, global quantities, such as eddy size, shape, shedding frequency and average velocity profiles.

3) As above, the accuracy primarily depends on the number of vortex elements. In addition, there was a critical empirical relationship between the number of elements and the time step.

We may interpret these results as follows. Within the laminar regime, the random vortex method accurately predicts pointwise, instantaneous measurements. Because it is a probabilistic method, one studies variances and means. Nonetheless, accurate calculation of the relevant quantities can be achieved. While vortex methods may not be the method of choice in this regime, they can be expected to converge to the correct solution under parameter refinement.

In the turbulent regime, our conclusions are somewhat more guarded. Results indicate that the larger scales (average velocity profiles, shedding frequency, size and shape of eddies) are accurately modeled, and converge as numerical parameters are refined. In this sense, vortex methods can be used to predict periodic flow dynamics, as well as large-scale averaged, intergrated quantities.

Note that one is getting more than might naturally come out of more traditional turbulence closure models approaches to the problem. Vortex methods allow one to compute the transitory, periodic dynamics of the flow, rather than just the average profile. Indeed, in many engineering situations, it is the large-scale transitory dynamics that are crucial for mixing and stress/pressure fluctuations. Turbulence-closure models can average out these effects, masking the most interesting features of the flow.

What does one get for refining numerical parameters? We suspect that the answer depends on the equations themselves, and summarize some the arguments discussed in Sethian[205]. Our experience indicates that one can obtain fairly good predictions of such integrated quantities as

average velocity profiles and eddy shedding size and frequencies. On the other hand, a tremendous amount of computational effort might be necessary to get a significant improvement.

We suggest that this insensitivity to further numerical refinement lies in the equations of motion as much as the method. In turbulent flow, small-scale perturbations organize themselves and pass energy on to the larger scales. Most finite difference approximations to turbulent flow oversmooth the calculation, inhibiting the natural growth of these perturbations. On the other hand, in a random vortex approximation, the number of vortex elements and particular string of random numbers chosen represent perturbations to the smaller scales. These perturbations organize themselves and pass energy to the larger scales. Thus, we argue that refining numerical parameters merely serves to inject a different set of small-scale perturbations to the flow. The most important requirement is to to calculate the flow with a resolution greater (that is, on a smaller scaler) than the "repeatability' scale of the flow. Put another way, if excitations below a certain size organize themselves into large structures somewhat independent of the particular excitation, then it might not be worth a tremendous amount of effort to excite the flow on a significantly finer scale.

VI. Concluding Remarks

In this article, we have described some of the central themes in the design and application of vortex methods. In the next few years, a major focus of interest will be the practical application of vortex methods to realistic three-dimensional flows. Due to the shear complexity of such flows, the intricate folding and stretching of vorticity will require a large number of discrete vortex elements to adequately resolve the solution. Thus, research in fast summation methods for efficiently computing vortex interactions will be an important element of a practical vortex algorithm, as well as some sort of vortex amalgamation technique, in which vortex motion on the smallest scales is homogenized to reduce the total number of vortex elements.

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