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New algebraic representations of quantum mechanics

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Contrary to the usual view that quantum mechanics in configuration space has, in general, only one algebaic representation(the Heisenberg algebraic representation), we have proved that quantum mechanics in configuration space has, in general, alternative algebraic representations: (i) In a finite domain of configuration space, it can be expressed in terms of su(2) algebra; (ii) in an infinite domain of configuration space, it can be expressed in terms of su(1,1) algebra. The above results open a new possibility to reformulate quantum mechanics and provide more mathematical tools to solve diverse physical problems. Nonlinear relations of different Lie algebas may imply a unification of Lie algebras and their physical implications (the quantum motion modes) in a deeper nonlinear domain. This observation raises challenging mathematical and physical questions.

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It is a common belief that quantum mechanics in configuration space, in general case, can only be expressed in terms of the Heisenberg(H) algebraic representation or the Heisenberg-Weyl(HW) algebraic representation[1,2]; while only in particular cases it can be expressed in terms of other Lie algebras[3], such as the so(4;2) algebra for the hydrogen atom, the u(3) algebra for the harmonic oscillator, the su(2) algebra for the Morse potential of diatomic molecules. Of course, even in H-algebraic or HW-algebraic representations, one still has the conjugate X- and P-representations for the space variables, and T- and E-representations for the time variable. It is evident that these conjugate representations have the same algebras: H-algebra or HW-algebra with the canonical coordinates or their conjugate momenta as variables. Here the H-algebra consists of $h = \{ X, \hat{P}, 1 \}$ with Heisenberg commutators as its algebraic relations, namely,

$$[X, \hat{P}] = i \quad (\hbar=1). \tag{1}$$

with the other commutators vanishing. The HW-algebra is defined by

$$hw = \{ a \ , a^{+} \ , a^{+}a \ , 1 \} \tag{2a}$$

with creation and annihilation operators defined as

$$a = 1/\sqrt{2} \ (X + i\hat{P}) \ , \quad a^{+} = 1/\sqrt{2} \ (X - i\hat{P}) \ , \tag{2b}$$

and satisfying the following commutation relations

$$[a \ , a^{+}] = 1, \ [a^{+}a \ , a \ ] = - a, \ [a^{+}a \ , a^{+}] = = a^{+}, \tag{3}$$

with the remaining commutators vanishing.

In short, the popular point of view is that in general, quantum mechanics in configuration space, has only one algebraic representation, i.e., the Heisenberg algebraic representation, or its equivalence, the Heisenberg-Weyl algebraic representation. In this note we shall indicate that the above popular viewpoint is only partly true,and that quantum mechanics in configuration space, in general, can find its alternative algebraic representaions. This opens the new possibility to reformulate quantum mechanics and provides more mathematical tools to solve diverse physical problems. In what follows, we

shall prove that (i) Quantum mechanics in a finite domain of configuration space can be expressed in terms of su(2) algebra; (ii) Quantum mechanics in an infinite domain of configuration space can be expressed in terms of su(1,1) algebra.

The proof depends on a novel and profound fact: Lie algebras with different mathematical structures may have nonlinear relations, i.e., different Lie algebras may reach their unification in a nonlinear domain. This observation may have important physical consequences. In view of the well-known fact that a Lie algebra may describe a hierarchy of quantum motion modes with raising and lowering operators representing elementary excitations and deexcitations respectively, the above Lie algebra unification may imply a corresponding physical unification of different quantum motion modes in a deeper nonlinear domain.

Our proof is given for the one-dimensional case. The generalization to multiple dimensions is straightforward. Since quantum mechanics in configuration space is conventionally expressed in terms of the Heisenberg algebra, our proof is thus equivalent to proving: (i) H-algebra in a finite domain can be expressed in terms of su(2) algebra; (ii) H-algebra in an infinite domain can be expressed in terms of su(1,1) algebra.

(i) Consider H-algebra in a finite domain, namely,

$$\{ \theta , \hat{P}_\theta , 1 \} , \tag{4}$$

with

$$\hat{P}_\theta = -i\frac{\partial}{\partial\theta} \tag{5a}$$

$$\theta \in [0 , 2\pi] . \tag{5b}$$

Let

$$J_3 = \hat{P}_\theta = -i\frac{\partial}{\partial\theta} , \tag{6a}$$

which is one of the su(2) operators. Introduce the other su(2) operators as follows,

$$J_1 = j \cos\theta - \sin\theta\frac{\partial}{\partial\theta} , \tag{6b}$$

$$J_2 = j \sin\theta + \cos\theta\frac{\partial}{\partial\theta} , \tag{6c}$$

or

$$J_+ = J_1 + i\, J_2 = \exp\{+i\theta\}(j - J_3), \qquad (6d)$$

$$J_- = J_1 - i\, J_2 = \exp\{-i\theta\}(j + J_3). \qquad (6e)$$

It is not difficult to prove the following su(2) commutators,

$$[\, J_i \,,\, J_j \,] = i\,\epsilon_{ijk}\, J_k, \qquad (7a)$$

or

$$[\, J_3 \,,\, J_\pm \,] = \pm\, J_\pm, \qquad (7b)$$

$$[\, J_+ \,,\, J_- \,] = 2\, J_3 \quad . \qquad (7c)$$

It is easy to show that just as in the H-algebra case, the Casimir operator of the su(2) representation, (following from equations (6a-e)), is also a constant,

$$J^2 = J_1^2 + J_2^2 + J_3^2 = j(j+1), \qquad (8)$$

where j is the irreducible label of su(2) algebra.

From eqs.(7a-c), we have

$$\exp\{\pm i\theta\} = J_\pm / (j \mp J_3), \qquad (9)$$

which leads to

$$\cos\theta = 1/2\, \{\, J_+/(j - J_3) + J_-/(j + J_3) \,\}, \qquad (9a)$$

or

$$\theta = \arccos\{1/2[J_+/(j-J_3)+J_-/(j+J_3)]\} \,. \qquad (9b)$$

Therefore the H-algebra can be expressed in terms of the su(2) generators,

$$\theta = \arccos\{1/2[J_+/(j-J_3)+J_-/(j+J_3)]\}, \qquad (10a)$$

$$\hat{P}_\theta = J_3 \quad . \qquad (10b)$$

The Hamiltonian is now

$$H(\theta,\hat{P}_\theta) = H(\arccos\{1/2[J_+/(j-J_3)+J_-/(j+J_3)]\},\ J_3)$$

$$= H(su(2)) \ . \qquad (11a)$$

For the N-dimensional case,

$$H(X_i,\ \hat{P}_i) = H(\prod_i^N \otimes h(i))$$

$$= H(\prod_i^N \otimes su_i(2)) \ . \qquad (11b)$$

Thus quantum mechanics in a finite domain can be expressed in terms of su(2) algebra[4].

(ii) Now consider H-algebra in an infinite domain, namely,

$$\{ X,\ \hat{P},\ 1 \} , \qquad (12a)$$

where

$$X \in [-\infty, +\infty] , \qquad (12b)$$

$$\hat{P} = -i\frac{\partial}{\partial X} \ . \qquad (12c)$$

Introduce Su(1,1) generators as follows,

$$K_1 = 1/2(X\hat{P}^2 - X),\ K_2 = X\hat{P},\ K_3 = 1/2(X\hat{P}^2 + X) , \qquad (13a)$$

or

$$K_\pm = K_1 \pm iK_2 , \quad K_0 = K_3 \ . \qquad (13b)$$

It is easy to show that K_i constitute su(1,1) algebra, namely,

$$[K_1, K_2] = -iK_3 ,\ [K_2, K_3] = iK_1 ,\ [K_3, K_1] = iK_2 , \qquad (14a)$$

or

$$[K_0, K_\pm] = \pm K_\pm , \quad [K_+, K_-] = -2K_0 , \qquad (14b)$$

and

$$K^2 = K_1^2 + K_2^2 - K_3^2 = 0 \ . \qquad (14c)$$

The inverse is

$$\hat{P}^2 = (K_3 - K_1)^{-1}(K_3 + K_1) \ , \quad \hat{P} = \pm \{ (K_3 - K_1)^{-1}(K_3 + K_1) \} \ , \qquad (15a)$$

$$X = (K_3 - K_1) \ . \qquad (15b)$$

Thus quantum mechanics in an infinite domain of configuration space can be expressed in terms of su(1,1) algebra,

$$H(X,\hat{P}) = H((K_3 - K_1), \pm\{ (K_3 - K_1)^{-1}(K_3 + K_1) \})$$

$$= H(su(1,1)) . \qquad (16a)$$

For the N-dimensional case,

$$H(X_i, \hat{P}_i) = H(\prod_i^N \otimes su_i(1,1)) \ . \qquad (16b)$$

Before deepening our discussion of basic physical-mathematical problems, we would like to give several examples to illustrate the above general formalism. Since this work was stimulated by our study of quantum chaos in an attempt to reformulate several famous models in terms of familiar Lie groups and to make the Dynamical Group Approach to quantum irregular motions workable, we would like to give two examples from four famous models [5] .

(A) The Kicked Quantum Rotator Model (KQRM) [6], whose classical correspondence is the famous standard mapping in the study of classical chaos, has the following Hamiltonian in the H-algebraic representation,

$$H = \hbar^2/2I \ (-i\frac{\partial}{\partial\theta})^2 + \varepsilon_0 \cos\theta \sum_n \delta(t-nT) \ . \qquad (17a)$$

But in terms of su(2) algebra, it reads

$$H = \hbar^2/2I \ J_3^2 + \varepsilon_0/2 \ \{ J_+ /(j-J_3) + J_- /(j+J_3) \} \sum_n \delta(t-nT) \ . \qquad (17b)$$

(B) The One Dimension Hydrogen Atom (ODHA) is another model extensively studied by theorists and the results can be tested by experiments [7]. In the H-algebraic representation, its Hamiltonian is

$$H = 1/2 \ \hat{P}^2 - 1/Z + Z \ E \cos\omega t \ . \qquad (18a)$$

In the su(1,1) algebraic representation, it reads

$$H = 1/2\ (K_3 - K_1)^{-1}(K_3 + K_1) - (K_3 - K_1)^{-1} + (K_3 - K_1)\ E \cos\omega t\ . \quad (18b)$$

The advantage of the alternative algebraic representations resides in that, firstly they provide more mathematical tools to solve a given physical problem. It may happen that in one algebraic representation the problem seems difficult to solve, while in other algebraic representation it becomes easier. Secondly, it is likely that the new algebraic representation may provide a new insight into the physical problems. For example, in the KQRM, since the H-algebra is noncompact and its unitary irreducible representation is of infinite dimensions, any truncation to finite dimensional Hilbert space always leads to an approximation and in general one doesn't possess a criterion for judging whether a truncation is good or not, short of performing numerical calculations.However,in the su(2) algebraic representation,any j-irreducible representation space is a reasonable subspace from the point of view of su(2) algebra.Therefore a good truncation scheme is naturally that it should assume $m = -j, -j+1, \ldots\ldots j-1, j$, since this is an invariant subspace according to the su(2) algebra. Yet, the dissipative properties of the KQRM can be understood even better in the su(2)-algebraic representation. It is not difficult to prove in the su(2) representation that the KQRM Hamiltonian (17b) possesses no conserved quantity except the constant Casimir $J^2 = j(j+1)$. It is the kicking term that destroys the constant of motion by the ladder operators $J_\pm$. As the dynamical breaking term is strong enough to reach the excitation energy, i.e., $\mathcal{E}_0 T / I \geqslant 1$, the dynamical symmetry of the system is thus seriously broken and the system therefore begins to be driven to chaotic motion by the perturbation $V(t)$[6] . Since the kinetic energy is $(\hbar^2/2I)m^2$, both J_+ and J_- play the same role of excitations if the system is initially in a small m_0 states (for instance, $m_0 = 0$). Each kick brings the system a step far away from its starting point and increases its excitation energy.As j is finite, long term kicks will bring the system to a certain kind of stable distribution $P(m)$. This leads to the saturation of the energy dissipation. As j approaches infinite, the increase of the excitation energy due to each kick will continue for ever.

The above intriguing results also raise challenging questions.

In a nonlinear domain of Lie algebras, there is no absolute gap between different Lie algebras. Yet there even exists a nonlinear relations between a compact Lie algebra (su(2)) and a noncompact Lie algebra (H-algebra). This observation makes us conjecture that in the nonlinear domain of Lie algebras, there may exist a unification or equivalence among different Lie algebras. The challenging question is how one can establish such a nonlinear algebraic domain and explore its unification, classification, and how the classical Lie algebras are related to the nonlinear algebraic domain.

The physical significance of the above mathematical results are as follows. Suppose the Lie algebra L_1 dscribes a quantum system Q_1 and produces energy(mass) spectrum $E_1(M_1)$, the Lie algebra L_2 describe a different quantum system Q_2 and produces energy(mass) spectrum $E_2(M_2)$. A nonlinear relation between L_1 and L_2 implies a corresponding nonlinear relation between the two different quantum systems Q_1 and Q_2, as well as their energy(mass) spectra $E_1(M_1)$ and $E_2(M_2)$. If this conjecture is true, it will be an attractive and exciting field to explore.

Let us look at how the nonlinear transformations eqs.(6a-e) and (9) fill the gap between compact Lie algebra su(2) and noncompact Heiseberg Lie algebra h. Since the su(2) representation, eq.(6a-e), is not in a Hermitian form (but it is related to Hermitian forms through non-unitary-similarity transformations) its irreducible bases constitute a set of bi-orthogonal bases[8].The eigensolutions of J_3 are

$$|j\ m\rangle = [(2j)!\ /(j-m)!\ (j+m)!\ ]^{\frac{1}{2}}\ \mathrm{Exp}\{+im\theta\}\ , \qquad (19a)$$

and their orthonormal duals are

$$\widetilde{|j\ m\rangle} = [(j-m)!\ (j+m)!\ /(2j)!\ ]^{\frac{1}{2}}\ /\ 2\pi\ \ \mathrm{Exp}\{+im\theta\}\ . \qquad (19b)$$

The bi-orthonormal condition is

$$\langle \widetilde{j\ m}\,|\,j\ m'\rangle = \delta_{mm'}\ . \qquad (19c)$$

It is straightforward to check that,

$$J^2|j\ m\rangle = j(j+1)|j\ m\rangle\ , \qquad (20a)$$

$$J_3 \,|\, j\ m\rangle = m \,|\, j\ m\rangle \ , \tag{20b}$$

$$J_\pm \,|\, j\ m\rangle = [(j \mp m)(j \pm m + 1)]^{\frac{1}{2}} \,|\, j\ m \pm 1\rangle \ . \tag{20c}$$

The compactness of SU(2) is manifested by the ladder operators $J_\pm$, which contain proper cutoff factors. The cutoff factors $(j \mp J_3)$ in eqs.(6d,e) and the cutoff factors $[\ (j \mp m)\ ]^{\frac{1}{2}}$ in eq.(20c) give rise of an automatic cutoff of the irreducible bases and thus guarantee the finite dimensional property of the su(2) irreducible representations. On the contrary, the nonlinear transformation (9) clearly indicates that the cutoff factors of $J_\pm$ are exactly canceled by the denominators $(j \mp J_3)$ and makes the transformed operators lose the cutoff property. Thus the gap between the compact su(2) algebra and the noncompact H-algebra is filled up by eliminating the cutoff factors. This can only be realized by nonlinear transformations. Therefore we are led to the observation that gaps in the linear case can be filled up in the nonliear case. This indicates again that the nonliear mathematics is powerful in the unification of the physical-mathematical world.

We should note that the nonlinear transformations from H-algebra to the su(2) and su(1,1), i.e., eqs.(6a-c) and (13a), are also valid at the classical level. It is obvious that the H-algebra in the form of quantum commutators is still valid in the form of Poisson brackets. Here we show you that the nonlinear expressions of su(2) and su(1,1) algebras still preserve their algebraic relations at the classical level. From eqs.(6a-c) and (13a), it is not difficult to confirm the algebraic relations of su(2) and su(1,1) in the form of Poisson brackets, namely,

$$\{\ J_i^c\ ,\ J_j^c\ \} = \epsilon_{ijk}\ J_k^c, \tag{21}$$

$$\{\ K_1^c\ ,\ K_2^c\ \} = -\ K_3^c\ ,\quad \{\ K_2^c\ ,\ K_3^c\ \} = K_1^c\ ,\quad \{\ K_3^c\ ,\ K_1^c\ \} = K_2^c\ . \tag{22}$$

Where J_i^c and K_i^c are classical quantities and related to classical canonical variables $\{\ \theta\ ,\ P_\theta\ \}$ and $\{\ X,\ P\ \}$ through the same nonlinear transformations (6a-c) and (13a). The existence of nonlinear relations of different Lie algebras at the level of poisson brackets may have further implications.

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