LBL-29312

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Materials & Chemical Sciences Division

Current and Potential Distributions on a Cylinder Electrode

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(M.S. Thesis)

July 1990

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Current and Potential Distributions on a Cylinder Electrode

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This work was supported by the Assistant Secretary for Conservation and Renewable Energy, Deputy Assistant Secretary for Utility Technologies, Office of Energy Management, Advanced Utility Concepts Division of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

Current and Potential Distributions on a Cylinder Electrode

By

Anthony Joseph Grabowski

B.S. (University of Michigan) 1987

THESIS

Submitted in partial satisfaction of the requirements for the degree of

MASTER OF SCIENCE

in

CHEMICAL ENGINEERING

in the

GRADUATE DIVISION

of the

UNIVERSITY OF CALIFORNIA at BERKELEY

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ACKNOWLEDGEMENTS

I would like to thank John Newman for his invaluable suggestions and insight. I would also like to thank my colleagues in our research group; Vince Battaglia, Alan West, Tom Fuller, Paul Shain and Mei-Hui Wang, for all of their help.

I am also grateful to Dawn Maxon for both her stunning artistry, evident in figures 1, 2a and 2b, and for staying by me over the years.

I would like to thank my family for all they have done for me, and I would like to thank Vince for being my pal.

This work was supported by the Assistant Secretary for Conservation and Renewable Energy, Deputy Assistant Secretary for Utility Technologies, Office of Energy Management, Advanced Utility Concepts Division, of the U. S. Department of Energy under Contract No. DE-AC03-76SF00098.

SECTION 1: INTRODUCTION

This work presents the numerical solution for the current and potential distributions of a cylindrical electrode. In particular, it investigates the primary current distribution, the secondary current distribution with linearized kinetics, and the potential distribution due to constant current density on the working electrode. Figure 1 illustrates the idealized electrochemical cell geometry and its geometric parameters: the working electrode of length $2l_o$ and radius r_o is imbedded in an infinite insulating cylinder, while the counterelectrode is at infinity.

1.1 THE PRIMARY AND SECONDARY CURRENT DISTRIBUTIONS

Laplace's equation describes the potential variation in dilute electrolytic solutions with negligible concentration gradients.

$$\nabla^2 \Phi = 0 \qquad 1.1\text{-}1$$

Ohm's law gives the current density under these conditions.

$$i = -\kappa \nabla \Phi \qquad 1.1\text{-}2$$

Equations 1.1-1 and 1.1-2 approximate behavior in electrochemical systems with negligible mass transfer resistance. In particular, they apply to the bulk solution for electrochemical systems in which convection dominates mass transport in the bulk and concentration gradients are confined to a thin boundary layer near the electrode surface.

1.1.1 Electrode Kinetics

The Butler-Volmer equation describes many electrochemical reactions at an electrode. For constant reactant concentration, this equation becomes:

$$i_n = i_o\left(\exp\left(\frac{\alpha_a F}{RT}\eta_s\right) - \exp\left(-\frac{\alpha_c F}{RT}\eta_s\right)\right), \qquad \eta_s = V - \Phi_o . \qquad 1.1\text{-}3,\ 1.1\text{-}4$$

In this equation, i_n is the current density at the electrode due to the reaction, i_o is a kinetic parameter called the exchange current density, α_a and α_c are kinetic parameters called the apparent transfer coefficients, **F** is Faraday's constant, and **R** is the universal gas constant. The surface overpotential, η_s, is the potential difference between the electrode potential, V, and the potential in solution adjacent to the electrode, Φ_o, measured with a reference electrode identical to the working electrode.

For small surface overpotentials, one can linearize equation 1.1-3 around η_s=0, yielding:

$$i_n = (\alpha_a + \alpha_c)\frac{i_o F}{RT}\eta_s, \quad \eta_s \rightarrow 0. \qquad 1.1\text{-}5$$

For large surface overpotentials one of the terms in equation 1.1-3 becomes negligible, depending upon the sign of η_s, resulting in the so-called Tafel approximation.

1.1.2 The Primary Current Distribution

Negligible electrode kinetic resistance implies negligible surface overpotential. Assuming no ohmic resistance in the electrode, the boundary condition at the electrode becomes:

$$\Phi_o = V, \qquad 1.1\text{-}6$$

where V is a constant. The current distribution governed by Laplace's equation in the electrolytic solution and constant Φ_o at the electrode is called the primary current distribution. The primary distribution is limited only by ohmic resistance in the solution. It is characterized solely by geometric parameter ratios (1). Ohmic resistance, calculated from the primary distribution, gives the relationship between potential drop and total current in a cell limited by electrolyte ohmic resistance.

Near an electrode-insulator edge, the following equation describes the primary current distribution:

$$\frac{i_n}{P_o} = x^{\left(\pi/2\alpha - 1\right)}, \; x \rightarrow 0, \qquad 1.1\text{-}7$$

where x is the distance from the electrode-insulator edge, α is the electrode-insulator intersection angle, and P_o is a function of the system geometry. Current density is zero at the edge for acute angles, a constant for right angles, and infinite for obtuse angles.

1.1.3 The Secondary Current Distribution

The current distribution governed by Laplace's equation in solution and finite electrode kinetics is called the secondary current distribution. This distribution depends not only upon geometric parameter ratios, but also upon the absolute size of the system.

If one uses the full Butler-Volmer equation to describe electrode kinetics, the secondary current distribution depends upon three parameters in addition to the geometric parameters of the primary distribution (2). Due to the large number of parameters, this system is difficult to analyze. The linear, or Tafel kinetics, approximation simplifies the problem. The secondary current distribution with linear electrode kinetics introduces only one additional parameter, J_h:

$$J_h = (\alpha_a + \alpha_c)\frac{i_o F h}{\kappa R T}, \qquad 1.1\text{-}8$$

where h is an important geometric length in the system. J_h represents the ratio of ohmic resistance to kinetic resistance. As J_h approaches zero, kinetic resistance predominates, rendering the current distribution uniform on the electrode. Conversely, as J_h approaches infinity, ohmic resistance predominates, and the current distribution approaches the primary current distribution. Tafel kinetics also involves only one additional parameter, which is proportional to the average current density on the electrode instead of the exchange current density.

1.2 POTENTIAL DISTRIBUTION DUE TO CONSTANT ELECTRODE CURRENT DENSITY

The potential due to uniform i_n provides an estimate of the maximum possible potential variation on an electrode. Oxygen mass transport generally limits corrosion processes. Consequently, these constant current density calculations are relevant to cathodic protection technology, in which sacrificial anodes protect metal objects from corroding. In these systems, the largest potential difference on the protected metal object, $\Delta\Phi_O$, must be kept within a finite range to avoid both corrosion and hydrogen evolution. This restriction dictates, for example, the maximum size of the object which a counterelectrode at infinity can protect.

1.3 PREVIOUS WORK

Most nontrivial, analytic solutions to current distribution on finite cylindrical electrodes have been obtained for concentric electrodes with symmetric insulators at right angles to the axis of symmetry. Weisselberg (3) solved the primary current distribution analytically in this geometry for an inner electrode with finite ohmic resistance. Waber (4) followed with an analytic solution to the secondary current distribution with linearized kinetics. Finally, Alkire and Varjian (5) solved the problem numerically, with full Butler-Volmer kinetics, for a thin resistive wire inner electrode.

More closely related to the geometry under consideration, Strommen and Rodland (6) developed a fairly sophisticated finite difference method for calculating potential distributions on underwater pipelines protected by exterior sacrificial anodes.

SECTION 2: MATHEMATICAL METHOD

2.1 RING SOURCE IMBEDDED IN AN INSULATING CYLINDER

The following expression gives the potential at z and r, due to a differentially thin ring at z' and r_o, emitting total current $2\pi(2i_n)r_o dz'$ (7).

$$d\Phi = \frac{2r_o}{\pi\kappa} \frac{K(m)\, i_n\, dz'}{\left((z-z')^2 + (r+r_o)^2\right)^{1/2}} \qquad 2.1\text{-}1$$

Here K is the elliptic integral of the first kind and m is defined as:

$$m = \frac{4r\, r_o}{(z-z')^2 + (r+r_o)^2}. \qquad 2.1\text{-}2$$

Figure 2a shows the ring source described by equation 2.1-1.

One can nondimensionalize the differential potential as follows:

$$G(R^*,Z) = \frac{2\,K(m)}{\left(Z^2 + (R^*+1)^2\right)^{1/2}}. \qquad 2.1\text{-}3$$

G, R*, and Z are the following groupings:

$$G(R^*,Z) = \frac{\pi\kappa}{i_n\, dz'} d\Phi, \quad Z = \frac{z-z'}{r_o}, \quad R^* = \frac{r}{r_o}. \qquad 2.1\text{-}4,5,6$$

To obtain the potential due to a ring source imbedded in an infinite insulating cylinder, G*, as shown in figure 2b, Mak (8) added the following integrated sum of orthogonal functions to G:

$$G^* = G + F\,, \qquad F(R^*,Z) = \int_0^\infty B(\zeta)\, K_0(\zeta R^*) \cos(\zeta Z)\, d\zeta\,. \qquad 2.1\text{-}7,8$$

Here K_0 is the modified Bessel function of the second kind of order 0. Appendix A provides a detailed derivation of equation 2.1-8 and Mak's calculation of F and $B(\zeta)$.

The following equation approximates F's behavior on the cylinder surface:

$$F \approx -\frac{1}{2}\pi \frac{1}{\sqrt{1+Z^2}}\,. \qquad 2.1\text{-}9$$

G^* behaves in three fairly distinct fashions on the cylinder, depending on distance from the ring source. For small Z, m approaches one, and K approaches infinity logarithmically.

$$K(m) = \frac{1}{2}\ln\left(\frac{16}{1-m}\right), \quad m \to 1 \qquad 2.1\text{-}10$$

In this region, the ring source appears as a line source imbedded in an infinite insulating plane, and G^* behaves accordingly:

$$G^* = \frac{1}{2}\ln(64) - \frac{\pi}{2} - \frac{1}{2}\ln(Z)^2\,, \qquad Z \to 0\,. \qquad 2.1\text{-}11$$

For distances of one radius to a finite number of radii, the ring appears as a three dimensional object and requires the full form of equation 2.1-7 to describe the potential variation. Finally, as Z approaches infinity, K(m) approaches $\pi/2$, and the ring behaves as a point source. Equation 2.1-7 becomes:

$$G^* = \frac{1}{2}\pi \frac{1}{|Z|}\,, \quad |Z| \to \infty\,. \qquad 2.1\text{-}12$$

2.2 SUPERPOSITION OF RING SOURCES

Ring sources described by equation 2.1-7 can be superimposed, giving the potential as a function of current density on a cylinder electrode.

$$\Phi\left(\frac{r}{r_o},\frac{z}{r_o}\right)=\frac{1}{\pi\kappa}\int_{-l_o}^{l_o} G^*(R^*,Z)\, i_n(z')dz' \qquad 2.2\text{-}1$$

We can then adjust i_n to satisfy electrode boundary conditions.

With the following dimensionless variables,

$$\Phi^*=\frac{\Phi}{V},\quad Z^*=\frac{z}{r_o},\quad Z'=\frac{z'}{r_o}, \qquad 2.2\text{-}2,3,4$$

equation 2.2-1 becomes:

$$\Phi^*=\frac{1}{\pi}\int_{-\frac{l_o}{r_o}}^{\frac{l_o}{r_o}} G^*(R^*,Z^*-Z')\left(-\frac{\partial\Phi^*}{\partial R^*}\right)dZ'. \qquad 2.2\text{-}5$$

Boundary conditions for the primary distribution and linear kinetics (equations 1.1-6 and 1.1-5) become:

$$1=\Phi^* \text{ at } R^*=1,\qquad 1=\Phi^*+\frac{1}{J_r}\left(-\frac{\partial\Phi^*}{\partial R^*}\right) \quad \text{at } R^*=1\,, \qquad 2.2\text{-}6,7$$

respectively. One can propose a linear combination of electrode surface current density functions to satisfy these boundary conditions:

$$-\frac{\partial\Phi^*}{\partial R^*} = \sum_j a_j \left(-\frac{\partial\Phi^*}{\partial R^*}\right)_j , \qquad 2.2\text{-}8$$

where the a_j's are the adjustable constants. Substituting 2.2-8 into 2.2-5 provides an expression for the potential as a linear combination of potential functions.

$$\Phi^*(R^*,Z^*) = \sum_j a_j \frac{1}{\pi} \int_{-\frac{l_o}{r_o}}^{\frac{l_o}{r_o}} G^*(R^*,Z^*-Z') \left(\frac{\partial\Phi^*}{\partial R^*}\right)_j dZ' = \sum_j a_j \Phi^*_j . \qquad 2.2\text{-}9$$

When equation 2.2-9 is substituted into equation 2.2-6, the boundary condition for the primary distribution becomes:

$$1 = \sum_j a_j \Phi^*_j \text{ at } R^*=1. \qquad 2.2\text{-}10$$

Inserting 2.2-8 and 2.2-9 into 2.2-7, the boundary condition for the linear kinetics becomes:

$$1 = \sum_j a_j \left(\Phi^*_j + \frac{1}{J_r}\left(\frac{-\partial\Phi^*}{\partial R^*}\right)_j\right) \quad \text{at } R^*=1. \qquad 2.2\text{-}11$$

The first current density function in equation 2.2-8 is chosen to approximate the expected current distribution behavior. The rest of the current functions are even Legendre polynomials.

$$-\frac{\partial\Phi^*}{\partial R^*} = a_{-1}\left(-\frac{\partial\Phi^*}{\partial R^*}\right)_{-1} + \sum_{j=0}^{19} a_j P_{2j}\left(\frac{r_o}{l_o}Z'\right) \qquad 2.2\text{-}12$$

For the primary distribution, the first current density function simulates the current singularity at the edge. Wagner (9) has given the primary current distribution for an infinite strip imbedded in an infinite insulating plane. This expression is used as the first function for small l_o/r_o.

$$\left(-\frac{\partial\Phi^*}{\partial R^*}\right)_{-1} = \frac{1}{\sqrt{1 - Z'^2}} \qquad 2.2\text{-}13$$

For large l_o/r_o, the following function is used to approximate the primary current distribution:

$$\left(-\frac{\partial\Phi^*}{\partial R^*}\right)_{-1} = \left(\left(\frac{r_o i_{center}}{V\kappa}\right)^b - \left(\frac{P_r\sqrt{2}}{\sqrt{l_o/r_o}}\right)^b + \left(\frac{P_r\sqrt{2\,l_o/r_o}}{\sqrt{\left(l_o/r_o\right)^2 - Z'^2}}\right)^b\right)^{1/b}, \qquad 2.2\text{-}14$$

where b is a constant between one and two. The parameters b, $r_o i_{center}/V\kappa$, and P_r are fit using the results from the primary distribution employing equation 2.2-13. At the center of the electrode, equation 2.2-14 approaches:

$$\left(-\frac{\partial\Phi^*}{\partial R^*}\right)_{-1} = \frac{r_o i_{center}}{V\kappa}. \qquad 2.2\text{-}15$$

At the electrode edge equation 2.2-14 becomes:

$$\left(-\frac{\partial\Phi^*}{\partial R^*}\right)_{-1} = \frac{P_r}{\sqrt{\frac{l_o}{r_o} - Z'}}, \quad Z' \rightarrow \frac{l_o}{r_o}. \qquad 2.2\text{-}16$$

The first function in equation 2.2-12 is also tailored to the expected results for the secondary current distribution. The secondary current distribution becomes uniform over the electrode as the polarization parameter becomes small. Consequently, in this

limit, the first Legendre polynomial, $P_0=1$, approximates current distribution behavior well, and the first function was set to zero.

$$\left(-\frac{\partial\Phi^*}{\partial R^*}\right)_{-1} = 0 \qquad 2.2\text{-}17$$

Nisancioglu and Newman (10) solved the secondary current distribution governed by linear kinetics, in the electrode-insulator edge region, for high polarization parameters and an electrode-insulator intersection angle of π. When their results are couched in terms of the cylinder-electrode problem, the following grouping is a universal function of stretched edge distance in the electrode-insulator edge region:

$$\frac{-\dfrac{\partial\Phi^*}{\partial R^*}}{P_r J_r^{1/2}} = f_{univ}\left((x^*)^{1/2}\right), \quad J_r \to \infty, J_l \to \infty, \quad x^* = J_r \frac{l_o - z}{r_o}. \qquad 2.2\text{-}18,19$$

In this equation f_{univ} equals 1.75 at $z=l_o$ while approaching the primary distribution in the edge region, $x^{*-1/2}$, as x^* approaches infinity. Both J_r and J_l must be large for equation 2.2-18 to be valid. Equation 2.2-14 and equation 2.2-18 form the basis for the following expression, valid for high polarization parameters, and used for all l_o/r_o.

$$\left(-\frac{\partial\Phi^*}{\partial R^*}\right)_{-1} = \left(\left(\frac{r_o i_{center}}{V\kappa}\right)^b - \left(P_r J_r^{1/2} f_{univ}\left(\left(J_l/2\right)^{1/2}\right)\right)^b + \left(P_r J_r^{1/2} f_{univ}\left((y')^{1/2}\right)\right)^b\right)^{1/b} \qquad 2.2\text{-}20$$

$$y' = \frac{J_l}{2}\left(1 - \left(\frac{r_o}{l_o} Z'\right)^2\right) \qquad 2.2\text{-}21$$

Equation 2.2-20 behaves like equation 2.2-18 in the small edge region, where kinetic limitations are significant for high-polarization parameters. It behaves like equation

2.2-14 away from the edge region, where kinetic limitations are not significant for high polarization parameters.

To solve for the a_j's in equation 2.2-8, a collocation technique was employed. For n current functions, equation 2.2-10 or 2.2-11 was evaluated at n points on the electrode, generating the following n x n matrix for the primary and secondary current distributions, respectively:

$$[\,1\,] = \left[\Phi^*_{ij}\right]\left[a_j\right], \qquad 2.2\text{-}22$$

$$[\,1\,] = \left[\frac{1}{J_r}\left(-\frac{\partial\Phi^*}{\partial R^*}\right)_{ij} + \Phi^*_{ij}\right]\left[a_j\right]. \qquad 2.2\text{-}23$$

The a_j's were obtained by inverting matrix 2.2-22 or 2.2-23. Using the zero's of the Legendre polynomials and the electrode edge as collocation points provided the best results. For n current functions, the zeros of the 2(n-1) order Legendre polynomials were used along with the edge point.

Current functions were integrated in equation 2.2-9 using Simpson's rule. The methods for integrating singularities inherent in equations 2.2-13, 2.2-14, 2.2-20, and G* are given in Appendix B.

2.2.1 Convergence Criterion

Results were considered accurate when boundary conditions 2.2-10 or 2.2-11 were satisfied to within 0.01% at each sampled electrode boundary point. In addition, the primary current distributions, using equations 2.2-13 and 2.2-14 as first functions, were compared for $l_o/r_o=10$ and $l_o/r_o=100$. For both aspect ratios, the difference between the results for the two functions was within 0.1% . Secondary current distribution results, using 2.2-17 and 2.2-20 as first functions, for $l_o/r_o=1$ and $J_r=10$ were also compared. These results also differed by less than 0.1%.

SECTION 3: RESULTS AND DISCUSSION

3.1 PARAMETER RANGE OF RESULTS

Both the primary current distribution and the potential distribution due to constant current density on the electrode were calculated for $l_o/r_o=0$ to $l_o/r_o=1000$. Beyond $l_o/r_o=1000$ the method broke down. This breakdown probably occurred because the Legendre polynomial correction functions were unable to satisfy boundary condition in the edge region of $O(r_o)$ where the distributions vary greatly as l_o/r_o becomes large.

Figure 3 shows the parameter space for which the numerical method accurately calculated the secondary current distribution. For aspect ratios from zero to ten, current distributions could be calculated for the entire polarization parameter range. Beyond $l_o/r_o=10$, results could only be obtained for large J_r using equation 2.2-20, and small J_r using equation 2.2-17. When used as first current density functions in this range, neither equation yielded accurate results for intermediate polarization parameters.

3.2 PRIMARY CURRENT DISTRIBUTION

Ohmic resistance in solution solely determines the primary current distribution . As a result, primary distributions are a unique function of cell geometry (11). In the cylinder electrode case, the ratio l_o/r_o uniquely determines the primary current distribution.

Figure 4 shows the results for the primary current distribution. For all l_o/r_o, the current density approaches infinity inversely proportional to the square root of distance from electrode-insulator edge, as described by equation 1.1-7.

Here we examine the primary distribution's behavior for asymptotically large and small l_o/r_o. In the limit of small l_o/r_o, with finite l_o, the cylinder loses its curvature, appearing as an infinite strip imbedded in an infinite plane. The primary distribution for an infinite strip has been treated by Wagner (12) giving:

$$\frac{i_n}{i_{n,avg}} = \frac{2}{\pi} \frac{1}{\left(1 - \left(\frac{z}{l_o}\right)^2\right)^{1/2}} . \qquad 3.2\text{-}1$$

In the opposite extreme, as l_o/r_o becomes large, the current distribution appears to approach infinity in an edge region which grows increasingly small when compared with l_o. As l_o/r_o approaches infinity, the mid-region carries all the current, and the singularity at the edge appears as a spike of current. The current distributions for finite l_o/r_o fall between these two cases.

To understand further the singular behavior in the edge region, one can examine the asymptotic coefficient of equation 1.1-7 as a function of l_o/r_o. Equation 3.2-1 describes the primary distribution as l_o/r_o approaches zero. So in this limit it is reasonable to examine the behavior of $P_{avg,l}$ given by:

$$\frac{i_n}{i_{n,avg}} = P_{avg,l} \frac{1}{\left(1 - \left|\frac{z}{l_o}\right|\right)^{1/2}} . \qquad 3.2\text{-}2$$

As l_o/r_o approaches infinity with finite r_o, the electrode appears as a semi-infinite cylinder. It is reasonable to expect that the singular edge region has a length corresponding to a finite number of radii. Thus, in this limit, we examine $P_{avg,r}$ given by;

$$\frac{i_n}{i_{n,avg}} = P_{avg,r} \frac{1}{\left(\frac{l_o}{r_o} - \left|\frac{z}{r_o}\right|\right)^{1/2}} . \qquad 3.2\text{-}3$$

Figure 5 shows $P_{avg,l}$ and $P_{avg,r}$ as functions of l_o/r_o. As l_o/r_o approaches zero, the thin strip limit, $P_{avg,l}$ approaches $2^{1/2}/\pi$, as evidenced by equation 3.2-1. $P_{avg,r}$ on the other hand continues to increase as l_o/r_o approaches infinity. Pavg,r's functional behavior at large aspect ratios could not be determined.

3.2.1 Ohmic Resistance

The primary distribution ohmic resistance determines the relationship between potential difference and total current in a cell limited by ohmic resistance. It is convenient to use two dimensionless resistances for the cylinder electrode system, $R\kappa r_o$ and $R\kappa l_o$, related by l_o/r_o. As l_o/r_o approaches zero, for finite r_o, the cylinder electrode shrinks to a differentially thin ring. As a result, $R\kappa r_o$ approaches infinity in this limit. At the other extreme, as l_o/r_o approaches infinity, for finite l_o, the cylinder shrinks to a differentially thin needle. In this limit $R\kappa l_o$ approaches infinity.

$R\kappa r_o$ approaches infinity logarithmically as the geometric aspect ratio shrinks according to the following expression derived in Appendix C.

$$R\kappa r_o = \frac{1}{2\pi^2}\left(\ln(16) - \frac{\pi}{2} - \ln\left(\frac{l_o}{r_o}\right)\right), \quad \frac{l_o}{r_o} \to 0 \qquad 3.2\text{-}4$$

For asymptotically large l_o/r_o, it has already been shown that the current density is essentially constant over the electrode except for a negligible spike of current at the edges. Thus, for resistance calculations, it is reasonable to assume constant current density on the electrode. At a distance from the electrode much greater than l_o, the electrode behaves as a point source. In Appendix C, the potential for a cylinder source and a point source are assumed equal at a distance of $O(l_o)$, and the following estimate is derived for $R\kappa l_o$, as l_o/r_o approaches infinity.

$$R\kappa l_o \cong \frac{1}{4\pi}\left(1 + \ln\left(\frac{l_o}{r_o}\right)\right), \quad \frac{l_o}{r_o} \to \infty \qquad 3.2\text{-}5$$

Here $R\kappa l_o$ approaches infinity logarithmically as l_o/r_o becomes large.

Figure 6 is a graph of the ohmic resistance, along with asymptotic equations 3.2-4 and 3.2-5, as functions of l_o/r_o.

3.3 SECONDARY CURRENT DISTRIBUTION

The secondary current distribution takes into account finite electrode kinetics and ohmic resistance in solution. Thus, in addition to geometric parameters which govern the primary distribution on an electrode, additional kinetic parameters determine the secondary current distribution. For small surface overpotentials, the linear kinetics approximation can be invoked introducing the polarization parameter, J_h, as a governing parameter.

The secondary current distribution on the cylinder electrode for linear kinetics is dictated by the geometric ratio l_o/r_o and either J_r or J_l. J_r and J_l are polarization parameters containing lengths r_o and l_o, respectively. They are related by l_o/r_o. Figure 7 illustrates the thirteen distinct asymptotic regions resulting from these governing parameters.

Figure 8 portrays the results for the secondary current distribution for $l_o/r_o=1$, while figure 9 shows the location of the curves in figure 8 in parameter space. J_l (or J_r) represents the ratio of ohmic resistance to kinetic resistance. As J_l approaches infinity, ohmic resistance predominates, and the current distribution approaches the primary distribution. Conversely, as J_l approaches zero, electrode kinetic resistance predominates, and the current density becomes uniform over the entire electrode. Current distributions for finite J_l fall between these two extremes.

3.3.1 Secondary Current Distribution as the Primary Current Distribution is Approached

It is of interest to examine the way the current distribution approaches the primary distribution as the polarization parameters approach infinity (regions two ,three and four in figure 7). Nicancioglu and Newman (13) and West and Newman (14) examined the secondary current distribution with linear kinetics, for high polarization parameters. Nicancioglu and Newman solved the current distribution problem in the electrode-insulator edge region with intersection angle π. West and Newman solved the problem for an arbitrary intersection angle. These results showed that when ohmic effects dominate the secondary current distribution governed by linear kinetics, the current distribution near an electrode-insulator interface deviates from the primary current distribution only in a small edge region of order σ, where

$\sigma=((\alpha_a+\alpha_c)Fi_o/RT\kappa)^{-1}$. In this edge region, the current distribution is relatively independent of the details of the cell's geometry. "Relatively independent" will be quantified shortly. Away from this edge region, toward the center of the electrode, the current distribution approaches the primary distribution. This edge region can be expressed in terms of a dimensionless distance by dividing by h, one of the important geometric lengths. This gives $\sigma/h=1/J_h$.

In the cylinder electrode problem, the edge region can be expressed in two dimensionless forms, $\sigma/r_o=1/J_r$ and $\sigma/l_o=1/J_l$. The current distribution will be relatively independent of cell geometry in the cylinder edge region only if both σ/r_o and σ/l_o are small, so that the edge region is small relative to both r_o and l_o. Figure 10 illustrates the secondary current distribution at high J_l (or J_r) and $l_o/r_o=1$, confirming deviation from the primary current distribution at $(l_o-z)/l_o=O(1/J_l)$. Figure 9 shows the location of the curves in figure 10 in parameter space.

Nisancioglu and Newman solved the aforementioned problem in the edge region. When their results are cast in terms of the cylinder electrode problem (see Appendix D), one obtains the following grouping as a universal function of stretched distance from the edge:

$$\frac{1}{J_l^{1/2}P_{avg,l}}\frac{i_n}{i_{n,avg}} = f_{univ}\left(\left(\frac{l_o-z}{\sigma}\right)^{1/2}\right) = f_{univ}\left(\left(\left(1-\frac{z}{l_o}\right)J_l\right)^{1/2}\right), \qquad 3.3\text{-}1$$

or equivalently,

$$\frac{1}{J_r^{1/2}P_{avg,r}}\frac{i_n}{i_{n,avg}} = f_{univ}\left(\left(\frac{l_o-z}{\sigma}\right)^{1/2}\right) = f_{univ}\left(\left(\left(\frac{l_o}{r_o}-\frac{z}{r_o}\right)J_r\right)^{1/2}\right). \qquad 3.3\text{-}2$$

Again, these relationships hold only when both J_l and J_r are large. At the electrode edge, f_{univ} equals 1.75; as the stretched variable approaches infinity, f_{univ} approaches the edge region primary distribution, $(l_o-z)/\sigma)^{-1/2}$ Figure 11 shows the secondary current distribution for $l_o/r_o=1$ and high J_l, plotted with the variables given in equation 3.3-1.

Since equations 3.3-1 and 3.3-2 equal 1.75 at the electrode edge, they can be used along with $P_{avg,r}$ and $P_{avg,l}$ (given in figure 5) to estimate the edge current for high polarization parameters:

$$\frac{i_{n,edge}}{i_{n,avg}} = 1.75 P_{avg,l} J_l^{1/2} \quad , J_l \to \infty \, , \, J_r \to \infty \qquad 3.3\text{-}3$$

and equivalently,

$$\frac{i_{n,edge}}{i_{n,avg}} = 1.75 P_{avg,r} J_r^{1/2}, \qquad J_r \to \infty, \, J_l \to \infty. \qquad 3.3\text{-}4$$

3.3.2 Secondary Current Distribution as l_o/r_o Approaches zero

As l_o/r_o nears zero (regions four, five, six, seven and eight in figure 7), the secondary current distribution approaches the infinite strip limit, leaving J_l as the only governing parameter. Figure 12 illustrates how the current distribution becomes a sole function of J_l in the limit of small l_o/r_o. Figure 9 shows the location of the curves in figure 12 in parameter space. Equation 3.3-3 and equation 3.2-1 can be combined to give an expression for the current density at the electrode edge for high J_l and vanishing l_o/r_o:

$$\frac{i_{n,edge}}{i_{n,avg}} = \frac{1.75\sqrt{2}}{\pi} J_l^{1/2} \, , \, J_l \to \infty \, , \, \frac{l_o}{r_o} \to 0 \, . \qquad 3.3\text{-}5$$

One can insert the result for the small l_o/r_o, uniform i_n, potential distribution, equation 3.4-6 (derived in section 3.4.1), into the linear kinetics boundary condition to obtain an estimate for edge current density in the limit of small l_o/r_o:

$$\frac{i_{n,edge}}{i_{n,avg}} = 1 + \frac{2\ln 2}{\pi} J_l \, , \quad J_l \to 0 \, , \, \frac{l_o}{r_o} \to 0. \qquad 3.3\text{-}6$$

Thus, when J_l approaches 0, in the limit of small l_o/r_o, the current distribution becomes uniform.

Wagner (15) numerically solved the secondary current distribution governed by linear kinetics on a thin strip for finite J_l. His results apply for vanishing l_o/r_o.

A current distribution can be considered uniform if the current density varies by less than one percent from the average current density over the entire electrode. In the cylinder electrode case this implies, $i_{n,edge}/i_{n,avg}<1.01$. Equation 3.3-6 shows that in the small l_o/r_o limit, J_l must be less than roughly 0.01 to achieve a uniform current distribution. Thus, in figure 7, the line separating uniform current distribution parameter space from non uniform current distribution parameter space, as l_o/r_o approaches 0, is given by $J_l=O(10^{-2})$.

3.3.3 Secondary Current Distribution as l_o/r_o Approaches Infinity

As l_o/r_o approaches infinity (regions two, ten, eleven, twelve and thirteen in figure 7), one might suspect that only J_r governs the secondary current distribution in the electrode edge region. However, numerical results show that l_o/r_o also influences the current distribution. Figure 13 illustrates the current distribution for large l_o/r_o and $J_r=10$, while figure 9 portrays the location of the curves in figure 13 in parameter space. Figure 13 shows the edge current density increasing with l_o/r_o. Equation 3.3-4 further illustrates this point, for large J_r. Figure 5 shows $P_{avg,r}$ increasing as l_o/r_o approaches infinity. Thus, for large J_r, i_{edge}/i_{avg} increases as l_o/r_o approaches infinity. Unfortunately, the numerical method can not produce accurate results for intermediate J_r in the limit of large l_o/r_o (regions twelve and thirteen in figure 7). This hampers one's understanding of the current distribution for large l_o/r_o.

One can estimate the boundary between uniform and non-uniform current distribution parameter space for large l_o/r_o in the same manner as the small l_o/r_o boundary is estimated. Inserting equation 3.4-15 (developed in section 3.4.2.4) into the linear kinetics boundary condition, one obtains the following expression for large l_o/r_o and vanishing J_r:

$$\frac{i_{n,edge}}{i_{n,avg}} \cong 1+\frac{1}{2}\ln\left(\frac{l_o}{2\beta r_o}\right)J_r\ ,\ \ J_r \rightarrow 0\ ,\ \frac{l_o}{r_o} \rightarrow \infty. \qquad 3.3\text{-}7$$

where β=O(1). Equations 3.3-6 and 3.3-7, along with computer generated results for l_o/r_o=O(1), give the boundary between uniform and non-uniform current distribution parameter space illustrated in figure 7.

3.4 POTENTIAL DISTRIBUTION DUE TO CONSTANT ELECTRODE CURRENT DENSITY

The potential distribution due to constant i_n gives an estimate of the maximum possible potential variation on an electrode. In the cylinder electrode system, l_o/r_o solely governs this potential distribution. Here, we examine behavior in the extreme limits of this ratio.

3.4.1 Potential Distribution Due to Constant i_n as l_o/r_o Approaches Zero

As l_o/r_o approaches zero, one can imagine four regions in the solution in which the potential due to constant i_n varies in distinctly different fashions. In a region between the electrode surface and distances of O(l_o) from the electrode surface, the electrode appears to have no curvature and finite width. When one moves further out into solution--to distances between O(l_o) and O(r_o)--the electrode still appears to have no curvature, but now appears as a line source. Here, the potential varies logarithmically according to equation 2.1-11. When one moves still further out--to distances from order r_o to a finite number of radii--the electrode appears as a ring source. Consequently, the full ring source expression, equation 2.1-7, determines the potential distribution. Finally, at distances approaching infinity, the electrode appears as a point source, and equation 2.1-12 governs the potential distribution.

The following equations (developed in Appendix E) express the potential variation on the electrode and insulator, up to distances of O(l_o), for small l_o/r_o. On the electrode one obtains:

$$\frac{\Phi_o \kappa}{i_c l_o} = \frac{1}{\pi}\left((\ln(64)) + 2 - \pi - 2\ln\left(\frac{l_o}{r_o}\right) - \left(1 - \frac{z}{l_o}\right)\ln\left(1 - \frac{z}{l_o}\right) - \left(1 + \frac{z}{l_o}\right)\ln\left(1 + \frac{z}{l_o}\right)\right) + O\left(\frac{l_o}{r_o}\right), \qquad 3.4\text{-}1$$

and on the insulator,

$$\frac{\Phi_O\kappa}{i_c l_o} = \frac{1}{\pi}\left((\ln(64)) + 2 - \pi - 2\ln\left(\frac{l_o}{r_o}\right) + \left(\frac{z}{l_o} - 1\right)\ln\left(\frac{z}{l_o} - 1\right) - \left(1 + \frac{z}{l_o}\right)\ln\left(1 + \frac{z}{l_o}\right)\right) + O\left(\frac{l_o}{r_o}\right), \qquad 3.4\text{-}2$$

where i_c is the constant current density on the electrode. These equations approach infinity as l_o/r_o approaches zero in the same manner as equations 3.2-4. Equation 3.2-4 grossly approximates equations 3.4-1 and 3.4-2. It gives a rough estimate of the relationship between potential drop and total current. Subtracting equation 3.2-4 from equation 3.4-1 and 3.4-2, one obtains finite dimensionless potential expressions:

$$\frac{\Phi_O\kappa}{i_c l_o} - 4\pi R\kappa r_o = \frac{1}{\pi}\left(2 - 2\ln 2 - \left(1 - \frac{z}{l_o}\right)\ln\left(1 - \frac{z}{l_o}\right) - \left(1 + \frac{z}{l_o}\right)\ln\left(1 + \frac{z}{l_o}\right)\right), \frac{l_o}{r_o} \to 0, \qquad 3.4\text{-}3$$

and,

$$\frac{\Phi_O\kappa}{i_c l_o} - 4\pi R\kappa r_o = \frac{1}{\pi}\left(2 - 2\ln 2 + \left(\frac{z}{l_o} - 1\right)\ln\left(\frac{z}{l_o} - 1\right) - \left(1 + \frac{z}{l_o}\right)\ln\left(1 + \frac{z}{l_o}\right)\right), \frac{l_o}{r_o} \to 0, \qquad 3.4\text{-}4$$

respectively. Subtracting equation 3.2-4 from the asymptotic ring source equation (2.1-11) yields:

$$\frac{d\Phi\kappa}{i_c\frac{dz}{2}} - 4\pi R\kappa r_o = -\frac{2}{\pi}\ln\left(\frac{2z}{l_o}\right). \qquad 3.4\text{-}5$$

In the limit as z/l_o approaches infinity, equation 3.4-4 approaches equation 3.4-5.

Figure 15 shows the potential distribution on the cylinder, due to constant i_n, for vanishing l_o/r_o. Equation 3.4-3 and 3.4-4 predict the potential distribution well to about $z=2l_o$, as expected. As z/l_o increases further, the potential distribution begins to approach the ring source asymptote described by equation 3.4-5. Thus, at distances from the electrode greater than order l_o, the potential distribution is described by the ring source equation 2.1-7.

The potential difference between the center and edge of the electrode follows from equation 3.4-1. As l_o/r_o approaches zero, this difference becomes $(2\ln 2)/\pi$.

$$\Delta\frac{\Phi_o\kappa}{i_c l_o} = \frac{2}{\pi}\ln 2 + O\left(\frac{l_o}{r_o}\right) \qquad 3.4\text{-}6$$

Figure 16 shows the center-edge potential difference as a function of l_o/r_o.

3.4.2 Potential Distribution Due to Constant i_n, as l_o/r_o Approaches Infinity

For the large l_o/r_o potential distribution, as with the thin ring potential distribution, we can identify four fairly distinct regions, on the cylinder and insulator, where potential varies in different fashions. From the middle of the cylinder to an edge distance of $O(r_o)$, one would expect the potential distribution to be fairly flat, behaving like an infinitely long cylinder electrode. For distances from the edge of $O(r_o)$, the electrode appears as a semi-infinite electrode, and the potential should behave correspondingly. On the insulator, for distances from the edge between $O(r_o)$ and a finite number of electrode lengths, the electrode appears as a three dimensional object. Finally, at an infinite distance, the cylinder behaves as a point source.

As mentioned previously, the elliptic integral K(m) in equation 2.1-1 approaches infinity logarithmically as m approaches one. Consequently, at large l_o/r_o, the portion of the integrand where $z' >> r_o$ provides the major contribution to the integral in equation 2.2-1. To obtain an estimate for the potential on the electrode surface and insulator, due to constant i_n, for large l_o/r_o, one can approximate this integral as:

$$\Phi_o \cong \int_{-l_o}^{l_o} \frac{2\pi r_o i_c}{4\pi\kappa\sqrt{(z-z')^2 + 4(\beta r_o)^2}}\,dz'. \qquad 3.4\text{-}7$$

Here the ring-like behavior at small to finite z/r_o is lumped into the constant β. One can derive an integrated expression for the above integral (as outlined in Appendix E) yielding:

$$\frac{\Phi_o \kappa}{i_c r_o} \cong \frac{1}{2}\left(\ln\left(\frac{l_o - z}{2\beta r_o} + \sqrt{\left(\frac{l_o - z}{2\beta r_o}\right)^2 + 1}\right)\right) + \frac{1}{2}\left(\ln\left(\frac{l_o + z}{2\beta r_o} + \sqrt{\left(\frac{l_o + z}{2\beta r_o}\right)^2 + 1}\right)\right). \qquad 3.4\text{-}8$$

This equation equals $\ln(l_o/\beta r_o)$ at the middle of the electrode, $1/2\ln(2l_o/\beta r_o)$ at the electrode edge, and approaches $1/(z/l_o)$ at an infinite distance from the electrode.

3.4.2.1 Potential Distribution Due to Constant i_n, for Large l_o/r_o, on the Electrode Mid-Region

For $l_o - z = O(l_o)$ on the electrode, equation 3.4-8 reduces to:

$$\frac{\Phi_o \kappa}{i_c r_o} - \ln\left(\frac{l_o}{\beta r_o}\right) \cong \frac{1}{2}\ln\left(1 - \left(\frac{z}{l_o}\right)^2\right). \qquad 3.4\text{-}9$$

(See Appendix E.) Approaching the electrode edge, equation 3.4-9 becomes:

$$\frac{\Phi_o \kappa}{i_c r_o} - \ln\left(\frac{l_o}{\beta r_o}\right) \cong \frac{1}{2}\ln\left(2\left(1 - \frac{z}{l_o}\right)\right). \qquad 3.4\text{-}10$$

This equation approaches negative infinity logarithmically as z/l_o approaches one. Figure 17 shows the computer-generated potential and equation 3.4-9, on the electrode mid-region. Computer-generated results for potentials at the center of the electrode were used to fit a β-value of 0.44. Equation 3.4-9 and the numerical results agreed, as expected, away from the edge region.

3.4.2.2 Potential Distribution Due to Constant i_n, for Large l_o/r_o, on the Electrode Edge Region

For $l_o - z = O(r_o)$, equation 3.4-8 reduces to:

$$\frac{\Phi_o \kappa}{i_c r_o} - \frac{1}{2}\ln\left(\frac{2l_o}{\beta r_o}\right) \cong \frac{1}{2}\ln\left(\frac{l_o - z}{2\beta r_o} + \sqrt{\left(\frac{l_o - z}{2\beta r_o}\right)^2 + 1}\right). \qquad 3.4\text{-}11$$

(See Appendix E.). As $(l_o-z)/2\beta r_o$ approaches infinity (the electrode mid-region), equation 3.4-11 approaches equation 3.4-10:

$$\frac{\Phi_o \kappa}{i_c r_o} - \ln\left(\frac{l_o}{\beta r_o}\right) \cong \frac{1}{2}\ln\left(2\left(1 - \frac{z}{l_o}\right)\right). \qquad 3.4\text{-}10$$

As $(l_o-z)/2\beta r_o$ approaches negative infinity (away from the insulator-edge region), equation 3.4-11 approaches minus infinity logarithmically:

$$\frac{\Phi_o \kappa}{i_c r_o} \cong -\frac{1}{2}\ln\left(\frac{1}{2}\left(\frac{z}{l_o} - 1\right)\right). \qquad 3.4\text{-}12$$

Figure 18 portrays the computer generated potential distributions and equation 3.4-11 in the edge region. Equation 3.4-11 poorly predicts the numerically generated potentials for edge distances of $O(r_o)$. $\partial\Phi/\partial z$ approaches infinity logarithmically as the electrode insulator edge region is approached (See Appendix E.). However equation 3.4-11 predicts finite $\partial\Phi/\partial z$ at the electrode edge. This discrepancy occurs because equation 3.4-7 poorly approximates equation 2.2-1 for $l_o-z=O(r_o)$. Equation 3.4-11 describes the potential distribution more accurately for edge distances several radii away because equation 3.4-7 is accurate for z at distances of $O(l_o)$ or greater from the electrode edge.

3.4.2.3 Potential Due to a Constant i_n, for Large l_o/r_o, on the Insulator, away from the Edge Region

If $z-l_o=O(l_o)$, on the insulator, equation 3.4-8 reduces to:

$$\frac{\Phi_o \kappa}{i_c r_o} \cong \frac{1}{2} \ln\left(\frac{1 + \frac{z}{l_o}}{\frac{z}{l_o} - 1}\right). \qquad 3.4\text{-}13$$

(See Appendix E.) As z/l_o approaches one (the electrode edge region), equation 3.4-13 approaches equation 3.4-12.

$$\frac{\Phi_o \kappa}{i_c r_o} \cong -\frac{1}{2} \ln\left(\frac{1}{2}\left(\frac{z}{l_o} - 1\right)\right) \qquad 3.4\text{-}12$$

As z/l_o approaches infinity, equation 3.4-13 assumes the form of a point source potential.

$$\frac{\Phi_o \kappa}{i_c r_o} = \frac{1}{\left(\frac{z}{l_o}\right)}, \quad \frac{z}{l_o} \rightarrow \infty \qquad 3.4\text{-}14$$

Figure 19 shows the potential variation on the insulator. Equation 3.4-13 agrees with the computer-generated potential variation away from the electrode insulator edge region.

3.4.2.4 Difference In Potential between Electrode Center and Edge, for Constant i_n, as l_o/r_o Approaches Infinity

The potential difference between the electrode center and edge can be estimated from equation 3.4-8.

$$\Delta\frac{\Phi_o \kappa}{i_c r_o} \cong \frac{1}{2} \ln\left(\frac{l_o}{2\beta r_o}\right), \quad \frac{l_o}{r_o} \rightarrow \infty \qquad 3.4\text{-}15$$

Figure 16 shows this asymptote along with numerically generated potential differences.

SECTION 4: SUMMARY

The primary current distribution, secondary current distribution with linearized kinetics, and the potential distribution due to constant electrode current density, were calculated numerically for a cylindrical electrode by superimposing ring current sources to satisfy relevant boundary conditions. Results were obtained for length-to-diameter ratios from zero to one thousand for both the primary current distribution and the potential distribution calculations. The secondary current distribution was solved for the full range of polarization parameters, for aspect ratios ranging from zero to ten. For aspect ratios greater than ten the numerical method could only yield solutions for very high and low J_r.

For small aspect ratios, the primary distribution on the cylinder electrode approached that of a thin strip embedded in a insulating wall; for large aspect ratios the primary distribution's functional form could not be deduced. The primary ohmic resistances were calculated and asymptotic forms given for large and small aspect ratios.

The secondary current distribution also approached the infinite strip limit in the small aspect ratio limit, while its functional form at large aspect ratios remained unclear. Asymptotic expressions were developed for the electrode edge current density in the high and low polarization parameter limit.

Asymptotic expressions for the potential on the electrode and insulator, due to constant electrode current density, were derived for high and low aspect ratios. In addition, correlations describing the maximum potential difference on the electrode were produced for this boundary condition.

fig.1: Electrode Geometry

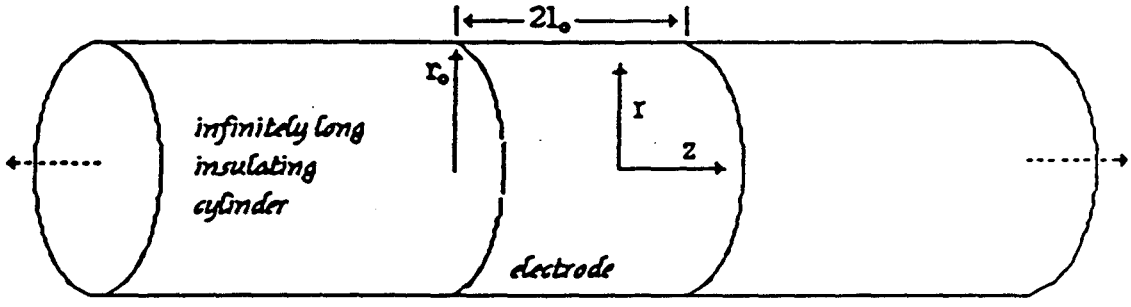
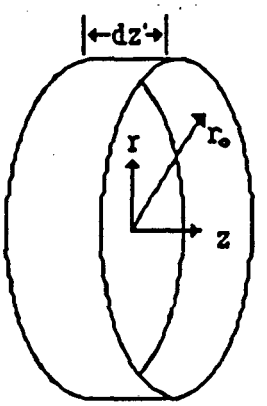


fig. 2a: (above) Differential Ring Current Source

fig. 2b: (below) Differential Ring Current Source Imbedded in an Infinite Insulating Cylinder



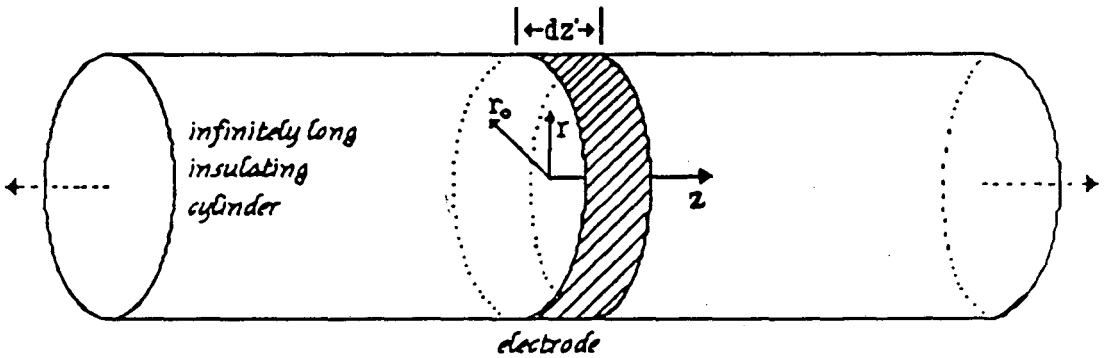


fig. 3: Current Distribution Parameter Range of Results

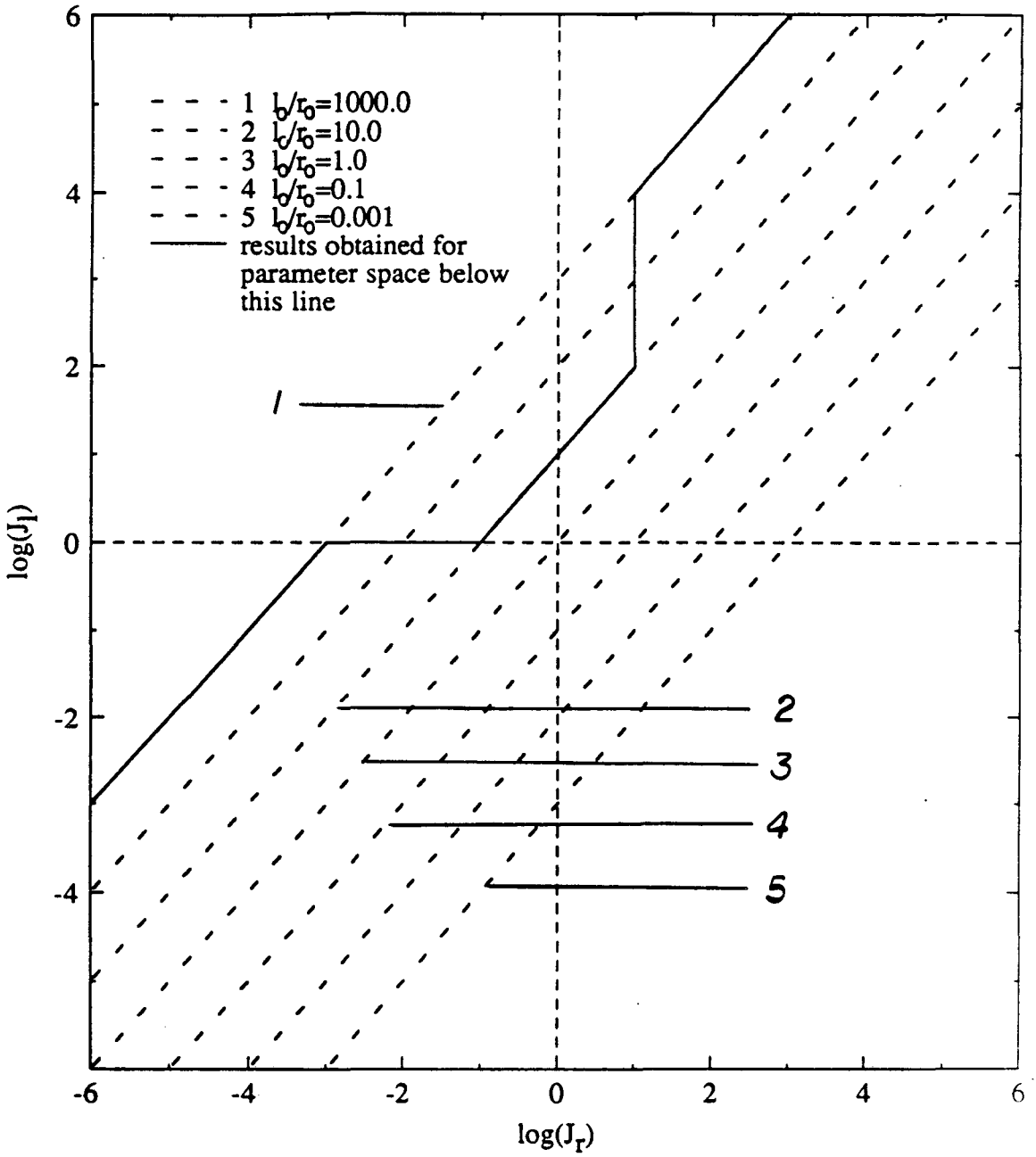


fig. 4: Primary Current Distribution

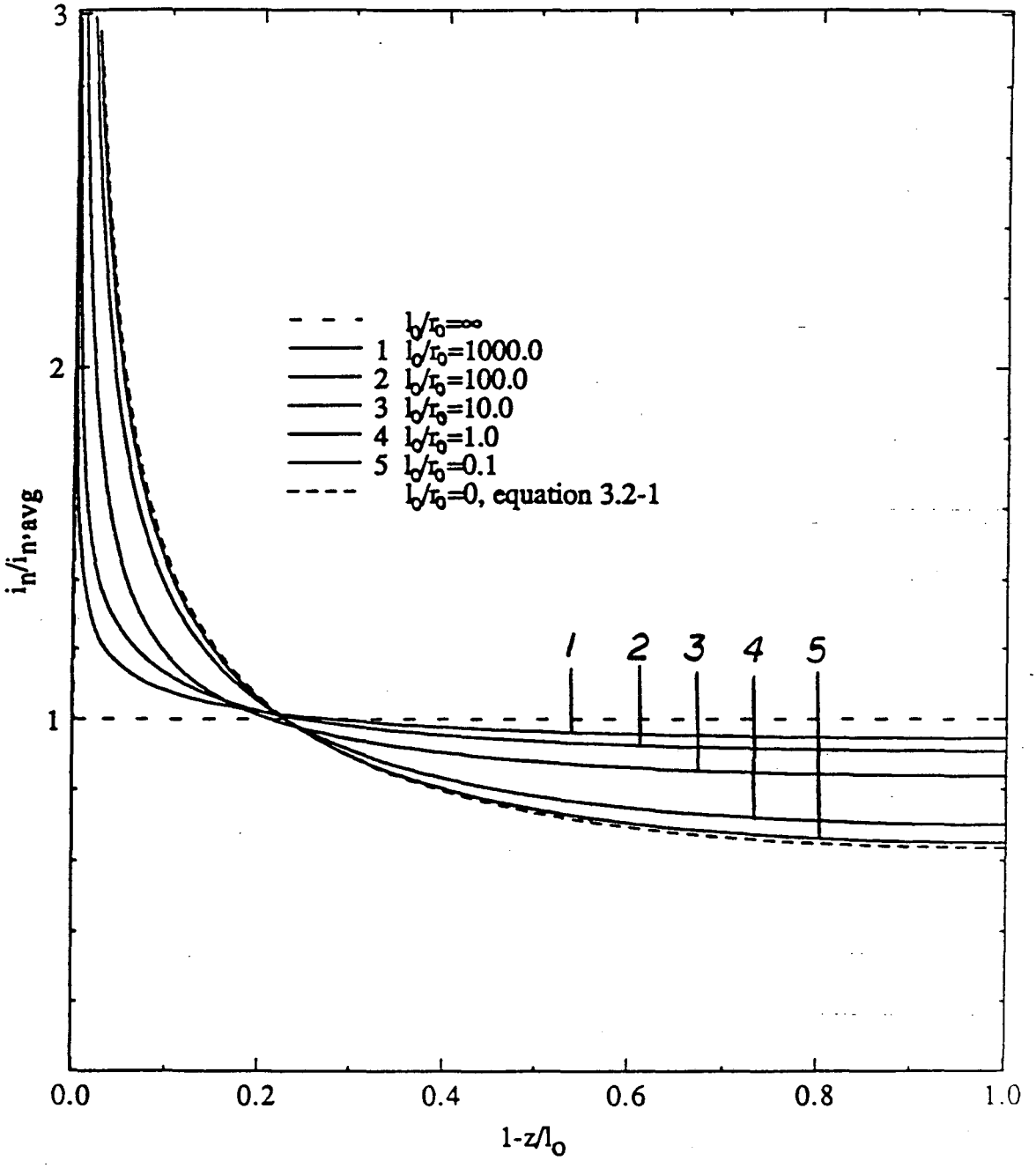


fig. 5: Primary Current Distribution Asymptotic Coefficients

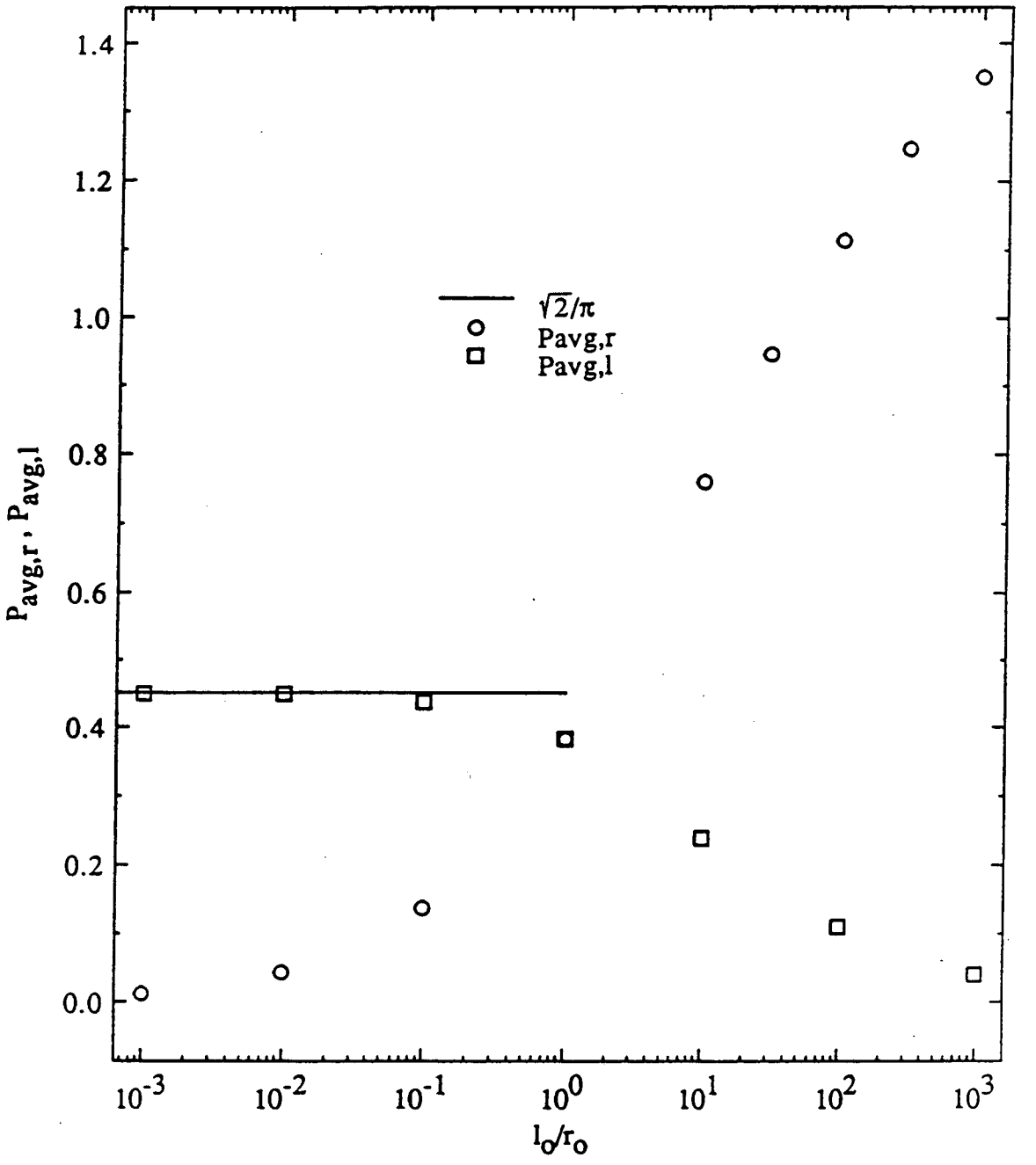


fig. 6: Ohmic Resistance

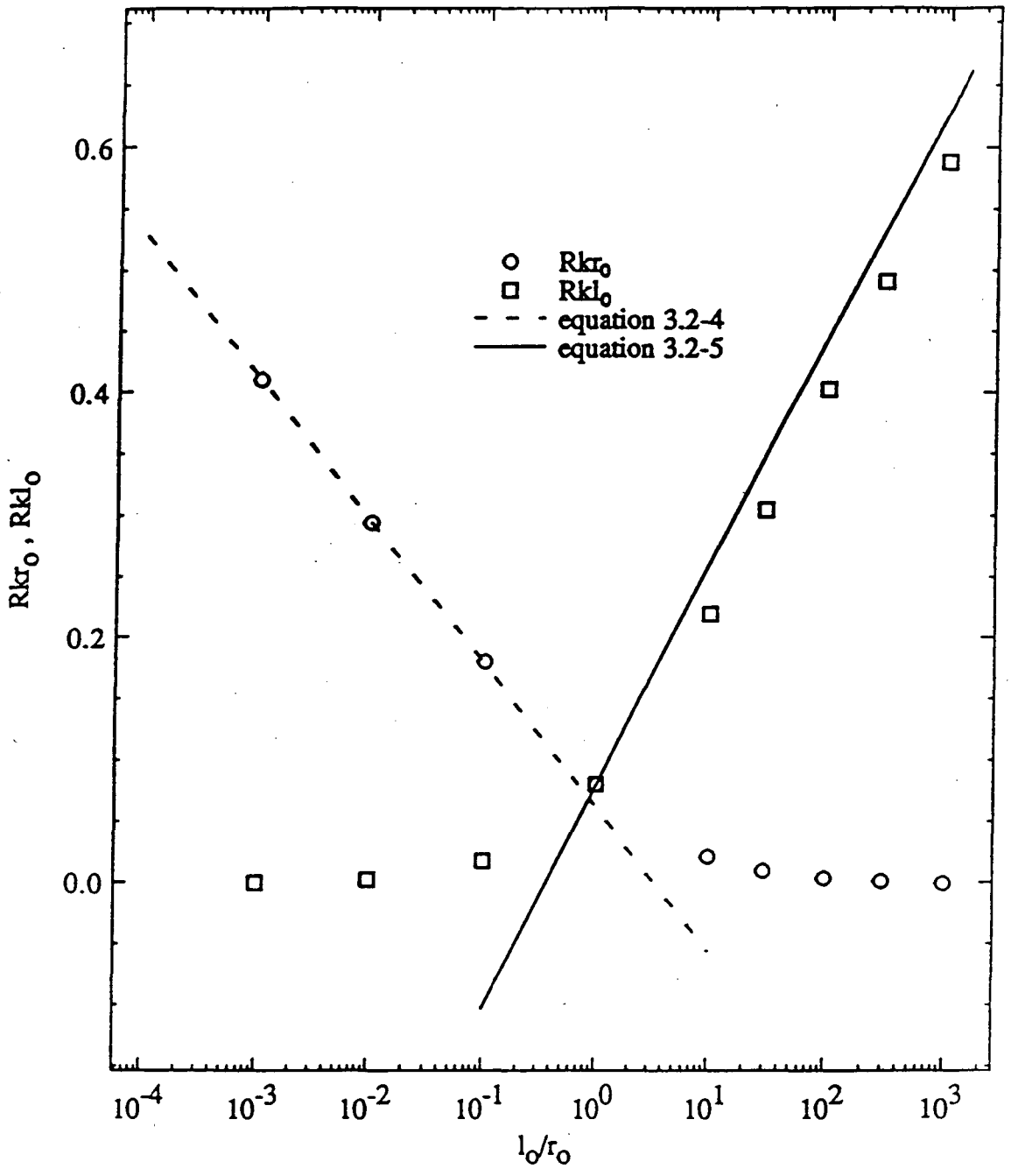


fig. 7: Asymptotic Regions in Current Distribution Parameter Space

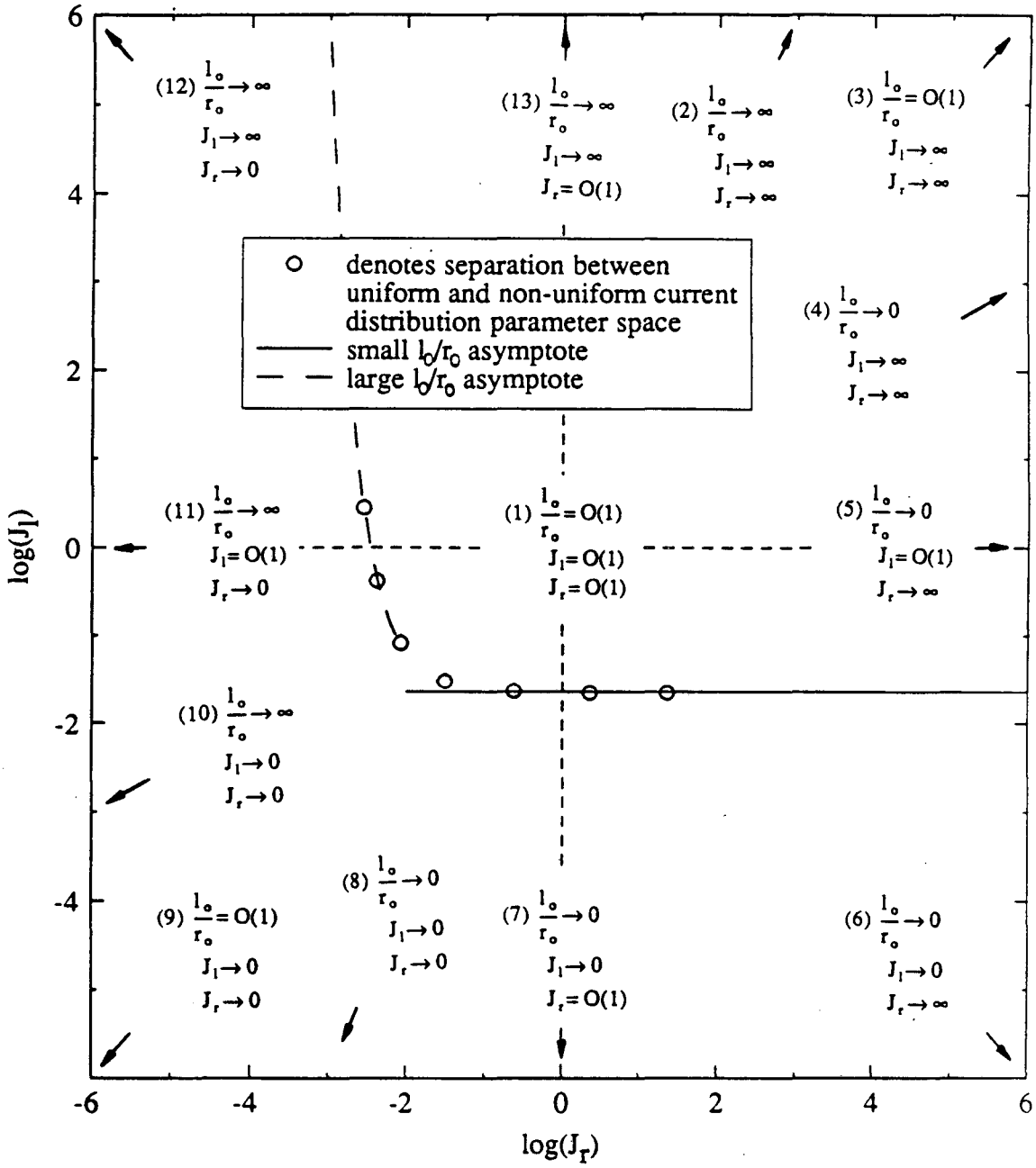


fig. 8: Secondary Current Distribution, $l_o/r_o = 1.0$

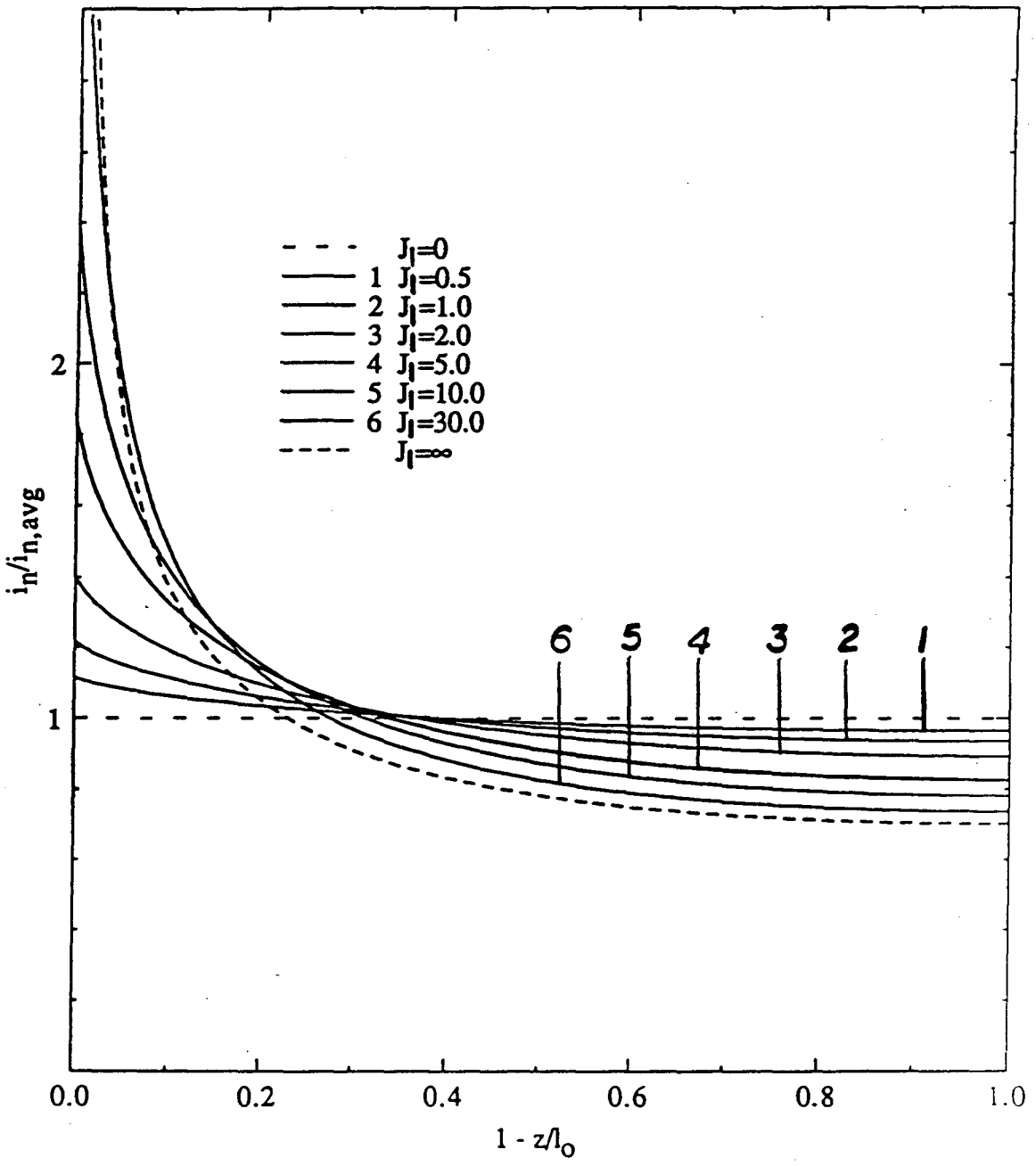


fig. 9: Current Distribution Parameter Space

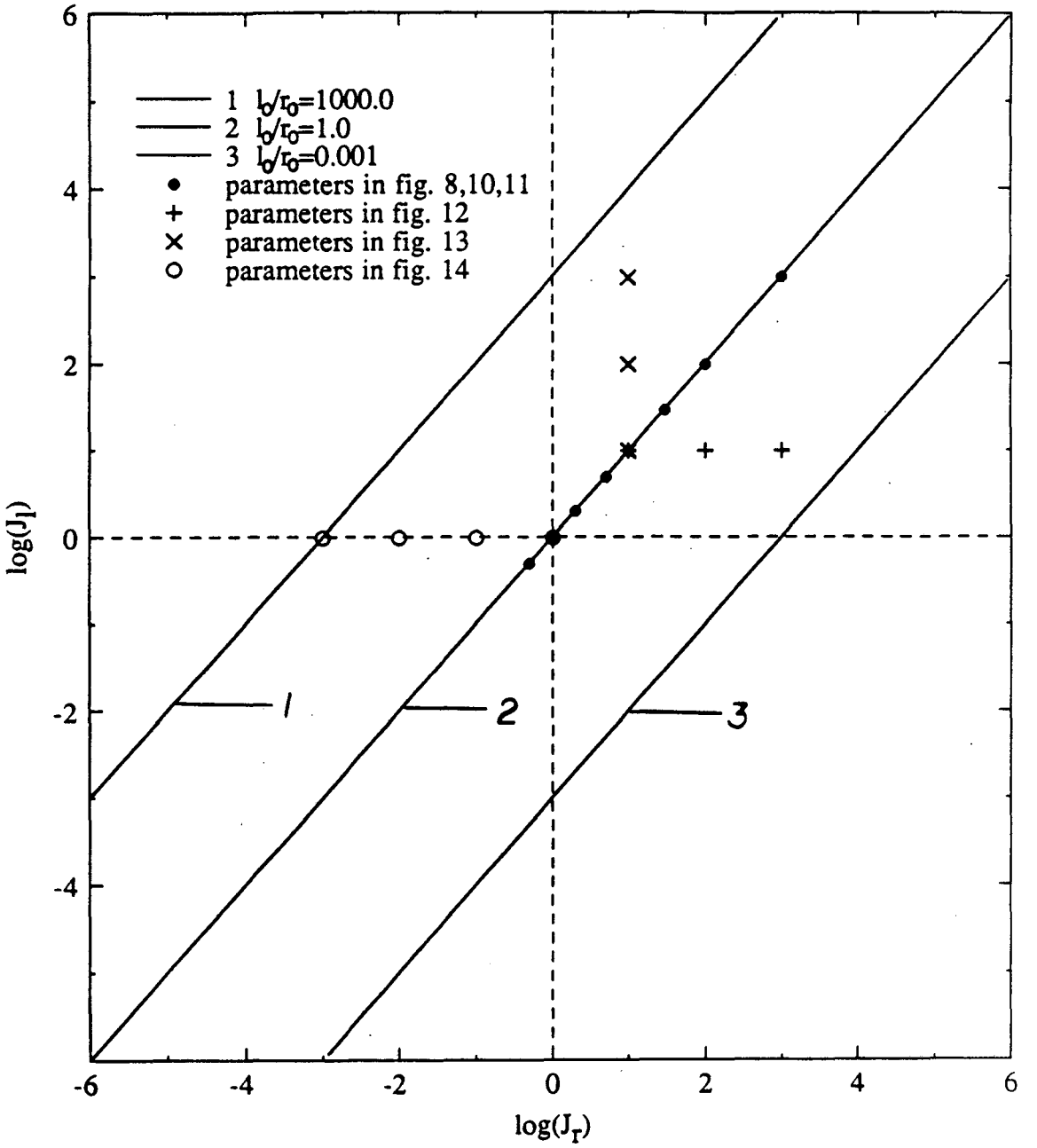


fig. 10: Secondary Current Distribution for Large J_l, $l_o/r_o = 1.0$

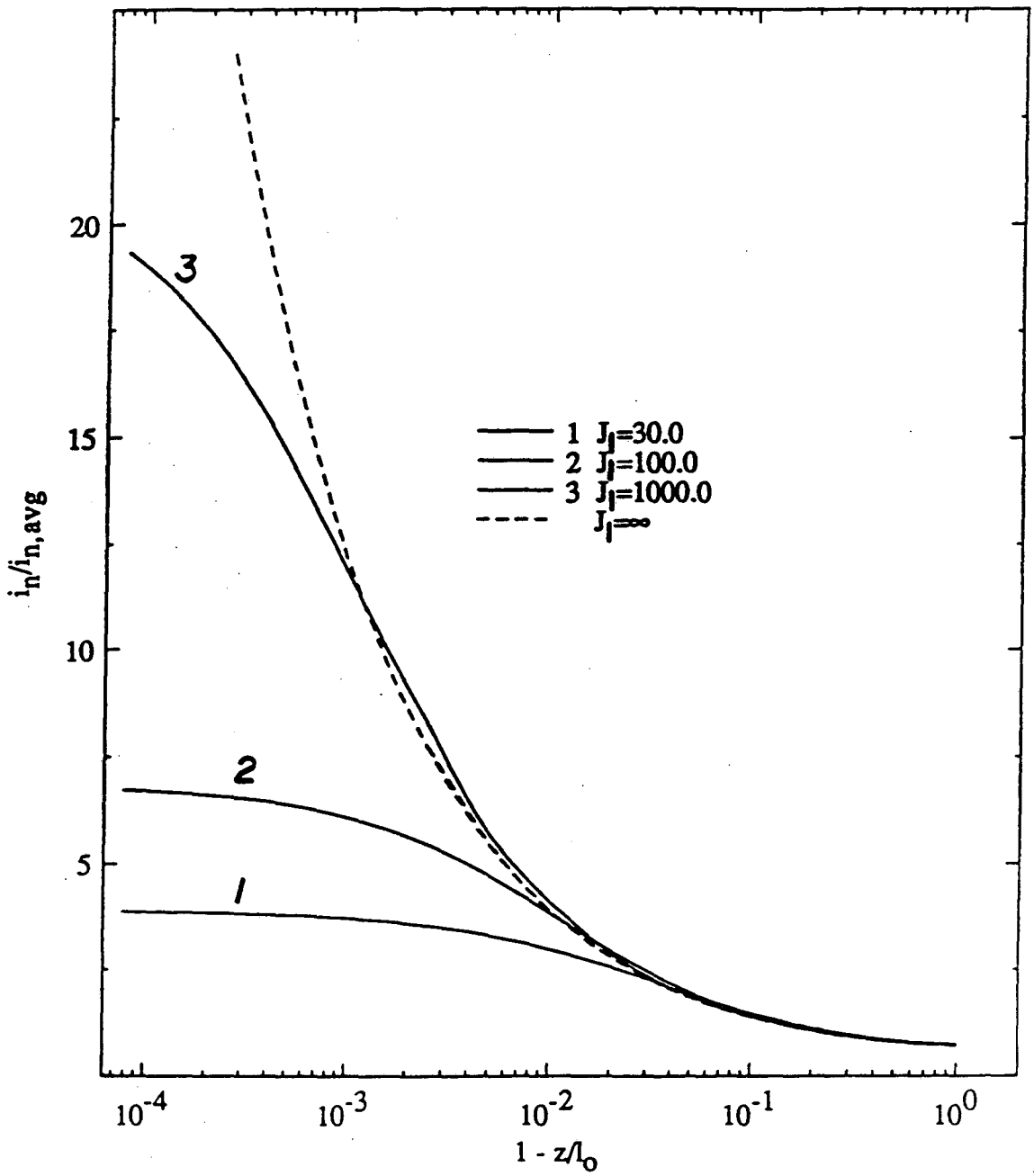


fig. 11: Secondary Current Distribution for Large J_l, $l_o/r_o = 1.0$

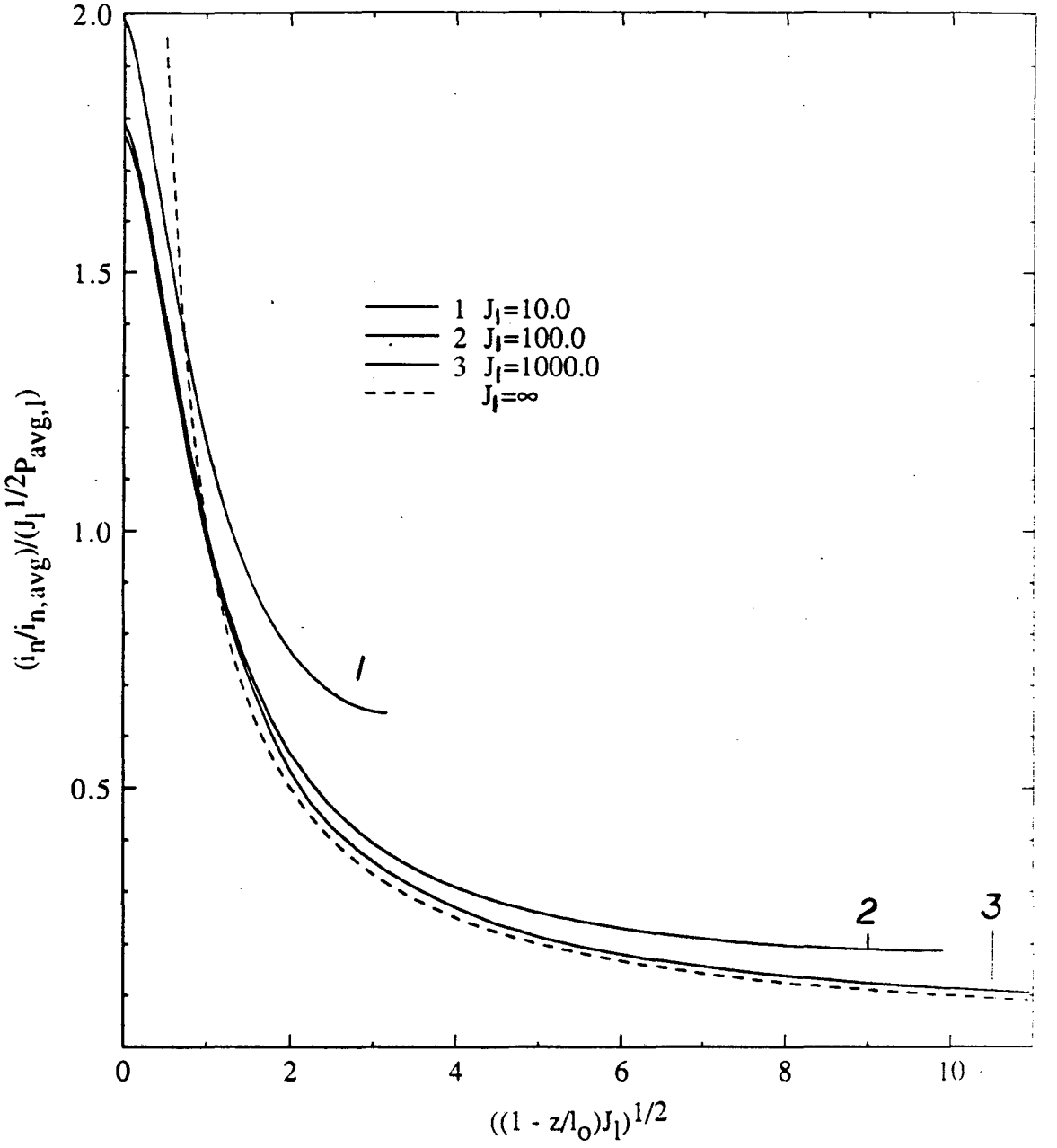


fig. 12: Secondary Current Distribution for Small l_o/r_o, J_l= 10.0

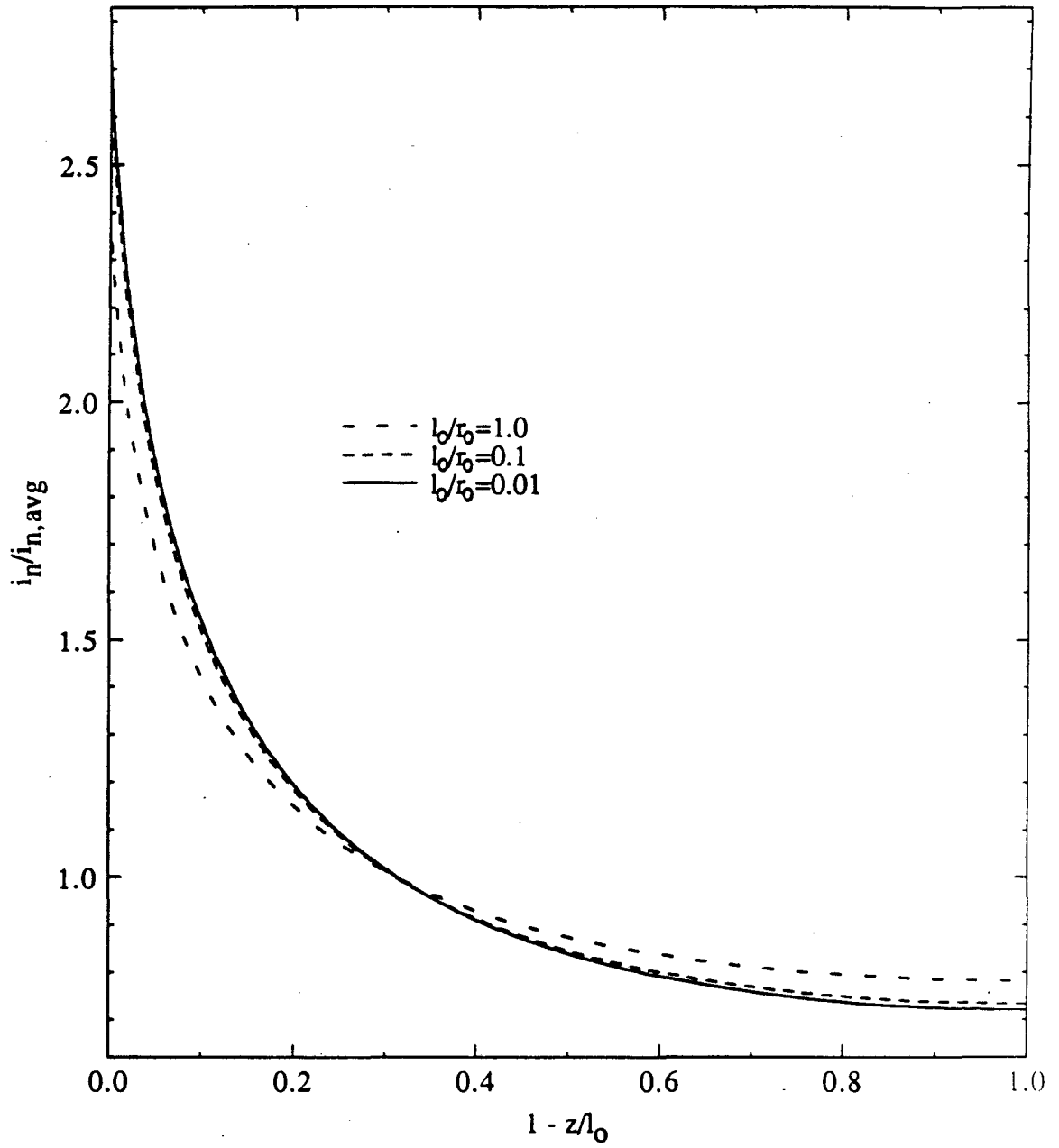


fig. 13: Secondary Current Distribution for Large l_o/r_o, J_r= 10.0

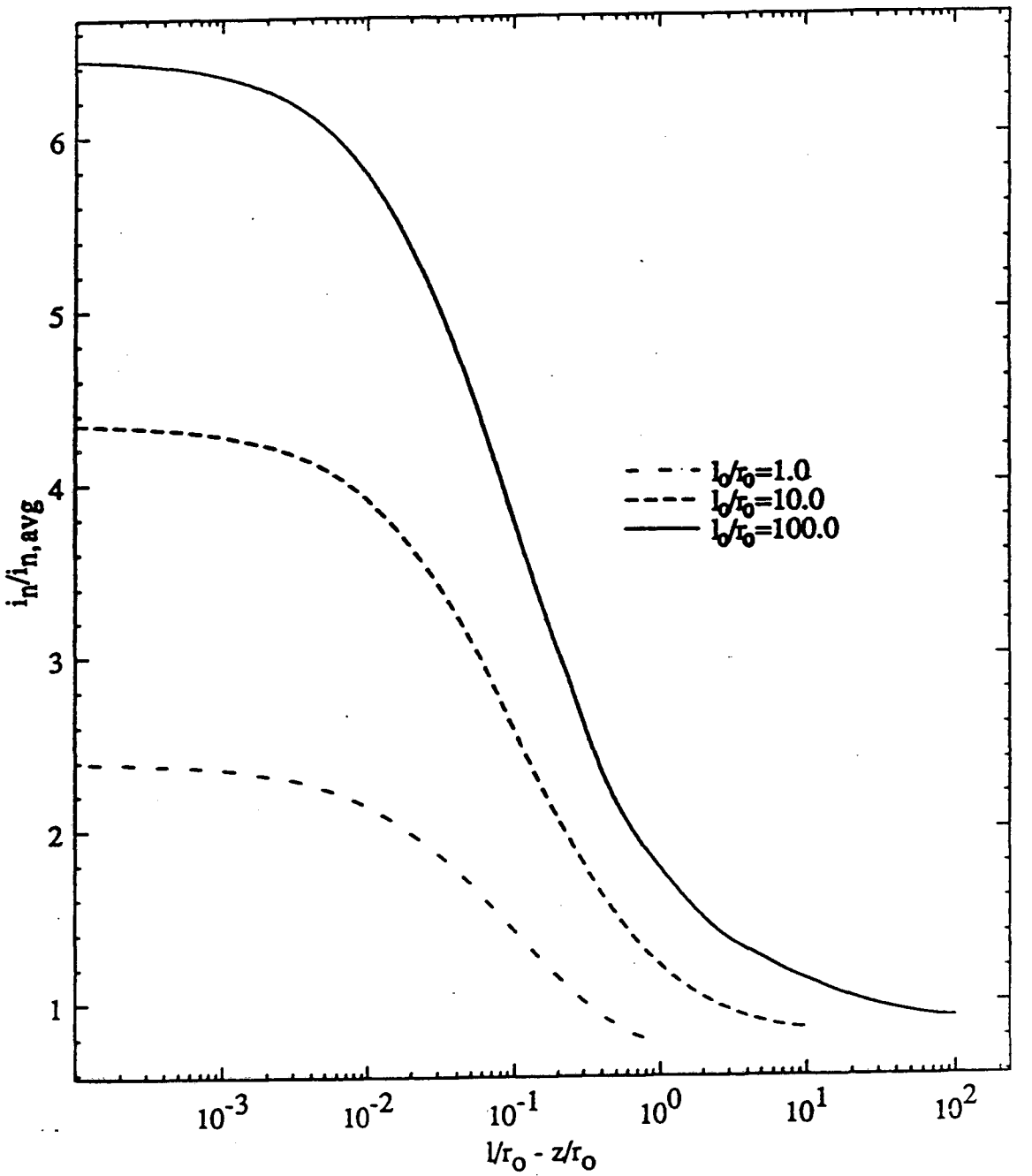


fig. 14: Secondary Current Distribution for Small J_r, J_l= 1.0

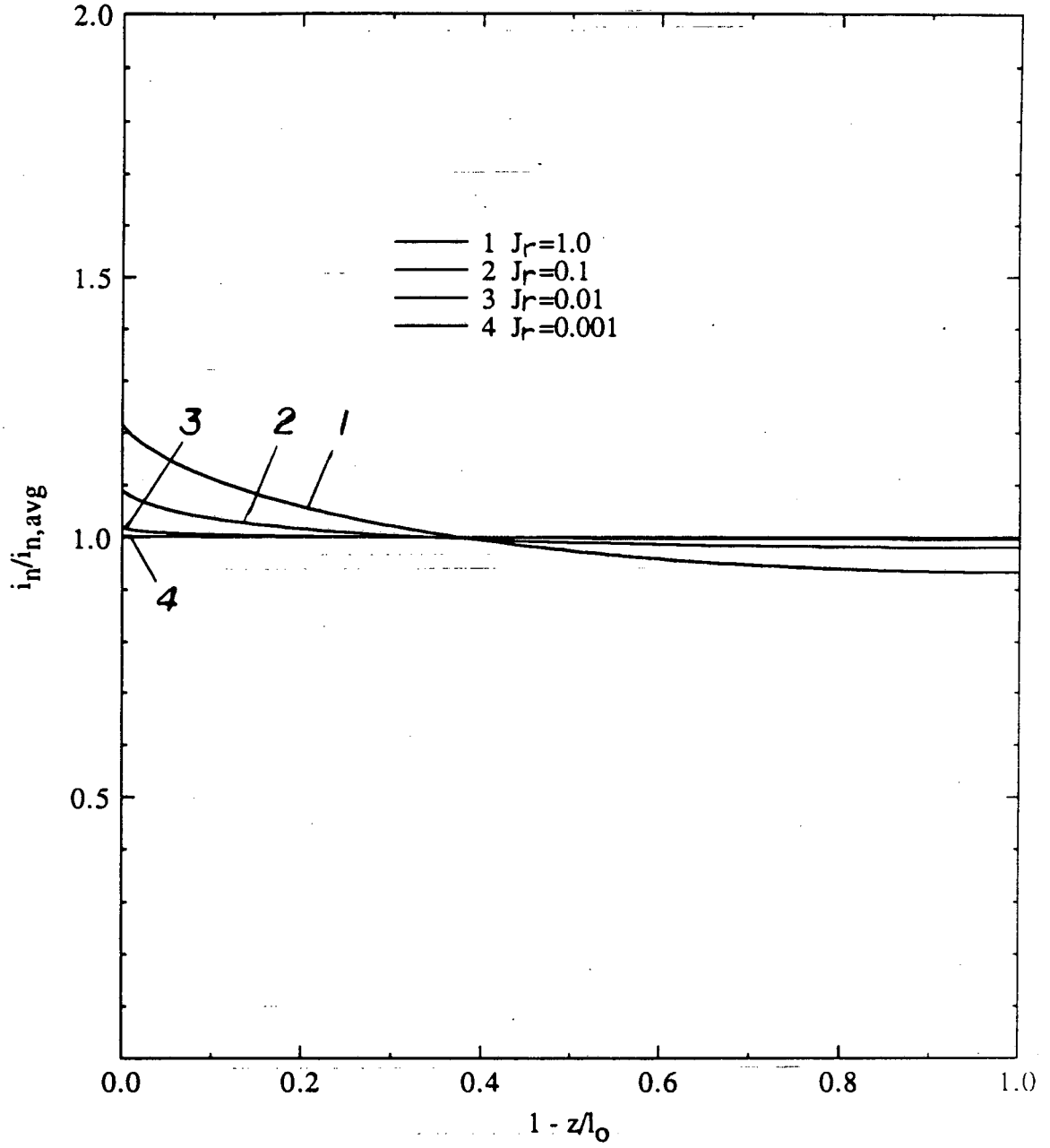


fig. 15: Potential Distribution for Uniform i_n and Small l_o/r_o

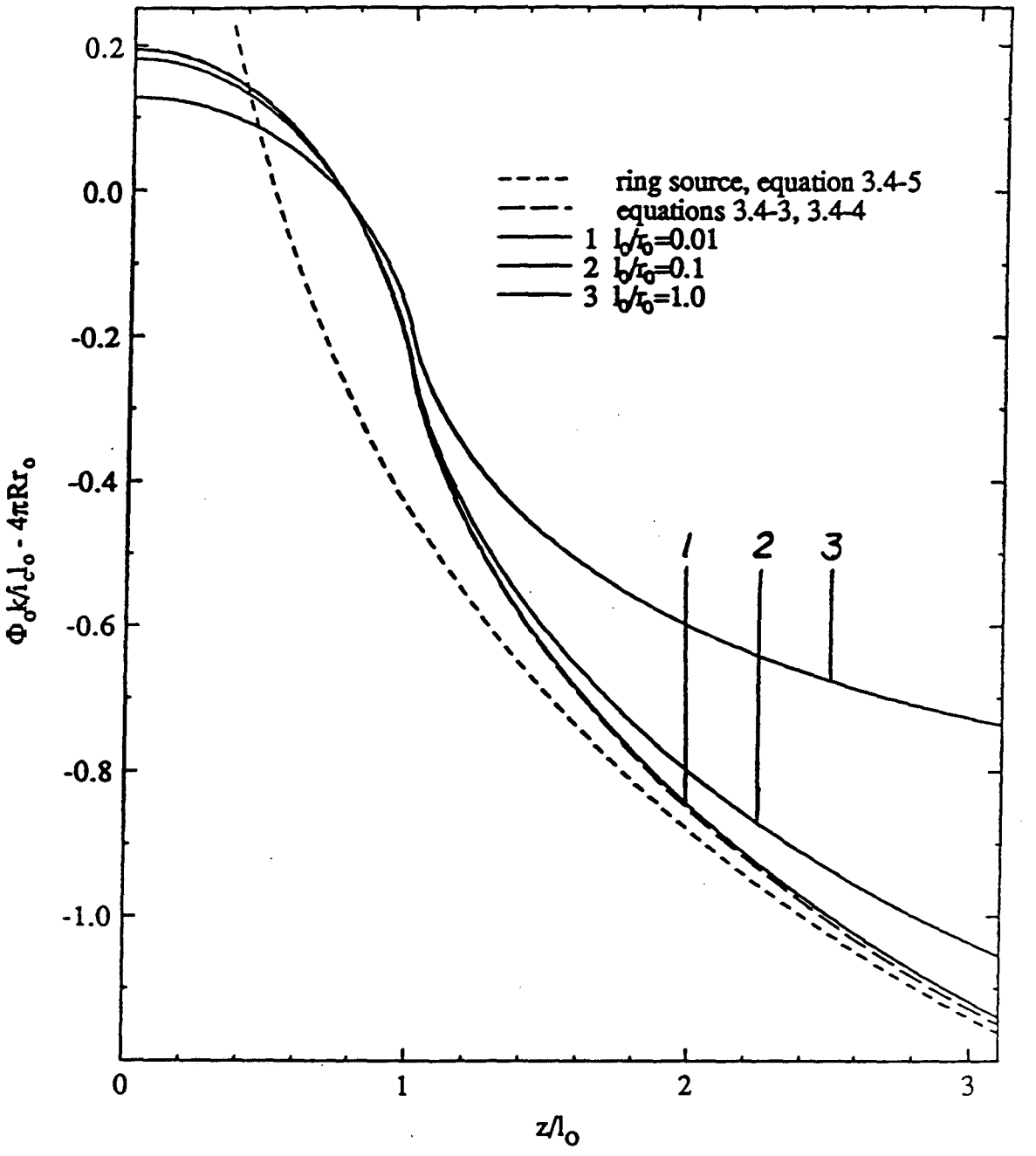


fig. 16: Maximum Potential Difference on the Electrode for Uniform i_n

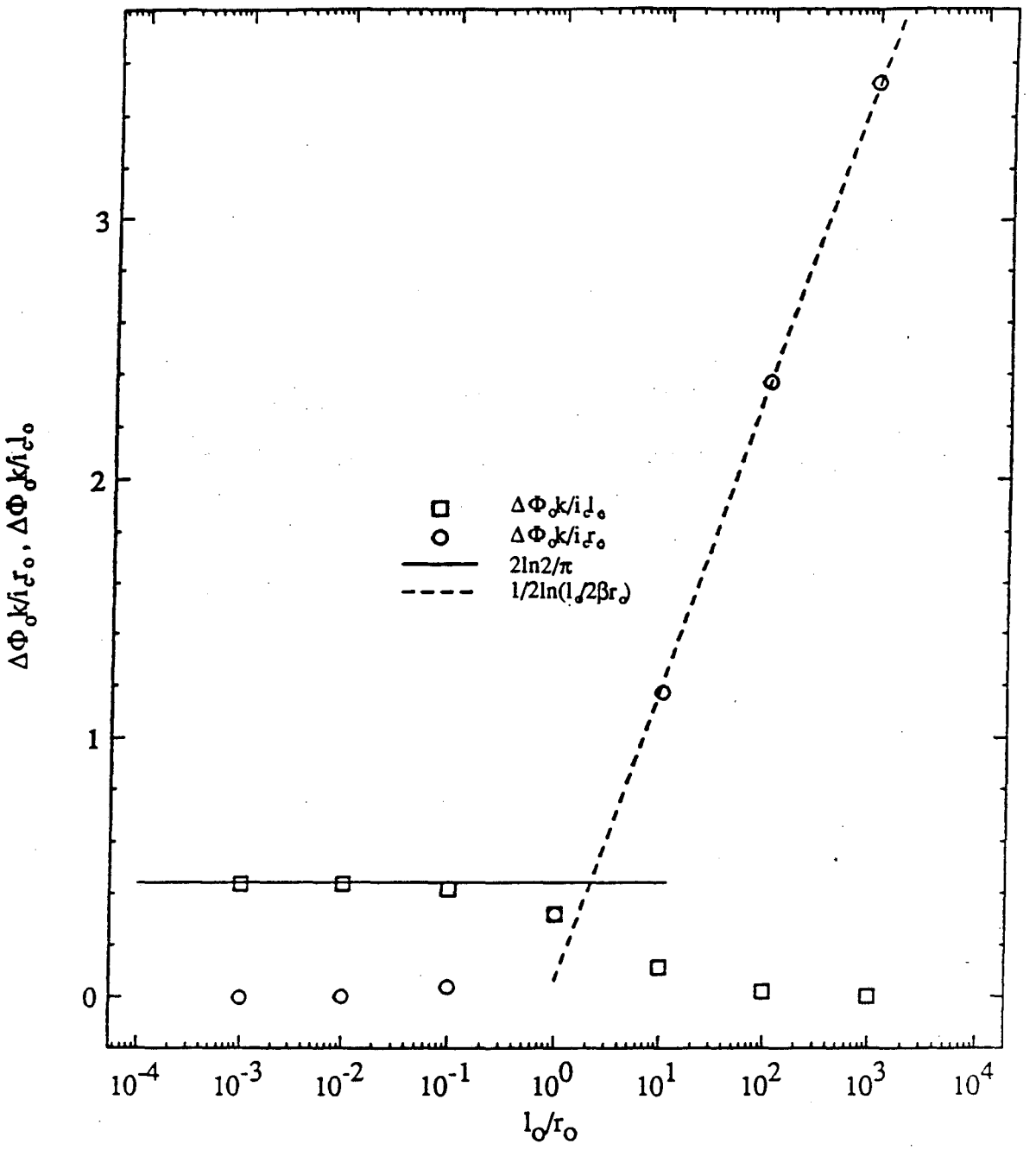


fig 17: Potential Distribution on the Electrode for Uniform i_n and Large l_o/r_o

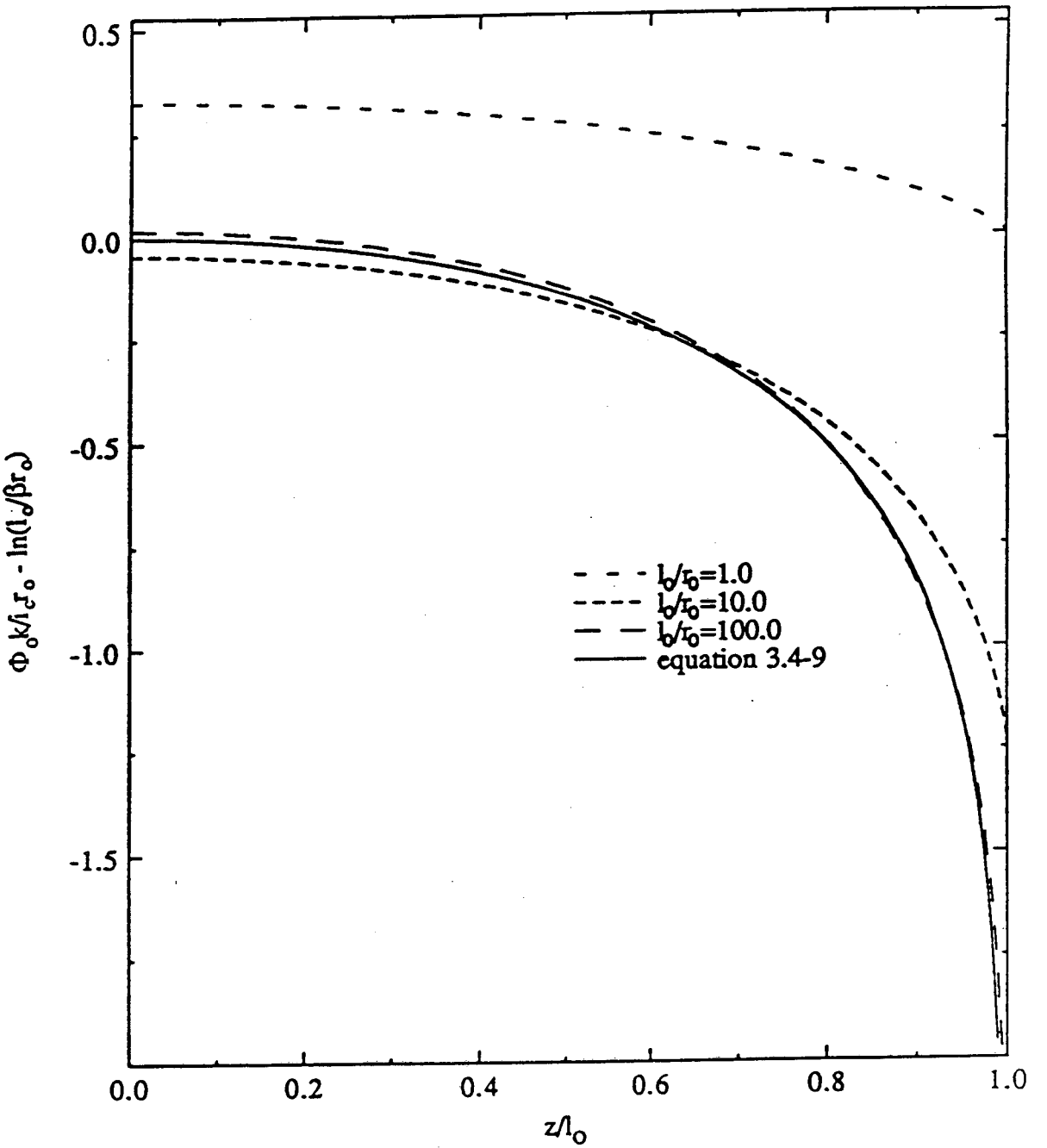


fig 18: Potential Distribution on the Edge Region for Uniform i_n and Large l_o/r_o

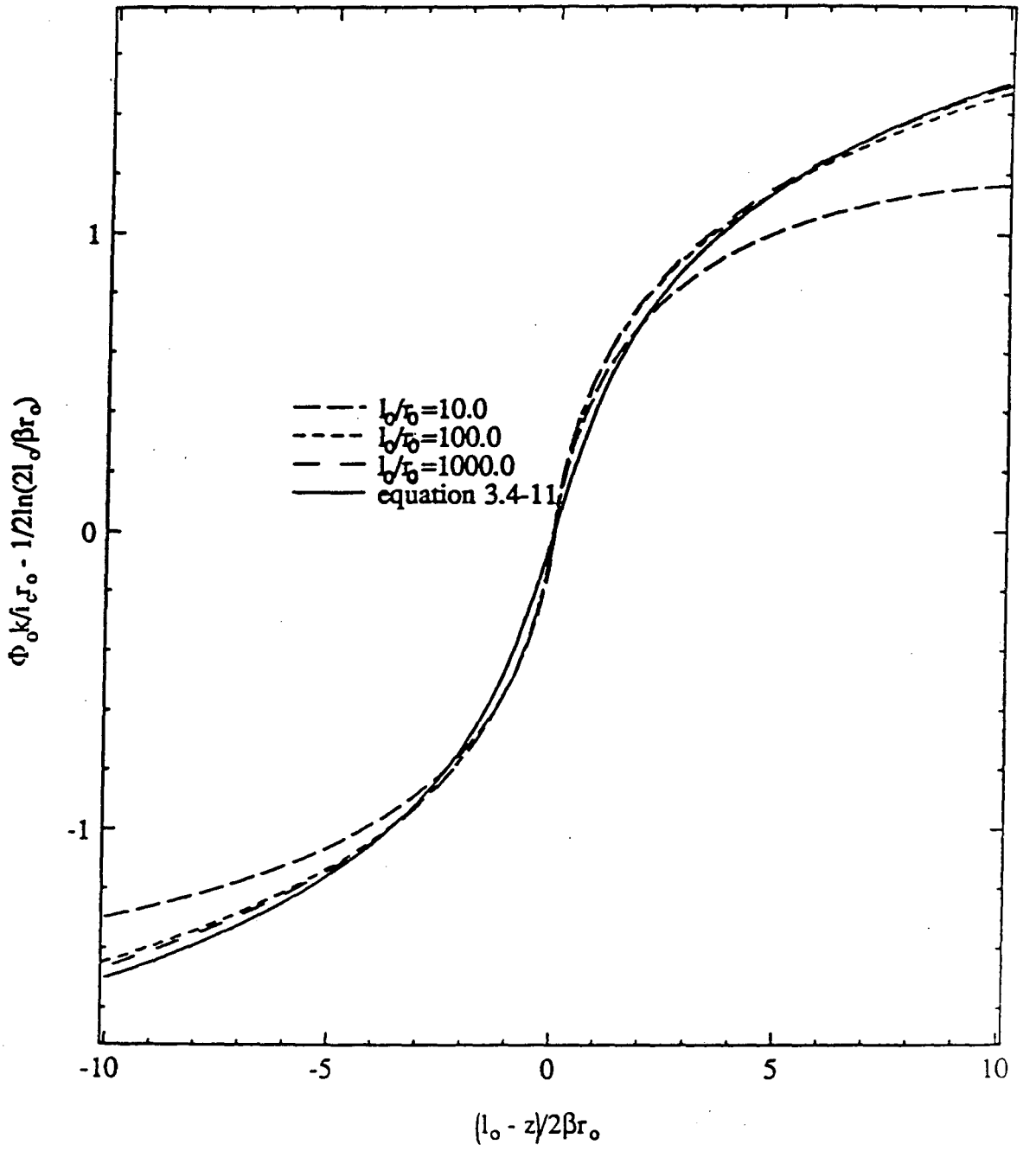
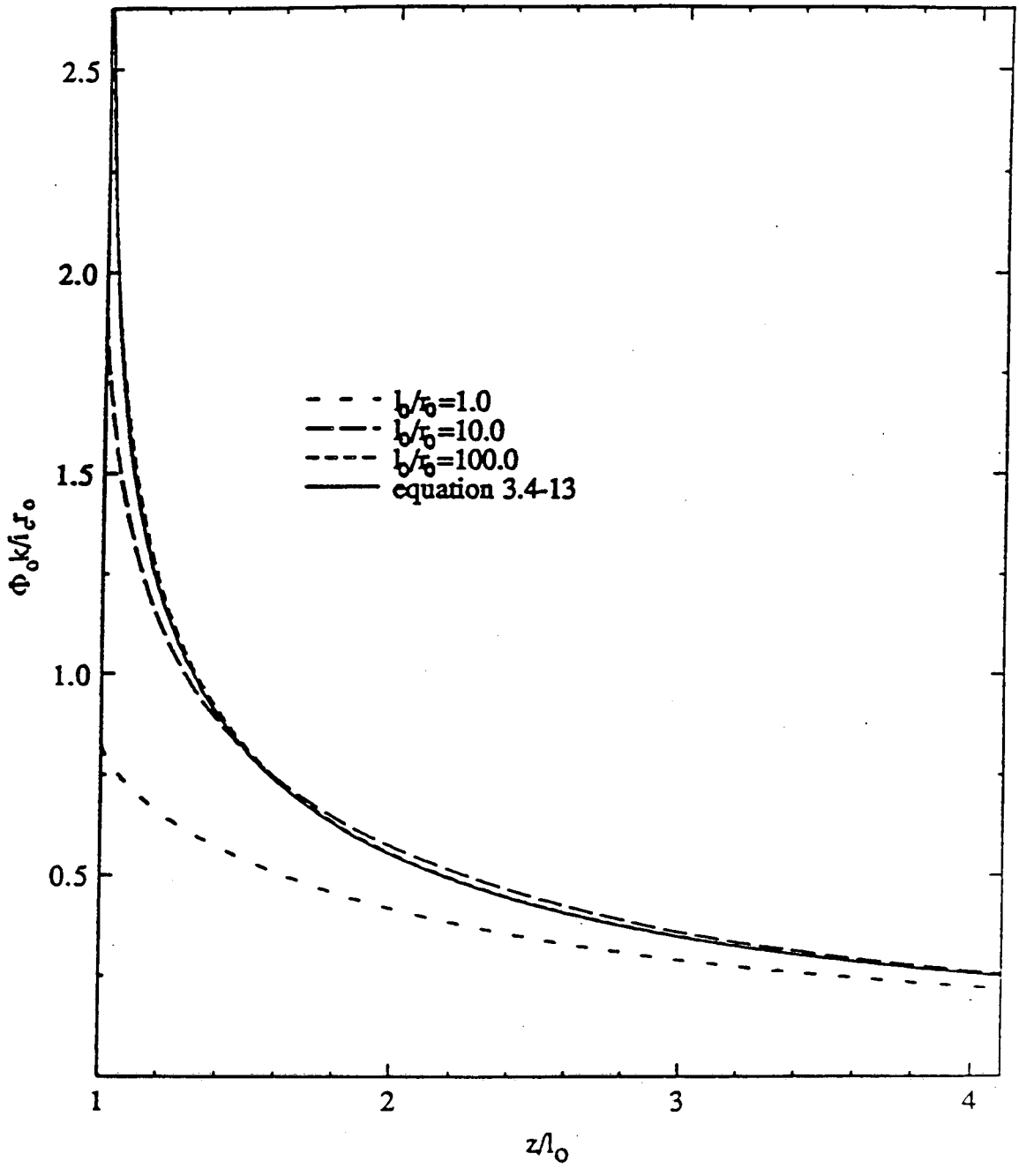


fig 19: Potential Distribution on the Insulator for Uniform i_n and Large l_o/r_o



NOMENCLATURE

a_j	dimensionless current function coefficient
b	dimensionless exponent
B	dimensionless ring source correction coefficient
F	dimensionless ring source correction function
F	Faraday's constant, 96,487 coulombs/equivalent
G	dimensionless ring source potential
G*	dimensionless ring source potential, includes F
h	important geometric length, cm
i	current density, A/cm^2
i_c	constant current density on the electrode surface, A/cm^2
i_{center}	current density at the electrode center, A/cm^2
i_n	current density on the electrode surface, A/cm^2
$i_{n,avg}$	average current density on the electrode surface, A/cm^2
$i_{n,edge}$	current density at the electrode edge, A/cm^2
i_o	exchange current density, A/cm^2
J_h	dimensionless polarization parameter containing geometric length h
J_l	dimensionless polarization parameter containing geometric length l_o
J_r	dimensionless polarization parameter containing geometric length r_o
K(m)	elliptic integral of the second kind
K_o	modified Bessel function of the second kind of order 0
l_o	electrode center to edge distance , cm
m	dimensionless elliptic integral variable
$P_{avg,l}$	dimensionless primary distribution asymptotic edge coefficient, when the edge region is stretched with l_o
$P_{avg,r}$	dimensionless primary distribution asymptotic edge region coefficient, when the edge region is stretched with r_o
P_{2j}	even Legendre Polynomial of order 2j
P_o	primary distribution asymptotic edge region coefficient, $A/cm^{(1+\pi/2\alpha)}$
P_r	dimensionless primary distribution asymptotic edge region coefficient, when the edge region is stretched with r_o
r	radial distance coordinate, cm
r_o	electrode radius, cm

R	electrolyte ohmic resistance, ohm
$\mathbf{R}$	universal gas constant, 8.3143 J/mol-K
R^*	dimensionless radial distance coordinate
T	absolute temperature, K
V	electrode potential, V
x	distance from the electrode edge, cm
x^*	dimensionless stretched distance from the electrode edge
y'	dimensionless stretched distance variable in equation 2.2-20
z	axial distance coordinate, cm
z'	axial location of ring source, cm
Z	dimensionless axial distance from ring source
Z'	dimensionless axial ring source location
Z^*	dimensionless axial distance
α	insulator-electrode intersection angle, radians
α_a, α_c	kinetic transfer coefficient for the anodic or cathodic reaction
β	dimensionless parameter characterizing ring source behavior near the ring
ζ	separation of variables constant
η_s	surface overpotential, V
κ	electrolyte conductivity, $ohm^{-1}cm^{-1}$
σ	edge region where finite electrode kinetics limit the electrode current density for high polarization parameters, cm
$d\Phi$	differential electric potential, V
Φ	electric potential, V
Φ^*	dimensionless electric potential
Φ_o	electric potential on the electrode or insulator, V

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APPENDIX A: CALCULATION OF F

F must satisfy Laplaces's equation in cylindrical coordinates.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) + \frac{\partial^2 F}{\partial z^2} = 0. \qquad \text{A-1}$$

Or equivalently, in dimensionless coordinates:

$$\frac{1}{R^*}\frac{\partial}{R^*}\left(R^*\frac{\partial F}{\partial R^*}\right) + \frac{\partial^2 F}{\partial Z^2} = 0. \qquad \text{A-2}$$

Using the separation of variables method, one assumes; F=X(R*)Y(Z). Equation A-2 becomes:

$$\frac{1}{R^*}\frac{1}{X}\frac{\partial}{R^*}\left(R^*\frac{\partial X}{\partial R^*}\right) = -\frac{1}{Y}\frac{\partial^2 Y}{\partial Z^2} = \zeta^2. \qquad \text{A-3}$$

Where ζ must be a constant.

The solution to the differential equation for Y is:

$$Y = c_1\cos(\zeta Z) + c_2\sin(\zeta Z). \qquad \text{A-4}$$

Y must be symmetrical about Z=0 so c_2 must be 0. The solution for X is:

$$X = c_3 K_o(\zeta R^*) + c_4 I_o(\zeta R^*) . \qquad \text{A-5}$$

K_O is the modified Bessel function of the first kind of order zero and I_O is the modified Bessel function of the second kind of order zero. As R approaches infinity, I_O

approaches infinity exponentially and K_0 approaches zero exponentially. The potential must be zero at infinity so c_4 is zero.

Next the solutions to equation A-3 are integrated to create a general solution.

$$F(R^*,Z)=\int_0^\infty B(\zeta)\, K_0(\zeta R^*)\cos(\zeta Z)\, d\zeta \qquad 2.1\text{-}8$$

The insulator boundary condition determines $B(\zeta)$.

$$\frac{\partial G^*}{\partial R^*}=\frac{\partial G}{\partial R^*}+\frac{\partial F}{\partial R^*}=0 \quad \text{at } R^*=1 \;\; Z\neq 0 \qquad \text{A-6}$$

Using equations 2.1-8, A-6, 2.1-3 and the following identity:

$$\frac{\partial K(m)}{dm}=-\frac{K(m)}{2m}+\frac{E(m)}{2m(1-m)}, \qquad \text{A-7}$$

where E(m) is the elliptical integral of the second kind, one finds:

$$\frac{\partial F}{\partial R^*}=-\int_0^\infty B(\zeta)\zeta K_1(\zeta)\cos(\zeta Z)\, d\zeta=-\frac{\partial G}{\partial R^*}=\frac{K\left(\frac{4}{4+Z^2}\right)-E\left(\frac{4}{4+Z^2}\right)}{\left(4+Z^2\right)^{\frac{1}{2}}} \quad \text{at } R^*=1. \qquad \text{A-8}$$

Where $K_1(\zeta)$ is the modified Bessel function of the second kind of order 1. Equation A-8 is in the form of a Fourier cosine integral which can be inverted to obtain $B(\zeta)$.

$$B(\zeta)=\frac{2}{\pi\zeta\, K_1(\zeta)}\int_0^\infty \frac{E\left(\frac{4}{t^2+4}\right)-K\left(\frac{4}{t^2+4}\right)}{\left(t^2+4\right)^{\frac{1}{2}}}\cos(\zeta t)\, dt \qquad \text{A-9}$$

Mak integrated A-9 and 2.1-8 numerically to obtain F values on the surface of the cylinder. Their results were fit with the following function:

$$F = -\frac{0.5\pi}{1+|Z|} \pm .0001 \quad 0.003 > Z > 20.0 \qquad \text{A-10a}$$

$$F = \frac{1}{\pi}\left(-\frac{0.5}{1+|Z|} - 1.1083 + .29872\Theta - 0.63683\Theta^2 \right. \qquad \text{A-10b}$$

$$\left. + 1.5519\Theta^3 - 2.4429\Theta^4 + 1.2281\Theta^5 \right) \pm 0.0001 \ , \quad 0.003 \leq Z \leq 20$$

where;

$$\Theta = \frac{|Z|}{1-|Z|}. \qquad \text{A-11}$$

APPENDIX B: INTEGRATING SINGULARITIES

To integrate equations 2.2-13, 2.2-14, 2.2-20, and Legendre polynomials as current functions in equation 2.2-9, one must deal with the electrode edge singularities in equations 2.2-13, 2.2-14, 2.2-20, and the logarithmic singularity in K(m) at $Z'=Z^*$. The following change of variables renders equations 2.2-13, 2.2-14 and 2.2-20 numerically integrable:

$$U'=\sqrt{\left(2\frac{l_o}{r_o}\right)^{\frac{1}{2}}-\left(\frac{l_o}{r_o}-Z'\right)^{\frac{1}{2}}} \qquad \text{B-1}$$

In general, the logarithmic singularity inherent in K(m) was subtracted out and integrated analytically.

Substituting equation 2.2-13 in equation 2.2-5, we have:

$$\Phi^* = \frac{1}{\pi}\int_{-\frac{l_o}{r_o}}^{\frac{l_o}{r_o}} G^*(R^*=1,Z^*-Z')\frac{1}{\frac{r_o}{l_o}\sqrt{\left(\frac{l_o}{r_o}\right)^2-Z'^2}}dZ' \ . \qquad \text{B-2}$$

Substituting equation B-1 into equation B-2, subtracting out the limiting behavior of K(m) embodied in equation 2.1-11, and using the identity

$$\int_{-1}^{1}\frac{\ln(x-x')^2}{\sqrt{1-x'^2}}dx' = -2\pi\ln 2 \ , \qquad \text{B-3}$$

one obtains the following numerically integrable expression:

$$\Phi^*_{-1} = \frac{4}{\pi}\int_0^{\left(2\frac{l_o}{r_o}\right)^{\frac{1}{4}}} \frac{G^* + \frac{1}{2}\ln(Z^*-Z')^2}{\frac{r_o}{l_o}\sqrt{2\left(2\frac{l_o}{r_o}\right)^{\frac{1}{2}} - U'^2}}\, dU' \; - \frac{1\, l_o}{2\, r_o}\left(2\ln\left(\frac{l_o}{r_o}\right) - 2\ln 2\right) \quad . \qquad \text{B-4}$$

Substituting equation 2.2-14 into equation 2.2-5 yields:

$$\Phi^* = \frac{1}{\pi}\int_{-\frac{l_o}{r_o}}^{\frac{l_o}{r_o}} G^*(R^*=1, Z^*-Z')\left(i_{const} + \left(\frac{P_r\sqrt{2\,{}^{l_o}/_{r_o}}}{\sqrt{\left({}^{l_o}/_{r_o}\right)^2 - Z'^2}}\right)^b\right)^{\frac{1}{b}} dZ' \;, \qquad \text{B-5}$$

where i_{const} is

$$i_{const} = \left(\frac{r_0 i_{center}}{Vk}\right)^b - \left(\frac{P_r\sqrt{2}}{\sqrt{{}^{l_o}/_{r_o}}}\right)^b . \qquad \text{B-6}$$

By substituting equation B-1 into equation B-5, then subtracting out equation 2.1-11, multiplied by the now numerically integrable current expression evaluated at Z*, one obtains the following numerically integrable expression for equation B-5.

$$\Phi^*_{-1} = \frac{4}{\pi}\int_0^{\left(2\frac{l_o}{r_o}\right)^{\frac{1}{4}}} G^* i_{U'} - \left(3\ln 2 - \frac{\pi}{2} - \frac{1}{2}\ln(Z^* - Z')^2\right) i_{U^*}\; dU'$$

$$+\frac{4}{\pi} i_{U^*}\left(\left(3\ln 2-\frac{\pi}{2}\right)\left(2\frac{l_o}{r_o}\right)^{1/4}-\left(\left(2\frac{l_o}{r_o}\right)^{1/4}-U^*\right)\ln\left(\left(2\frac{l_o}{r_o}\right)^{1/4}-U^*\right)\right.$$

$$-\left(\left(2\frac{l_o}{r_o}\right)^{1/4}+U^*\right)\ln\left(\left(2\frac{l_o}{r_o}\right)^{1/4}+U^*\right)$$

$$+\left(\sqrt{2\left(2\frac{l_o}{r_o}\right)^{1/2}-U^{*2}}-\left(2\frac{l_o}{r_o}\right)^{1/4}\right)\ln\left(\sqrt{2\left(2\frac{l_o}{r_o}\right)^{1/2}-U^{*2}}-\left(2\frac{l_o}{r_o}\right)^{1/4}\right)$$

$$-\left(\left(2\frac{l_o}{r_o}\right)^{1/4}+\sqrt{2\left(2\frac{l_o}{r_o}\right)^{1/2}-U^{*2}}\right)\ln\left(\left(2\frac{l_o}{r_o}\right)^{1/4}+\sqrt{2\left(2\frac{l_o}{r_o}\right)^{1/2}-U^{*2}}\right)$$

$$\left.+4\left(2\frac{l_o}{r_o}\right)^{1/4}\right) \qquad \text{B-7}$$

In this equation, U* is given by equation B-1 with Z* replacing Z′, $i_{U'}$ is expressed as:

$$i_{U'}=\left(\left(\left(2\frac{l_o}{r_o}\right)^{1/2}-U'^2\right)^b U'^b i_{const}+\left(\frac{P_r\left(2\frac{l_o}{r_o}\right)^{1/2}}{\sqrt{2\left(2\frac{l_o}{r_o}\right)^{1/2}-U'^2}}\right)^b\right)^{\frac{1}{b}}, \qquad \text{B-8}$$

while equation B-8 with U′ replaced by U* gives i_{U^*}.

We can deal with the singularity in equation 2.2-20 in the same manner as the singularities in equation 2.2-14. When equation 2.2-20 is substituted into equation 2.2-5, one obtains:

$$\Phi^* = \frac{1}{\pi}\int_{-\frac{l_o}{r_o}}^{\frac{l_o}{r_o}} G^*(R^*=1, Z^*-Z')\left(i_{const,J} + \left(P_r J_r f_{univ}(y')\right)^b\right)^{\frac{1}{b}} dZ' \ . \qquad \text{B-9}$$

In this equation,

$$i_{const,J} = \left(\frac{r_0 i_{center}}{Vk}\right)^b - \left(P_r J_r^{1/2} f_{univ}\left(J_{1/2}\right)\right)^b \ . \qquad \text{B-10}$$

Substituting equation B-1 into equation B-9, then subtracting equation 2.1-11, multiplied by the (now well-behaved) current function evaluated at Z*, results in the following expression, which we can easily integrate numerically.

$$\Phi^*_{-1} = \frac{4}{\pi}\int_0^{\left(2\frac{l_o}{r_o}\right)^{\frac{1}{4}}} G^* i_{y'} - \left(3\ln 2 - \frac{\pi}{2} - \frac{1}{2}\ln(Z^* - Z')^2\right) i_{y^*} \ dU'$$

$$+ \frac{4}{\pi} i_{y^*}\left(3\ln 2 - \frac{\pi}{2}\right)\left(2\frac{l_o}{r_o}\right)^{1/4} - \left(\left(2\frac{l_o}{r_o}\right)^{1/4} - U^*\right)\ln\left(\left(2\frac{l_o}{r_o}\right)^{1/4} - U^*\right)$$

$$- \left(\left(2\frac{l_o}{r_o}\right)^{1/4} + U^*\right)\ln\left(\left(2\frac{l_o}{r_o}\right)^{1/4} + U^*\right)$$

$$+ \left(\sqrt{2\left(2\frac{l_o}{r_o}\right)^{1/2} - U^{*2}} - \left(2\frac{l_o}{r_o}\right)^{1/4}\right)\ln\left(\sqrt{2\left(2\frac{l_o}{r_o}\right)^{1/2} - U^{*2}} - \left(2\frac{l_o}{r_o}\right)^{1/4}\right)$$

$$- \left(\left(2\frac{l_o}{r_o}\right)^{1/4} + \sqrt{2\left(2\frac{l_o}{r_o}\right)^{1/2} - U^{*2}}\right)\ln\left(\left(2\frac{l_o}{r_o}\right)^{1/4} + \sqrt{2\left(2\frac{l_o}{r_o}\right)^{1/2} - U^{*2}}\right)$$

$$+ 4\left(2\frac{l_o}{r_o}\right)^{1/4} \quad) \qquad \text{B-11}$$

Here, y* is the same as y′ (equation 2.2-21) if Z′ is replaced with Z*. In addition, $i_{y'}$ is:

$$i_{y'} = \left(i_{const,J} + \left(P_r J_r f_{univ}(y')\right)^b\right)^{\frac{1}{b}} \left(\left(\frac{2l_o}{r_o}\right)^{1/2} - U'^2\right) U', \qquad \text{B-12}$$

while i_{y*} is given by equation B-12, with y′ and U′ replaced by y* and U*, respectively.

If we substitute a Legendre polynomial into equation 2.1-5, we obtain:

$$\Phi* = \frac{1}{\pi} \int_{-\frac{l_o}{r_o}}^{\frac{l_o}{r_o}} G*(R*=1, Z*-Z') P_{2j}\left(\frac{r_o}{l_o} Z'\right) dZ' . \qquad \text{B-13}$$

Since Legendre polynomials are well-behaved, we only have to deal with the G*'s logarithmic singularity in this case. If we subtract out equation 2.1-11, multiplied by P_{2j} evaluated at Z*, we obtain the following numerically integrable expression.

$$\Phi^*_{2j} = \frac{1}{\pi} \int_{-\frac{l_o}{r_o}}^{\frac{l_o}{r_o}} G P_{2j}\left(\frac{r_o}{l_o} Z'\right) + \frac{1}{2} \ln(Z*-Z')^2 P_{2j}\left(\frac{r_0}{l_o} Z*\right) \; dZ'$$

$$- \frac{1}{\pi} P_{2j}\left(\frac{r_o}{l_o} Z*\right)\left(\left(\frac{r_o}{l_o} - Z*\right) \ln\left(\frac{r_o}{l_o} - Z*\right) + \left(\frac{r_o}{l_o} + Z*\right) \ln\left(\frac{r_o}{l_o} + Z*\right)\right) \qquad \text{B-14}$$

APPENDIX C: RESISTANCE FOR LARGE AND SMALL l_o/r_o

C.1 RESISTANCE FOR SMALL l_o/r_o

If we take equation 2.1-2, and make the following substitutions,

$$x = \frac{z}{l_o},\ x' = \frac{z'}{l_o},\ \varepsilon = \frac{l_o}{r_o}\ , \qquad \text{C.1-1a,b,c}$$

we find:

$$1 - m = \frac{\varepsilon^2 (x - x')^2}{\varepsilon^2 (x - x')^2 + 4}\ . \qquad \text{C-.1-2}$$

Assuming equation 2.2-13, the thin ring limit, approximates the current distribution, we can substitute it into equation 2.2-1 to obtain:

$$\Phi_o = \frac{l_o}{\pi\kappa} \int_{-1}^{1} \left(\frac{2K(m)}{\left(\varepsilon^2 (x - x')^2 + 4\right)^{\frac{1}{2}}} + F \right) \frac{1}{\sqrt{1 - x'^2}} dx'. \qquad \text{C.1-3}$$

As ε approaches zero, K(m) behaves as follows (16):

$$K(m) = -\frac{1}{2}\ln\left(\frac{1-m}{16}\right) + O\left(\varepsilon^2 \ln(\varepsilon)\right), \qquad \text{C.1-4}$$

or,

$$K(m) = \ln\left(\frac{4}{\varepsilon}\right) + \frac{1}{2}\ln\left(\varepsilon^2 (x - x')^2 + 4\right) - \ln|x - x'| + O\left(\varepsilon^2 \ln(\varepsilon)\right)\ . \qquad \text{C.1-5}$$

Substituting equations C.1-1a,b,c into equation 2.1-9, we find that the following equation roughly describes F:

$$F \approx -\frac{\pi}{2}\frac{1}{\sqrt{1+\varepsilon^2(x-x')^2}} \quad \text{C.1-6}$$

Therefore, it is reasonable to state, for small ε,

$$F = -\frac{1}{2}\pi + O(\varepsilon). \quad \text{C.1-7}$$

We can make two further simplifications for small ε:

$$\left(\varepsilon^2(x-x')^2+4\right)^{-\frac{1}{2}} = \frac{1}{2} + O\left(\varepsilon^2\right), \quad \ln\left(\varepsilon^2(x-x')^2+4\right) = \ln(4) + O\left(\varepsilon^2\right). \quad \text{C.1-8,9}$$

Substituting equations C.1-5, C.1-7 and equations C-1.8,9 into equation C.1-3 and taking advantage of equation B-3, we find:

$$\Phi_o = \frac{2l_o}{\kappa}\left(\frac{1}{2}\ln\left(\frac{4}{\varepsilon}\right) + \ln 2 - \frac{\pi}{4}\right) + O(\varepsilon). \quad \text{C.1-10}$$

It should be noted that equation C.1-10 is not strictly valid because equation 2.2-13 applies only for the limit as l_o/r_o approaches zero.

Integrating over the cylinder surface, we can solve for the total current :

$$i_{total} = 2\pi r_o \int_{-l_o}^{l_o} \frac{1}{\sqrt{1-x^2}}dz = 4\pi l_o r_o \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx = 2\pi^2 l_o r_o . \quad \text{C.1-11}$$

Dividing equation C.1-10 by C.1-11 results in the following expression of resistance.

$$Rr_o\kappa = \frac{1}{2\pi^2}\left(\ln(16) - \frac{\pi}{2} - \ln\left(\frac{l_o}{r_o}\right)\right), \quad \frac{l_o}{r_o} \to 0 \qquad 3.2\text{-}4$$

C.2 RESISTANCE FOR LARGE l_o/r_o

For large l_o/r_o, we postulate a constant current on the electrode for the primary distribution, and solve Laplace's equation for an infinitely long cylinder.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) = 0 \qquad \Phi = \Phi_o,\ i = i_{avg} \quad \text{at } r = r_o \qquad \text{C.2-1}$$

The solution to equation C.2-1 is:

$$\Phi = \Phi_o - \frac{i_{avg}r_o}{\kappa}\ln\left(\frac{r}{r_o}\right). \qquad \text{C.2-2}$$

For distances in solution much greater than l_o, the electrode appears as a point source.

$$\Phi = \frac{I}{4\pi\kappa\sqrt{r_o^2 + z^2}}, \quad \text{I=total current} = 4\pi l_o r_o i_{avg} \qquad \text{C.2-3}$$

Setting equation C.2-2 equal to equation C.2-3 at a distance of l_o from the middle of the electrode, one obtains:

$$\frac{i_{avg}r_o}{\kappa\Phi_o} = \frac{1}{1 + \ln\left(\frac{l_o}{r_o}\right)}. \qquad \text{C-.2-4}$$

A resistance approximation for large l_o/r_o follows from this equation.

$$\frac{4\pi\Phi_o\kappa l_o}{4\pi l_o r_o i_{avg}} = 4\pi R\kappa l_o \cong \frac{1}{1 + \ln\left(\frac{l_o}{r_o}\right)} \ , \ \frac{l_o}{r_o} \to \infty \qquad 3.2\text{-}5$$

APPENDIX D: EQUATIONS DESCRIBING THE SECONDARY CURRENT DISTRIBUTION FOR LARGE POLARIZATION PARAMETERS

West and Newman solved the secondary current distribution in the electrode-insulator edge region for large polarization parameters. They showed that current in the edge region is described by:

$$\frac{i_n(x)}{P_o} = -\left(\frac{1}{\sigma}\right)^{\left(1-\frac{\pi}{2\alpha}\right)} \bar{\phi} \; . \qquad \text{D-1}$$

where α is the electrode-insulator intersection angle, P_o is the constant in equation 1.1-7, $\sigma=((\alpha_a+\alpha_c)Fi_o/RT\kappa)^{-1}$, and the stretched potential is

$$-\bar{\phi} = -\left(\Phi_o - V\right)\frac{\kappa}{P_o}\left(\frac{1}{\sigma}\right)^{\frac{\pi}{2\alpha}} = f_{univ}\left(\left(\frac{x}{\sigma}\right)^{\frac{\pi}{2\alpha}}\right) . \qquad \text{D-2}$$

Here, f_{univ} is a sole function of α and the stretched distance. It approaches the primary distribution in the edge region, $(x/\sigma)^{(\pi/2\alpha-1)}$, as x/σ approaches infinity.

For the cylinder electrode geometry, $\alpha = \pi$, and it follows from equation 3.2-2 and 3.2-3 that:

$$P_o = i_{avg}\, l_o^{\frac{1}{2}} P_{avg,l} = i_{avg}\, r_o^{\frac{1}{2}} P_{avg,r} \cdot \qquad \text{D-3a,b}$$

For $\alpha = \pi$, f_{univ}= 1.75 at x/σ=0. Combining equations D-1, D-2 and D-3a,b yields:

$$\frac{1}{J_l^{\frac{1}{2}} P_{avg,l}} \frac{i_n}{i_{n,avg}} = \frac{1}{J_r^{\frac{1}{2}} P_{avg,r}} \frac{i_n}{i_{n,avg}} = f_{univ}\left(\left(\frac{l_o - z}{\sigma}\right)^{1/2}\right) . \qquad \text{3.3-1,2}$$

APPENDIX E: POTENTIAL DUE TO A CONSTANT CURRENT FOR SMALL AND LARGE l_o/r_o

E.1 POTENTIAL FOR SMALL l_o/r_o

Substituting equations C.1-1a,b,c and C.1-2 into equation 2.2-1 for a constant current density, i_c, at the electrode surface, one derives the following expression for the potential on the electrode and insulator:

$$\frac{\Phi_o \kappa}{i_c l_o} = \frac{1}{\pi} \int_{-1}^{1} \left(\frac{2K(m)}{\left(\varepsilon^2 (x - x')^2 + 4 \right)^{\frac{1}{2}}} + F \right) dx' \ . \qquad \text{E.1-1}$$

Substituting equations C.1-5, C.1-7, C.1-8 and C.1-9 into equation E.1-1 one obtains the following expression for the potential on the electrode and insulator respectively:

$$\frac{\Phi_o \kappa}{i_c l_o} = \frac{1}{\pi} \left((\ln(64)) + 2 - \pi - 2\ln\left(\frac{l_o}{r_o}\right) - \left(1 - \frac{z}{l_o}\right) \ln\left(1 - \frac{z}{l_o}\right) - \left(1 + \frac{z}{l_o}\right) \ln\left(1 + \frac{z}{l_o}\right) \right) + O\left(\frac{l_o}{r_o}\right), \qquad 3.4\text{-}1$$

$$\frac{\Phi_o \kappa}{i_c l_o} = \frac{1}{\pi} \left((\ln(64)) + 2 - \pi - 2\ln\left(\frac{l_o}{r_o}\right) + \left(\frac{z}{l_o} - 1\right) \ln\left(\frac{z}{l_o} - 1\right) - \left(1 + \frac{z}{l_o}\right) \ln\left(1 + \frac{z}{l_o}\right) \right) + O\left(\frac{l_o}{r_o}\right). \qquad 3.4\text{-}2$$

E.2 POTENTIAL FOR LARGE l_o/r_o

The potential variation due to a point source at the origin is described by:

$$\Phi = \frac{I}{4\pi\kappa\sqrt{r^2 + z^2}} \ , \qquad \text{E.2-1}$$

where I is the total current emitted from the source. For large l_o/r_o, a modified point source equation can be proposed for purposes of integration which gives ring source behavior for distances much greater than $O(r_o)$ accurately but lumps the complicated three dimensional potential variation near the ring source into a functional form suggested by equation 2.1-1. For potentials on the cylinder surface due to a ring source one postulates:

$$\Phi \cong \frac{I}{4\pi\kappa\sqrt{(z-z')^2 + 4(\beta r_o)^2}}, \qquad \text{E.2-2}$$

where β is assumed to be O(1). Potential behavior near the ring source has been lumped into a "hump" near the ring.

A differential ring source imbedded in an insulating cylinder has total current $2\pi i_c r_o dz'$. Summing the approximated ring sources over the entire electrode produces:

$$\Phi_o \cong \int_{-l_o}^{l_o} \frac{2\pi r_o i_c}{4\pi\kappa\sqrt{(z-z')^2 + 4(\beta r_o)^2}}\, dz'. \qquad \text{3.4-7}$$

Equation 3.4-7 is formulated to approximate equation 2.2-1 for l_o-z=O(l_o) or greater. It is expected to break down in the edge region, where the three dimensional nature of the ring source is important in determining potential variation.

Equation 3.4-7 can be written:

$$\frac{\Phi_o \kappa}{r_o i_c} \cong \int_{-\frac{l_o+z}{2\beta r_o}}^{\frac{l_o-z}{2\beta r_o}} \frac{1}{\sqrt{y^2+1}}\, dy, \quad y = \frac{(z'-z)}{2\beta r_o}. \qquad \text{E.2-3}$$

Equation E.2-3 assumes the following analytic form:

$$\frac{\Phi_o\kappa}{i_c r_o} \cong \frac{1}{2}\left(\ln\left(\frac{l_o-z}{2\beta r_o}+\sqrt{\left(\frac{l_o-z}{2\beta r_o}\right)^2+1}\right)\right)+\frac{1}{2}\left(\ln\left(\frac{l_o+z}{2\beta r_o}+\sqrt{\left(\frac{l_o+z}{2\beta r_o}\right)^2+1}\right)\right). \qquad 3.4\text{-}8$$

E.1.1 <u>Asymptotic Expressions for Equation 3.4-8</u>

For the electrode mid-region; $z>0$, $l_o>>r_o$, $l_o-z>0$ and $l_o-z=O(l_o)$ implies:

$$\frac{l_o+z}{2\beta r_o}>>1\,,\quad \frac{l_o-z}{2\beta r_o}>>1\,,$$

and equation 3.4-8 becomes:

$$\frac{\Phi_o\kappa}{i_c r_o} \cong \frac{1}{2}\left(\left(\frac{l_o-z}{\beta r_o}\right)\left(\frac{l_o+z}{\beta r_o}\right)\right), \qquad E.2\text{-}4$$

or:

$$\frac{\Phi_o\kappa}{i_c r_o}-\ln\left(\frac{l_o}{\beta r_o}\right) \cong \frac{1}{2}\ln\left(1-\left(\frac{z}{l_o}\right)^2\right). \qquad 3.4\text{-}9$$

For the electrode-insulator edge region; $z>0$, $l_o>>r_o$, and $l_o-z=O(r_o)$, implying:

$$\frac{l_o+z}{2\beta r_o}>>1\,.$$

Equation 3.4-8 becomes:

$$\frac{\Phi_o\kappa}{i_c r_o}-\frac{1}{2}\ln\left(\frac{2l_o}{\beta r_o}\right) \cong \frac{1}{2}\ln\left(\frac{l_o-z}{2\beta r_o}+\sqrt{\left(\frac{l_o-z}{2\beta r_o}\right)^2+1}\right). \qquad 3.4\text{-}11$$

After multiplying the numerator and denominator inside the first logarithm of equation 3.4-8 by:

$$\sqrt{\left(\frac{l_o-z}{2\beta r_o}\right)^2+1} - \frac{l_o-z}{2\beta r_o},$$

equation 3.4-8 becomes:

$$\frac{\Phi_o\kappa}{i_c r_o} \cong \frac{1}{2}\left(\ln\left(\frac{\frac{l_o+z}{2\beta r_o}+\sqrt{\left(\frac{l_o+z}{2\beta r_o}\right)^2+1}}{\frac{z-l_o}{2\beta r_o}+\sqrt{\left(\frac{l_o-z}{2\beta r_o}\right)^2+1}}\right)\right). \qquad \text{E.2-5}$$

For the insulator region away from the edge region; $z>l_o$, $l_o>>r_o$ and $z-l_o=O(l_o)$ or greater; equation E.2-5 becomes:

$$\frac{\Phi_o\kappa}{i_c r_o} = \frac{1}{2}\ln\left(\frac{1+\frac{z}{l_o}}{\frac{z}{l_o}-1}\right). \qquad \text{3.4-13}$$

E.3 POTENTIAL DISTRIBUTION ON THE EDGE REGION FOR CONSTANT i_n

For large l_o/r_o, and $l_o-z<<r_o$, the cylinder electrode-insulator edge region appears to lose its curvature. The potential distribution, for constant i_n, is given by the solution to Laplace's equation for an infinite planar electrode-insulator configuration. In cylindrical coordinates we have:

$$\frac{1}{r}\frac{\partial}{r}\left(r\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}=0, \qquad \text{E.3-1}$$

$$\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = 0 \text{ at } \theta = 0 \quad , \quad \frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -\frac{i_c}{\kappa} \text{ at } \theta = \pi \ . \qquad \text{E.3-2}$$

The solution to this set of equations is:

$$\frac{\Phi\kappa\pi}{i_c} = -\, r\ln(r)\cos\theta + r\theta\sin\theta \ . \qquad \text{E.3-3}$$

For $\theta=\pi/2$ we have:

$$\frac{\kappa\pi}{i_c}\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = 1 + \ln(r) \ . \qquad \text{E.3-4}$$

Equation E.3-3 and E.3-4 break down at l_o-z=O(r_o) for the large l_o/r_o electrode edge region.

APPENDIX F: COMPUTER PROGRAMS

F.1 CURVE FIT FOR F

$dq = (z - z')/r_o$

```
      function pif(dq)
        implicit double precision(a-h,o-z)
     abvdz=dabs(dq)
      zeta=abvdz/(1.0d0 + abvdz)
       b=0.5d0/(1.0d0 + abvdz)
       c=-1.1083d-3
      d=0.29872d0*zeta
        e= -0.63683d0*zeta**2
       f= 1.5519d0*zeta**3
       g= -2.4429d0*zeta**4
      h= 1.2281d0*zeta**5
     if (abvdz .ge. 20.0d0 .or. abvdz .le. 0.003d0) then
            ef=b
            goto 10
        end if
     ef=b + c + e + f + d + g + h
10        pi=3.1415926535898d0
      pif=pi*ef
     return
    end
```

F.2 FUNCTION CALCULATING f_{univ}

$sqxbar = y'^{1/2} = (J_l(1 - (z'/l_o)^2)/2)^{1/2}$

```
      function feebar(sqxbar)
       implicit double precision(a-h,o-z)
c
      if (sqxbar/2.0d0 .ge. 0.5d0) then
          fractin=1.0d0/dcosh(1.0d0/(sqxbar/2.0d0)**5)
    else
      fractin=0.0d0
    end if
       fractout=1.0d0-fractin
c
      b0=    1.75302518293d0
       b1=  -0.195234399525d0
       b2=  -2.05934224121d0
      b3=    2.63679433677d0
       b4=  -1.66817831037d0
```

```
      b5=    0.629696102982d0
      b6=   -0.147232218361d0
      b7=    0.0208781607515d0
      b8=   -0.00164370726206d0
      b9=    0.0000550849360993d0
c
      if (sqxbar .le. 6.0d0) then
          feebarin=b0+b1*sqxbar+b2*sqxbar**2+b3*sqxbar**3+b4*sqxbar**4
         1  +b5*sqxbar**5+b6*sqxbar**6+b7*sqxbar**7+b8*sqxbar**8
      2 +b9*sqxbar**9
     else
      feebarin=0.0d0
     end if
c
      if (sqxbar .ge. 0.1d0) then
      feebarout=1.0d0/sqxbar
     else
      feebarout=0.0d0
     end if
c
       feebar=fractout*feebarin + fractin*feebarout
c
      return
     end
```

F.3 FUNCTION CALCULATING EQUATION 2.2-20

z = Z′

a = A_r

power = b

rldr0 = l_o/r_o

rjay = J_r

curcent = $r_o i_{center}/Vk$

```
       function ai(z,a,power,rldr0,rjay,curcent)
      implicit double precision(a-h,o-z)
          ybar=0.5d0/rldr0*rjay*(rldr0**2  -  z**2)
       ybarcent=0.5d0*rldr0*rjay
      sqybar=sqrt(abs(ybar))
     sqybarcent=sqrt(ybarcent)
     afeebar=feebar(sqybar)
     cfeebar=feebar(sqybarcent)
      edge=a*sqrt(rjay)*afeebar
      edgecent=a*sqrt(rjay)*cfeebar
      ain=curcent**power + edge**power - edgecent**power
       ai=ain**(1.0d0/power)
c
     return
```

```
end
```

F.4 FUNCTION CALCULATING LEGENDRE POLYNOMIALS

$x = z'/l_o$

n is the order of the Legendre Polynomial

```
      function P(n,x)
        implicit double precision(a-h,o-z)
     p1=1.d0
     p2=x
        if(n-1)1,2,3
1     P=p1
      return
2     P=p2
      return
3      nm1=n-1
     do 4 nu=1,nm1
            P=(x*dfloat(2*nu+1)*p2-dfloat(nu)*p1)/dfloat(nu+1)
     p1=p2
4     p2=P
      return
    end
```

F.5 FUNCTION CALCULATING THE ELLIPTICAL INTEGRAL OF THE FIRST KIND

w = 1 - m

```
    function e1(w)
    implicit double precision (a-h,o-z)
    dimension a(5),b(5)
     data a / 1.38629436112d0,.09666344259d0,.03590092383d0
      2     ,.03742563713d0,.01451196212d0/
     data b / .5d0,.12498593597d0,.06880248576d0,
      3     .03328355346d0,.00441787012d0/
        d=a(1)+a(2)*w+a(3)*w**2+a(4)*w**3+a(5)*w**4

e1=d+(b(1)+b(2)*w+b(3)*w**2+b(4)*w**3+b(5)*w**4)*dlog(1.0d0/w)
    return
   end
```

F.6 FUNCTION CALCULATING EQUATION B-8

u = U′
a = P_r
power = b
rldr0 = l_o/r_o

```
      function fcurnt(u,a,power,rldr0)
      implicit double precision(a-h,o-z)
       param=sqrt(2.0d0*rldr0)
        curcent=(1.0d0/(1.0d0 + log(rldr0)))**power
         curcent=curcent - (a*sqrt(2.0d0)/sqrt(rldr0))**power
         curcent=(abs(param - u**2))**power*(abs(u))**power*curcent
         curend=(a*param/sqrt(2.0d0*param-u**2))**power
       fcurnt=(curcent + curend)**(1.0d0/power)
c
     return
    end
```

F.7 FUNCTION CALCULATING EQUATION B-12

u = U′
a = P_r
power = b
rldr0 = l_o/r_o
rjay = J_r
curcent = $r_o i_{center}/Vk$

```
      function acurnt(u,a,power,rldr0,rjay,curcent)
      implicit double precision(a-h,o-z)
       param=sqrt(2.0d0*rldr0)
        z=-rldr0 + 2.0d0*param*u**2 - u**4
        curnt=ai(z,a,power,rldr0,rjay,curcent)
        acurnt=curnt*(param -u**2)*u

c
     return
    end
```

F.8 SUBROUTINE CALCULATING THE INTEGRAND IN EQUATION B-14

di is the value of the integrand in equation B-14
rldr0 = l_o/r_o
zs = Z*

zp = Z′
nleg is the order of the legendre polynomial

```
      subroutine  dileg(di,rldr0,zs,zp,nleg)
      implicit double precision (a-h,o-z)
            pi=3.14159265358979d0
      dz=zs-zp
c
      if (abs(dz) .ge. 1.0d-4)  then
       wm=4.0d0/(dz**2  +4.0d0)
      w=1.0d0 - wm
       elipt=e1(w)
              f=-pif(dz)
      g=2.0d0*elipt/(sqrt(dz**2 + 4.0d0))   +   f
c
            zpdl=zp/rldr0
       zsdl=zs/rldr0
      pnzp=p(nleg,zpdl)
      pnzs=p(nleg,zsdl)
     gpnzp=g*pnzp
       aspnzs=0.5d0*pnzs*log((zs-zp)**2)
      di=(gpnzp + aspnzs)/pi
c
c
         else
c
            zsdl=zs/rldr0
      pnzs=p(nleg,zsdl)
              di=1.0d0/pi*(3.0d0*log(2.0d0) - pi/2.0d0)*pnzs
c
      end if
       return
     end
```

F.9 SUBROUTINE CALCULATING THE INTEGRAND IN EQUATION B-4

di is the value of the integrand in equation B-4
rldr0 = l_o/r_o
zs = Z*
zp = Z′

```
      subroutine  dinf(di,rldr0,zs,zp)
      implicit double precision (a-h,o-z)
            pi=3.14159265358979d0
      dz=zs-zp
c
```

```
   if (abs(dz) .ge. 1.0d-4)  then
    wm=4.0d0/(dz**2  +4.0d0)
   w=1.0d0 - wm
    elipt=e1(w)
            f=-pif(dz)
   g=2.0d0*elipt/(sqrt(dz**2  +  4.0d0))   +   f
c           u=sqrt(sqrt(2.0d0*rldr0)   -   sqrt(rldr0 - zp))
    u1=sqrt(2.0d0*rldr0)
    u2=sqrt(abs(rldr0-zp))
   pctu1=abs((u1  -  u2)/u1)
   if (pctu1  .ge. .999999999d0 ) then
       u=sqrt(u1)
   else
           u=sqrt(u1 - u2)
   end if
    denom  =  sqrt(2.0d0*sqrt(2.0d0*rldr0)  -  u**2)
    dnum=g  +  0.5d0*log((zs-zp)**2)
          di=4.0d0*rldr0/pi*dnum/denom
c
      else
c
c            u=sqrt(sqrt(2.0d0*rldr0)   - sqrt(rldr0 - zp))
    u1=sqrt(2.0d0*rldr0)
    u2=sqrt(abs(rldr0-zp))
   pctu1=abs((u1  -  u2)/u1)
   if (pctu1  .ge. .999999999d0 ) then
       u=sqrt(u1)
   else
           u=sqrt(u1 - u2)
   end if
c
    denom=sqrt(2.0d0*sqrt(2.0d0*rldr0)  -  u**2)
    dnum=4.0d0*rldr0/pi*(3.0d0*log(2.0d0)  -  0.5d0*pi)
   di=dnum/denom
c
   end if
c       return
```

F.10 SUBROUTINE CALCULATING THE INTEGRAND IN EQUATION B-7

di is the value of the integrand in equation B-7

rldr0 = l_o/r_o

zs = Z^*

zp = Z'

a = P_r

power = b

```
      subroutine  fdinf(di,rldr0,zs,zp,a,power)
      implicit double precision (a-h,o-z)
            pi=3.14159265358979d0
      dz=zs-zp
c
      if (abs(dz) .ge. 1.0d-4)  then
       wm=4.0d0/(dz**2  +4.0d0)
      w=1.0d0 - wm
       elipt=e1(w)
              f=-pif(dz)
      g=2.0d0*elipt/(sqrt(dz**2 + 4.0d0))   +   f
              u=sqrt(abs(sqrt(2.0d0*rldr0)   -   sqrt(abs(rldr0 - zp))))
              us=sqrt(abs(sqrt(2.0d0*rldr0)   -   sqrt(abs(rldr0 - zs))))
c
          dicurp=fcurnt(u,a,power,rldr0)
          dicurs=fcurnt(us,a,power,rldr0)
        di=g*dicurp
          di=di - (3.0d0*log(2.0d0) - pi/2.0d0 - log(abs(dz)))*dicurs
         di=4.0d0/pi*di
c
       else

           u=sqrt(abs(sqrt(2.0d0*rldr0)   -   sqrt(abs(rldr0 - zp))))
        di=0.0d0
c
       end if
c
      return
     end
```

F.11 SUBROUTINE CALCULATING THE INTEGRAND IN EQUATION B-11

di is the value of the integrand in equation B-11

rldr0 = l_o/r_o

zs = Z^*

zp = Z'

a = P_r

power = b

rjay = J_r

curcent = $r_o i_{center}/Vk$

```
      subroutine  adinf(di,rldr0,zs,zp,a,power,rjay,curcent)
      implicit double precision (a-h,o-z)
            pi=3.14159265358979d0
      dz=zs-zp
c
```

```
    if (abs(dz) .ge. 1.0d-6)  then
     wm=4.0d0/(dz**2  +4.0d0)
    w=1.0d0 - wm
     elipt=e1(w)
            f=-pif(dz)
    g=2.0d0*elipt/(sqrt(dz**2 + 4.0d0))   +   f
            u=sqrt(abs(sqrt(2.0d0*rldr0)   -   sqrt(abs(rldr0 - zp))))
            us=sqrt(abs(sqrt(2.0d0*rldr0)   -   sqrt(abs(rldr0 - zs))))
c
         dicurp=acurnt(u,a,power,rldr0,rjay,curcent)
         dicurs=acurnt(us,a,power,rldr0,rjay,curcent)
      di=g*dicurp
        di=di - (3.0d0*log(2.0d0) - pi/2.0d0 - log(abs(dz)))*dicurs
       di=4.0d0/pi*di
c
     else
c

         u=sqrt(abs(sqrt(2.0d0*rldr0)   -   sqrt(abs(rldr0 - zp))))
      di=0.0d0
c
     end if
c
    return
   end
```

F.12 SUBROUTINE CALCULATING POTENTIAL FUNCTIONS WITH EQUATION 2.2-13 AS A FIRST CURRENT FUNCTION

zs = Z*

rldr0 = l_o/r_o

m is the number of mesh spacings for numerical integration

nleg is the order of the legendre polynomial

theta is the value of the potential function

```
          subroutine trap(zs,rldr0,m,nleg,theta)
    implicit double precision (a-h,o-z)
     pi=3.1415926535898d0
    theta=0.0d0
c
         total=2.0d0*rldr0
    part=rldr0 + zs
     fract=part/total
     m1=int(fract*dfloat(m))  +  1
     m2=int((1.0d0  -  fract)*dfloat(m))  +  1
c
    if (nleg .eq. -1) then
```

```
      if (m1 .ge. 2) then
              delta1=sqrt(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs)))
           eps1=delta1/dfloat(m1-1)
        call  dinf(di,rldr0,zs,-rldr0)
      or1=di
c
      do 1 j=2,m1
        u=eps1*dfloat(j-1)
         zp=-rldr0  +  2.0d0*sqrt(2.0d0*rldr0)*u**2  -  u**4
       call  dinf(di,rldr0,zs,zp)
         or2=di
      theta=theta + eps1*(or2 + or1)/2.0d0
      or1=or2
  1     continue
c
        else
c
        call  dinf(di,rldr0,zs,-rldr0)
     beg=di
           call dinf(di,rldr0,zs,zs)
     end=di
         uend=sqrt(sqrt(2.0d0*rldr0)-sqrt(abs(rldr0-zs)))
      theta=theta + (end + beg)/2.0d0*abs(uend)
c
     end if
c
      if (m2 .ge. 2) then
         delta2=sqrt(sqrt(2.0d0*rldr0))
        1         - sqrt(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0-zs)))
       eps2=delta2/dfloat(m2-1)
       call  dinf(di,rldr0,zs,zs)
      or1=di
c
      do 3 j=2,m2
         u=sqrt(sqrt(2.0d0*rldr0)  -  sqrt(abs(rldr0-zs)))
      1        + eps2*dfloat(j-1)
               zp=-rldr0  +  2.0d0*sqrt(2.0d0*rldr0)*u**2 - u**4
       call  dinf(di,rldr0,zs,zp)
      or2=di
       theta=theta + eps2*(or2 + or1)/2.0d0
      or1=or2
   3    continue
c
     else
c
           call dinf(di,rldr0,zs,zs)
     beg=di
        call  dinf(di,rldr0,zs,rldr0)
     end=di
         ubeg=sqrt(sqrt(2.0d0*rldr0)-sqrt(abs(rldr0-zs)))
        uend=sqrt(sqrt(2.0d0*rldr0))
```

```
      theta=theta + (end+beg)/2.0d0*(abs(uend - ubeg))
c
    end if
c
        theta2=  -0.5d0*rldr0*(2.0d0*log(rldr0)
     1            -2.0d0*log(2.0d0))
     theta=theta + theta2
c
  else
c
         if (m1 .ge. 2) then
     delta1=rldr0 + zs
      eps1=delta1/dfloat(2*m1)
       call  dileg(di,rldr0,zs,-rldr0,nleg)
     theta=theta + di*eps1/3.0d0
      call  dileg(di,rldr0,zs,zs,nleg)
     theta=theta + di*eps1/3.0d0
c
        do 2 j=1, m1
      zp=-rldr0  +  eps1*dfloat(2*j  -  1)
      call  dileg(di,rldr0,zs,zp,nleg)
      theta=theta  +  4.0d0/3.0d0*eps1*di
  2     continue
     do 21 j=1, m1-1
      zp=-rldr0  +  eps1*dfloat(2*j)
          call dileg(di,rldr0,zs,zp,nleg)
     theta=theta  +  2.0d0/3.0d0*eps1*di
 21      continue
c
    else
c
       call  dileg(di,rldr0,zs,-rldr0,nleg)
    beg=di
      zpmid=(-rldr0  +  zs)/2.0d0
      call  dileg(di,rldr0,zs,zpmid,nleg)
    amid=di
          call dileg(di,rldr0,zs,zs,nleg)
    end=di
      eps1=abs(zs  +  rldr0)/2.0d0
     theta=theta  +  eps1*(beg  +  4.0d0*amid  +  end)/3.0d0
c
    end if
c
     if (m2 .ge. 2) then
      delta2=2.0d0*rldr0  -  (rldr0  +  zs)
      eps2=delta2/dfloat(2*m2)
      call  dileg(di,rldr0,zs,zs,nleg)
          theta=theta + eps2*di/3.0d0
      call  dileg(di,rldr0,zs,rldr0,nleg)
     theta=theta  +  eps2*di/3.0d0
c
```

```
      do 4 j=1,m2
       zp=zs + eps2*dfloat(2*j -1)
       call dileg(di,rldr0,zs,zp,nleg)
      theta=theta + 4.0d0/3.0d0*eps2*di
  4     continue
         do 41 j=1, m2-1
          zp=zs + eps2*dfloat(2*j)
          call dileg(di,rldr0,zs,zp,nleg)
          theta=theta + 2.0d0/3.0d0*eps2*di
  41     continue
c
    else
c
          call dileg(di,rldr0,zs,zs,nleg)
    beg=di
      zpmid=(zs + rldr0)/2.0d0
      call dileg(di,rldr0,zs,zpmid,nleg)
    amid=di
      call dileg(di,rldr0,zs,rldr0,nleg)
    end=di
      eps2=(rldr0 - zs)/2.0d0
     theta=theta + eps2*(beg + 4.0d0*amid + end)/3.0d0
c
    end if
c
      zsdl=zs/rldr0
     pnzs=p(nleg,zsdl)
      a=rldr0-zs
      b=rldr0+zs
       if (abs(a) .le. 1.0d-11) then
          theta2=-1.0d0/pi*pnzs*(b*log(b) - 2.0d0*rldr0)
          else if (abs(b) .le. 1.0d-11) then
          theta2=-1.0d0/pi*pnzs*(a*log(a) - 2.0d0*rldr0)
          else
                theta2=-1.0d0/pi*pnzs*(a*log(a)+b*log(b)-2.0d0*rldr0)
        end if
         theta=theta + theta2
      end if
  return
  end
```

F.13 SUBROUTINE CALCULATING POTENTIAL FUNCTIONS WITH EQUATION 2.2-14 AS A FIRST CURRENT FUNCTION

zs = Z^*

rldr0 = l_o/r_o

m is the number of mesh spacings for numerical integration

nleg is the order of the legendre polynomial

theta is the value of the potential function

a = P_r

power = b

```
      subroutine fsim(zs,rldr0,m,nleg,theta,a,power)
   implicit double precision (a-h,o-z)
    pi=3.1415926535898d0
   theta=0.0d0
   if (nleg .eq. -2) then
         total=sqrt(sqrt(2.0d0*rldr0))
    part=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
    fract=part/total
    m1=int(fract*dfloat(m)) + 1
    m2=int((1.0d0 - fract)*dfloat(m)) + 1
c
  else
c
      total=2.0d0*rldr0
   part=rldr0 + zs
    fract=part/total
    m1=int(fract*dfloat(m)) + 1
    m2=int((1.0d0 - fract)*dfloat(m)) + 1
c
     end if
c
   if (nleg .eq. -2) then
     if (m1 .ge. 2) then
      delta1=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
      eps1=delta1/dfloat(2*m1)
       call fdinf(di,rldr0,zs,-rldr0,a,power)
    up=di
      call fdinf(di,rldr0,zs,zs,a,power)
    down=di
     theta=theta + (up + down)*eps1/3.0d0
c
     do 1 j=1,m1
       u=eps1*dfloat(2*j-1)
        zp=-rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*u**2 - u**4
      call fdinf(di,rldr0,zs,zp,a,power)
     theta=theta + di*eps1*4.0d0/3.0d0
 1   continue
     do 12 j=1,m1-1
      u=eps1*dfloat(2*j)
        zp=-rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*u**2 - u**4
      call fdinf(di,rldr0,zs,zp,a,power)
     theta=theta + di*eps1*2.0d0/3.0d0
 12  continue
c
```

```
      else
c
        call fdinf(di,rldr0,zs,-rldr0,a,power)
    beg=di
        umid=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))/2.0d0
        zpmid=-rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*umid**2 - umid**4
      call fdinf(di,rldr0,zs,zpmid,a,power)
    rmid=di
      call fdinf(di,rldr0,zs,zs,a,power)
    rend=di
        uend=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
    eps1=uend/2.0d0
      theta=theta + (beg + 4.0d0*rmid + rend)*eps1/3.0d0
c
    end if
c
    if (m2 .ge. 2) then
      delta2=sqrt(sqrt(2.0d0*rldr0))
      1          - sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0-zs))))
      eps2=delta2/dfloat(2*m2)
      call fdinf(di,rldr0,zs,zs,a,power)
    up=di
      call fdinf(di,rldr0,zs,rldr0,a,power)
    down=di
      theta=theta + (up + down)*eps2/3.0d0
    do 3 j=1,m2
        u=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
      1 + eps2*dfloat(2*j-1)
        zp=-rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*u**2 - u**4
      call fdinf(di,rldr0,zs,zp,a,power)
      theta= theta + di*eps2*4.0d0/3.0d0
 3  continue
    do 31 j=1,m2-1
        u=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
      1 + eps2*dfloat(2*j)
        zp=-rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*u**2 - u**4
      call fdinf(di,rldr0,zs,zp,a,power)
      theta= theta + di*eps2*2.0d0/3.0d0
31  continue
c
    else
c
      call fdinf(di,rldr0,zs,zs,a,power)
    beg=di
      umid= (sqrt(sqrt(2.0d0*rldr0)) +
      1 sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs)))))/2.0d0
        zpmid=-rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*umid**2 - umid**4
      call fdinf(di,rldr0,zs,zpmid,a,power)
    rmid=di
      call fdinf(di,rldr0,zs,rldr0,a,power)
    rend=di
```

```
        ubeg=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
       uend=sqrt(sqrt(2.0d0*rldr0))
     eps2=(uend - ubeg)/2.0d0
      theta=theta + (beg + 4.0d0*rmid + rend)*eps2/3.0d0
c
    end if
c
       const=sqrt(sqrt(2.0d0*rldr0))
        us=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
c
     if (abs(const - us) .ge. 1.0d-12) then
    a1=const - us
    a2=const + us
       a3=sqrt(2.0d0*const**2 - us**2) - const
       a4=sqrt(2.0d0*const**2 - us**2) +const
       theta2=a1*log(a1) + a2*log(a2) - a3*log(a3)
     1  + a4*log(a4) -4.0d0*const
    else
       theta2=4.0d0*const*log(2.0d0*const) - 4.0d0*const
    end if
c
       theta2=(3.0d0*log(2.0d0) - pi/2.0d0)*const - theta2
       curnts=fcurnt(us,a,power,rldr0)
       theta2=4.0d0/pi*curnts*theta2
    theta=theta + theta2
c
  else
c
        if (m1 .ge. 2) then
    delta1=rldr0 + zs
     eps1=delta1/dfloat(2*m1)
      call  dileg(di,rldr0,zs,-rldr0,nleg)
    theta=theta + di*eps1/3.0d0
     call  dileg(di,rldr0,zs,zs,nleg)
    theta=theta + di*eps1/3.0d0
c
       do 2 j=1, m1
     zp=-rldr0  +  eps1*dfloat(2*j  -  1)
     call  dileg(di,rldr0,zs,zp,nleg)
    theta=theta  +  4.0d0/3.0d0*eps1*di
  2     continue
    do 21 j=1, m1-1
     zp=-rldr0  +  eps1*dfloat(2*j)
         call dileg(di,rldr0,zs,zp,nleg)
    theta=theta  +  2.0d0/3.0d0*eps1*di
 21     continue
c
   else
c
     call  dileg(di,rldr0,zs,-rldr0,nleg)
   beg=di
```

```
      zpmid=(-rldr0 + zs)/2.0d0
      call dileg(di,rldr0,zs,zpmid,nleg)
     amid=di
          call dileg(di,rldr0,zs,zs,nleg)
     end=di
      eps1=abs(zs + rldr0)/2.0d0
      theta=theta + eps1*(beg + 4.0d0*amid + end)/3.0d0
c
     end if
c
      if (m2 .ge. 2) then
       delta2=2.0d0*rldr0 - (rldr0 + zs)
       eps2=delta2/dfloat(2*m2)
      call dileg(di,rldr0,zs,zs,nleg)
          theta=theta + eps2*di/3.0d0
       call dileg(di,rldr0,zs,rldr0,nleg)
      theta=theta + eps2*di/3.0d0
c
      do 4 j=1,m2
       zp=zs + eps2*dfloat(2*j -1)
       call dileg(di,rldr0,zs,zp,nleg)
       theta=theta + 4.0d0/3.0d0*eps2*di
 4      continue
         do 41 j=1, m2-1
          zp=zs + eps2*dfloat(2*j)
          call dileg(di,rldr0,zs,zp,nleg)
          theta=theta +  2.0d0/3.0d0*eps2*di
 41      continue
c
     else
c
          call dileg(di,rldr0,zs,zs,nleg)
     beg=di
      zpmid=(zs + rldr0)/2.0d0
      call dileg(di,rldr0,zs,zpmid,nleg)
     amid=di
       call dileg(di,rldr0,zs,rldr0,nleg)
     end=di
      eps2=(rldr0 - zs)/2.0d0
      theta=theta + eps2*(beg + 4.0d0*amid + end)/3.0d0
c
     end if
c
      zsdl=zs/rldr0
      pnzs=p(nleg,zsdl)
      aa=rldr0-zs
      b=rldr0+zs
       if (abs(aa) .le. 1.0d-11) then
          theta2=-1.0d0/pi*pnzs*(b*log(b) - 2.0d0*rldr0)
          else if (abs(b) .le. 1.0d-11) then
         theta2=-1.0d0/pi*pnzs*(aa*log(aa) - 2.0d0*rldr0)
```

```
          else
              theta2=-1.0d0/pi*pnzs*(aa*log(aa)+b*log(b)-2.0d0*rldr0)
        end if
         theta=theta + theta2
     end if
  return
  end
```

F.14 SUBROUTINE CALCULATING POTENTIAL FUNCTIONS WITH EQUATION 2.2-20 AS A FIRST CURRENT FUNCTION

zs = Z*

rldr0 = l_o/r_o

m is the number of mesh spacings for numerical integration

nleg is the order of the legendre polynomial

theta is the value of the potential function

a = P_r

power = b

rjay = J_r

curcent = $r_o i_{center}/Vk$

```
          subroutine asim(zs,rldr0,m,nleg,theta,a,power,rjay,curcent)
   implicit double precision (a-h,o-z)
    pi=3.1415926535898d0
   theta=0.0d0
c
   if (nleg .eq. -2) then
          total=sqrt(sqrt(2.0d0*rldr0))
    part=sqrt(abs(sqrt(2.0d0*rldr0)  -  sqrt(abs(rldr0  -  zs))))
    fract=part/total
    m1=int(fract*dfloat(m))  +  1
    m2=int((1.0d0  -  fract)*dfloat(m))  +  1
c
   else
c
         total=2.0d0*rldr0
   part=rldr0  +  zs
    fract=part/total
    m1=int(fract*dfloat(m))  +  1
    m2=int((1.0d0  -  fract)*dfloat(m))  +  1
c
      end if
c
   if (nleg .eq. -2) then
     if (m1 .ge. 2) then
         delta1=sqrt(abs(sqrt(2.0d0*rldr0)  -  sqrt(abs(rldr0  -  zs))))
```

```
      eps1=delta1/dfloat(2*m1)
        call adinf(di,rldr0,zs,-rldr0,a,power,rjay,curcent)
    up=di
       call adinf(di,rldr0,zs,zs,a,power,rjay,curcent)
    down=di
      theta=theta + (up + down)*eps1/3.0d0
c
     do 1 j=1,m1
       u=eps1*dfloat(2*j-1)
         zp=-rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*u**2 - u**4
       call adinf(di,rldr0,zs,zp,a,power,rjay,curcent)
      theta=theta + di*eps1*4.0d0/3.0d0
 1   continue
      do 12 j=1,m1-1
       u=eps1*dfloat(2*j)
         zp=-rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*u**2 - u**4
       call adinf(di,rldr0,zs,zp,a,power,rjay,curcent)
      theta=theta + di*eps1*2.0d0/3.0d0
 12  continue
c
      else
c
        call adinf(di,rldr0,zs,-rldr0,a,power,rjay,curcent)
    beg=di
        umid=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))/2.0d0
        zpmid=-rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*umid**2 - umid**4
       call adinf(di,rldr0,zs,zpmid,a,power,rjay,curcent)
     rmid=di
       call adinf(di,rldr0,zs,zs,a,power,rjay,curcent)
    rend=di
        uend=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
     eps1=uend/2.0d0
      theta=theta + (beg + 4.0d0*rmid + rend)*eps1/3.0d0
c
    end if
c
     if (m2 .ge. 2) then
       delta2=sqrt(sqrt(2.0d0*rldr0))
      1          - sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0-zs))))
      eps2=delta2/dfloat(2*m2)
       call adinf(di,rldr0,zs,zs,a,power,rjay,curcent)
    up=di
        call adinf(di,rldr0,zs,rldr0,a,power,rjay,curcent)
    down=di
      theta=theta + (up + down)*eps2/3.0d0
     do 3 j=1,m2
        u=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
      1 +  eps2*dfloat(2*j-1)
        zp=-rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*u**2 - u**4
      call adinf(di,rldr0,zs,zp,a,power,rjay,curcent)
      theta= theta + di*eps2*4.0d0/3.0d0
```

```
 3  continue
     do 31 j=1,m2-1
        u=sqrt(abs(sqrt(2.0d0*rldr0)  -  sqrt(abs(rldr0  -  zs))))
     1 + eps2*dfloat(2*j)
        zp=-rldr0  +  2.0d0*sqrt(2.0d0*rldr0)*u**2  -  u**4
       call adinf(di,rldr0,zs,zp,a,power,rjay,curcent)
      theta= theta + di*eps2*2.0d0/3.0d0
 31  continue
c
    else
c
       call adinf(di,rldr0,zs,zs,a,power,rjay,curcent)
    beg=di
       umid= (sqrt(sqrt(2.0d0*rldr0)) +
       1  sqrt(abs(sqrt(2.0d0*rldr0)  -  sqrt(abs(rldr0  -  zs)))))/2.0d0
        zpmid=-rldr0  +  2.0d0*sqrt(2.0d0*rldr0)*umid**2  -  umid**4
       call adinf(di,rldr0,zs,zpmid,a,power,rjay,curcent)
     rmid=di
       call adinf(di,rldr0,zs,rldr0,a,power,rjay,curcent)
    rend=di
       ubeg=sqrt(abs(sqrt(2.0d0*rldr0)  -  sqrt(abs(rldr0  -  zs))))
       uend=sqrt(sqrt(2.0d0*rldr0))
     eps2=(uend - ubeg)/2.0d0
      theta=theta + (beg + 4.0d0*rmid + rend)*eps2/3.0d0
c
    end if
c
       const=sqrt(sqrt(2.0d0*rldr0))
        us=sqrt(abs(sqrt(2.0d0*rldr0)  -  sqrt(abs(rldr0  -  zs))))
c
     if (abs(const - us) .ge. 1.0d-12) then
    a1=const - us
    a2=const + us
       a3=sqrt(2.0d0*const**2  -  us**2)  -  const
       a4=sqrt(2.0d0*const**2  -  us**2)  +const
       theta2=a1*log(a1)  +  a2*log(a2)  -  a3*log(a3)
     1  + a4*log(a4)  -4.0d0*const
    else
       theta2=4.0d0*const*log(2.0d0*const)  -  4.0d0*const
    end if
c
       theta2=(3.0d0*log(2.0d0)  -  pi/2.0d0)*const  -  theta2
       curnts=acurnt(us,a,power,rldr0,rjay,curcent)
       theta2=4.0d0/pi*curnts*theta2
    theta=theta + theta2
c
  else
c
       if (m1 .ge. 2) then
    delta1=rldr0 + zs
     eps1=delta1/dfloat(2*m1)
```

```
      call dileg(di,rldr0,zs,-rldr0,nleg)
     theta=theta + di*eps1/3.0d0
      call dileg(di,rldr0,zs,zs,nleg)
     theta=theta + di*eps1/3.0d0
c
        do 2 j=1, m1
      zp=-rldr0 + eps1*dfloat(2*j - 1)
      call dileg(di,rldr0,zs,zp,nleg)
     theta=theta + 4.0d0/3.0d0*eps1*di
 2      continue
     do 21 j=1, m1-1
      zp=-rldr0 + eps1*dfloat(2*j)
          call dileg(di,rldr0,zs,zp,nleg)
     theta=theta + 2.0d0/3.0d0*eps1*di
 21     continue
c
    else
c
      call dileg(di,rldr0,zs,-rldr0,nleg)
    beg=di
      zpmid=(-rldr0 + zs)/2.0d0
      call dileg(di,rldr0,zs,zpmid,nleg)
    amid=di
          call dileg(di,rldr0,zs,zs,nleg)
    end=di
     eps1=abs(zs + rldr0)/2.0d0
     theta=theta + eps1*(beg + 4.0d0*amid + end)/3.0d0
c
    end if
c
    if (m2 .ge. 2) then
     delta2=2.0d0*rldr0 - (rldr0 + zs)
     eps2=delta2/dfloat(2*m2)
     call dileg(di,rldr0,zs,zs,nleg)
         theta=theta + eps2*di/3.0d0
      call dileg(di,rldr0,zs,rldr0,nleg)
     theta=theta + eps2*di/3.0d0
c
    do 4 j=1,m2
     zp=zs + eps2*dfloat(2*j -1)
     call dileg(di,rldr0,zs,zp,nleg)
     theta=theta + 4.0d0/3.0d0*eps2*di
 4      continue
         do 41 j=1, m2-1
          zp=zs + eps2*dfloat(2*j)
          call dileg(di,rldr0,zs,zp,nleg)
          theta=theta + 2.0d0/3.0d0*eps2*di
 41      continue
c
    else
c
```

```
          call dileg(di,rldr0,zs,zs,nleg)
      beg=di
        zpmid=(zs + rldr0)/2.0d0
        call dileg(di,rldr0,zs,zpmid,nleg)
      amid=di
         call dileg(di,rldr0,zs,rldr0,nleg)
      end=di
        eps2=(rldr0 - zs)/2.0d0
       theta=theta + eps2*(beg + 4.0d0*amid + end)/3.0d0
c
      end if
c
       zsdl=zs/rldr0
      pnzs=p(nleg,zsdl)
       aa=rldr0-zs
       b=rldr0+zs
        if (abs(aa) .le. 1.0d-11) then
            theta2=-1.0d0/pi*pnzs*(b*log(b) - 2.0d0*rldr0)
            else if (abs(b) .le. 1.0d-11) then
           theta2=-1.0d0/pi*pnzs*(aa*log(aa) - 2.0d0*rldr0)
            else
                 theta2=-1.0d0/pi*pnzs*(aa*log(aa)+b*log(b)-2.0d0*rldr0)
          end if
           theta=theta + theta2
       end if
  return
  end
```

APPENDIX G: TABLES OF NUMERICAL RESULTS

G.1 P_{avg}

l_o/r_o	$P_{avg,l}$	$P_{avg,r}$
0.001	0.450	0.0142
0.01	0.449	0.0449
0.1	0.438	0.138
1.0	0.383	0.383
10.0	0.240	0.761
30.0	0.173	0.947
100.0	0.111	1.11
300.0	0.0721	1.25
1000.0	0.0421	1.35

G.2 RESISTANCE

l_o/r_o	$R\kappa l_o$	$R\kappa r_o$
0.001	0.000411	0.411
0.01	0.00295	0.295
0.1	0.0181	0.181
1.0	0.0809	0.0809
10.0	0.220	0.0220
30.0	0.306	0.0102
100.0	0.403	0.00403
300.0	0.492	0.00164
1000.0	0.589	0.000589

G.3 POTENTIAL DIFFERENCE BETWEEN ELECTRODE CENTER AND EDGE FOR UNIFORM i_n

l_o/r_o	$\Delta\Phi_o\kappa/l_o i_c$	$\Delta\Phi_o\kappa/r_o i_c$
0.001	0.441	0.000441
0.01	0.439	0.00439
0.1	0.419	0.0419
1.0	0.322	0.322
10.0	0.118	1.18
100.0	0.0237	2.37
1000.0	0.00353	3.53

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