Submitted to Journal Physical Review C

LBL-2958
Preprint c. 2

COUPLED-CHANNEL PION-NUCLEUS CHARGE EXCHANGE REACTIONS

M. A. Nagarajan and W. L. Wang

June 1974

Prepared for the U. S. Atomic Energy Commission
under Contract W-7405-ENG-48

TWO-WEEK LOAN COPY

This is a Library Circulating Copy which may be borrowed for two weeks. For a personal retention copy, call Tech. Info. Division, Ext. 5545

LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY

LBL-2958
c. 2

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

COUPLED-CHANNEL PION-NUCLEUS CHARGE EXCHANGE REACTIONS*

M. A. Nagarajan and W. L. Wang

Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

June 1974

ABSTRACT

We discuss a coupled-channel formulation for single-charge exchange reactions in pion-nucleus scattering. Within an "adiabatic approximation" and the framework of the isobar-doorway model, we obtain an algebraic solution for the scattering amplitudes. We also introduce, as our second approximation, a separable expansion for the partial-wave form factors and obtain a set of linear equations for quantities which appear in the T-matrice for elastic and charge exchange interactions.

NUCLEAR REACTIONS: Pion-Nucleus Scattering, (3,3) Resonance Charge Exchanges, Coupled-Channel Calculations

I. INTRODUCTION

In this report, we shall discuss a coupled-channel approach to pion-nucleus charge-exchange reactions near the (3,3) resonance region. These processes are of particular interest because they are dominated by the $\Delta(1231)$ isobar formation in the nucleus. In the distorted wave Born approximation (DWBA), the effects of the (3,3) resonance on the single and double charge exchange reactions have been studied in detail.[1] Within similar approximations, we shall show that the channel coupling effects may also be treated without much complication. The simplfying hypotheses are quite similar to these used in the isobar-doorway model,[2] which was used in the DWBA calculation of Ref. 1. In Section II, we shall discuss a coupled-channel formalism using the concept of isobar-doorway states in the elastic and charge exchange channels. We use an adiabatic approximation to simplify our coupled equations, and obtain a formal expression of the T-matrix in terms of a resonant and a nonresonant contributions. We then explicitly introduce the approximations of the isobar-doorway model in Section III, where we obtain solutions of the coupled-channel equations algebraically in a closed form. The resonant couplings may be treated exactly by an N-point integration, or approximately by introducing a separable expansion of the particle wave nuclear form factors. The separable approximation is discussed in the appendices. We draw some formal conclusions and remarks on the formalism in Section IV.

II. COUPLED-CHANNEL FORMALISM

Our formulation with the doorway concept is best carried out by using the projection-operator techniques of Feshbach;[3] this has been discussed in detail in Ref. 1. We shall recall some important definitions here. Let us define (1) P_1-space as the incident channel and P_2-space the exit (charge exchanged) channel, with the respective nuclei at their ground states, (2) the Q-space as the isobar compound state where a nucleon is changed into a stable $\Delta(1231)$ isobar, and (3) the q-space as to include the complementary spage; i.e. $q=1-Q-P_1-P_2$. Thus the q-space contains all the reaction channels except the elastic (P_1) and the charge exchange (P_2) channels. These spaces are coupled to one another, except by the doorway-state hypothesis there is no direct coupling between the P ($P \equiv P_1+P_2$) and the q-spaces. The nonresonant π-nucleon charge exchange interactions couple P_1 and P_2 directly, but only the resonant π-nucleon interaction can create an isobar in the nucleus, which may decay into q, P_1 or P_2 spaces.

With the above separation of spaces, we obtain the following coupled equations

$$(E - \mathcal{H}_{11})\, P_1\, \Psi = \mathcal{H}_{12}\, P_2 \Psi \quad , \tag{1}$$

and

$$(E - \mathcal{H}_{22})\, P_2\, \Psi = \mathcal{H}_{21}\, P_1 \Psi \quad , \tag{2}$$

where E and Ψ are the energy and the total wave function of the system. The projected wave function in the i space is $P_i \Psi$ The effective Hamiltonians in Eqs. (1) and (2) are defined as $\mathcal{H}_{ij} = P_i \mathcal{H} P_j$ where

$$\mathcal{H} = H + HQ(E-\tilde{H}_{QQ})^{-1} QH \quad , \tag{3}$$

which is modified due to coupling to the Q-space. The q-space has been suppressed through the use of $\tilde{H}$

$$\tilde{H} = H + Hq(E-H_{qq})^{-1} qH \quad , \tag{4}$$

where $H_{qq} = qHq$ and $H_{QQ} = QHQ$. The total Hamiltonian H may be written as

$$H = H_A + T + V \quad , \tag{5}$$

where H_A is the baryon Hamiltonian (i.e. the nuclear Hamiltonian including a possible isobar), T is the kinetic energy operator of the pion. The π-nucleus interaction V may be separated into a nonresonant and a resonant parts, V^{NR} and V^R respectively, and the Coulomb interaction.

In order to reduce the many-body equations, Eqs. (1) and (2), to two-body coupled equations, we introduce the product wave function:

$$P_i \Psi = \Psi_i \Phi_i \quad , \tag{6}$$

where Ψ_i describes the pion state and Φ_i the nuclear state in channel i. By multiplying on the left of Eqs. (1) and (2) by Φ_1^+ and Φ_2^+ respectively and integrating over nuclear co-ordinates, we obtain

$$[E - E_1 - T - V_{11}] \Psi_1 = V_{12} \Psi_2 \quad , \tag{7}$$

$$[E - E_2 - T - V_{22}] \Psi_2 = V_{21} \Psi_1 \quad , \tag{8}$$

where E_i is the nuclear ground state energy in channel i, i.e.

$$E_i = \langle \Phi_i | H_A | \Phi_i \rangle \quad . \tag{9}$$

The diagonal V_{ii} and coupling V_{ij} interactions are given as

$$V_{ij} = V_{ij}^{NR} + V_{ij}^{R} + V_{ii}^{C} \delta_{ij} \quad , \tag{10}$$

where

$$V_{ij}^{NR} = \langle \Phi_i | V^{NR} | \Phi_j \rangle \quad , \tag{11}$$

$$V_{ij}^{R} = \langle \Phi_i | HQ(E-\tilde{H}_{QQ})^{-1} QH | \Phi_j \rangle \quad , \tag{12}$$

and V_{ii}^{C} is the Coulomb interaction. Eqs. (10) and (11) show that we are interested only in the single-charge exchange (SCX) reactions, which may go through one intermediate $\Delta(1231)$ resonance. Eqs. (6) and (7) are the desired coupled equations for our discussion.

To simplify the coupled equations further, let us now introduce an "adiabatic approximation" for SCX reactions. For the nuclei near the doubly closed shells, e.g. ^{44}Ca, the final nucleus will differ from the initial nucleus only by the Coulomb energy. This difference is partly compensated by the change of the charge state of the pion. For this reason, we may assume

$$E_1 + V_{11}^{C} = E_2 + V_{22}^{C} \quad , \tag{13}$$

or we may take $E_i + V_{ii}^{C} = 0$ (the reference energy). The nonresonant π-nucleus interactions are rather small compared to the pion kinetic energy and will

depend little on the isospin of the target, so we may assume

$$V_{11}^{NR} = V_{22}^{NR} \quad . \tag{14}$$

The most important difference between the left-hand sides of Eqs. (7) and (8) is perhaps in the resonant diagonal interaction V_{ii}^{R}. From Eq. (12) and the approximation used in the isobar-doorway model [1,2], we have

$$V_{ij}^{R}(\vec{k}',\vec{k}) = \frac{\eta_{ij}}{2\pi}\sqrt{\frac{\Gamma_{\pi N}(k')}{\Gamma_{\pi N}(k)}}\,\left|\tilde{t}_{33}(E)\right|\hat{k}'.\hat{k}\;F_{ij}(\vec{k}',\vec{k}) \tag{15}$$

where $|\tilde{t}_{33}(k)|$ is the modified π-nucleon (3,3) resonance amplitude in the medium, and $F_{ij}(\vec{k}'-\vec{k})$ is the nuclear form factor. The factor η_{ij} depends on the charge state of the pion. For the reactions and nuclei under consideration, we expect $|\tilde{t}_{33}(E)|$ and $F_{ii}(\vec{k}'-\vec{k})$ to be independent of i. So the major difference in the optical potentials in the two channels is due to the factor η_{ii}. For a target nucleus A with neutron number N and proton number Z, we have

for π^+ $\quad \eta_{ii} = Z + \frac{N}{3}$, (16)

for π^o $\quad \eta_{ii} = \frac{2}{3}(Z + N)$, (17)

and for π^- $\quad \eta_{ii} = \frac{Z}{3} + N$. (18)

For SCX reaction on A, we expect a change in the resonant optical potential as

$$\Delta\eta_{12} \doteq \left|\frac{\eta_{11}(\pi^{\pm}) - \eta_{22}(\pi^{o})}{\eta_{22}(\pi^{o})}\right| = \frac{T_o}{A} \tag{19}$$

where T_o is the isospin projection of the target, $T_o \equiv \frac{1}{2}(N - Z)$. From this estimate, we know that even the resonant optical potentials in both channels are not too different if $T_o \ll A$.

With the above considerations, we may now assume the charge exchange reaction to be adiabatic and write the coupled equations as

$$[E - T - V_o^{NR} - V_o^R]\Psi_1 = [V_{12}^{NR} + V_{12}^R]\Psi_2 \quad , \tag{20}$$

$$[E - T - V_o^{NR} - V_o^R]\Psi_2 = [V_{21}^{NR} \quad V_{21}^R]\Psi_1 \quad , \tag{21}$$

or as a matrix equation

$$\{(E - H_o)\underset{\approx}{I}\}\underset{\sim}{\Psi} = \underset{\approx}{V}\underset{\sim}{\Psi} \quad , \tag{22}$$

with

$$H_o = T + V_o^{NR} + V_o^R \quad . \tag{23}$$

In Eq. (22) $\underset{\approx}{I}$ is a 2 x 2 unit matrix, $\underset{\sim}{\Psi}$ are two component vectors

$$\underset{\sim}{\Psi} = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} \quad . \tag{24}$$

The interaction matrix $\underset{\approx}{V}$ is defined as

$$\underset{\approx}{V} = \underset{\approx}{V_{12}^{NR}} + \underset{\approx}{V_{12}^R} \quad , \tag{25}$$

where the nonresonant interaction matrix

$$\underset{\approx}{V_{12}^{NR}} = \begin{bmatrix} 0 & V_{12}^{NR} \\ V_{21}^{NR} & 0 \end{bmatrix} \tag{26}$$

and $$\underset{\approx}{V}_{12}^{R} = \begin{bmatrix} 0 & V_{12}^{R} \\ V_{21}^{R} & 0 \end{bmatrix} \quad . \tag{27}$$

We therefore may treat Eq. (22) as a standard two potential scattering problem for $\underset{\approx}{V}_{12}^{NR}$ and $\underset{\approx}{V}_{12}^{R}$. The complete T-matrix may be written as[4]

$$\underset{\approx}{T} = \underset{\approx}{T}^{R} + \underset{\approx}{\Omega}_{R}^{(-)\dagger} \underset{\approx}{T}^{NR} \underset{\approx}{\Omega}_{R}^{(+)} \quad , \tag{28}$$

where $\underset{\approx}{T}^{R}$ is the solutions of Eq. (22) without $\underset{\approx}{V}_{12}^{NR}$. This is our resonant interaction T-matrix, T^{R}, is a 2x2 matrix describing the elastic and the charge exchange scattering processes. The wave matrices $\underset{\approx}{\Omega}^{(\pm)}$ for the resonant interaction are defined as

$$\underset{\approx}{\Omega}_{R}^{(\pm)} = \underset{\approx}{I} + \frac{1}{E - H_{o} \pm i\eta} \underset{\approx}{T}^{R} \quad . \tag{29}$$

Formally, we may write the solutions of T^{R} and T^{NR} as given by

$$\underset{\approx}{T}^{R} = \underset{\approx}{V}_{12}^{R} + \underset{\approx}{V}_{12}^{R} (E - H_{o} + i\eta)^{-1} \underset{\approx}{T}^{R} \quad , \tag{30}$$

and

$$\underset{\approx}{T}^{NR} = \underset{\approx}{V}_{12}^{NR} + \underset{\approx}{V}_{12}^{NR} \left[(E - H_{o} + i\eta) \underset{\approx}{I} + \underset{\approx}{V}_{12}^{R} \right]^{-1} \underset{\approx}{T}^{NR} \quad . \tag{31}$$

Equation (30) is the formal coupled-channel equation for the resonant charge exchange interactions, but with full resonant and nonresonant distortions in both channels. However, Eq. (31) also contain resonant charge exchange interaction $\underset{\approx}{V}_{12}^{R}$, besides the nonresonant charge exchange interactions.

III. ISOBAR-DOORWAY APPROXIMATIONS

Next we will simplify the above formal solutions in the isobar-doorwas models. The approximation we have introduced so far is essentially the adiabatic approximation, i.e. we assume the diagonal optical potentials in both channels to be the same. This approximation makes it possible to solve the coupled equations, Eqs. (7) and (8), in a simple way, if V_{12} and V_{21} also have proper symmetry. Clearly, this approximation was not introduced in case of the DWBA calculation in Ref. 1, but, to include the coupled-channel effects in a simple way, it is well justified to use the adiabatic approximation in our present formulation. The complete coupled-channel calculation using Eqs. (20) and (21) is still complicated. Nevertheless we shall attempt to solve only the coupling due to the resonant interaction V_{12}^R exactly in the isobar-doorway model. The nonresonant coupling interaction may be solved by using a DWBA, as described in Ref. 1, or by a full coupled-channel calculation using the usual optical model. We also discuss the nonresonant interaction for completeness, since it is also affected by the resonant interactions.

The resonant coupled-equations are given as

$$[E - T - V_o^{NR} - V_o^R]\ \Psi_1^R = V_{12}^R\ \Psi_2^R \quad , \tag{32}$$

$$[E - T - V_o^{NR} - V_o^R]\ \Psi_2^R = V_{21}^R\ \Psi_1^R \quad , \tag{33}$$

where we have the superscript R to denote the effects of the resonant charge exchange coupling. Since $V_{12}^R = V_{21}^R$, these equations may be solved by using the techniques referred to as the "exact resonance approximation."[5] Equations (32) and (33) may be combined to give the following form

$$[E - K - V_o^{NR}]\ \Psi_\pm = [V_o^R \pm V_{12}^R]\ \Psi_\pm \quad , \tag{34}$$

where we have defined

$$\Psi_\pm = \Psi_1^R \pm \Psi_2^R \quad . \tag{35}$$

We now instead solve for $\Psi_\pm$ from Eq. (34). The proper boundary conditions for Eqs. (32) and (33) should also be incorporated in the solutions for $\Psi_\pm$. Only Ψ_1^R has both incoming and outgoing waves; the charge-exchange Ψ_2^R channel has purely outgoing waves. Let us first define the nonresonant wave functions in Ψ_1^R as

$$[E - T - V_o^{NR}]\phi = 0 \quad , \tag{36}$$

which includes a plane wave solution plus a spherical outgoing wave due the nonresonant potential V_o^{NR}. Since V_o^{NR} is small, ϕ may be taken to be plane waves, (see also Ref. 1). The complete solution of Eq. (34) in the momentum representation may be written as

$$\Psi_{\vec{k}_o(\pm)}(\vec{k}) = \phi_{\vec{k}_o}(\vec{k}) + \frac{1}{E - E_k + i\eta} \int d\vec{k}'[V_o^R(\vec{k},\vec{k}') \pm V_{12}^R(\vec{k},\vec{k}')]\Psi_{\vec{k}_o(\pm)}(\vec{k}') \tag{37}$$

where $\vec{k}_o$ denotes the incident momentum, E_k is the energy associated with momentum $\vec{k}$. The diagonal matrix element $V_o^R(\vec{k},\vec{k}')$ has been given by Eq. (15), with i = j. The off-diagonal matrix element as given by Eq. (15) involves $F_{12}(\vec{k},\vec{k}')$, the transition nuclear form factor from the initial state 1 to the final state 2, and the coefficient η_{12} depending on the charge states of the pions and the isospin of the target. In Eq. (37), we have taken the nonresonant propagator $(E - T - V_o^{NR})^{-1}$ of Eq. (36) to be a plane wave propagator, i.e., neglecting the effects of the nonresonant potential on the resonant interactions. This approximation is reasonable near the resonance, and greatly simplifies the solution.

To obtain a partial-wave expansion of Eq. (37), we introduce the following expressions: (1) For the wave functions, we have

$$\Psi_{\vec{k}_o(\pm)}(\vec{k}) = 4\pi \sum_{LM} Y^*_{LM}(\hat{k}) Y_{LM}(\hat{k}_o) F_{L(\pm)}(k,k_o) \quad , \tag{38}$$

and

$$\phi_{\vec{k}_o}(\vec{k}) = 4\pi \sum_{LM} Y^*_{LM}(\hat{k}) Y_{LM}(\hat{k}_o) \mathcal{F}_L(k,k_o) \quad . \tag{39}$$

(2) For the nuclear form factors

$$F_{11}(\vec{k},\vec{k}') = 4\pi \sum_{\ell m} Y^*_{\ell m}(\hat{k})\, Y_{\ell m}(\hat{k}')\, \rho_\ell(k,k') \quad , \tag{40}$$

and

$$F_{12}(\vec{k},\vec{k}') = 4\pi \sum_{\ell m} Y^*_{\ell m}(\hat{k})\, Y_{\ell m}(\hat{k}')\, \tilde{\rho}_\ell(k,k') \quad . \tag{41}$$

The L-th partial-wave component of the complete solution, Eq. (37), becomes

$$F_{L(\pm)}(k,k_o) = \mathcal{F}_L(k,k_o) + \frac{1}{E - E_k + i\eta} \frac{2\left|\tilde{t}_{33}(k)\right|}{\sqrt{\Gamma_{\pi N}(k)}} \sum_\ell (2\ell+1) \begin{pmatrix} \ell & 1 & L \\ 0 & 0 & 0 \end{pmatrix}^2$$

$$\int k'^2 dk' \sqrt{\Gamma_{\pi N}(k')} \left[\eta_{11}\rho_\ell(k,k') \pm \eta_{12}\tilde{\rho}_\ell(k,k')\right] F_{L(\pm)}(k',k_o) \quad . \tag{42}$$

where the round bracket denotes a Wigner 3-j symbol. This is an integral equation for $F_{L\pm}(k,k_o)$, which can be solved by the following two methods.

By using N-point integration technique, we may write Eq. (42) as

$$F_{L(\pm)}(k,k_o) = f_L(k,k_o) + \sum_{n=1}^{N} W(k_n)\, k_n^{\,2}\, K_L^{(\pm)}(k,k_n)\, F_{L(\pm)}(k_n,k_o) \quad , \tag{43}$$

where $W(k_n)$ are the weight factors for the N-point integration. The kernel $K_L^{(\pm)}(k,k')$ is defined as

$$K_L^{(\pm)}(k,k') = \frac{1}{E - E_k + i\eta} \frac{2\left|\tilde{t}_{33}(k)\right|}{\sqrt{\Gamma_{\pi N}(k)}} \sum_{\ell} (2\ell+1) \begin{pmatrix} \ell & 1 & L \\ 0 & 0 & 0 \end{pmatrix}^2$$

$$\sqrt{\Gamma_{\pi N}(k')} \left[\eta_{11}\rho_\ell(k,k') \pm \eta_{12}\tilde{\rho}_\ell(k,k')\right], \tag{44}$$

If we take $k = k_m$ in Eq. (43), we have the following matrix equation:

$$\sum_{n=1}^{N} \left[\delta_{mn} - W(k_n) k_n^{\,2}\, K_L^{(\pm)}(k_m,k_n)\right] F_{L(\pm)}(k_n,k_o) = f_L(k_m,k_o) \quad , \tag{45}$$

which gives

$$F_{L(\pm)}(k_n,k_o) = \sum_m \left[M_{(\pm)}^{-1}\right]_{nm} f_L(k_m,k_o) \quad , \tag{46}$$

where the inverse matrix is defined as

$$\left[M_{(\pm)}\right]_{mn} = \delta_{mn} - W(k_n)k_n^{\,2}\, K_L^{(\pm)}(k_m,k_n) \quad . \tag{47}$$

The above method is exact. The second approach begins with a separable expansion for the density in Eq. (42). We have, for the ℓ-th components (see Appendix I)

$$\rho_\ell(k,k') = \sum_{m=o}^{\infty} U_\ell^{(m)}(k)\, U_\ell^{(m)}(k') \quad , \tag{48}$$

and

$$\tilde{\rho}_\ell(k,k') = \sum_{m=o}^{\infty} V_\ell^{(m)}(k)\, V_\ell^{(m)}(k') \quad , \tag{49}$$

where $U_\ell^{(m)}(k)$ and $V_\ell^{(m)}(k)$ are orthogonal functions. We may now rewrite Eq. (42) as

$$F_{L(\pm)}(k,k_o) = \mathcal{F}_L(k,k_o) + \frac{1}{E - E_k + i\eta}\, \frac{2\left|\tilde{t}_{33}(k)\right|}{\sqrt{\Gamma_{\pi N}(k)}} \sum_{\ell m} (2\ell+1) \begin{pmatrix} \ell & 1 & L \\ 0 & 0 & 0 \end{pmatrix}^2 \left[\eta_{11} U_\ell^{(m)}(k)\, \Lambda^{(m)}_{\ell L(\pm)}(k_o) \pm \eta_{12} V_\ell^{(m)}(k)\, \Omega^{(m)}_{\ell L(\pm)}(k_o)\right], \tag{50}$$

where

$$\Lambda^{(m)}_{\ell L(\pm)}(k_o) = \int k'^{\,2} dk' U_\ell^{(m)}(k') \sqrt{\Gamma_{\pi N}(k')}\; F_{L(\pm)}(k',k_o) \quad , \tag{51}$$

and

$$\Omega^{(m)}_{\ell L(\pm)}(k_o) = \int k'^2 dk' \; V_\ell^{(m)}(k') \sqrt{\Gamma_{\pi N}(k')} \; F_{L(\pm)}(k',k_o) \quad . \qquad (52)$$

If we multiply both sides of Eq. (50) by $k^2 U_\ell^{(m)}(k)\sqrt{\Gamma_{\pi N}(k)}$ and $k^2 V_\ell^{(m)}(k)\sqrt{\Gamma_{\pi N}(k)}$ respectively and integrate over k, we obtain the following set of linear equations for $\Lambda^{(m)}_{\ell L(\pm)}(k)$ and $\Omega^{(m)}_{\ell L(\pm)}(k)$:

$$\sum_{\ell' m'} \left\{ \left[\delta_{mm'}\delta_{\ell\ell'} - \eta_{11} M^{mm'}_{\ell\ell'} \right] \Lambda^{(m')}_{\ell' L(\pm)}(k_o) \mp \eta_{12} N^{mm'}_{\ell\ell'} \, \Omega^{(m')}_{\ell' L(\pm)}(k_o) \right\}$$

$$= \omega^{(m)}_{\ell L}(k_o) \quad , \qquad (53)$$

and

$$\sum_{\ell' m'} \left\{ \left[\delta_{mm'}\delta_{\ell\ell} \mp \eta_{12} P^{mm'}_{\ell\ell'} \right] \Omega^{(m')}_{\ell' L(\pm)}(k_o) - \eta_{11} N^{mm'}_{\ell\ell'} \, \Lambda^{(m')}_{\ell' L(\pm)}(k_o) \right\}$$

$$= \lambda^{(m)}_{\ell L}(k_o) \quad , \quad , \qquad (54)$$

where we have defined

$$M^{mm'}_{\ell\ell'} = 2(2\ell'+1) \begin{pmatrix} \ell' & 1 & L \\ 0 & 0 & 0 \end{pmatrix}^2 \int_0^\infty \frac{k^2 dk}{E-E_k + i\eta} \left| \tilde{t}_{33}(k) \right| U_{\ell'}^{m'}(k) \; U_\ell^{m}(k) \qquad (55)$$

$$N^{mm'}_{\ell\ell'} = 2(2\ell'+1) \begin{pmatrix} \ell' & 1 & L \\ 0 & 0 & 0 \end{pmatrix}^2 \int_0^\infty \frac{k^2 dk}{E-E_k + i\eta} \left| \tilde{t}_{33}(k) \right| V_{\ell'}^{m'}(k) \; U_\ell^{m}(k) \, ,$$

(56)

$$P_{\ell\ell'}^{mm'} = 2(2\ell'+1)\begin{pmatrix} \ell' & 1 & L \\ 0 & 0 & 0 \end{pmatrix}^2 \int \frac{k^2 dk}{E-E_k + i\eta} \left|\tilde{t}_{33}(k)\right| V_{\ell'}^{m'}(k)\, V_{\ell}^{m}(k) \quad , \tag{57}$$

$$\omega_{\ell L}^{(m)}(k_o) = \int k^2 dk \sqrt{\Gamma_{\pi N}(k)}\; U_{\ell}^{(m)}(k) \fint_L (k,k_o) \quad , \tag{58}$$

and

$$\lambda_{\ell L}^{(m)}(k_o) = \int k^2 dk \sqrt{\Gamma_{\pi N}(k)}\; V_{\ell}^{(m)}(k) \fint_L (k,k_o) \quad . \tag{59}$$

We may substitute the solutions of Eqs. (53) and (54) into Eq. (50) and obtain the amplitude, $F_{L(\pm)}(k,k_o)$.

To obtain the T-matrix elements from the channel wave functions, we need to consider Ψ_1 and Ψ_2 individually. Let us expand the channel wave functions in momentum space representation as

$$\Psi_{i,\vec{k}_o}(\vec{k}) = 4\pi \sum_{LM} Y_{LM}^*(\hat{k})\, Y_{LM}(\hat{k}_o)\, \chi_{i,L}(k,k_o) \quad , \tag{60}$$

where the partial waves $\chi_{i,L}(k,k_o)$ are related to the amplitudes $F_{L(\pm)}$ as

$$\chi_{1,L}(k,k_o) = \frac{1}{2}\left[F_{L(-)}(k,k_o) + F_{L(-)}(k,k_o)\right] \quad , \tag{61}$$

and

$$\chi_{2,L}(k,k_o) = \frac{1}{2}\left[F_{L(+)}(k,k_o) - F_{L-)}(k,k_o)\right] \quad . \tag{62}$$

From the asymptotic behavior of $\chi_{1,L}$ and $\chi_{2,L}$, we may obtain the elastic scattering and charge exchange amplitudes respectively. First, we may introduce the elastic scattering T-matrix T_{el} and the SCX T-matrix T^R_{SCX}. These T-matrices may be expanded as

$$\langle \vec{k}_o' | T_{el} | \vec{k}_o \rangle = \sum_L (2L+1)\ T_{el}(L)\ P_L(\cos\theta), \quad , \tag{63}$$

and $\cos\theta = \hat{k}_o' \cdot \hat{k}_o$, and

$$\langle \vec{k}_o' | T^R_{SCX} | \vec{k}_o \rangle = \sum_L (2L+1)\ T^R_{SCX}(L)\ P_L(\cos\theta) \quad , \tag{64}$$

where $T_{el}(L)$ and $T^R_{SCX}(L)$ are the L-th partial wave amplitudes. The superscript R denotes the SCX reactions through the resonant interaction. The forms of these T-matrix amplitudes depend on the methods we use to solve Eq. (42). With the N-point integration technique, we obtain numerical solutions for $F_{L(\pm)}(k,k_o)$, so the elastic and SCX amplitudes are given as

$$T_{el}(L) = \frac{(E-E_k + i\eta)}{2} \int \left[K_L^{(+)}(k_o,k')\ F_{L(+)}(k',k_o) + K_L^{(-)}(k_o,k')\ F_{L(-)}(k',k_o) \right] dk' \ , \tag{65}$$

$$T^R_{SCX}(L) = \frac{(E-E_k + i\eta)}{2} \int \left[K_L^{(+)}(k_o,k')\ F_{L(+)}(k',k_o) - K_L^{(-)}(k_o,k')\ F_{L(-)}(k',k_o) \right] dk' \tag{66}$$

where the integration may be carried out with the same N-point integration. With the method of separable expansion, these amplitudes are given as

$$T_{el}(L) = \frac{\left|\tilde{t}_{33}(k_o)\right|}{\sqrt{\Gamma_{\pi N}(k_o)}} \sum_{\ell m} (2\ell+1) \begin{pmatrix} \ell & 1 & L \\ 0 & 0 & 0 \end{pmatrix}^2 \Big\{ \eta_{11} U_\ell^{(m)}(k_o) \left[\Lambda^{(m)}_{\ell L(+)}(k_o) + \Lambda^{(m)}_{\ell L(-)}(k_o) \right]$$

$$+ \eta_{12} U_\ell^{(m)}(k_o) \left[\Omega^{(m)}_{\ell L(+)}(k_o) \; \Omega^{(m)}_{\ell L(-)}(k_o) \right] \Big\} \quad , \qquad (67)$$

and

$$T^R_{SCX}(L) = \frac{\left|\tilde{t}_{33}(k_o)\right|}{\sqrt{\Gamma_{\pi N}(k_o)}} \sum_{\ell m} (2\ell+1) \begin{pmatrix} \ell & 1 & L \\ 0 & 0 & 0 \end{pmatrix}^2 \Big\{ \eta_{11} U_\ell^{(m)}(k_o) \left[\Lambda^{(m)}_{\ell L(+)}(k_o) - \Lambda^{(m)}_{\ell L(-)}(k_o) \right]$$

$$+ \eta_{12} U_\ell^{(m)}(k_o) \left[\Omega^{(m)}_{\ell L(+)}(k_o) - \Omega^{(m)}_{\ell L(-)}(k_o) \right] \Big\} \quad . \qquad (68)$$

Finally, we may write the total SCX T-matrix as

$$\langle \vec{k}' | T_{SCX} | \vec{k} \rangle = \langle \vec{k}' | T^R_{SCX} | \vec{k} \rangle + \langle \Psi_{2,\vec{k}'} | T^{NR}_{SCX} | \Psi_{1,\vec{k}} \rangle \quad , \qquad (69)$$

where the nonresonant part is, explicitly,

$$\langle \Psi_{2,\vec{k}'} | T^{NR}_{SCX} | \Psi_{1,\vec{k}} \rangle = \iint d\vec{q} \; d\vec{q}' \Psi_{2,\vec{k}'}(\vec{q}') \langle \vec{q}' | T^{NR}_{SCX} | \vec{q} \rangle \Psi_{1,\vec{k}}(\vec{q}) \quad , \qquad (70)$$

with $\langle \vec{q}' | T^{NR}_{SCX} | \vec{q} \rangle$ determined from solving Eq. (31) either in a DWBA or a coupled channel calculation.

In this section we have solved the coupled-channel equation in the isobar-doorway model. We have shown that the couplings through the isobar compound states may be treated to all orders. The essential approximations are the adiabatic approximation discussed in the previous section and the isobar-doorway model. Under these approximations, we obtain the integral equation, Eq. (42), for the scattering amplitudes $F_{L(\pm)}(k',k_o)$, which may be solved exactly by using an N-point integration method as shown by Eq. (46). We also introduce separable expansions for the nuclear form factors and obtain an explicit solution for $F_{L(\pm)}(k',k_o)$. Since the first approach is exact, the validity of the second approach may be directly checked by comparing their results. (Some qualitative argument for using the separable expansion is given in Appendix I).

As we have discussed in the appendix, the simple separable form of density, Eq. (48) may not be sufficient. An exact separable expansion is introduced in Eq. (I.5), by which our results in Eq. (54) will be modified to a more complicated form. This point will be discussed in a further investigation.

IV. CONCLUDING REMARKS

We have discussed a simple model for pion-nucleus single change exchange reactions, where the main couplings between the elastic and the SCX channel are treated in a closed form. These coulped-channel effects may be important due to the (3,3) resonance in the continuum. Our observation begins with the "exact resonance" approximations since the optical potentials felt by the incident and the outgoing pions are quite similar. This is particularly true for nuclei with $|N-Z| \ll A$. We then make the usual isobar-doorway assumptions for the resonant elastic and charge exchange interactions, and finally reduce the coupled-channel equations to a pair of integral equations which may be solved exactly. These integral equations may also be solved by using separable expansions of the nuclear form factors.

It is of interest to study the effects of the channel couplings on the elastic and charge exchange scattering by comparing the result of the present formulation with the DWBA as proposed in Ref. (1). We should note, however, that the coupled-channel formalism here depends on the "exact resonance" approximation. Otherwise, the two formalisms are identical. We may diminish the uncertainty due to the exact resonance approximation by applying the theory to larger nuclei, and study the true channel-coupling effects.

We have also proposed the separable expansions for the nuclear form factors. The separable representation shall also be of interest in various particle-nucleus scattering formulations, where a separable particle-particle T-matrix is used to construct a particle-nucleus optical model potential (first-order optical potentials). If the density is separable, the particle-nucleus optical potential is also separable. The formulation may be greatly simplified.

FOOTNOTES AND REFERENCES

* Work performed under the auspices of the U. S. Atomic Energy Commission.

1. M. A. Nagarajan and W. L. Wang, Lawrence Berkeley Laboratory Report - 2934 (June 1974).

2. L. S. Kisslinger and W. L. Wang, Phys. Rev. Letters 30, 1071 (1973).

3. H. Feshbach, A. K. Kerman and R. H. Lemmer, Ann. Phys. (New York) 41, 230 (1967).

4. M. Gell-Mann and M. L. Goldberger, Phys. Rev. 91 398 (1953).

5. See, for example, N. F. Mott and H. S. W. Massey, The Theory of Atomic Collisions, Oxford University Press (1965).

Appendix I. Separable Expansion of $\rho_\ell(k,k')$

We first discuss the expansion of diagonal partial wave form factor $\rho_\ell(k,k')$ of Eq. (48), which is defined as

$$\rho_\ell(k,k') = 4\pi\int r^2 j_\ell(kr) j_\ell(k'r)\rho_{oo}(r)\,dr \quad , \tag{I.1}$$

where $j_\ell(kr)$ are the spherical Bessel functions and $\rho_{oo}(r)$ is the (spherical) ground state particle density distribution. Usually $\rho_{oo}(r)$ is large only in the region of the size of the nucleus, hence one needs an adequate representation of of $j_\ell(kr)$ only for the range $0 \leqslant r \leqslant R$, where R is to be considered as an equivalent radius of a spherical density function. Within the range R, we may use a Fourier-Bessel expansion for $j_\ell(kr)$

$$j_\ell(kr) = \sum_n A_n^{(\ell)}(k)\; j_\ell(\alpha_n^{(\ell)} r) \quad , \tag{I.2}$$

where $\alpha_n^{(\ell)}$ are the roots of the following equation

$$j_\ell(\alpha_n^{(\ell)} R) = 0 \quad . \tag{I.3}$$

The functions $j_\ell(\alpha_n^{(\ell)} r)$ form a complete orthnormal set in the region $0 \leqslant r \leqslant R$. The coefficients $A_n^{(\ell)}(k)$ may therefore be obtained as

$$A_n^{(\ell)}(k) = \frac{2}{R^3 |j_\ell'(\alpha_n^{(\ell)} R)|^2} \int_o^R r^2 dr\; j_\ell(\alpha_n^{(\ell)} r)\; j_\ell(kr) \quad , \tag{I.4}$$

where $j_\ell'(\chi)$ stands for the derivative $\frac{d}{d\chi}[j_\ell(\chi)]$. We may then rewrite Eq. (I.1) as

$$\rho_\ell(k,k') = 4\pi \sum_{n,m} A_n^{(\ell)}(k)\, A_m^{(\ell)}(k')\, I_{nm}(\alpha_n^{(\ell)},\alpha_m^{(\ell)}) \quad , \tag{I.5}$$

where

$$I_{nm}(\alpha_n^{(\ell)},\alpha_m^{(\ell)}) = \int_0^\infty r^2 dr\; j_\ell(\alpha_n^{(\ell)} r)\; j_\ell(\alpha_m^{(\ell)} r)\; \rho_{oo}(r) \quad . \tag{I.6}$$

Since the integral is an oscillatory function of r, if $m \neq n$, we expect the off-diagonal ($n \neq m$) elements of I_{nm} to be small.

If we assume $\rho_{oo}(r)$ to be constant ($= \rho_o$) for $r \leqslant R$ and zero outside, we have

$$I_{nm}(\alpha_n^{(\ell)},\alpha_m^{(\ell)}) = I_{nn}(\alpha_n^{(\ell)},\, \alpha_n^{(\ell)})\delta_{nm} \quad , \tag{I.7}$$

where

$$I_{nn}(\alpha_n^{(\ell)},\, \alpha_n^{(\ell)}) = \frac{\rho_o R^3}{2}\, |j_\ell'(\alpha_n^{(\ell)} R)|^2 \quad , \tag{I.8}$$

and

$$\rho_\ell(k,k') = 4\pi\rho_o \sum_n A_n^{(\ell)}(k)\, A_n^{(\ell)}(k') \tag{I.9}$$

The coefficients $A_n^{(\ell)}(k)$ are given as

$$A_n^{(\ell)}(k) = \frac{2\alpha_n^{(\ell)}\; j_\ell(kR)}{R[\alpha_n^{(\ell)} + k]\;[\alpha_n^{(\ell)} - k]\; j_\ell'(\alpha_n^{(\ell)} R)} \quad . \tag{I.10}$$

One should note that there is no singularity at $k = \alpha_n^{(\ell)}$, since

$$\lim_{k \to \alpha_n^{(\ell)}} \frac{j_\ell(kR)}{(\alpha_n^{\ell} - k)} = -j'_\ell(\alpha_n^{(\ell)} R) \quad . \tag{I.11}$$

We have shown that for a uniform density distribution the partial wave form factor $\rho_\ell(k,k')$ is exactly separable. For a more general density, we would expect some corrections, or we have the following separable form

$$\rho_\ell(k,k') = \sum_n U_\ell^{(n)}(k)\, U_\ell^{(n)}(k') + \text{corrections} \quad , \tag{I.12}$$

where

$$U_\ell^{(n)}(k) = \sqrt{4\pi}\; A_n^{(\ell)}(k) \sqrt{I_{nn}(\alpha_n^{(\ell)}, \alpha_n^{(\ell)})} \tag{I.13}$$

We now turn to the transition form factors. We may follow the above argument to obtain a separable representation. The approximation will be good when we consider charge exchange transition to the analogue state of the target. In this case, the transition density will be products of two single particle wave functions of the outmost shell-model levels. This density will be peaked near the nuclear surface. If we take

$$\tilde{\rho}(r) = \tilde{\rho}_o \delta(r-R) \quad , \tag{I.14}$$

then, from Eq. (I.1), the partial wave density will be separable.

We therefore have shown that in two limits of the density distributions, the partial wave density exactly factorized. In a more general distribution,

we have to use the exact separable form of Eq. (I.5) with double summation. We have numerically tested our discussions here by performing the calculations for a Woods-Saxon density with a half-radius 4 fm and a diffuseness a = 0.6 fm. For typical pion momenta near the resonance: $k = 1.5\ \mathrm{fm}^{-1}$ and $k' = 2.0\ \mathrm{fm}^{-1}$, we find it necessary to use Eq. (I.5). An accuracy to within a few percent may be obtained by keeping $n \geq 5$ terms in Eq. (I.2). A complete investigation of this method will be reported elsewhere.

Appendix II. A simple separable approximation

In this appendix, we show our calculation in its crudest form. We assume that numerically the nuclear form factors may be represented by

$$\rho(k,k') = \tilde{\rho}_\ell^{(o)}(k)\ R(k') \quad , \tag{II.1}$$

and

$$\tilde{\rho}_\ell(k,k') = \tilde{\rho}_\ell^{(o)}(k)\ R(k') \quad , \tag{II.2}$$

where the function $R(k')$ is independent of ℓ. Then the L-th partial-wave component of Eq. (42) becomes

$$F_{L(\pm)}(k,k_o) = \mathcal{F}_L(k,k_o) + \frac{1}{E - E_k + i\eta} \frac{\left|\tilde{t}_{33}(k)\right|}{\sqrt{\Gamma_{\pi N}(k)}}\ M_{L(\pm)}(k)\Lambda_{L(\pm)}(k_o), \tag{II.3}$$

where

$$M_{L(\pm)}(k) = \sum_\ell (2\ell+1)\begin{pmatrix} 1 & \ell & L \\ 0 & 0 & 0 \end{pmatrix}^2 [\eta_{11}\rho_\ell^{(o)}(k) \pm \eta_{12}\tilde{\rho}_\ell^{(o)}(k)] \quad , \tag{II.4}$$

and

$$\Lambda_{L(\pm)}(k_o) = \int_0^\infty \sqrt{\Gamma_{\pi N}(k)}\ F_{L(\pm)}(k,k_o)\ R(k)\ k^2\ dk \quad . \tag{II.5}$$

Multiplying on the left of Eq. (I.3) by $\Gamma_{\pi N}(k)\ R(k)k^2$, and integrating over k, we obtain the desired solution for $\Lambda_{L(\pm)}$,

$$\Lambda_{L(\pm)}(k_o) = \frac{\lambda_L(k_o)}{1 - P_{L(\pm)}(k_o)} \quad , \tag{II.6}$$

where we have

$$\lambda_L(k_o) = \int \sqrt{\Gamma_{\pi N}(k)} \, f_L(k,k_o) R(k) \; k^2 dk \quad , \tag{II.7}$$

and

$$P_{L(\pm)}(k_o) = \frac{4\pi}{3} \int \frac{k^2 dk}{E - E_k + i\eta} \; \frac{|\tilde{t}_{33}(k)|}{\sqrt{\Gamma_{\pi N}(k)}} \; R(k) \; M_{L(\pm)}(k) \quad . \tag{II.8}$$

The partial wave T-matrix elements are given as

$$T_{el}(L) = \frac{N_L(k_o)}{2} \; [\Lambda_{L(+)}(k_o) + \Lambda_{L(-)}(k_o)] \quad , \tag{II.9}$$

and

$$T^R_{SCX}(L) = \frac{\tilde{N}_L(k_o)}{2} \; [\Lambda_{L(+)}(k_o) - \Lambda_{L(-)}(k_o)] \quad ; \tag{II.10}$$

where

$$N_L(k) = \frac{\eta_{11}}{2\pi} \frac{|\tilde{t}_{33}(k)|}{\sqrt{\Gamma_{\pi N}(k)}} \sum_{\ell} (2\ell+1) \begin{pmatrix} 1 & \ell & L \\ 0 & 0 & 0 \end{pmatrix}^2 \rho_\ell^{(o)}(k) \quad , \tag{II.11}$$

and

$$\tilde{N}_{L(k)} = \frac{\eta_{12}}{2\pi} \frac{|\tilde{t}_{33}(k)|}{\sqrt{\Gamma_{\pi N}(k)}} \sum_{\ell} (2\ell+1) \begin{pmatrix} 1 & \ell & L \\ 0 & 0 & 0 \end{pmatrix}^2 \tilde{\rho}_\ell^{(o)}(k) \quad . \tag{II.12}$$

LEGAL NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

TECHNICAL INFORMATION DIVISION
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720