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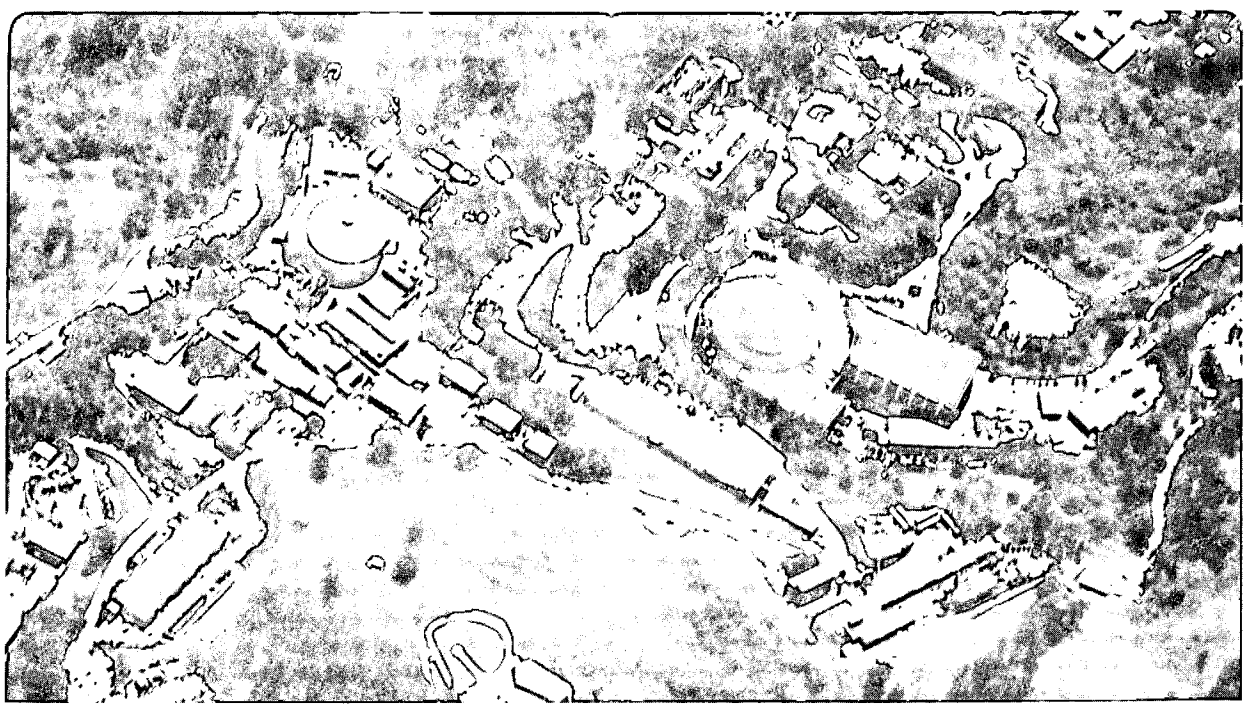

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CONVERGENCE OF THE RANDOM VORTEX METHOD IN THREE DIMENSIONS[1]

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Convergence of the Random Vortex Method in Three Dimensions *

(Revised)

Ding-Gwo Long

Abstract

The convergence of the random vortex method in $\mathbf{R}^3$ is proved. An almost optimal rate of convergence is obtained. The convergence follows from the consistency and the stability of the method. Since the motion of the vortices is random, the major task of the paper is to incorporate appropriately the stochastic elements of the method in all parts of the proof. The framework established earlier for proving the convergence of the random vortex method in $\mathbf{R}^2$ is adapted to treat vortex stretching, a mechanism absent in two dimensional fluid flows.

*1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 60F10, 65M15; Secondary 76C05.

1 Introduction

The random vortex method was introduced by Chorin [6] to simulate viscous incompressible fluid flows governed by Navier-Stokes equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)\, u = -\nabla p + \nu \nabla^2 u \tag{1.1}$$

$$\nabla \cdot u = 0 \tag{1.2}$$

where u is the velocity field, p is the pressure, and ν is the kinematic viscosity. By taking the curl of equation (1.1), the vorticity field $\omega = \nabla \times u$ satisfies the equation

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = (\nabla u) \cdot \omega + \nu \nabla^2 \omega \tag{1.3}$$

and the velocity is determined by the vorticity through the Biot-Savart Law

$$u(x,t) = \int_{\mathbf{R}^3} K(x-y) \cdot \omega(y,t)\, dy \tag{1.4}$$

where K is a matrix-valued kernel given by

$$K(x,t) = -\frac{1}{4\pi} \frac{x}{|x|^3} \times . \tag{1.5}$$

The terms $(u \cdot \nabla)\omega$, $(\nabla u) \cdot \omega$, and $\nu \nabla^2 \omega$ in (1.3) represent convection, stretching, and diffusion of the vorticity, respectively. In two dimensions ω is perpendicular to u and there is no stretching of the vorticity. The random vortex method uses finitely many particles called vortices each carrying a vorticity vector to approximate the vorticity field. The vortices evolve according to the approximate velocity field which in turn is determined by the vortices through a discrete analogue of (1.4). The viscous diffusion is simulated by adding random perturbations to the motion of the vortices. The random perturbations are independent Gaussian random walks or independent Brownian motions (Wiener processes) depending on the time being discrete or continuous. If the boundary is present, then the non-slip boundary condition can be satisfied by creating vorticity on the boundary. For a general introduction of the method, see [8]. The random vortex method is particularly suitable for flows at high Reynolds number since in these flows the vorticity is usually concentrated in regions which are much smaller compared to the total fluid volume. Moreover, unlike the difference methods, the step of simulating the viscous diffusion by random walks does not introduce numerical viscosity which may swamp the effects of physical viscosity.

For numerical computations using the random vortex method, see e. g. [6], [7], [9], [13], [25].

Various partial results on the convergence of the vortex method have been proved. For inviscid flows (i. e. $\nu = 0$) in two and three dimensions without boundaries, the convergence results have been established by the works of Hald [17], Beale and Majda [2, 3], Cottet [10, 11], Anderson and Greengard [1], and Beale [5]. In this case the evolution of the vortices is described by a finite system of ODE's. There are two versions of the inviscid vortex method in $\mathbf{R}^3$. Their difference lies in the updating of the vortex stretching. Beale and Majda [2, 3] proposed and proved a version in which the vortex stretching is incorporated through a Lagrangian update. Later Anderson and Greengard [1] proposed the other version in which the vortex stretching is obtained by differentiating the computed velocity field. Its convergence was proved by Beale [5].

It is difficult to generalize the version in [2, 3] to a random vortex method yet the version in [1] can be converted to a random vortex method simply by adding independent Brownian motions to the motion of the vortices. Thus the evolution of the vortices is described by a finite system of stochastic differential equations (abbreviated as SDE's). Esposito and Pulvirenti [12] proved a "propagation of chaos" (law of large numbers) type of result on the convergence in $\mathbf{R}^3$ which is similar to an earlier result by Marchioro and Pulvirenti [23] in two dimensions. Their results are not satisfactory from the point of view of numerical computation since there is no rate of convergence. The satisfactory results should be of "large deviation" type. Moreover, the analysis of the random vortex method should generalize that of the inviscid vortex method since the former is a random perturbation of the latter. Goodman [16] considered and proved the convergence of a version of the random vortex method in which the initial positions of the vortices are randomly chosen. This version is not used in actual computation and the approach in [16] is not generalized from the analysis of the inviscid vortex method. As a consequence the rate of convergence obtained in [16] is not optimal. Based on the works in the inviscid vortex method mentioned above, the author [20] constructed a unified framework to analyze the random vortex method. He used a large deviation estimate (Bennett's inequality) to prove an almost optimal rate of convergence in $\mathbf{R}^2$.

In this paper we consider the random vortex method in $\mathbf{R}^3$ with continuous time and prove almost optimal results on the convergence. Aspects of time discretization will be discussed in [21]. The approach in the present paper is based on the framework in [20] and the analysis of the vortex

stretching in [5]. However, more advanced probability theory is required to treat the random vortex stretching. In [5] and [20] the convergence follows immediately from the consistency and the stability. Here in the proof of convergence one needs to use stochastic calculus and continuous martingales to estimate the error of vorticity in the negative Sobolev space. The main results are stated in Section 3 and they are uniform with respect to the viscosity $\nu \in [0, \nu_0]$ for arbitrary $\nu_0 > 0$.

The consistency error consists of three components: the moment error, the discretization error, and the statistical error. They are of the order δ^m, $\delta(h/\delta)^L$, and $h(h/\delta)^{1/2}|\ln h|$, respectively. h is the lattice spacing and it is proportional to $N^{-1/3}$ where N is the number of vortices used in the computation. δ is required to be of the order h^q with $0 < q < 3/5$. The positive integers m and L will be defined in Section 2. For smooth flows one can choose m and L to be large. Consequently the statistical error is the dominant error in terms of order. It follows from central limit theorem that the estimate for the statistical error is almost optimal—within a factor of $|\ln h|$. One expects that the statistical error decreases to zero as the viscosity $\nu \to 0$. The detailed ν-dependence of the statistical error will be treated in [22].

The rest of the paper is organized as follows. Section 2 contains a brief summary of the inviscid vortex method and its convergence results. In Section 3 we formulate the random vortex method and state the main theorem on the convergence. The theorem is proved in Section 7, following the proofs of the consistency and the stability in Sections 5 and 6, respectively. Several estimates frequently used in proving the consistency and the stability are gathered in Section 4.

2 Inviscid Vortex Method

The vorticity-stream formulation of inviscid incompressible fluid flows in $\mathbf{R}^3$ is

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = (\nabla u) \cdot \omega \tag{2.6}$$

$$u(x,t) = \int_{\mathbf{R}^3} K(x-y) \cdot \omega(y,t)\, dy$$

where K is defined in (1.5). The evolution of the vorticity field expressed in terms of Lagrangian coordinates is given by

$$\frac{d\omega}{dt}(t;\alpha) = [\nabla u(x(t;\alpha),t)] \cdot \omega(t;\alpha) \tag{2.7}$$

where the particle trajectory $x(t;\alpha)$ is the solution of the ODE

$$\frac{dx}{dt}(t;\alpha) = u(x(t;\alpha),t) \tag{2.8}$$

with the initial data

$$x(0;\alpha) = \alpha.$$

By expressing u and ∇u in terms of $x(t;\alpha)$ and $\omega(t;\alpha)$, we have the particle trajectory formulation

$$\frac{dx}{dt}(t;\alpha) = \int_{\mathbf{R}^3} K(x(t;\alpha) - x(t;\alpha')) \cdot \omega(t;\alpha')\, d\alpha' \tag{2.9}$$

$$\frac{d\omega}{dt}(t;\alpha) = [\int_{\mathbf{R}^3} \nabla K(x(t;\alpha) - x(t;\alpha')) \cdot \omega(t;\alpha')\, d\alpha'] \cdot \omega(t;\alpha) \tag{2.10}$$

By discretizing (2.9) and (2.10), Anderson and Greengard [1] proposed the following vortex method

$$\frac{d\tilde{x}_i}{dt} = \sum_j K_\delta(\tilde{x}_i - \tilde{x}_j) \cdot \tilde{\omega}_j h^3 \tag{2.11}$$

$$\frac{d\tilde{\omega}_i}{dt} = [\sum_j \nabla K_\delta(\tilde{x}_i - \tilde{x}_j) \cdot \tilde{\omega}_j h^3] \cdot \tilde{\omega}_i \tag{2.12}$$

with the initial data

$$\tilde{x}_i(0) = \alpha_i, \quad \tilde{\omega}_i(0) = \omega(\alpha_i, 0)$$

where $\alpha_i = h \cdot i$ with $i \in \mathbf{Z}^3$ are lattice points of spacing $h > 0$, K_δ is a smoothed kernel with

$$K_\delta = K * \psi_\delta, \quad \psi_\delta(x) = \delta^{-3}\psi(\delta^{-1}x), \quad \int_{\mathbf{R}^3} \psi(x)\, dx = 1, \quad \delta > 0,$$

and the initial vorticity $\omega(\cdot, 0)$ is assumed to have bounded support. The choice of the smoothing function ψ is closely related to the accuracy of the method. We denote that $\psi \in M^{L,m}$ if it satisfies the following three conditions:

(i) $\int_{\mathbf{R}^3} \psi(x)\, dx = 1.$

(ii) $\int_{\mathbf{R}^3} x^\beta \psi(x)\, dx = 0$, for all multi-indices β with $1 \le |\beta| \le m - 1$.

(iii) $\psi \in C^L(\mathbf{R}^3)$ and ψ decreases rapidly at infinity.

Examples of ψ can be found in [4] and [5].

The convergence of the method was proved by Beale [5]. The main result in [5] is summarized in the rest of this section. Before stating the main result, we need to introduce certain notations. The computed velocity is denoted by

$$\tilde{u}^h(x,t) = \sum_i K_\delta(x - \tilde{x}_i(t)) \cdot \tilde{\omega}_i(t) h^3. \tag{2.13}$$

To analyze the method, we introduce a reference velocity

$$u^h(x,t) = \sum_i K_\delta(x - x_i(t)) \cdot \omega_i(t) h^3 \tag{2.14}$$

which is obtained from the exact particle paths

$$\frac{dx_i}{dt}(t) = u(x_i(t), t), \quad x_i(0) = \tilde{x}_i(0) = h \cdot i$$

and the exact vorticity vectors

$$\frac{d\omega_i}{dt}(t) = [\nabla u(x_i(t), t)] \cdot \omega_i(t).$$

The discrete velocities evaluated at $\tilde{x}_i(t)$ and $x_i(t)$ are denoted by

$$\tilde{u}_i^h(t) = \tilde{u}^h(\tilde{x}_i(t), t) \tag{2.15}$$

$$u_i^h(t) = u^h(x_i(t), t), \tag{2.16}$$

respectively. The gradients of the velocity fields $\nabla\widetilde{u}^h$, ∇u^h, $\nabla\widetilde{u}_i^h$, ∇u_i^h are defined as in (2.13), (2.14), (2.15), and (2.16) with K_δ replaced by ∇K_δ. The error $\widetilde{x}_i - x_i$ is estimated in the discrete L_h^p-norms defined by

$$\|f_i\|_{0,p,h} = \{\sum_i |f_i|^p h^3\}^{1/p}.$$

The error $\widetilde{\omega}_i - \omega_i$ is estimated in the discrete negative Sobolev space $W_h^{-1,p}$ with the norm

$$\|f_i\|_{-1,p,h} = \sup_{g_i \in W_h^{1,p^*}} \frac{|(f_i, g_i)_h|}{\|g_i\|_{1,p^*,h}}$$

where

$$(f_i, g_i)_h = \sum_i f_i g_i h^3$$

and

$$\|g_i\|_{1,p^*,h}^{p^*} = \|g_i\|_{0,p^*,h}^{p^*} + \sum_{l=1}^{3} \|D_l^+ g_i\|_{0,p^*,h}^{p^*}$$

is the norm of the discrete Sobolev space W_h^{1,p^*} with D_l^+ being the forward-difference operator in the lth coordinate direction and $(1/p) + (1/p^*) = 1$.

The main results on the convergence are stated in the next theorem.

Theorem 1 *Assume that the velocity field $u(x,t)$ is smooth enough, that the initial vorticity is supported in a bounded domain Ω, and that $\psi \in M^{L,m}$ with $m \geq 4$. Then for all sufficiently small h and δ, $\delta = c_0 h^q$ with c_0, q fixed and $1/3 < q < 1$, we have the following estimates*

(1) *Convergence of particle paths*

$$\max_{0 \leq t \leq T} \|\widetilde{x}_i(t) - x_i(t)\|_{0,p,h} \leq C[\delta^m + (h/\delta)^L]$$

(2) *Convergence of discrete velocity*

$$\max_{0 \leq t \leq T} \|\widetilde{u}_i^h(t) - u(x_i(t), t)\|_{0,p,h} \leq C[\delta^m + (h/\delta)^L]$$

(3) *Convergence of continuous velocity*

$$\max_{0 \leq t \leq T} \|\widetilde{u}^h(\cdot, t) - u(\cdot, t)\|_{L^p(B(R_0))} \leq C[\delta^m + (h/\delta)^L]$$

where the constant C only depends on T, L, m, p, q, R_0, the diameter of Ω, and the bounds for a finite number of derivatives of the velocity field.

The convergence results follow from the following two lemmas.

Lemma 2.1 (Consistency Lemma) *Under the same assumptions as in Theorem 1, we have*

(C1) $|u^h(x,t) - u(x,t)| \leq C[\delta^m + \delta(h/\delta)^L]$

(C2) $|\nabla u_i^h(t) - \nabla u(x_i(t),t)| \leq C[\delta^m + (h/\delta)^L]$

uniformly in x and t for $|x| \leq R_0$ and $0 \leq t \leq T$ where the constant C only depends on T, L, m, R_0, the diameter of $\mathrm{supp}\omega_0$, *and the bounds for a finite number of derivatives of $u(x,t)$.*

Lemma 2.2 (Stability Lemma) *Under the same assumptions as in Theorem 1, choose ϵ and p so that $0 < \epsilon < q/2$, $2\epsilon < 3q - 1$, and $p > 3/\epsilon$. If*

$$\eta(t) \equiv \|\widetilde{x}_i(t) - x_i(t)\|_{0,p,h} + \|\widetilde{\omega}_i(t) - \omega_i(t)\|_{-1,p,h} \leq h^{mq-\epsilon}$$

for some time t with $0 \leq t \leq T$, then the following estimates

(S1) $\|\widetilde{u}_i^h(t) - u_i^h(t)\|_{0,p,h} \leq C\eta(t)$

(S2) $\|\nabla \widetilde{u}_i^h(t)\widetilde{\omega}_i(t) - \nabla u_i^h(t)\omega_i(t)\|_{-1,p,h} \leq C\eta(t)$

hold where the constant C is independent of t.

Remark One may ask the following two questions.

1. Can one use $W_h^{-1,p}$-norms to estimate the consistency error in (C2) instead of the pointwise estimate so that the discretization error can be improved by a factor of δ?

2. Is the constraint $1/3 < q$ in Theorem 1 necessary?

The answer for the first question is yes and the answer for the second question is no. More details can be found in Sections 3 and 7.

3 Random Vortex Method

The natural generalization of equation (2.8) for particle paths is the SDE

$$dX(t;\alpha) = u(X(t;\alpha),t)\,dt + \sqrt{2\nu}\,dW(t) \tag{3.1}$$

with the initial data

$$X(0;\alpha) = \alpha$$

where $W(t)$ is a standard Brownian motion in $\mathbf{R}^3$. $W(t)$ is a stochastic process characterized by the following properties:

1. $W(0) = 0$.
2. For $0 \le t_0 \le t_1 \le \cdots \le t_n$, the increments

   $$W(t_n) - W(t_{n-1}),\ W(t_{n-1}) - W(t_{n-2}), \ldots,\ W(t_1) - W(t_0)$$

   are independent.
3. The three components of the increment $W(t) - W(s)$ are independent Gaussian random variables with mean 0 and variance $t - s$.
4. The sample paths of $W(t)$ are continuous.

The equation for vorticity stretching remains in the same form:

$$\frac{d\omega}{dt}(t;\alpha) = [\nabla u(X(t;\alpha),t)] \cdot \omega(t;\alpha) \tag{3.2}$$

with the initial data

$$\omega(0;\alpha) = \omega(\alpha,0).$$

Since the diffusion coefficient $\sqrt{2\nu}$ is a constant, the SDE (3.1) is equivalent to the integral equation

$$X(t;\alpha) = \alpha + \int_0^t u(X(s;\alpha),s)\,ds + \sqrt{2\nu}W(t). \tag{3.3}$$

The integral equation (3.3) can be solved sample path by sample path. Since each sample path of $W(t)$ is continuous, the integral equation (3.3) has a unique continuous solution by the method of successive approximation. Since $\nabla \cdot u = 0$, the map $\alpha \mapsto X(t;\alpha)$ is a volume preserving diffeomorphism for each sample path w.

It turns out that the velocity can be expressed as

$$\begin{aligned} u(x,t) &= \int_{\mathbf{R}^3} K(x-y)\cdot\omega(y,t)\,dy \\ &= \int_{\mathbf{R}^3} E[K(x-X(t;\alpha))\cdot\omega(t;\alpha)]\,d\alpha \end{aligned} \tag{3.4}$$

where E denotes the expectation with respect to the Wiener measure. (3.4) is a generalized version of the Feynman-Kac formula. For the sake of completeness we give a derivation of (3.4). The derivation is based on the Trotter product formula. It is sufficient to show that for any vector function $f \in L^2(\mathbf{R}^3)\cap L^\infty(\mathbf{R}^3)$,

$$(f,\omega) = \int f(x)\cdot\omega(x,t)\,dx = \int E[f(X(t;\alpha))\cdot\omega(t;\alpha)]\,d\alpha.$$

We can write the vorticity equation (1.3) as

$$\frac{\partial\omega}{\partial t} = (A+B)\omega$$

where

$$A(x,t) = -[u(x,t)\cdot\nabla] + \nu\nabla^2$$

is the operator for convection and diffusion while

$$B(x,t) = \nabla u(x,t)$$

denotes the operator for stretching. The equation

$$\frac{\partial\varpi}{\partial t} = A\varpi \tag{3.5}$$

is both a backward and a forward equation since $\nabla\cdot u = 0$. Therefore the solutions of (3.5) satisfy the maximum principle and the fundamental solution $G(x,t;y,s)$ is the transition probability density of the diffusion process $X(t)$. Let $X(t_0) = x_0$ be fixed. Then it follows from the Markov property of $X(t)$ that for $t_n > t_{n-1} > \cdots > t_1 > t_0$, the joint probability density of $X(t_n),\ X(t_{n-1}),\ldots,X(t_1)$ is

$$\prod_{k=1}^{n} G(x_k,t_k;x_{k-1},t_{k-1}). \tag{3.6}$$

Let $S(t,s)$ denote the solution operator of equation (3.5). i. e.

$$\varpi(x,t) = S(t,s)\varpi(\cdot,s) = \int G(x,t;y,s)\varpi(y,s)\,dy.$$

The equation

$$\frac{\partial \omega}{\partial t}(x,t) = B(x,t)\cdot\omega(x,t) \tag{3.7}$$

can be regarded as a system of ODE's with x being parameters. Let $\Phi(t,s;x)$ be the fundamental matrix of (3.7) with $\Phi(s,s;x) = I$. Let $\omega_t(x) = \omega(x,t)$. By the Trotter product formula,

$$(f,\omega_t) = \lim_{n\to\infty}(f,[\prod_{k=1}^{n} S_k\Phi_k]\cdot\omega_0)$$

where $S_k = S(t_k,t_{k-1})$, $\Phi_k = \Phi(t_k,t_{k-1};x)$, $t_k = k\Delta t$, $0 \le k \le n$, and $\Delta t = t/n$. It follows from (3.6) that

$$\begin{aligned}
&(f,[\prod_{k=1}^{n} S_k\Phi_k]\cdot\omega_0)\\
&= \int\cdots\int\int f(x_n)[\prod_{k=1}^{n} G(x_k,t_k;x_{k-1},t_{k-1})\Phi(t_k,t_{k-1};x_{k-1})]\\
&\qquad\qquad \cdot\omega_0(x_0)\,dx_0dx_1\cdots dx_n\\
&= \int E[f(X(t_n))\cdot\Phi_n(X(t_{n-1}))\cdots\Phi_1(X(t_0))]\omega_0(x_0)\,dx_0.
\end{aligned}$$

Therefore

$$\begin{aligned}
(f,\omega_t) &= \lim_{n\to\infty}\int E[f(X(t_n;\alpha))\prod_{i=1}^{n-1}\Phi(t_i,t_{i-1};X(t_i;\alpha))\cdot\omega(\alpha,0)]\,d\alpha\\
&= \int E[f(X(t;\alpha))\cdot\omega(t;\alpha)]\,d\alpha.
\end{aligned}$$

Equations (3.1), (3.2), and (3.4) lead to the random vortex method

$$d\widetilde{X}_i(t) = [\sum_j K_\delta(\widetilde{X}_i(t)-\widetilde{X}_j(t))\cdot\widetilde{\omega}_j(t)h^3]\,dt + \sqrt{2\nu}\,dW_i(t) \tag{3.8}$$

$$\frac{d\widetilde{\omega}_i(t)}{dt} = [\sum_j \nabla K_\delta(\widetilde{X}_i(t)-\widetilde{X}_j(t))\cdot\widetilde{\omega}_j(t)h^3]\cdot\widetilde{\omega}_i(t) \tag{3.9}$$

with the initial data

$$\widetilde{X}_i(0) = \alpha_i = h\cdot i, \quad \widetilde{\omega}_i(0) = \omega(\alpha_i,0) \tag{3.10}$$

where $W_i(t)$ are independent standard Brownian motions in $\mathbf{R}^3$. There are three stages of approximation in (3.8) and (3.9):

1. the kernel K being replaced by a mollified one K_δ,
2. the integral (3.4) being approximated by the discrete sum in (3.8),
3. using the summation of the random vectors to approximate the summation of their expectations provided that the number of the random vectors is large enough.

Accordingly the error committed in each approximation is called the moment error, the discretization error, and the statistical error, respectively. They are the three components of the consistency error.

To analyze the method we follow the strategy in the inviscid case by introducing the auxiliary processes

$$dX_i(t) = u(X_i(t),t)\,dt + \sqrt{2\nu}\,dW_i(t) \tag{3.11}$$

$$\frac{d\omega_i(t)}{dt} = [\nabla u(X_i(t),t)]\cdot\omega_i(t) \tag{3.12}$$

with the same initial data (3.10). Notice that equations (3.8) and (3.9) are coupled while equations (3.11) and (3.12) are not. Moreover, the motion and the stretching of different vortices in (3.11) and (3.12) are independent of one another. The main results are stated in the following theorem.

Theorem 2 *Assume that the velocity field $u(x,t)$ is smooth enough, that the initial vorticity is supported in a bounded domain Ω, and that $\psi \in M^{L,m}$ with $m \geq 4$. Then for all sufficiently small h and δ, $\delta = c_0 h^q$ with c_0, q fixed and $0 < q < 3/5$, we have the following estimates*

(1) *Convergence of particle paths*

$$\max_{0\leq t\leq T} \|\widetilde{X}_i(t) - X_i(t)\|_{0,p,h} \leq C[\delta^m + h(h/\delta)^{1/2}|\ln h|]$$

(2) *Convergence of discrete velocity*

$$\max_{0\leq t\leq T} \|\widetilde{u}_i^h(t) - u(X_i(t),t)\|_{0,p,h} \leq C[\delta^m + h(h/\delta)^{1/2}|\ln h|]$$

(3) *Convergence of continuous velocity*

$$\max_{0\leq t\leq T} \|\widetilde{u}^h(\cdot,t) - u(\cdot,t)\|_{L^p(B(R_0))} \leq C[\delta^m + h(h/\delta)^{1/2}|\ln h|]$$

except for an event of probability less than $h^{C''C}$ provided that $C \geq C'$ where the constants C', $C'' > 0$ only depend on T, L, m, p, q, R_0, the diameter of Ω, and the bounds for a finite number of derivatives of the velocity field.

Notation Following the notation in [20], we will use the symbol $a \preceq b$ to denote that $a \leq b$ except for an event of probability approaching to zero faster than any polynomial rate by choosing the constant C sufficiently large.

The convergence follows from the consistency and the stability which are stated in Lemmas 3.1 and 3.2. Before stating the lemmas, we need to define certain quantities. Let

$$\begin{aligned} e_i(t) &= \widetilde{X}_i(t) - X_i(t) \\ \varepsilon_i(t) &= \widetilde{\omega}_i(t) - \omega_i(t) \end{aligned}$$

be the errors in positions and vorticities, respectively. e_i will be estimated in the space L_h^p as in [20]. In the inviscid case ε_i was estimated in the space $W_h^{-1,p}$ which is defined on a Lagrangian lattice with spacing h. This approach does not work for the random vortex method since $\widetilde{X}_i(t)$ and $X_i(t)$ are random. It is the same problem encountered in proving the stability lemma in two dimensions. We follow the approach in [20] by averaging the relevant quantities in Eulerian coordinates. Let ϕ be a radially symmetric C^∞ function with compact support in the unit ball and

$$\int \phi(x)\, dx = 1. \tag{3.13}$$

For example, we may choose

$$\phi(x) = \begin{cases} a \cdot \exp\{1/(|x|^2 - 1)\} & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.14}$$

with the constant a determined by (3.13). We define

$$\varepsilon(x,t) = \sum_i \phi_\lambda(x - X_i(t))\, \varepsilon_i(t)\, h^3$$

where

$$\phi_\lambda(x) = \lambda^{-3}\phi(\lambda^{-1}x)$$

with $\lambda = h^{q'}$, $0 < q < q' < 3/5$. In the proof of convergence it is more convenient to estimate the continuous error $\varepsilon(x,t)$ in the potential space $\mathcal{L}^{-1,p}$ which is equivalent to the negative Sobolev space $W^{-1,p}$. The $\mathcal{L}^{-1,p}$-norm is defined by

$$\|\varepsilon\|_{\mathcal{L}^{-1,p}} = \|H * \varepsilon\|_p$$

where H is the Bessel potential with its Fourier transform

$$\hat{H}(x) = \frac{1}{(1 + 4\pi^2|x|^2)^{1/2}}.$$

H has a singularity at 0 with

$$H(x) = c_1|x|^{-2} + o(|x|^{-2}) \quad \text{as } |x| \to 0$$

and H decreases rapidly at infinity with

$$H(x) = O(e^{-c_2|x|}) \quad \text{as } |x| \to \infty$$

for some constants c_1 and c_2. See e. g. [27]. We will also need to estimate the error ε_i in the discrete space L^p_h. The $\mathcal{L}^{-1,p}$ and L^p norms of any other quantity f_i associated with X_i are defined in the same fashion. i. e.

$$\begin{aligned} \|f_i\|_{-1,p} &\equiv \|\sum_i \phi_\lambda(x - X_i(t))\, f_i h^3\|_{\mathcal{L}^{-1,p}} \\ \|f_i\|_{0,p} &\equiv \|\sum_i \phi_\lambda(x - X_i(t))\, f_i h^3\|_{L^p}. \end{aligned}$$

For example, $f_i = \nabla u_i^h(t)\omega_i(t) - \nabla u(X_i(t),t)\omega_i(t)$ in (C3) of Lemma 3.1.

Lemma 3.1 (Consistency) *Under the same assumptions as in Theorem 2, we have*

(C1) $|u^h(x,t) - u(x,t)| \preceq C[\delta^m + h(h/\delta)^{1/2}|\ln h|]$

(C2) $|\partial^\beta u^h(x,t) - \partial^\beta u(x,t)| \preceq C[\delta^m + \delta^{-|\beta|}h(h/\delta)^{1/2}|\ln h|]$

(C3) $\|\nabla u_i^h(t)\omega_i(t) - \nabla u(X_i(t),t)\omega_i(t)\|_{-1,p} \preceq C[\delta^m + h(h/\delta)^{1/2}|\ln h|]$

for $C \geq C'$ where the constant C' only depends on the same parameters as in Theorem 2.

Remark In [5] only the consistency estimate (C2) was used. One can afford to lose a power of δ in the inviscid case since it can be compensated by large value of L. For the random vortex method, however, one has to do more refined analysis like (C3) in order to obtain optimal results.

Lemma 3.2 (Stability) *Let*

$$\eta(t) = \|e_i(t)\|_{0,p,h} + \|\varepsilon(x,t)\|_{-1,p} + \lambda\|\varepsilon_i(y)\|_{0,p,h}.$$

In addition to the assumptions in Theorem 2, if

$$\max_{0\leq t\leq T} \max_i |e_i(t)| \preceq \lambda^2 \tag{3.15}$$

$$\max_{0\leq t\leq T} \max_i |\varepsilon_i(t)| \preceq \lambda, \tag{3.16}$$

then the following estimates

(S1) $\|\widetilde{u}_i^h(t) - u_i^h(t)\|_{0,p,h} \preceq C\eta(t)$

(S2) $\|\partial^\beta \widetilde{u}_i^h(t) - \partial^\beta u_i^h(t)\|_{0,p,h} \preceq C\delta^{-|\beta|}[\|e_i(t)\|_{0,p,h} + \delta\|\varepsilon_i(t)\|_{0,p,h}]$, $|\beta| \geq 1$

(S3) $\|\nabla \widetilde{u}_i^h(t)\widetilde{\omega}_i(t) - \nabla u_i^h(t)\omega_i(t)\|_{-1,p} \preceq C\eta(t)$

hold where the constant C only depends on the same parameters as in Theorem 2 excluding m and R_0. In particular, C is independent of T.

4 Basic Estimates

We list several estimates which will be applied throughout the rest of the paper. Lemma 4.1 was used to estimate the discretization error of the inviscid vortex method. Lemma 4.2, a generalization of Lemma 4.1, will be needed to estimate the discretization error of the random vortex method. It turns out that Lemma 4.2 is also needed in estimating the variances occuring in the statistical error and in the stability estimate.

Lemma 4.1 *If $F \in C_0^L(\mathbf{R}^d)$ with $L \geq d+1$, then the following estimate*

$$|\sum_{i\in\mathbf{Z}^d} F(h\cdot i)\,h^d - \int_{\mathbf{R}^d} F(\alpha)\,d\alpha| \leq Ch^L \cdot \max(\|\partial_1^L F\|_{L^1}, \cdots, \|\partial_d^L F\|_{L^1})$$

for the quadrature error holds and the constant C only depends on d.

See [1] for a proof by using Poisson's summation formula. When applied to the inviscid vortex method in computing the velocity field the function F is of the form

$$F(\alpha) = K_\delta(x - x(t;\alpha)) \cdot \omega(t;\alpha)$$

where t, x are fixed and α is the Lagrangian coordinates. For random vortex method the function F is the expectation of the functional

$$\mathcal{F}(\alpha|w) = K_\delta(x - X(t;\alpha)) \cdot \omega(t;\alpha). \tag{4.1}$$

where $X(t;\alpha)$ is determined by the integral equation

$$X(t;\alpha) = \alpha + \int_0^t u(X(s;\alpha),s)\,ds + \sqrt{2\nu}w(t) \tag{4.2}$$

with $w(t) \in C[0,T]$ being sample paths of a standard Brownian motion. Notice that $\omega(t;\alpha)$ is a smooth function of the initial position α and a functional of the sample path w. The form of the functional $\mathcal{F}$ leads to the formulation of the following lemma.

Lemma 4.2 *Let $X(t;\alpha)$ be the solution of the SDE*

$$dX(t;\alpha) = u(X(t;\alpha),t)\,dt + \sqrt{2\nu}\,dW(t)$$

in $\mathbf{R}^d$ with the initial data $X(0;\alpha) = \alpha \in \mathbf{R}^d$ where $\nabla\cdot u = 0$, $u \in C^L(\mathbf{R}^d \times [0,T])$, $L \geq d+1$, and all the spatial derivatives of $u(\cdot,t)$ up to the order L are uniformly bounded. Let $F(\alpha,t)$ be the expectation of the functional

$$\mathcal{F}(\alpha,t|w) = f(X(t;\alpha)) \cdot g(\alpha|w)$$

where $f \in C_b^L(\mathbf{R}^d)$, $L \geq d+1$, and g as a function of α is supported in a bounded domain Ω. Then we have the following estimate for the quadrature error

$$\max_{0\leq t\leq T} |\sum_{i\in\mathbf{Z}^d} F(h\cdot i,t)\,h^d - \int_{\mathbf{R}^d} F(\alpha,t)\,d\alpha|$$
$$\leq \; Ch^L \max_w \|g\|_{L,\infty} \sum_{0\leq|\beta|\leq L} \{\int_{|x|\leq R} |\partial^\beta f(x)|\,dx + \sup_{|x|>R} |\partial^\beta f(x)|\}$$

where $R>0$ is arbitrary and the constant C only depends on d, T, L, the diameter of Ω, and $\max_{1\leq|\beta|\leq L} \|\partial^\beta u\|_{L^\infty(\mathbf{R}^d\times[0,T])}$.

Proof: After applying Lemma 4.1 with respect to each sample path and then taking the expectation, we have

$$|\sum_{i\in\mathbf{Z}^d} E[\mathcal{F}(h\cdot i|w)]\,h^d - \int_{\mathbf{R}^d} E[\mathcal{F}(\alpha|w)]\,d\alpha| \leq Ch^L \cdot \sup_w \max_{1\leq l\leq d} \|\partial_l^L \mathcal{F}(\cdot|w)\|_{L^1}$$

where the constant C only depends on d. We then estimate $\|\partial_l^L \mathcal{F}\|_{L^1}$ for $l=1,\cdots,d$. By direct differentiation we know that $\partial_l^L F$ are sums of finite terms of the form

$$E[(\partial^\beta f)(X(t;\alpha))\cdot \prod_{1\leq|\gamma|\leq L} (\partial^\gamma X(t;\alpha))^{\kappa(\gamma)}\cdot \partial^\mu g(t;\alpha)] \tag{4.3}$$

where β,γ,κ,μ are multiple indices with $0\leq|\beta|,|\gamma|,|\kappa|,|\mu|\leq L$. To apply lemma 4.1 we need to show that all the partial derivatives $\partial^\gamma X(t;\alpha)$ up to order L are uniformly bounded with respect to w. By differentiating (4.2) with respect to α, the first order partial derivatives of $X(t;\alpha)$ satisfy the integral equations

$$\frac{\partial}{\partial\alpha_l} X(t;\alpha) = \hat{v}_l + \int_0^t [\nabla u(X(s;\alpha),s)]\cdot[\frac{\partial}{\partial\alpha_l} X(s;\alpha)]\,ds \tag{4.4}$$

where $\hat{v}_l$ is the unit vector in the lth coordinate direction with $l=1,\cdots,d$. By repeated differentiation of (4.4), the integral equations for higher order derivatives are

$$\partial^\beta X(t;\alpha) = \int_0^t Y(s)\,ds + \int_0^t [\nabla u(X(s;\alpha),s)]\cdot\partial^\beta X(s;\alpha)\,ds \tag{4.5}$$

where $Y(s)$ only contains the derivatives of u and X of orders lower than $|\beta|$. Notice that (4.4) and (4.5) do not have explicit dependence on w. By applying Gronwall's inequality to (4.4), we have

$$\Big|\frac{\partial}{\partial\alpha_l}X(t;\alpha)\Big| \le \exp\left[t\cdot\|\nabla u\|_{L^\infty}\right].$$

By induction and Gronwall's inequality,

$$\max_{1\le|\beta|\le L}\max_{0\le t\le T}|\partial^\beta X(t;\alpha)| \le C \tag{4.6}$$

where C only depends on L, T, and $\max_{1\le|\beta|\le L}\|\partial^\beta u\|_{L^\infty}$. It follows from (4.3) and (4.6) that

$$\begin{aligned}\|\partial_l^L\mathcal{F}\|_{L^1} &= \int_\Omega\Big|\frac{\partial^L}{\partial\alpha_l^L}\mathcal{F}(\alpha)\Big|\,d\alpha\\ &\le C\max_{0\le|\beta|\le L}\|\partial^\beta g\|_{L^\infty}\sum_{0\le|\beta|\le L}\int_\Omega|\partial^\beta f(X(t;\alpha))|\,d\alpha\end{aligned}$$

and

$$\int_\Omega|\partial^\beta f(X(t;\alpha))|\,d\alpha \tag{4.7}$$

$$= \int_{X(t;\Omega)}|\partial^\beta f(x)|\,dx \tag{4.8}$$

$$\le \int_{|x|\le R}|\partial^\beta f(x)|\,dx + \mathrm{Area}(\Omega)\cdot\sup_{|x|>R}|\partial^\beta f(x)|. \tag{4.9}$$

where (4.8) follows from the map $\alpha\mapsto X(t;\alpha)$ being a volume preserving diffeomorphism and (4.9) is obtained by splitting the domain $X(t;\Omega)$ into two parts: inside and outside the sphere $|x|=R$. This finishes the proof of the lemma.

Since the Navier-Stokes equations are nonlinear, we need a uniform bound on the velocities u with respect to the viscosity ν ranging in a compact set (say, $0\le\nu\le\nu_0$) so that in later sections we can apply Lemma 4.2 to obtain estimates uniform in $[0,\nu_0]$. The next lemma from Kato [19] provides the bound.

Lemma 4.3 *Let u^ν denote a family of solutions of the Navier-Stokes equations with the same initial data $u_0\in H^s(\mathbf{R}^3)$, $\|u_0\|_{H^s}\le M$, and s sufficiently large. Then the solution exists in $C([0,T_0];H^s(\mathbf{R}^3))$ for $T_0<(C_sM)^{-1}$ with*

$$\|u^\nu(\cdot,t)\|_{H^s}\le M(1-C_sMt)^{-1}$$

where C_s depends on s but not on ν.

By Lemma 4.3 and Sobolev's lemma, $\max_{1\le|\beta|\le L}\|\partial^\beta u\|_{L^\infty(\mathbf{R}^3\times[0,T])}$ is uniformly bounded with respect to the viscosity ν if the initial data is smooth enough.

We also need the following estimates about K_δ and its derivatives.

Lemma 4.4

(i) $|\partial^\beta K_\delta(x)| \le C\delta^{-2-|\beta|}$, $\forall x \in \mathbf{R}^3$.

(ii) $|\partial^\beta K_\delta(x)| \le C|x|^{-2-|\beta|}$, $\forall |x| \ge \delta$.

See Lemma 5.1 on p.21 of [2].

Lemma 4.5

(i) $\displaystyle\int_{|x|\le R} |K_\delta(x)|\,dx \le CR$, $\forall\delta < 1$.

(ii) $\displaystyle\int_{|x|\le R} |\partial^\beta K_\delta(x)|\,dx \le C\ln(R/\delta)$, *for* $|\beta| = 1$, $\delta < 1$.

(iii) $\displaystyle\int_{|x|\le R} |\partial^\beta K_\delta(x)|\,dx \le C_R\delta^{1-|\beta|}$, *for* $|\beta| > 1$, $\delta < 1$.

Lemma 4.5 follows from Lemma 4.4 by applying the pointwise estimates (i) and (ii) to the regions $|x| \le \delta$ and $\delta \le |x| \le R$, respectively. Notice that H_λ satisfies the same inequalities in Lemmas 4.4 and 4.5 with δ replaced by λ since H and K has the same order of singularity at 0.

Lemma 4.6 (Generalized Young's Inequality) *Let (U,μ) and (V,ν) be two measure spaces and J be a measurable function on the product space $(U\times V, \mu\times\nu)$. Let*

$$f(x) = \int_V J(x,y)g(y)\,d\nu(y) \tag{4.10}$$

where g is a measurable function on (V,ν) such that the integral (4.10) exists a. e. μ. For $1 \le p < \infty$, define

$$|||J|||_p = \max\Big\{\sup_y\Big(\int_U |J(x,y)|^p\,d\mu(x)\Big)^{1/p},\ \sup_x\Big(\int_V |J(x,y)|^p\,d\nu(y)\Big)^{1/p}\Big\}.$$

Then we have

(i) $\|f\|_p \le |||J|||_1\,\|g\|_p$

(ii) $\|f\|_p \leq |||J|||_p \, \|g\|_1$

(iii) $\|f\|_r \leq |||J|||_p \, \|g\|_s$ *with* $r^{-1} = p^{-1} + s^{-1} - 1$.

Proof: The proofs of (i) and (ii) are similar. See Section 0.C in [14] for a proof of (i). Here we give a proof of (ii). We have

$$\begin{aligned} |f(x)| &\leq \int |J(x,y)||g(y)| \, d\nu(y) \\ &\leq \Big(\int |J(x,y)|^p |g(y)| \, d\nu(y) \Big)^{1/p} \cdot \Big(\int |g(y)| \, d\nu(y) \Big)^{1/p^*} \end{aligned}$$

by Hölder's inequality where $(1/p) + (1/p^*) = 1$. Therefore

$$\begin{aligned} \|f\|_p &\leq \Big(\int_U \int_V |J(x,y)|^p |f(y)| \, d\nu(y) \, d\mu(x) \Big)^{1/p} \cdot \|g\|_1^{1/p^*} \\ &\leq |||J|||_p \, \|g\|_1 \end{aligned}$$

by Fubini's theorem. It follows from Hölder's inequality that

$$\|f\|_\infty \leq |||J|||_p \, \|g\|_{p^*}. \tag{4.11}$$

(iii) follows from (ii), (4.11), and Riesz's convexity theorem. See e. g. Section V.1 in [26].

Lemma 4.7 (Calderón-Zygmund Inequality) *Let* $\Phi \in C^1(\mathbf{R})$ *be homogeneous of degree zero. Assume that* Φ *satisfy the cancellation property*

$$\int_{S^{d-1}} \Phi(x) \, d\sigma = 0.$$

For $g \in L^p(\mathbf{R}^d)$, $1 < p < \infty$, *define*

$$f_\varepsilon(x) = \int_{|y| \geq \varepsilon} \frac{\Phi(x-y)}{|x-y|^d} g(y) \, dy.$$

Then $\lim_{\varepsilon \to 0} f_\varepsilon = f$ *exists in* L^p *and*

$$\|f\|_p \leq C \|g\|_p$$

where the constant C *only depends on* Φ *and* p.

See [27] for a proof. The next lemma is useful in estimating the statistical error. It is also needed in proving the stability lemma.

Lemma 4.8 (Bennett's Inequality) *Let $S = \sum_i Y_i$ be the sum of independent bounded random variables Y_i with mean zero and variance σ_i^2. Assume that $|Y_i| \leq M$ and $\sum_i \sigma_i^2 \leq V$. Then for all $\eta > 0$,*

$$P\{|S| \geq \eta\} \leq 2\exp\left[-\frac{1}{2}\eta^2 V^{-1} B(M\eta V^{-1})\right] \tag{4.12}$$

where $B(\lambda) = 2\lambda^{-2}[(1+\lambda)\ln(1+\lambda) - \lambda]$, $\lambda > 0$, $\lim_{\lambda \to 0^+} B(\lambda) = 1$, and $B(\lambda) \sim 2\lambda^{-1}\ln\lambda$ as $\lambda \to \infty$.

For random vectors Y_i in $\mathbf{R}^3$, we have

$$P\{|S| \geq \eta\} \leq 6\exp\left[-\frac{1}{6}\eta^2 V^{-1} B(M\eta V^{-1})\right]$$

by applying (4.12) to the three components.

5 Consistency

The proof of (C1) is almost the same as that in two dimensions. We decompose the error into

$$
\begin{aligned}
&|u^h(x,t) - u(x,t)| \\
&= \ |\sum_i K_\delta(x - X_i(t))\omega_i(t)h^3 - \int_{\mathbf{R}^3} K(x-y)\,\omega(y,t)\,dy| \\
&\leq \ |\sum_i K_\delta(x - X_i(t))\omega_i(t)h^3 - \sum_i E[K_\delta(x - X_i(t))\omega_i(t)]\,h^3| + \\
&\quad |\sum_i E[K_\delta(x - X_i(t))\omega_i(t)]\,h^3 - \int_{\mathbf{R}^3} E[K_\delta(x - X(t;\alpha))\omega(t;\alpha)]\,d\alpha| + \\
&\quad |\int_{\mathbf{R}^3} K_\delta(x-y)\,\omega(y,t)\,dy - \int_{\mathbf{R}^3} K(x-y)\,\omega(y,t)\,dy| \\
&= \ \text{statistical error} + \text{discretization error} + \text{moment error}
\end{aligned}
$$

where $\int E[K_\delta(x - X(t;\alpha))\cdot\omega(t;\alpha)]\,d\alpha = \int K_\delta(x-y)\cdot\omega(y,t)\,dy$ follows from the generalized Feynman-Kac formula derived in Section 3. The moment error is bounded by $C\delta^m$ by the same argument as in the inviscid case. It follows from Lemmas 4.2, 4.3, 4.4, and 4.5 that the discretization error is bounded by $C(h/\delta)^L\delta$. We begin to estimate the statistical error. Let

$$Y_i = h^3[K_\delta(x - X_i(t))\cdot\omega_i(t) - EK_\delta(x - X_i(t))\cdot\omega_i(t)].$$

We have $EY_i = 0$, $|Y_i| \leq Ch^3\delta^{-2} \equiv M$, and

$$
\begin{aligned}
&\sum_i \mathrm{Var}Y_i \\
&= \ h^3\sum_i \{E|K_\delta(x - X_i(t))\cdot\omega_i(t)|^2 - |EK_\delta(x - X_i(t))\cdot\omega_i(t)|^2\}\,h^3 \\
&\leq \ h^3\sum_i \{E|K_\delta(x - X_i(t))\cdot\omega_i(t)|^2\}\,h^3. \qquad (5.1)
\end{aligned}
$$

The summation in (5.1) can be approximated by the integral

$$\int_\Omega E|K_\delta(x - X(t;\alpha))\cdot\omega(t;\alpha)|^2\,d\alpha$$

with an error less than $C\delta^{-1}(h/\delta)^L$ by the equality $|K(x)|^2 = |\nabla^2 K(x)|$ and Lemmas 4.2, 4.4, and 4.5. Since $\omega(t;\alpha)$ is bounded,

$$\int_\Omega E|K_\delta(x - X(t;\alpha))\cdot\omega(t;\alpha)|^2\,d\alpha$$

$$
\begin{aligned}
&\leq \int_\Omega E|K_\delta(x - X(t;\alpha))|^2\, d\alpha \\
&= \int_\Omega \int_{\mathbf{R}^3} |K_\delta(x-y)|^2 G(y,t;\alpha,0)\, dy\, d\alpha \\
&= \int_{\mathbf{R}^3} |K_\delta(x-y)|^2 \varpi(y,t)\, dy
\end{aligned}
$$

where G is the fundamental solution of (3.5) and $\varpi(y,t)$ is the solution of (3.5) with the initial data

$$
\varpi(\alpha,0) = \begin{cases} 1 & \text{if } |\alpha| \in \Omega \\ 0 & \text{otherwise.} \end{cases}
$$

Since $|\varpi(y,t)| < 1$ and $\int \varpi(y,t)\, dy = \text{Area}(\Omega)$,

$$
\begin{aligned}
&\int_{\mathbf{R}^3} |K_\delta(x-y)|^2 \varpi(y,t)\, dy \\
&\leq \int_{|z|\leq R} |K_\delta(z)|^2\, dz + \text{Area}(\Omega) \cdot \sup_{|z|>R} |K_\delta(z)|^2 \\
&\leq C\delta^{-1}
\end{aligned}
$$

by Lemmas 4.4 and 4.5. Therefore $V = \sum_i \text{Var} Y_i \leq Ch^3\delta^{-1}$. By Bennett's inequality, we have

$$
\begin{aligned}
&P\{|\sum_i Y_i| \geq Ch(h/\delta)^{1/2}|\ln h|\} \\
&\leq 6\exp\{-\frac{1}{6}C^2h^3\delta^{-1}|\ln h|^2V^{-1}B[MCh(h/\delta)^{1/2}|\ln h|V^{-1}]\} \\
&\leq \exp\{-C_1C^2|\ln h|^2B[C_2C(h/\delta)^{3/2}|\ln h|]\} \\
&\leq \exp\{-C_3C|\ln h|^2\} \\
&\leq h^{C_3C|\ln h|}.
\end{aligned}
$$

This completes the proof of (C1). The proof of (C2) is the same as (C1) except that K_δ is replaced By $\partial^\beta K_\delta$. Therefore, the right hand side loses a factor of $\delta^{|\beta|}$ in both the discretization error and the statistical error.

(C1) and (C2) are estimates for a fixed point x. It follows from (C1) and (C2) that for lattice points $z_k = h^2 \cdot k$ in any ball $B(R)$,

$$
\max_k |u^h(z_k,t) - u(z_k,t)| \preceq C[\delta^m + h(h/\delta)^{1/2}|\ln h|] \tag{5.2}
$$

$$
\max_k |\partial^\beta u^h(z_k,t) - \partial^\beta u(z_k,t)| \preceq C[\delta^m + \delta^{-|\beta|} \cdot h(h/\delta)^{1/2}|\ln h|]. \tag{5.3}
$$

In Section 6 we will use (5.2) and (5.3) combining with the stability estimates (S1) and (S2) to obtain the L^p estimates:

$$\|u^h(x,t) - u(x,t)\|_{0,p} \preceq C[\delta^m + h(h/\delta)^{1/2}|\ln h|]. \tag{5.4}$$

$$\|\partial^\beta u^h(x,t) - \partial^\beta u(x,t)\|_{0,p} \preceq C[\delta^m + \delta^{-|\beta|}h(h/\delta)^{1/2}|\ln h|]. \tag{5.5}$$

In the convergence proof we will also need the following consistency estimates at the locales of the vortices:

$$\max_i |u_i^h(t) - u(X_i(t),t)| \preceq C[\delta^m + h(h/\delta)^{1/2}|\ln h|] \tag{5.6}$$

$$\max_i |\nabla u_i^h(t) - \nabla u(X_i(t),t)| \preceq C[\delta^m + (h/\delta)^{3/2}|\ln h|]. \tag{5.7}$$

To justify (5.6) and (5.7), we introduce an independent copy X_i' of X_i. By (C1), (C2), $K_\delta(0) = 0$, and $\nabla K_\delta(0) = 0$, we have

$$\begin{aligned} &|u_i^h(t) + K_\delta(X_i(t) - X_i'(t)) \cdot \omega_i h^3 - u(X_i(t),t)| \\ &\qquad \preceq C[\delta^m + h(h/\delta)^{1/2}|\ln h|] \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} &|\nabla u_i^h(t) + \nabla K_\delta(X_i(t) - X_i'(t)) \cdot \omega_i h^3 - \nabla u(X_i(t),t)| \\ &\qquad \preceq C[\delta^m + (h/\delta)^{3/2}|\ln h|]. \end{aligned} \tag{5.9}$$

Since $|K_\delta(x)| \le C\delta^{-2}$ and $|\nabla K_\delta(x)| \le C\delta^{-3}$, (5.6) and (5.7) follow from (5.8) and (5.9), respectively.

By definition the left hand side of (C3) is

$$\Big\| \sum_i \phi_\lambda(x - X_i)[\nabla u_i^h(t) - \nabla u(X_i,t)]\omega_i h^3 \Big\|_{-1,p}. \tag{5.10}$$

We expand $\nabla u_i^h(t)$ and $\nabla u(X_i,t)$ around x:

$$\begin{aligned} \nabla u^h(X_i,t) &= \nabla u^h(x,t) + [(X_i - x)\cdot\nabla]\nabla u^h(x,t) \\ &\quad + \frac{1}{2}[(X_i - x)\cdot\nabla]^2 \nabla u^h(x,t) \\ &\quad + \cdots + \frac{1}{n!}[(X_i - x)\cdot\nabla]^n \nabla u^h(x + Y_i,t) \\ &= \tilde{s}_i^{(0)} + \tilde{s}_i^{(1)} + \cdots + \tilde{s}_i^{(n)} \end{aligned}$$

$$\begin{aligned}\nabla u(X_i,t) &= \nabla u(x,t) + [(X_i - x)\cdot\nabla]\nabla u(x,t) \\ &\quad + \frac{1}{2}[(X_i - x)\cdot\nabla]^2\nabla u(x,t) \\ &\quad + \cdots + \frac{1}{n!}[(X_i - x)\cdot\nabla]^n\nabla u(x + Y_i', t) \\ &= s_i^{(0)} + s_i^{(1)} + \cdots + s_i^{(n)}\end{aligned}$$

where $[(X_i - x)\cdot\nabla]^l = \sum_{|\beta|=l}(X_i - x)^\beta\partial^\beta$ and we ignore the fact that Y_i and Y_i' may depend on the components. The l-th order term $(\widetilde{s}_i^{(l)} - s_i^{(l)})\omega_i$, $l = 0,\cdots,n-1$, in (5.10) is the summation over $|\beta| = l$ of the terms

$$\partial^\beta\nabla[u^h(x,t) - u(x,t)]\cdot\sum_i \phi_\lambda(x - X_i)\cdot(X_i - x)^\beta\omega_i h^3.$$

We need the following lemma.

Lemma 5.1 *For multi-indices β, let*

$$\omega^{\beta,h}(x,t) = \sum_i \phi_\lambda(x - X_i(t))\cdot(X_i(t) - x)^\beta\omega_i(t)h^3. \tag{5.11}$$

Then $\|\omega^{\beta,h}\|_{1,\infty} \preceq C\lambda^{|\beta|}$.

The proof of Lemma 5.1 will be given at the end of the section. It follows from Lemma 5.1 and (5.5) that

$$\begin{aligned}\|(\widetilde{s}_i^{(l)} - s_i^{(l)})\omega_i\|_{-1,p} &\le \sum_{|\beta|=l}\|\partial^\beta\nabla[u^h - u]\cdot\omega^{\beta,h}\|_{-1,p} \\ &\le \sum_{|\beta|=l}\|\omega^{\beta,h}\|_{1,\infty}\cdot\|\partial^\beta\nabla[u^h - u]\|_{-1,p} \\ &\preceq C_1\lambda^{|\beta|}\cdot\|\partial^\beta[u^h - u]\|_{0,p} \\ &\preceq C\lambda^\beta[\delta^m + \delta^{-|\beta|}h(h/\delta)^{1/2}|\ln h|] \\ &\le C[\delta^m + h(h/\delta)^{1/2}|\ln h|].\end{aligned}$$

The n-th order term will be estimated in L^p-norm. We consider the decomposition

$$\begin{aligned}&u^h(x + Y_i, t) - u(x + Y_i', t) \\ &\quad = [u^h(x + Y_i, t) - u(x + Y_i, t)] + [u(x + Y_i, t) - u(x + Y_i', t)].\end{aligned}$$

Since ϕ is supported in $|x| < 1$, only those X_i with $|X_i - x| \leq \lambda$ have contribution. For those X_i's we have $|Y_i - Y_i'| \leq 2\lambda$ and it follows from the smoothness of u that

$$\left\|[(X_i - x)\cdot\nabla]^n \cdot \nabla[u(x+Y_i,t) - u(x+Y_i',t)]\cdot\omega_i\right\|_{0,p} \leq C\lambda^{n+1}.$$

By (5.5),

$$\begin{aligned}\left\|[(X_i - x)\cdot\nabla]^n \cdot \nabla[u^h(x+Y_i,t) - u(x+Y_i,t)]\cdot\omega_i\right\|_{0,p} \\ \preceq\ C\lambda^n[\delta^m + \delta^{-n}(h/\delta)^{3/2}|\ln h|] \\ \leq\ C[\delta^m + h(h/\delta)^{1/2}|\ln h|]\end{aligned}$$

and hence

$$\|(\tilde{s}_i^{(n)} - s_i^{(n)})\omega_i\|_{-1,p} \leq \|(\tilde{s}_i^{(n)} - s_i^{(n)})\omega_i\|_{0,p} \preceq C[\delta^m + h(h/\delta)^{1/2}|\ln h|]$$

provided that we choose n large enough such that $\lambda^{n+1} \leq \delta^m + h(h/\delta)^{1/2}|\ln h|$, and $\lambda^n\delta^{-n} \leq h$. This completes the proof of (C3).

The proof of Lemma 5.1 requires the following lemma and Lemma 6.2. Lemma 5.2 asserts that the number density of the vortices is uniformly bounded with high probability. It turns out that Lemma 5.2 is also essential to the proof of the stability.

Lemma 5.2 *Let $\mathcal{N}(x,r,t)$ be the number of vortices $X_i(t)$ in the ball $B(x,r)$ at time t. Then*

$$h^3 \cdot \mathcal{N}(x,r,t) \preceq Cr^3$$

provided that $r \geq h|\ln h|$.

Proof. Define the function $I \in C_0^\infty(\mathbf{R}^3)$ by

$$I(y) = \begin{cases} 1 & \text{if } y \in B(x,r) \\ \phi(\frac{|y-x|}{r} - 1) & \text{if } r \leq |y-x| \leq 2r \\ 0 & \text{otherwise} \end{cases}$$

where $\phi(x) = \exp\{|x|^2/(|x|^2 - 1)\}$. It is clear that the partial derivatives $\partial^\beta I$ of order $|\beta| = L$ are bounded by Cr^{-L} where the constant C depends only on L. We have

$$\begin{aligned} h^3 \cdot \mathcal{N}(x,r,t) &\leq h^3 \sum_i I(X_i(t)) \\ &= h^3 \sum_i EI(X_i(t)) + h^3 \sum_i \{I(X_i(t)) - EI(X_i(t))\} \\ &= \text{expectation} + \text{fluctuation}. \end{aligned}$$

By applying Lemma 4.2 with $f(y) = I(y)$, $g(\alpha) = 1$, we can approximate the expectation $h^3 \sum_i EI(X_i(t))$ by the integral $\int_\Omega EI(X(t;\alpha))d\alpha$ within an error

$$C_1 h^L \cdot \frac{32}{3}\pi r^3 \sum_{l=0}^{L} r^{-l} \le C_2 r^3 h^L \{L \cdot [\min(1,r)]^{-L}\} \le Cr^3$$

provided that $h \le r$. Moreover,

$$\int_\Omega EI(X(t;\alpha))\, d\alpha \le \int_\Omega \int_{B(x,2r)} G(y,t;\alpha,0)\, dy\, d\alpha \le \int_{B(x,2r)} dy = \frac{32}{3}\pi r^3$$

where G is the fundamental solution of (3.5). Therefore the expectation is less than Cr^3 if $r \ge h$.

We use Bennett's inequality to estimate the fluctuation. Let

$$Y_i = h^3[I(X_i(t)) - EI(X_i(t))].$$

We have $EY_i = 0$, $|Y_i| \le h^3$, and

$$\sum_i \mathrm{Var} Y_i \le h^6 \sum_i E[I(X_i(t))]^2.$$

We apply Lemma 4.2 with $f(y) = E[I(X_i(t))]^2$, $g(\alpha) = 1$, to approximate $h^3 \sum_i E[I(X_i(t))]^2$ by the integral $\int_\Omega E[I(X(t;\alpha))]^2\, d\alpha$ within an error

$$C_1 h^L \cdot \frac{32}{3}\pi r^3 \sum_{l=0}^{L} r^{-l} \le C_2 r^3 h^L \{L \cdot [\min(1,r)]^{-L}\} \le Cr^3$$

provided that $h \le r$. Furthermore

$$\int_\Omega E[I(X(t;\alpha))]^2\, d\alpha \le \int_\Omega \int_{B(x,2r)} G(y,t;\alpha,0)\, dy\, d\alpha \le \int_{B(x,2r)} dy = \frac{32}{3}\pi r^3.$$

Therefore $\sum_i \mathrm{Var} Y_i \le Cr^3 h^3$ if $r \ge h$. According to Bennett's inequality,

$$\begin{aligned}
&P\{|\sum_i Y_i| \ge Cr^2 h|\ln h|\} \\
&\quad \le\ 2\exp\{-\frac{1}{2}(Cr^2 h|\ln h|)^2 \cdot C_1 r^{-3} h^{-3} \cdot B[h^3 Cr^2 h|\ln h| C_1 r^{-3} h^{-3}]\} \\
&\quad \le\ \exp\{-C_2 C^2 |\ln h|^2 \cdot B[C_2 C r^{-1} h|\ln h|]\} \\
&\quad \le\ \exp\{-C_3 C|\ln h|^2\} \\
&\quad =\ h^{C_3 C|\ln h|}
\end{aligned}$$

provided that $r^{-1}h|\ln h|$ stays bounded. Hence the fluctuation

$$|\sum_i Y_i| \preceq Cr^2 h|\ln h| \leq Cr^3$$

if $h|\ln h| \leq r$. This completes the proof of the lemma.

We begin to prove Lemma 5.1 by decomposing $\omega^{\beta,h}$ into the expectation $E\omega^{\beta,h}$ and the fluctuation $\omega^{\beta,h} - E\omega^{\beta,h}$. It suffices to show that

$$|\partial_l E\omega^{\beta,h}(x,t)| \leq C\lambda^{|\beta|} \tag{5.12}$$

and

$$\left|\partial_l[\omega^{\beta,h}(x,t) - E\omega^{\beta,h}(x,t)]\right| \preceq C\lambda^{|\beta|}. \tag{5.13}$$

$E\omega^{\beta,h}$ is a discrete approximation of the integral

$$\begin{aligned} \omega^\beta(x,t) &= \int_{\mathbf{R}^3} E[\phi_\lambda(x - X(t;\alpha)) \cdot (X(t;\alpha) - x)^\beta \cdot \omega(t;\alpha)]\, d\alpha \\ &= \int_{\mathbf{R}^3} \phi_\lambda(x-y) \cdot (y-x)^\beta \cdot \omega(y,t)\, dy \end{aligned} \tag{5.14}$$

where (5.14) follows from the generalized Feynman-Kac formula. let

$$f(x) = \partial_l[\phi_\lambda(x) \cdot (-x)^\beta].$$

f is supported in $|x| < \lambda$ and $|\partial^\gamma f| \leq C\lambda^{-4-|\gamma|+|\beta|}$. It follows from Lemma 4.2 that the discretization error

$$|\partial_l E\omega^{\beta,h}(x,t) - \partial_l\omega^\beta(x,t)| \leq Ch^L\lambda^{-(L+1)}\lambda^{|\beta|} \leq C\lambda^{|\beta|}.$$

Moreover, we have

$$\begin{aligned} &\left|\partial_l \int \phi_\lambda(x-y) \cdot (y-x)^\beta \cdot \omega(y,t)\, dy\right| \\ &= \left|\int \phi_\lambda(x-y) \cdot (y-x)^\beta \cdot \partial_l\omega(y,t)\, dy\right| \\ &\leq \lambda^{|\beta|} \cdot \|\omega(\cdot,t)\|_{1,\infty} \\ &\leq C\lambda^{|\beta|} \end{aligned}$$

since the flow is assumed to be smooth enough. Therefore

$$\|E\omega^{\beta,h}\|_{1,\infty} \leq C\lambda^{|\beta|}.$$

We apply Bennett's inequality to estimate the fluctuation which is the sum of the random vectors

$$Y_i = h^3\{f(x - X_i)\cdot\omega_i - E[f(x - X_i)\cdot\omega_i]\}.$$

We have $|Y_i| \le C_1\lambda^{|\beta|-4}h^3$ and

$$\sum_i \mathrm{Var}\, Y_i \le h^3 \sum_i E\{[f(x - X_i)]^2\cdot\omega_i^2\}h^3. \tag{5.15}$$

The summation on the right hand side of (5.15) can be approximated by the integral

$$\int_\Omega E\{[f(x - X(t;\alpha))]^2\cdot\omega^2(t;\alpha)\}\, d\alpha$$

with an error less than $Ch^L\lambda^{2|\beta|-(L+5)}$. Since $\omega(t;\alpha)$ is bounded,

$$\begin{aligned}
&\int_\Omega E\{[f(x - X(t;\alpha))]^2\cdot\omega^2(t;\alpha)\}\, d\alpha \\
&\le\ C\int_\Omega E[f(x - X(t;\alpha))]^2\, d\alpha \\
&=\ \int_\Omega\int_{\mathbf{R}^3}[f(x-y)]^2 G(y,t;\alpha,0)\, dy\, d\alpha \\
&\le\ \int_{\mathbf{R}^3}[f(x-y)]^2\, dy \\
&\le\ C\lambda^{2|\beta|-5}.
\end{aligned}$$

Therefore $V = \sum_i \mathrm{Var} Y_i \le C_2h^3\lambda^{2|\beta|-5}$. By Bennett's inequality,

$$\begin{aligned}
P\{|\sum_i Y_i| \ge C\lambda^{|\beta|}\} &\le 6\exp\{-\frac{1}{6}C^2C_2^{-1}h^{-3}\lambda^5 B(CC_1C_2^{-1}\lambda)\} \\
&\le \exp\{-C_3C^2(\lambda^5/h^3)\}
\end{aligned}$$

which goes to zero faster than polynomial rate since $\lambda = h^{q'}$ with $0 < q' < 3/5$. This proves (5.13) for a fixed point x. For the lattice points $z_k = h^2\cdot k$ in any ball $B(R)$, we have

$$\max_k |\partial_l\omega^{\beta,h}(z_k,t)| \preceq C\lambda^{|\beta|}.$$

If the radius R is sufficiently large (say, $R = c_0|\ln h|$), then it follows from Lemma 6.2 that $\max_i|X_i(t)| \preceq R-1$. We have $\omega^{\beta,h}(x,t) = 0$ for $|x| > R$ under the event $\max_i|X_i(t)| \le R-1$ since ϕ_λ is supported in $|x| < \lambda$. For

$x \in B(R)$, let z_k be the lattice point closest to x. By the Mean Value Theorem,

$$\begin{aligned} &\partial_l \omega^{\beta,h}(x,t) - \partial_l \omega^{\beta,h}(z_k,t) \\ &= \nabla[\partial_l \omega^{\beta,h}(z_k + y_k, t)] \cdot (x - z_k) \\ &= \nabla[\sum_i f(z_k + Y_{ik} - X_i)\omega_i h^3] \cdot (x - z_k) \end{aligned} \tag{5.16}$$

where we ignore the fact that Y_{ik} may depend on the components. Since f is supported in $|x| < \lambda$, only those X_i's with $|z_k + Y_{ik} - X_i| < \lambda$ has contribution to the summation in (5.16). It follows that these X_i's satisfy

$$|z_k - X_i| \leq |z_k + Y_{ik} - X_i| + |Y_{ik}| < 2\lambda.$$

By Lemma 5.2, $h^3 \cdot \mathcal{N}(z_k, 2\lambda, t) \preceq C\lambda^3$ which implies that

$$\left|\nabla[\sum_i f(z_k + Y_{ik} - X_i)\omega_i h^3] \cdot (x - z_k)\right| \preceq C(h/\lambda)^2 \lambda^{|\beta|}.$$

This completes the proof of Lemma 5.1.

6 Stability

The following lemma can be regarded as the discrete analogue of Lemma 4.5. The proof of Lemma 6.1 is based on Lemma 5.2. These two lemmas are fundamental in the proof of the stability.

Lemma 6.1 *Let* $M_{ij}^{(l)} = \max_{|y|\le C_0\delta} \max_{|\beta|=l} |\partial^\beta K_\delta(X_i - X_j + y)|$. *Then*

$$\sum_j M_{ij}^{(l)} h^3 \preceq \begin{cases} C|\ln\delta| & \text{if } l = 1 \\ C\delta^{1-l} & \text{if } l \ge 2. \end{cases}$$

Proof. We prove the case $l = 1$ and $C_0 = 1$ in detail. The proofs for general C_0 and $l \ge 2$ are almost the same. We write

$$\begin{aligned} \sum_j M_{ij}^{(1)} h^3 &= \sum_{|X_j - X_i|\le 2\delta} M_{ij}^{(1)} h^3 + \sum_{2\delta<|X_j-X_i|\le 2} M_{ij}^{(1)} h^3 + \sum_{|X_j-X_i|>2} M_{ij}^{(1)} h^3 \\ &= \Xi^{(1)} + \Xi^{(2)} + \Xi^{(3)}. \end{aligned}$$

$\Xi^{(1)} \preceq C_1 \cdot (2\delta)^3 \cdot C_2\delta^{-3} \le C$, by Lemmas 4.4 and 5.2.

In order to estimate $\Xi^{(2)}$ and $\Xi^{(3)}$, we notice that $|X_j - X_i| \ge 2\delta$ implies that

$$|X_j - X_i + y| \ge |X_j - X_i| - |y| \ge \frac{1}{2}|X_j - X_i|.$$

It follows that

$$\Xi^{(3)} \le C_1 \sum_{|X_j - X_i|\ge 2} |X_j - X_i|^{-3} h^3 \le C_2 \sum_{|X_j-X_i|>2} h^3 \le C$$

and

$$\Xi^{(2)} \le C \sum_{2\delta<|X_j-X_i|\le 2} |X_j - X_i|^{-3} h^3. \tag{6.1}$$

To estimate (6.1) we decompose the shell $\mathcal{S} = \{x : 2\delta < |x - X_i| \le 2\}$ into $N-2$ concentric shells $\mathcal{S}_n = \{x : (n+1)\delta < |x - X_i| \le (n+2)\delta\}$, $1 \le n \le N-2$ where $N = [2/\delta]$ is the least integer greater or equal to $2/\delta$. Let $a_n = \mathcal{N}(X_i, (n+1)\delta, t)$ be the number of the vortices in the ball $B(X_i, (n+1)\delta)$.

$$\Xi^{(2)} \le C_1 \sum_{X_j\in\mathcal{S}} |X_j - X_i|^{-3} h^3$$

$$
\begin{aligned}
&= C_1 \sum_{n=1}^{N-2} \sum_{X_j \in S_n} |X_j - X_i|^{-3} h^3 \\
&\le C_2 \sum_{n=2}^{N-1} (n\delta)^{-3} (a_n - a_{n-1}) h^3 \\
&= C_2 \{ h^3 [\frac{a_{N-1}}{N^3\delta^3} - \frac{a_1}{\delta^3}] - \sum_{n=1}^{N-1} [\frac{1}{(n+1)^3\delta^3} - \frac{1}{n^3\delta^3}] a_n h^3 \} \qquad (6.2) \\
&\preceq C_2 \{ C_3 + \sum_{n=1}^{N-1} [\frac{1}{\delta^3} \cdot \frac{(3n^2+3n+1)}{n^3(n+1)^3} \cdot C_4 (n+1)^3 \delta^3] \} \qquad (6.3) \\
&\le C_2 \{ C_3 + C_5 \sum_{n=1}^{N-1} \frac{1}{n} \} \\
&\le C_6 (1 + \ln N) \\
&\le C |\ln \delta|
\end{aligned}
$$

where (6.2) is obtained through summation by parts and (6.3) follows from Lemma 5.2. This completes the proof of the lemma.

We begin to prove (S1) for a fixed time. Consider the decomposition

$$
\begin{aligned}
\tilde{u}_i^h - u_i^h &= \sum_j [K_\delta(\tilde{X}_i - \tilde{X}_j) - K_\delta(X_i - X_j)] \cdot \omega_j h^3 + \\
&\quad \sum_j K_\delta(X_i - X_j) \cdot \varepsilon_j h^3 + \\
&\quad \sum_j [K_\delta(\tilde{X}_i - \tilde{X}_j) - K_\delta(X_i - X_j)] \cdot \varepsilon_j h^3 \\
&= v_i^{(1)} + v_i^{(2)} + v_i^{(3)}.
\end{aligned}
$$

$v_i^{(1)}$ can be estimated by the same arguments as in two dimensions. $v_i^{(1)}$ can be further decomposed into

$$
\begin{aligned}
v_i^{(1)} &= \sum_j [K_\delta(X_i - \tilde{X}_j) - K_\delta(X_i - X_j)] \cdot \omega_j h^3 + \\
&\quad \sum_j [K_\delta(\tilde{X}_i - \tilde{X}_j) - K_\delta(X_i - \tilde{X}_j)] \cdot \omega_j h^3 \\
&= v_i^{(11)} + v_i^{(12)}.
\end{aligned}
$$

By the Mean Value Theorem,

$$
v_i^{(11)} = \sum_j \nabla K_\delta(X_i - X_j + Y_{ij}) \cdot e_j \omega_j h^3
$$

where we ignore the fact that Y_{ij} may depend on the components. Let $Z_i \in \lambda \cdot \mathbf{Z}^3$ be the lattice point nearest to X_i. Then

$$v_i^{(11)} = \sum_j \nabla K_\delta(Z_i - Z_j) \cdot e_j \omega_j h^3 + r_i^{(1)}$$

where

$$r_i^{(1)} = \sum_j [\nabla K_\delta(X_i - X_j + Y_{ij}) - \nabla K_\delta(Z_i - Z_j)] \cdot e_j \omega_j h^3.$$

For each $k \in \mathbf{Z}^3$, let Q_k be the cube centered at z_k and with side length λ. We define f_k to be the average of all $e_j \omega_j$ with $X_j \in Q_k$. i. e.

$$f_k = \lambda^{-3} \sum_{X_j \in Q_k} e_j \omega_j h^3.$$

It follows from Lemma 5.2 that

$$\|f_k\|_{0,p,\lambda} \preceq C \|e_j \omega_j\|_{0,p,h} \tag{6.4}$$

and

$$\|\sum_j \nabla K_\delta(Z_i - Z_j) \cdot e_j \omega_j h^3\|_{0,p,h} \preceq C \|\sum_{k'} \nabla K_\delta(z_k - z_{k'}) \cdot f_{k'} \lambda^3\|_{0,p,\lambda}. \tag{6.5}$$

See [20] for detailed proofs. Let $g_k = \sum_{k'} \nabla K_\delta(z_k - z_{k'}) f_{k'} \lambda^3$. To show

$$\|g_k\|_{0,p,\lambda} \leq C \|f_k\|_{0,p,\lambda}, \tag{6.6}$$

we express the sum g_k as the integral

$$g(x) = \int_{\mathbf{R}^3} \mathcal{K}(x, x') f(x')\, dx'$$

where f and g are piecewise constant functions defined by

$$\begin{aligned} f(x) &= f_k \\ g(x) &= \sum_{k'} \nabla K_\delta(z_k - z_{k'})\, f_{k'} \lambda^3 \end{aligned}$$

for $x \in Q_k$ and

$$\mathcal{K}(x, x') = \nabla K_\delta(z_k - z_{k'}), \qquad x \in Q_k,\ x' \in Q_{k'}.$$

Since $\|f\|_{0,p} = \|f_k\|_{0,p,\lambda}$ and $\|g\|_{0,p} = \|g_k\|_{0,p,\lambda}$, (6.6) is equivalent to $\|g\|_{0,p} \leq C\|f\|_{0,p}$. We write

$$\mathcal{K}(x,x') = \nabla K_\delta(x-x') + \mathcal{R}(x,x')$$

with $\mathcal{R} = \mathcal{K} - \nabla K_\delta$. By the Mean Value Theorem,

$$\mathcal{R}(x,x') = [(z_k - x) + (x' - z_{k'})] \cdot \nabla^2 K_\delta(z_k - z_{k'} + y_{kk'}).$$

By Calderón-Zygmund inequality and Young's inequality,

$$\|\nabla K_\delta * f\|_{0,p} \leq C\|f\|_{0,p}.$$

Furthermore, Lemma 6.1 is applicable to $\nabla^2 K_\delta(z_k - z_{k'} + y_{kk'})$ since the only property of X_i used in the proof of the lemma is about the density of the points (Lemma 5.2). Hence it follows from generalized Young's inequality and Lemma 6.1 that

$$\Big\| \int \mathcal{R}(x,x') f(x')\, dx' \Big\|_{0,p} \leq C\lambda\delta^{-1}\|f\|_{0,p} \leq C\|f\|_{0,p}.$$

This proves (6.6). By (6.4), (6.5), and (6.6), we have

$$\|\sum_j \nabla K_\delta(Z_i - Z_j) \cdot e_j\omega_j h^3\|_{0,p,h} \preceq C'\|e_j\omega_j\|_{0,p,h} \leq C\|e_j\|_{0,p,h}.$$

To estimate $r_i^{(1)}$ we apply the Mean Value Theorem to write

$$r_i^{(1)} = \sum_j [\nabla^2 K_\delta(X_i - X_j + Y_{ij}'') \cdot Y_{ij}'] \cdot e_j\omega_j h^3$$

where $Y_{ij}' = Y_{ij} + (X_i - Z_i) - (X_j - Z_j)$. Since $|Y_{ij}'| \leq 2\lambda$ and $|Y_{ij}''| \leq 3\lambda$, it follows from generalized Young's inequality and Lemma 6.1 that

$$\|r_i^{(1)}\|_{0,p,h} \preceq C'\lambda\delta^{-1}\|e_j\omega_j\|_{0,p,h} \leq C\|e_j\|_{0,p,h}.$$

To estimate $v_i^{(12)}$ we appply the Mean Value Theorem to write

$$v_i^{(12)} = [\sum_j \nabla K_\delta(X_i - X_j + Y_{ij}) \cdot \omega_j h^3] \cdot (\widetilde{X}_i - X_i).$$

We will show that

$$\max_i \Big|\sum_j \nabla K_\delta(X_i - X_j + Y_{ij}) \cdot \omega_j h^3\Big| \preceq C \tag{6.7}$$

which implies that

$$\|v_i^{(12)}\|_{0,p,h} \preceq C\|e_i\|_{0,p,h}.$$

By the Mean Value Theorem,

$$\sum_j \nabla K_\delta(X_i - X_j + Y_{ij}) \cdot \omega_j h^3 = \sum_j \nabla K_\delta(X_i - X_j) \cdot \omega_j h^3 + r_i^{(2)}$$

where

$$r_i^{(2)} = \sum_j (Y_{ij} \cdot \nabla)[\nabla K_\delta(X_i - X_j + Y'_{ij})] \cdot \omega_j h^3$$

with $|Y_{ij}| \preceq 2\lambda^2$ and $|Y'_{ij}| \preceq 2\lambda^2$ by the assumption (3.15). It follows from generalized Young's inequality and Lemma 6.1 that $|r_i^{(2)}| \preceq C\lambda^2/\delta$. Moreover, since

$$|\sum_j \nabla K_\delta(X_i - X_j) \cdot \omega_j h^3 - \nabla u(X_i, t)| \preceq C[\delta^m + (h/\delta)^{3/2}|\ln h|]$$

by (C2) and we assume that $\nabla u(x,t)$ is bounded,

$$\max_i \Big|\sum_j \nabla K_\delta(X_i - X_j) \cdot \omega_j h^3\Big| \preceq C.$$

This proves (6.7).

To estimate $v_i^{(2)}$, we write

$$\begin{aligned} v_i^{(2)} &= \sum_j K_\delta(X_i - X_j) \cdot \varepsilon_j h^3 \\ &= \int_{\mathbf{R}^3} K_\delta(X_i - y)[\sum_j \phi_\lambda(y - X_j) \cdot \varepsilon_j\, h^3]\, dy + \\ &\quad \sum_j [K_\delta(X_i - X_j) - K_{\delta,\lambda}(X_i - X_j)] \cdot \varepsilon_j h^3 \\ &= v_i^{(21)} + v_i^{(22)} \end{aligned}$$

where $K_{\delta,\lambda} = K_\delta * \phi_\lambda$. To show that

$$\|v_i^{(21)}\|_{0,p,h} \preceq C\|\varepsilon_i\|_{-1,p}, \tag{6.8}$$

we follow a dualization procedure as in [5]. Let f_i be any element in $L_h^{p^*}$ and

$$b(y) = \sum_i K_\delta(X_i - y)\, f_i h^3.$$

Since

$$(v_i^{(21)}, f_i)_h = \sum_i \int_{\mathbf{R}^3} K_\delta(X_i - y)\, \varepsilon(y,t)\, dy f_i h^3 = (\varepsilon, b)$$

and

$$|(v_i^{(21)}, f_i)_h| \le \|\varepsilon\|_{-1,p} \|b\|_{1,p^*},$$

the estimate (6.8) follows from

$$\|b\|_{1,p^*} \preceq C \|f_i\|_{0,p^*,h}. \tag{6.9}$$

By applying Calderón-Zygmund inequality in a similar fashion as for $v^{(11)}$, we obtain

$$\| \sum_i \partial_l K_\delta(X_i - y)\, f_i h^3 \|_{0,p^*} \preceq C \|f_i\|_{0,p^*,\lambda}.$$

Thus

$$\|v_i^{(21)}\|_{0,p,h} \preceq C \|\varepsilon_i\|_{-1,p}.$$

For $v_i^{(22)}$ we will show that

$$\sum_j \left| K_\delta(X_i - X_j) - K_{\delta,\lambda}(X_i - X_j) \right| h^3 \preceq C\lambda$$

so that by generalized Young's inequality,

$$\|v_i^{(22)}\|_{0,p,h} \le C\lambda \|\varepsilon_i\|_{0,p,h}.$$

By Taylor's expansion, we have

$$\begin{aligned} K_{\delta,\lambda}(x) &= \int K_\delta(y) \phi_\lambda(x-y)\, dy \\ &= \int \{K_\delta(x) + [(y-x)\cdot\nabla] K_\delta(x) + R(x,y)\} \phi_\lambda(x-y)\, dy \\ &= K_\delta(x) + \int R(x,y) \phi_\lambda(x-y)\, dy \end{aligned}$$

by $\int \phi(x)\, dx = 1$ and ϕ being symmetric, where the remainder

$$R(x,y) = \int_0^1 (1-s)[(y-x)\cdot\nabla]^2 K_\delta(x + s(y-x))\, ds.$$

Let

$$\Psi(x) = K_{\delta,\lambda}(x) - K_\delta(x) = \int R(x,y) \phi_\lambda(x-y)\, dy.$$

It is clear that Ψ behaves like $\lambda^2\nabla^2 K_\delta$. In particular, it follows from (i) of Lemma 4.4 that

$$|\Psi(x)| \le \int |R(x,y)|\phi_\lambda(x-y)\,dy \le C\lambda^2\delta^{-4}, \qquad \forall x$$

since ϕ_λ is a non-negative function supported in $|x| \le \lambda$. Moreover, by (ii) of Lemma 4.4, we have for $|x| \ge 2\delta$,

$$\begin{aligned} |\Psi(x)| &\le 9\lambda^2 \max_{|\beta|=2} \max_{|y-x|\le\lambda} |\partial^\beta K_\delta(y)| \int \phi_\lambda(x-y)\,dy \\ &\le C_1\lambda^2(|x|-\lambda)^{-4} \\ &\le C\lambda^2|x|^{-4} \end{aligned}$$

since $|x| - \lambda \ge |x| - \delta \ge |x|/2 \ge \delta$. Therefore $\lambda^{-2}\Psi$ satisfies the same estimates in Lemma 4.4 with $|\beta| = 2$. Since the proof of Lemma 6.1 is based on Lemma 4.4, the estimate in Lemma 6.1 with $|\beta| = 2$ is also applicable to $\lambda^{-2}\Psi$. Hence

$$\sum_j |\Psi(X_i - X_j)|\, h^3 \preceq C\lambda^2\delta^{-1} \le C\lambda.$$

This finishes the estimate of $v_i^{(2)}$.

We write $v_i^{(3)}$ as

$$v_i^{(3)} = \sum_j [\nabla K_\delta(X_i - X_j + Y_{ij}) \cdot (e_i - e_j)] \cdot \varepsilon_j h^3.$$

Since $|\varepsilon_j| \le \lambda$, it follows from generalized Young's inequality and Lemma 6.1 that

$$\|v_i^{(3)}\|_{0,p,h} \preceq C\|e_i\|_{0,p,h}.$$

This finishes the stability estimate (S1) for a fixed time.

For (S2) we will prove the case $|\beta| = 1$. The proof for $|\beta| > 1$ is similar. We consider the decomposition

$$\begin{aligned} \nabla\tilde{u}_i^h - \nabla u_i^h &= \sum_j [\nabla K_\delta(\tilde{X}_i - \tilde{X}_j) - \nabla K_\delta(X_i - X_j)] \cdot \omega_j h^3 + \\ &\quad \sum_j \nabla K_\delta(X_i - X_j) \cdot \varepsilon_j h^3 + \\ &\quad \sum_j [\nabla K_\delta(\tilde{X}_i - \tilde{X}_j) - \nabla K_\delta(X_i - X_j)] \cdot \varepsilon_j h^3 \\ &= \chi_i^{(1)} + \chi_i^{(2)} + \chi_i^{(3)}. \end{aligned}$$

We write

$$\begin{aligned}
\chi_i^{(1)} &= \sum_j [\nabla K_\delta(X_i - \widetilde{X}_j) - \nabla K_\delta(X_i - X_j)] \cdot \omega_j h^3 + \\
&\quad \sum_j [\nabla K_\delta(\widetilde{X}_i - \widetilde{X}_j) - \nabla K_\delta(X_i - \widetilde{X}_j)] \cdot \omega_j h^3 \\
&= \sum_j \nabla^2 K_\delta(X_i - X_j + Y_{ij}) \cdot e_j \omega_j h^3 + \\
&\quad e_i \cdot \sum_j \nabla^2 K_\delta(X_i - X_j + Z_{ij}) \cdot \omega_j h^3 \\
&= \chi_i^{(11)} + \chi_i^{(12)}.
\end{aligned}$$

By Lemma 6.1 and generalized Young's inequality, we have

$$\|\chi_i^{(11)}\|_{0,p,h},\ \|\chi_i^{(12)}\|_{0,p,h} \preceq C\delta^{-1}\|e_i\|_{0,p,h}.$$

For $\chi^{(2)}$ it follows from the same argument for $v_i^{(11)}$ that

$$\|\chi_i^{(2)}\|_{0,p,h} \preceq C\|\varepsilon_i\|_{0,p,h}.$$

Finally we write

$$\chi_i^{(3)} = \sum_j [\nabla^2 K_\delta(X_i - X_j + Y_{ij}) \cdot (e_i - e_j)] \cdot \varepsilon_j h^3.$$

Since $|\varepsilon_i| \preceq \lambda$, it follows from generalized Young's inequality and Lemma 6.1 that

$$\|\chi_i^{(3)}\|_{0,p,h} \preceq C\lambda\delta^{-1}\|e_i\|_{0,p,h} \leq C\|e_i\|_{0,p,h}.$$

This completes the proof of (S2) with $|\beta| = 1$ for a fixed time. In the proof of the convergence we will need the following variation of (S2):

$$\|\nabla \widetilde{u}_i^h \widetilde{\omega}_i - \nabla u_i^h \omega_i\|_{0,p,h} \preceq C[\delta^{-1}\|e_i\|_{0,p,h} + \|\varepsilon_i\|_{0,p,h}]. \tag{6.10}$$

To show (6.10) we write

$$\nabla \widetilde{u}_i^h \widetilde{\omega}_i - \nabla u_i^h \omega_i = (\nabla \widetilde{u}_i^h - \nabla u_i^h) \cdot \widetilde{\omega}_i + \nabla u_i^h \cdot \varepsilon_i.$$

It follows from the assumption (3.16) and the consistency estimate (C2) that $|\widetilde{\omega}_i| \preceq C$ and $|\nabla u_i^h| \preceq C$, respectively. Therefroe (6.10) follows from (S2) with $|\beta| = 1$.

To prove (S3), we consider the following decomposition

$$
\begin{aligned}
&\nabla \tilde{u}_i^h \tilde{\omega}_i - \nabla u_i^h \omega_i \\
&= \{\sum_j [\nabla K_\delta(\tilde{X}_i - \tilde{X}_j) - \nabla K_\delta(X_i - X_j)] \cdot \omega_j h^3\} \cdot \omega_i + \\
&\quad \sum_j \nabla K_\delta(X_i - X_j) \cdot (\tilde{\omega}_j \tilde{\omega}_i - \omega_j \omega_i) h^3 + \\
&\quad \sum_j [\nabla K_\delta(\tilde{X}_i - \tilde{X}_j) - \nabla K_\delta(X_i - X_j)] \cdot (\tilde{\omega}_j \tilde{\omega}_i - \omega_j \omega_i) h^3 \\
&= \rho_i^{(1)} + \rho_i^{(2)} + \rho_i^{(3)}.
\end{aligned}
$$

$\rho_i^{(1)}$ can be written as

$$
\begin{aligned}
\rho_i^{(1)} &= \{\sum_j [\nabla K_\delta(X_i - \tilde{X}_j) - \nabla K_\delta(X_i - X_j)] \cdot \omega_j h^3\} \cdot \omega_i + \\
&\quad \{\sum_j [\nabla K_\delta(\tilde{X}_i - \tilde{X}_j) - \nabla K_\delta(X_i - \tilde{X}_j)] \cdot \omega_j h^3\} \cdot \omega_i \\
&= \rho_i^{(11)} + \rho_i^{(12)}.
\end{aligned}
$$

We begin to estimate

$$
\|\rho_i^{(11)}\|_{-1,p} \equiv \Big\| \sum_i \phi_\lambda(x - X_i) \cdot \rho_i^{(11)} h^3 \Big\|_{-1,p}. \tag{6.11}
$$

By the Mean Value Theorem,

$$
\begin{aligned}
\rho_i^{(11)} &= [\sum_j \nabla^2 K_\delta(X_i - X_j + Y_{ij}) \cdot e_j \omega_j h^3] \cdot \omega_i \\
&= [\sum_j \nabla^2 K_\delta(X_i - X_j) \cdot e_j \omega_j h^3] \cdot \omega_i \\
&\quad + \{\sum_j [\nabla^2 K_\delta(X_i - X_j + Y_{ij}) - \nabla^2 K_\delta(X_i - X_j)] \cdot e_j \omega_j h^3\} \cdot \omega_i \\
&= \varrho_i^{(11)} + r_i^{(11)}
\end{aligned}
$$

with $|Y_{ij}| \preceq \lambda^2$. Since ϕ_λ is supported in $|x| < \lambda$, only those X_i's with $|X_i - x| < \lambda$ have contribution to the summation in (6.11). We use Taylor's expansion to obtain

$$
\begin{aligned}
\nabla^2 K_\delta(X_i - X_j) &= \sum_{l=0}^{n-1} \frac{1}{l!} [(X_i - x) \cdot \nabla]^l \nabla^2 K_\delta(x - X_j) \\
&\quad + \frac{1}{n!} [(X_i - x) \cdot \nabla]^n \nabla^2 K_\delta(x - X_j + Y'_{ij})
\end{aligned}
$$

and write $\varrho_i^{(11)} = \varrho_i^{(110)} + \varrho_i^{(111)} + \cdots + \varrho_i^{(11n)}$ accordingly. Notice that $|Y_{ij}'| < \lambda$ since $|X_i - x| < \lambda$. For $\varrho_i^{(110)}$ we have

$$\left\|\varrho_i^{(110)}\right\|_{-1,p} = \left\|\left[\sum_j \nabla^2 K_\delta(x - X_j) \cdot e_j \omega_j h^3\right] \cdot \omega^{0,h}\right\|_{-1,p}$$

where $\omega^{0,h} = \sum_i \phi_\lambda(x - X_i) \cdot \omega_i h^3$ as defined in (5.11). Since $\|\omega^{0,h}\|_{1,\infty} \preceq C$ by Lemma 5.1, it is sufficient to estimate

$$\left\|\sum_j \nabla^2 K_\delta(x - X_j) \cdot e_j \omega_j h^3\right\|_{-1,p}$$

or equivalently, $\|\sum_j \nabla K_\delta(x - X_j) \cdot e_j \omega_j h^3\|_{0,p}$. We have

$$\left\|\sum_j \nabla K_\delta(x - X_j) \cdot e_j \omega_j h^3\right\|_{0,p} \preceq C\|e_i\|_{0,p,h}$$

by applying Calderón-Zygmund inequality in the same fashion as for $v^{(11)}$. For the higher order terms $\varrho^{(11l)}$, $l = 1, \cdots, n-1$, we apply Lemma 5.1, generalized Young's inequality, and Lemma 6.1 to obtain

$$\begin{aligned}
\|\varrho_i^{(11l)}\|_{-1,p} &\preceq C_1 \lambda^l \left\|\sum_j \nabla^{2+l} K_\delta(x - X_j) \cdot e_j \omega_j h^3\right\|_{-1,p} \\
&\leq C_2 \lambda^l \left\|\sum_j \nabla^{1+l} K_\delta(x - X_j) \cdot e_j \omega_j h^3\right\|_{0,p} \\
&\preceq C(\lambda/\delta)^l \|e_i\|_{0,p,h} \\
&\leq C\|e_i\|_{0,p,h}.
\end{aligned}$$

We can estimate the last term $\varrho^{(11n)}$ in L^p-norm for n large enough. By Lemma 6.1 and generalized Young's inequality,

$$\|\varrho_i^{(11n)}\|_{0,p} \preceq C\lambda^n \delta^{-n-1} \|e_i\|_{0,p,h} \leq C\|e_i\|_{0,p,h}$$

provided that $\lambda^n \leq \delta^{n+1}$, i. e. $n \geq q/(q' - q)$. To estimate $r_i^{(11)}$ we apply Mean Value Theorem to write

$$r_i^{(11)} = \left[\sum_j \nabla^3 K_\delta(X_i - X_j + Z_{ij}) \cdot Y_{ij} \cdot e_j \omega_j h^3\right] \cdot \omega_i$$

with $|Z_{ij}| \leq |Y_{ij}| \preceq \lambda^2$. It follows from the generalized Young's inequality and Lemma 6.1 that

$$\|r_i^{(11)}\|_{-1,p} \leq C_1 \|r_i^{(11)}\|_{0,p} \preceq C(\lambda/\delta)^2 \|e_i\|_{0,p,h} \leq C\|e_i\|_{0,p,h}.$$

This finishes the estimate for $\rho^{(11)}$.

To estimate $\rho_i^{(12)}$ we apply the Mean Value Theorem to obtain

$$\rho_i^{(12)} = [\sum_j \nabla^2 K_\delta(X_i - X_j + Y_{ij}) \cdot \omega_j h^3] \cdot e_i \omega_i$$

with $|Y_{ij}| \preceq 2\lambda^2$ by the assumption (3.15). By following a similar argument as that for $v^{(12)}$,

$$\max_i |\sum_j \nabla^2 K_\delta(X_i - X_j + Y_{ij}) \cdot \omega_j h^3| \preceq C. \tag{6.12}$$

Therefore

$$\|\rho_i^{(12)}\|_{0,p,h} \preceq C\|e_i\|_{0,p,h}.$$

To estimate $\rho_i^{(2)}$, we consider the decomposition

$$\widetilde{\omega}_j \widetilde{\omega}_i - \omega_j \omega_i = \varepsilon_j \omega_i + \omega_j \varepsilon_i + \varepsilon_j \varepsilon_i$$

and write

$$\begin{aligned}
\rho_i^{(2)} &= [\sum_j \nabla K_\delta(X_i - X_j) \cdot \varepsilon_j h^3] \cdot \omega_i + [\sum_j \nabla K_\delta(X_i - X_j) \cdot \omega_j h^3] \cdot \varepsilon_i \\
&\quad + [\sum_j \nabla K_\delta(X_i - X_j) \cdot \varepsilon_j h^3] \cdot \varepsilon_i \\
&= \rho_i^{(21)} + \rho_i^{(22)} + \rho_i^{(23)}.
\end{aligned}$$

To estimate $\rho^{(21)}$, we follow the strategy for $\rho^{(11)}$ by considering the Taylor's expansion

$$\begin{aligned}
\nabla K_\delta(X_i - X_j) &= \sum_{l=0}^{n-1} \frac{1}{l!}[(X_i - x) \cdot \nabla]^l \nabla K_\delta(x - X_j) \\
&\quad + \frac{1}{n!}[(X_i - x) \cdot \nabla]^n \nabla K_\delta(x - X_j + Y_{ij})
\end{aligned}$$

with $|Y_{ij}| < \lambda$ and write $\rho_i^{(21)} = \rho_i^{(210)} + \rho_i^{(211)} + \cdots + \rho_i^{(21n)}$ accordingly. For $\rho_i^{(210)}$ we have

$$\begin{aligned}
\|\rho_i^{(210)}\|_{-1,p} &= \left\|[\sum_j \nabla K_\delta(x - X_j) \cdot \varepsilon_j h^3] \cdot \omega^{0,h}\right\|_{-1,p} \\
&\preceq C_1 \left\|\sum_j \nabla K_\delta(x - X_j) \cdot \varepsilon_j h^3\right\|_{-1,p} \\
&\leq C_2 \left\|\sum_j K_\delta(x - X_j) \cdot \varepsilon_j h^3\right\|_{0,p}
\end{aligned}$$

by Lemma 5.1. It follows from a similar argument for $v^{(2)}$ that

$$\Big\| \sum_j K_\delta(x - X_j) \cdot \varepsilon_j h^3 \Big\|_{0,p} \preceq C\|\varepsilon_i\|_{-1,p}.$$

For $\rho_i^{(211)}$ we can use Lemma 5.1 and the argument for $v^{(11)}$ to obtain

$$\begin{aligned} \|\rho_i^{(211)}\|_{-1,p} &\preceq C_1\lambda \Big\| \sum_j \nabla^2 K_\delta(x - X_j) \cdot \varepsilon_j h^3 \Big\|_{-1,p} \\ &\leq C_2\lambda \Big\| \sum_j \nabla K_\delta(x - X_j) \cdot \varepsilon_j h^3 \Big\|_{0,p} \\ &\preceq C\lambda \|\varepsilon_i\|_{0,p,h}. \end{aligned}$$

For the higher order terms $\rho^{(21l)}$, $l = 2, \cdots, n-1$, we can apply Lemma 5.1, generalized Young's inequality, and Lemma 6.1 to obtain

$$\begin{aligned} \|\rho_i^{(21l)}\|_{-1,p} &\preceq C_1\lambda^l \Big\| \sum_j \nabla^{1+l} K_\delta(x - X_j) \cdot \varepsilon_j h^3 \Big\|_{-1,p} \\ &\leq C_2\lambda^l \Big\| \sum_j \nabla^l K_\delta(x - X_j) \cdot \varepsilon_j h^3 \Big\|_{0,p} \\ &\preceq C(\lambda/\delta)^{l-1}\lambda \|\varepsilon_i\|_{0,p,h} \\ &\leq C\lambda \|\varepsilon_i\|_{0,p,h}. \end{aligned}$$

The last term $\rho^{(21n)}$ can be estimated in L^p-norm for n large enough. It follows from the generalized Young's inequality and Lemma 6.1 that

$$\|\rho_i^{(21n)}\|_{0,p} \preceq C\lambda^n\delta^{-n}\|\varepsilon_i\|_{0,p,h} \leq C\lambda\|\varepsilon_i\|_{0,p,h}$$

provided that $n \geq q'/(q' - q)$.

To estimate $\rho_i^{(22)}$ we again consider the Taylor's expansion

$$\begin{aligned} \nabla K_\delta(X_i - X_j) &= \sum_{l=0}^{n-1} \frac{1}{l!}[(X_i - x) \cdot \nabla]^l \nabla K_\delta(x - X_j) \\ &\quad + \frac{1}{n!}[(X_i - x) \cdot \nabla]^n \nabla K_\delta(x - X_j + Y_{ij}) \end{aligned}$$

with $|Y_{ij}| < \lambda$ and write $\rho_i^{(22)} = \rho_i^{(220)} + \rho_i^{(221)} + \cdots + \rho_i^{(22n)}$. For $\rho_i^{(220)}$ we have

$$\begin{aligned} \|\rho_i^{(220)}\|_{-1,p} &= \Big\| [\sum_j \nabla K_\delta(x - X_j) \cdot \omega_j h^3] \cdot \varepsilon(x,t) \Big\|_{-1,p} \\ &\leq \Big\| \sum_j \nabla K_\delta(x - X_j) \cdot \omega_j h^3 \Big\|_{1,\infty} \|\varepsilon\|_{-1,p} \\ &\preceq C\|\varepsilon\|_{-1,p} \end{aligned}$$

provided that

$$\Big|\sum_j \nabla^2 K_\delta(x - X_j)\cdot\omega_j h^3\Big| \preceq C. \tag{6.13}$$

(6.13) is the continuous version of (6.12). We know that (6.13) holds on the lattice points $z_k = h^2\cdot k$ in any ball $B(R)$. If the radius R is large enough, then (6.13) is true for any x outside $B(R)$ by Lemma 6.2. For any $x\in B(R)$, let z_k be the closest lattice point. We have

$$\begin{aligned}
&\Big|\sum_j \nabla^2 K_\delta(x - X_j)\cdot\omega_j h^3 - \sum_j \nabla^2 K_\delta(z_k - X_j)\cdot\omega_j h^3\Big| \\
&= |x - z_k|\cdot\Big|\sum_j \nabla^3 K_\delta(z_k - X_j + Y_{jk})\cdot\omega_j h^3\Big| \\
&\preceq C(h/\delta)^2
\end{aligned}$$

by the generalized Young's inequality and Lemma 6.1. This justifies (6.13). It follows from a similar argument that

$$\Big|\sum_j \nabla^l K_\delta(x - X_j)\cdot\omega_j h^3\Big| \preceq C\delta^{2-l} \tag{6.14}$$

for $l > 2$. For the higher order terms $\rho^{(22l)}$, $l = 1,\cdots,n-1$, we have

$$\begin{aligned}
\|\rho^{(22l)}\|_{-1,p} &\le C_1\|\rho^{(22l)}\|_{0,p} \\
&\le C_2\lambda^l\Big\|\sum_j \nabla^{1+l} K_\delta(x - X_j)\cdot\omega_j h^3\Big\|_{0,\infty}\|\varepsilon_i\|_{0,p,h} \\
&\preceq C(\lambda/\delta)^{l-1}\lambda\|\varepsilon_i\|_{0,p,h} \\
&\le C\lambda\|\varepsilon_i\|_{0,p,h}
\end{aligned}$$

by the generalized Young's inequality and (6.14). For the last term $\rho_i^{(22n)}$ we can apply the generalized Young's inequality and Lemma 6.1 to obtain

$$\|\rho^{(22n)}\|_{0,p} \preceq C(\lambda/\delta)^n\|\varepsilon_i\|_{0,p,h} \le C\lambda\|\varepsilon_i\|_{0,p,h}$$

provided that $n \ge q'/(q'-q)$. For $\rho_i^{(23)}$ we have

$$\begin{aligned}
\|\rho_i^{(23)}\|_{0,p} &\preceq C_1\Big\|\Big[\sum_j \nabla K_\delta(X_i - X_j)\cdot\varepsilon_j h^3\Big]\cdot\varepsilon_i\Big\|_{0,p,h} \\
&\preceq C_1\lambda\Big\|\sum_j \nabla K_\delta(X_i - X_j)\cdot\varepsilon_j h^3\Big\|_{0,p,h} \\
&\le C\lambda\|\varepsilon_i\|_{0,p,h}
\end{aligned}$$

by the generalized Young's inequality, the assumption (3.16), and the argument for $v^{(11)}$.

By the Mean Value Theorem,

$$\begin{aligned}\rho_i^{(3)} &= \sum_j \nabla^2 K_\delta(X_i - X_j + Y_{ij}) \cdot e_i r_{ij} h^3 - \\ &\quad \sum_j \nabla^2 K_\delta(X_i - X_j + Y_{ij}) \cdot e_j r_{ij} h^3 \\ &= \rho_i^{(31)} - \rho_i^{(32)}\end{aligned}$$

where $r_{ij} = \tilde{\omega}_i \tilde{\omega}_j - \omega_i \omega_j$ with $|r_{ij}| \preceq C\lambda$. We have

$$\begin{aligned}\|\rho^{(31)}\|_{0,p} &\preceq C_1 \Big\| [\sum_j \nabla^2 K_\delta(X_i - X_j + Y_{ij}) \cdot r_{ij} h^3] \cdot e_i \Big\|_{0,p,h} \\ &\leq C_1 \max_i \Big| \sum_j \nabla^2 K_\delta(X_i - X_j + Y_{ij}) \cdot r_{ij} h^3 \Big| \cdot \|e_i\|_{0,p,h} \\ &\preceq C(\lambda/\delta) \|e_i\|_{0,p,h} \\ &\leq C \|e_i\|_{0,p,h}\end{aligned}$$

by the generalized Young's inequality and Lemma 6.1. Finally,

$$\begin{aligned}\|\rho^{(32)}\|_{0,p} &\preceq C_1 \Big\| \sum_j \nabla^2 K_\delta(X_i - X_j + Y_{ij}) \cdot e_j r_{ij} h^3 \Big\|_{0,p,h} \\ &\preceq C_1 \Big\| \sum_j M_{ij}^{(2)} |e_j| |r_{ij}| h^3 \Big\|_{0,p,h} \\ &\preceq C(\lambda/\delta) \|e_i\|_{0,p,h} \\ &\leq C \|e_i\|_{0,p,h}\end{aligned}$$

by the generalized Young's inequality and Lemma 6.1. This finishes the the proof of (S3) for a fixed time.

To extend the stability estimates for a fixed time to all time $t \in [0, T]$, we divide the interval $[0, T]$ into N subintervals $[t_n, t_{n+1}]$, $n = 0, \cdots, N-1$ with $\Delta t = t_{n+1} - t_n = O(h^4)$. Since the stability estimates hold for any fixed time except for an event of probability less than $h^{C_1 C |\ln h|}$, they hold on t_n, $n = 0, \cdots, N-1$ except for an event of probability less than $C_2 h^{C_1 C |\ln h| - 4}$ which goes to zero faster than any polynomial rate by choosing the constant C large enough. We can extend the stability estimates at t_n to the stability estimates at any $t \in [t_n, t_{n+1}]$. Notice that the proof of the stability estimates for a fixed time is based on Lemmas 5.2 and 6.1. The only statistics of X_i that

is required in the proofs of Lemma 6.1 and the stability lemma is about the density of the vortices (Lemma 5.2). $\widetilde{X}$ is treated as small perturbation from X_i. If Δt is small, then we can regard $X_i(t)$ as small perturbation from $X_i(t_n)$, too. Hence in the proof of the stability lemma for any $t \in [t_n, t_{n+1}]$ we can follow exactly the same argument for t_n. With the help of Lemma 6.1, whenever we need to estimate a term involving $X_i(t)$, we can write $X_i(t)$ as $X_i(t_n) + Y_i(t)$ with $Y_i(t) = X_i(t) - X_i(t_n)$ and transform the estimate to time t_n provided that

$$\max_n \max_{t_n \le t \le t_{n+1}} |X_i(t) - X_i(t_n)| \preceq \delta \tag{6.15}$$

as required by Lemma 6.1. To prove (6.15), we need the following elementary property of Brownian motion.

Lemma 6.2 *Let $W(t)$ be a standard Brownian motion in $\mathbf{R}^d$. Then*

$$P\{\max_{t \le s \le t+\Delta t} |W(s) - W(t)| \ge b\} \le c_1(\sqrt{\Delta t}/b)\exp(-c_2 b^2/\Delta t)$$

where $b > 0$ and the positive constants c_1, c_2 only depend on d.

See p.18 in Freedman [7] for a proof in $\mathbf{R}$. Since

$$X_i(t) - X_i(s) = \int_s^t u(X_i(\tau), \tau)\, d\tau + \sqrt{2\nu}\,\{W(t) - W(s)\},$$

it follows that for all $t \in [t_n, t_{n+1}]$

$$\begin{aligned} |X_i(t) - X_i(t_n)| &\le C_1|t - t_n| + \sqrt{2\nu}\,|W(t) - W(t_n)| \\ &\le Ch^4 + \sqrt{2\nu}\,|W(t) - W(t_n)|. \end{aligned}$$

By Lemma 6.2

$$P\{\max_{t_n \le t \le t_{n+1}} |W(t) - W(t_n)| \ge h^{3/2}\} \le C_1 h^{1/2} \exp\left[-C_2 h^{-1}\right]$$

which implies that

$$P\{\max_n \max_{t_n \le t \le t_{n+1}} |W(t) - W(t_n)| \ge h\} \le C_3 h^{-7/2} \exp\{-C_2 h^{-1}\} \to 0$$

faster than any polynomial in h as $h \to 0$. Therefore

$$\max_n \max_{t_n \le t \le t_{n+1}} |X_i(t) - X_i(t_n)| \preceq Ch^4 + \nu^{1/2} h^{3/2} \tag{6.16}$$

which is more than enough to guarantee (6.15). This completes the proof of stability lemma for all time.

We can extend the consistency estimates to all time by combining the consistency estimates for t_n, $n = 0, \cdots, N-1$, and the stability estimates for all time. For each $t \in [t_n, t_{n+1}]$, we have

$$\begin{aligned}
&\|u_i^h(t) - u(X_i(t),t)\|_{0,p,h} \\
&\leq \|u_i^h(t) - u_i^h(t_n)\|_{0,p,h} + \|u_i^h(t_n) - u(X_i(t_n),t_n)\|_{0,p,h} \\
&\quad + \|u(X_i(t_n),t_n) - u(X_i(t),t)\|_{0,p,h} \\
&\preceq C_1\big[\|X_i(t) - X_i(t_n)\|_{0,p,h} + \|\omega_i(t) - \omega_i(t_n)\|_{-1,p} \\
&\quad + \lambda\|\omega_i(t) - \omega_i(t_n)\|_{0,p,h}\big] + C_2[\delta^m + h(h/\delta)^{1/2}|\ln h|] \\
&\preceq C[\delta^m + h(h/\delta)^{1/2}|\ln h|] \qquad (6.17)
\end{aligned}$$

by (6.16) and the assumption that u has bounded derivatives. Therefore

$$\max_{0\leq t\leq T} \|u_i^h(t) - u(X_i(t),t)\|_{0,p,h} \preceq C[\delta^m + h(h/\delta)^{1/2}|\ln h|]. \qquad (6.18)$$

By the same argument we can justify

$$\|\partial^\beta u_i^h(t) - \partial^\beta u(X_i(t),t)\|_{0,p,h} \preceq C[\delta^m + \delta^{-|\beta|}h(h/\delta)^{1/2}|\ln h|] \qquad (6.19)$$

and (C3) for all time $t \in [0,T]$.

The stability lemma can also be used to pass the pointwise consistency estimates (5.2) and (5.3) to the L^p estimates (5.4) and (5.5). If the radius R of the ball $B(R)$ is sufficiently large, then (5.4) and (5.5) are equivalent to the L^p estimates on $B(R)$ since $u \in L^p(\mathbf{R}^3)$ and $|X_i(t)| \preceq R/2$ by Lemma 6.2. For any $x \in B(R)$, let $z_k = h^2 \cdot k$ be the lattice point closest to x. We write

$$\begin{aligned}
u^h(x,t) - u(x,t) &= [u^h(x,t) - u^h(z_k,t)] + [u^h(z_k,t) - u(z_k,t)] + \\
&\quad [u(z_k,t) - u(x,t)] \\
&= u^{(1)} + u^{(2)} + u^{(3)}.
\end{aligned}$$

We have $|u^{(3)}| \leq Ch^2$ and $|u^{(2)}| \preceq C[\delta^m + h(h/\delta)^{1/2}|\ln h|]$. For $u^{(1)}$ we write

$$\begin{aligned}
u^h(x,t) - u^h(z_k,t) &= \sum_i [K_\delta(x - X_i) - K_\delta(z_k - X_i)] \cdot \omega_i h^3 \\
&= (x - z_k) \cdot \sum_i \nabla K_\delta(z_k - X_i + Y_{ik}) \cdot \omega_i h^3.
\end{aligned}$$

By the same argument as for (6.7),

$$\max_k \Big| \sum_i \nabla K_\delta(z_k - X_i + Y_{ik}) \cdot \omega_i h^3 \Big| \preceq C. \tag{6.20}$$

Hence

$$\|u^h(x,t) - u(x,t)\|_{L^p(B(R))} \preceq C[\delta^m + h(h/\delta)^{1/2} |\ln h|].$$

(5.5) can be proved in the same way. Finally, both L^p estimates can be extended to all time $t \in [0,T]$ by following the argument for the corresponding discrete estimates.

7 Convergence

To prove the convergence we need to assume that

$$\max_{0\le t\le T}\max_i |\varepsilon_i(t)| \preceq \lambda^{5/2}\delta^{-3/2} \tag{7.1}$$

and

$$\max_{0\le t\le T} \|\varepsilon(x,t)\|_{-1,p} \preceq h^{3/2}\delta^{1/2}\lambda^{-1}. \tag{7.2}$$

(7.1) is stronger than the condition (3.16) in the stability lemma. The assumptions (7.1) and (7.2) will be justified later. Let

$$\eta(t) = \|e_i(t)\|_{0,p,h} + \|\varepsilon(x,t)\|_{-1,p} + \lambda\|\varepsilon_i(t)\|_{0,p,h}.$$

We will show that $\eta(t) \preceq C[\delta^m + h(h/\delta)^{1/2}|\ln h|]$. For e_i we have

$$\begin{aligned}\frac{de_i}{dt} &= \widetilde{u}_i^h(t) - u(X_i(t),t)\\ &= [\widetilde{u}^h(\widetilde{X}_i(t),t) - u^h(X_i(t),t)] + [u^h(X_i(t),t) - u(X_i(t),t)].\end{aligned}$$

By applying (6.17) and (S1), we obtain

$$\|de_i/dt\|_{0,p,h} \preceq C[\eta(t) + \delta^m + h(h/\delta)^{1/2}|\ln h|]. \tag{7.3}$$

For ε_i we have

$$\begin{aligned}\frac{d\varepsilon_i}{dt} &= \nabla\widetilde{u}_i^h(t)\widetilde{\omega}_i(t) - \nabla u(X_i(t),t)\omega_i(t)\\ &= [\nabla\widetilde{u}_i^h(t)\widetilde{\omega}_i(t) - \nabla u_i^h(t)\omega_i(t)] + [\nabla u_i^h(t)\omega_i(t) - \nabla u(X_i(t),t)\omega_i(t)].\end{aligned}$$

It follows from (6.19) and (6.10) that

$$\lambda\|d\varepsilon_i/dt\|_{0,p,h} \preceq C[\eta(t) + \delta^m + h(h/\delta)^{1/2}|\ln h|]. \tag{7.4}$$

Let $\sigma = H * \varepsilon$. To compute the differential $d\|\varepsilon\|_{-1,p} = d\|\sigma\|_{0,p}$, we need to use Itô's formula (chain rule in stochastic calculus) for continuous semimartingales. A continuous semimartingale is the sum of a continuous martingale and a process of bounded variation. Brownian motion $W(t)$ is the canonical example of continuous martingales. The diffusion process $X(t)$ is a semimartingale since by (3.3) it is the sum of a differentiable process and a Brownian motion. A smooth function of a continuous semimartingale is again a continuous semimartingale and its differential is given in the next lemma.

Lemma 7.1 (Itô's Formula) *Let $f \in C^2(\mathbf{R}^2)$ and $Y(t)$ be a continuous semimartingale. Then*

$$df(Y(t)) = f'(Y(t))dY(t) + \frac{1}{2}f''(Y(t))d\langle Y\rangle(t). \tag{7.5}$$

where $\langle Y\rangle(t)$ is the quadratic variation of $Y(t)$.

The formal relation

$$d\langle Y\rangle(t) = [dY(t)]^2$$

is a convenient device in computing the differentials. See e. g. [18] for a proof of Itô's formula and precise definitions of continuous martingales, semimartingales, and quadratic variations. Since σ is a smooth function of the diffusion processes X_i, σ is a semimartingale. By Itô's formula,

$$d\|\sigma\|_{0,p}^p = p\|\sigma\|_{0,p}^{p-1}d\|\sigma\|_{0,p} + \frac{1}{2}p(p-1)\|\sigma\|_{0,p}^{p-2}[d\|\sigma\|_{0,p}]^2.$$

Thus we have

$$d\|\sigma\|_{0,p} = p^{-1}\|\sigma\|_{0,p}^{1-p}d\|\sigma\|_{0,p}^p + \frac{1}{2}(1-p)\|\sigma\|_{0,p}^{-1}[d\|\sigma\|_{0,p}]^2.$$

On the other hand,

$$d\|\sigma\|_{0,p}^p = \int_{\mathbf{R}^3} d|\sigma(x,t)|^p\,dx$$

with

$$\begin{aligned} d|\sigma(x,t)|^p &= p|\sigma(x,t)|^{p-2}\sigma(x,t)d\sigma(x,t) \\ &\quad + \frac{1}{2}p(p-1)|\sigma(x,t)|^{p-2}[d\sigma(x,t)]^2. \end{aligned}$$

Therefore

$$\begin{aligned} d\|\sigma\|_{0,p} &= \|\sigma\|_{0,p}^{1-p}\int |\sigma(x,t)|^{p-2}\sigma(x,t)d\sigma(x,t)\,dx \\ &\quad + \frac{1}{2}(p-1)\|\sigma\|_{0,p}^{1-p}\int |\sigma(x,t)|^{p-2}[d\sigma(x,t)]^2\,dx \\ &\quad + \frac{1}{2}(1-p)\|\sigma\|_{0,p}^{-1}[d\|\sigma\|_{0,p}]^2 \\ &= \theta^{(1)} + \theta^{(2)} + \theta^{(3)}. \end{aligned} \tag{7.6}$$

Notice that $\theta^{(3)} \leq 0$. We begin to estimate $\theta^{(1)}$. Let $H_\lambda = H * \phi_\lambda$. It follows from Itô's formula that

$$\begin{aligned} d\sigma(x,t) &= [\sum_i H_\lambda(x - X_i(t)) \cdot \frac{d\varepsilon_i}{dt} h^3]\, dt \\ &\quad -\{\sum_i [u(X_i(t),t) \cdot \nabla] H_\lambda(x - X_i(t)) \cdot \varepsilon_i(t)\, h^3\}\, dt \\ &\quad +[\nu \sum_i \nabla^2 H_\lambda(x - X_i(t)) \varepsilon_i(t)\, h^3]\, dt \\ &\quad -\sqrt{2\nu} \sum_i [\nabla H_\lambda(x - X_i(t)) \cdot dW_i(t)] \varepsilon_i(t)\, h^3 \\ &= \sigma^{(1)} dt + \sigma^{(2)} dt + \sigma^{(3)} dt + \sigma^{(4)} \end{aligned}$$

where $\sigma^{(1)}$, $\sigma^{(2)}$, $\sigma^{(3)}$, $\sigma^{(4)}$ are the terms representing stretching, convection, diffusion, and the statistical error due to the finite number of vortices, respectively. For the stretching term involving $\sigma^{(1)}$, it follows from Hölder's inequality that

$$\Big| \int |\sigma(x,t)|^{p-2} \sigma(x,t) \sigma^{(1)}(x,t)\, dx \Big| \leq \big\| |\sigma|^{p-1} \big\|_{0,p^*} \|\sigma^{(1)}\|_{0,p} = \|\sigma\|_{0,p}^{p-1} \|\sigma^{(1)}\|_{0,p}.$$

where $(1/p) + (1/p^*) = 1$. We have $\|\sigma^{(1)}\|_{0,p} = \|d\varepsilon_i/dt\|_{-1,p}$ with

$$\begin{aligned} \frac{d\varepsilon_i(t)}{dt} &= \nabla \tilde{u}_i^h(t) \tilde{\omega}_i(t) - \nabla u(X_i(t),t) \omega_i(t) \\ &= [\nabla \tilde{u}^h(\tilde{X}_i(t),t) \tilde{\omega}_i(t) - \nabla u^h(X_i(t),t) \omega_i(t)] + \\ &\quad [\nabla u^h(X_i(t),t) \omega_i(t) - \nabla u(X_i(t),t) \omega_i(t)]. \end{aligned}$$

By (C3) and (S3),

$$\|\sigma^{(1)}\|_{0,p} \preceq C[\eta(t) + \delta^m + h(h/\delta)^{1/2} |\ln h|]. \tag{7.7}$$

In the convection term $\sigma^{(2)}$ we can replace $u(X_i,t)$ by $u(X_i,t) - u(x,t)$ without changing the integral $\theta^{(1)}$ since

$$\begin{aligned} &\int_{\mathbf{R}^3} |\sigma(x,t)|^{p-2} \sigma(x,t) (u(x,t) \cdot \nabla) \sigma(x,t)\, dx \\ &= \int_{\mathbf{R}^3} (u(x,t) \cdot \nabla) |\sigma(x,t)|^p\, dx \\ &= \int_{\mathbf{R}^3} \nabla \cdot [|\sigma(x,t)|^p \cdot u(x,t)]\, dx \\ &= 0. \end{aligned}$$

by $\nabla \cdot u = 0$ and Gauss' theorem. We redefine $\sigma^{(2)}(x,t) = \sum_i \Gamma(x, X_i(t); t) \cdot \varepsilon_i(t)\, h^3$ where

$$\Gamma(x, y; t) = [\big(u(x,t) - u(y,t)\big) \cdot \nabla] H_\lambda(x - y).$$

By Hölder's inequality,

$$\Big| \int |\sigma(x,t)|^{p-2} \sigma(x,t) \cdot \sigma^{(2)}(x,t)\, dx \Big| \le \|\sigma\|_{0,p}^{p-1} \|\sigma^{(2)}\|_{0,p}.$$

We will show that

$$\|\sigma^{(2)}\|_{0,p} \preceq C[\|\varepsilon\|_{-1,p} + \lambda \|\varepsilon_i\|_{0,p,h}]. \tag{7.8}$$

The proof of (7.8) is quite similar to the estimate of $v_i^{(2)}$ in the stability lemma. The difference is that Γ contains the term $u(x,t) - u(y,t)$ so that $\sigma^{(2)}$ is not the convolution of ε_i with certain function. Since we consider $\|\sigma^{(2)}(\cdot, t)\|_{0,p}$ for each fixed time, the variable t in Γ will be dropped for simplicity. We can write

$$\begin{aligned} \sigma^{(2)}(x,t) &= \int \Gamma(x,y)[\sum_i \phi_\lambda(y - X_i(t)) \varepsilon_i(t)\, h^3]\, dy \\ &\quad + \sum_i [\Gamma(x, X_i(t)) - \Gamma_\lambda(x, X_i(t))] \varepsilon_i(t)\, h^3 \\ &= \sigma^{(21)} + \sigma^{(22)} \end{aligned}$$

where $\Gamma_\lambda(x,y) = \int \Gamma(x,z) \phi_\lambda(y - z)\, dz$.

We begin to show that

$$\|\sigma^{(21)}\|_{0,p} \le C \|\varepsilon\|_{-1,p}. \tag{7.9}$$

Let f be any element in $L^{p^*}(\mathbf{R}^3)$. By the dualization procedure as in estimating $v_i^{(21)}$, (7.9) follows from

$$\Big\| \int \Gamma(x,y) f(x)\, dx \Big\|_{1,p^*} \le C \|f\|_{0,p^*}. \tag{7.10}$$

Since Γ is anti-symmetric, (7.10) is equivalent to

$$\Big\| \int \Gamma(x,y) f(y)\, dy \Big\|_{1,p^*} \le C \|f\|_{0,p^*}.$$

Let ∂_l denote the partial derivative with respect to x_l, $l = 1,\ 2,\ 3$. We have

$$\begin{aligned}\partial_l\Gamma(x,y) &= \sum_{m=1}^{3}[\partial_l u_m(x,t)]\partial_m H_\lambda(x-y) + \\ &\quad \sum_{m=1}^{3}[u_m(x,t)-u_m(y,t)]\partial_l\partial_m H_\lambda(x-y) \\ &= \Phi^{(1)}(x,y)+\Phi^{(2)}(x,y).\end{aligned}$$

Notice that H_λ is a radial function which implies that $\partial_m H_\lambda$ is an odd function. Since u is bounded, it follows from Calderón-Zygmund inequality that

$$\Big\|\int \Phi^{(1)}(x,y)f(y)\,dy\Big\|_{0,p^*} \le C\|f\|_{0,p^*}.$$

For $\Phi^{(2)}$ we consider the expansion

$$u(x,t)-u(y,t) = [(x-y)\cdot\nabla]u(x,t) + R(x,y)$$

where $R(x,y) = -\int_0^1(1-s)[(y-x)\cdot\nabla]^2 u(x+s(y-x),t)\,ds$. We write

$$\begin{aligned}\Phi^{(2)}(x,y) &= \sum_{m=1}^{3}\{[(x-y)\cdot\nabla]u_m(x,t)\}\partial_l\partial_m H_\lambda(x-y) + \\ &\quad \sum_{m=1}^{3}R_m(x,y)\partial_l\partial_m H_\lambda(x-y) \\ &= \Phi^{(21)}(x,y)+\Phi^{(22)}(x,y).\end{aligned}$$

Since ∇u is bounded and $x_n\partial_l\partial_m H_\lambda$ is an odd function, it follows from Calderón-Zygmund inequality that

$$\Big\|\int \Phi^{(21)}(x,y)f(y)\,dy\Big\|_{0,p^*} \le C\|f\|_{0,p^*}.$$

Since $|R(x,y)| \le C|x-y|^2$, $\Phi^{(22)}$ is integrable with respect to the variable x or y. Therefore by generalized Young's inequality,

$$\Big\|\int \Phi^{(22)}(x,y)f(y)\,dy\Big\|_{0,p^*} \le C\|f\|_{0,p^*}.$$

This finishes the proof of (7.9).

For $\sigma^{(22)}$ we consider the expansion

$$\begin{aligned}\Gamma_\lambda(x,y) &= \int \Gamma(x,z)\phi_\lambda(y-z)\,dz \\ &= \int \{\Gamma(x,y) + [(z-y)\cdot\nabla]\Gamma(x,y) + R(x,y,z)\}\phi_\lambda(y-z)\,dz \\ &= \Gamma(x,y) + \int R(x,y,z)\phi_\lambda(y-z)\,dz\end{aligned}$$

where

$$R(x,y,z) = \int_0^1 (1-s)[(z-y)\cdot\nabla]^2\Gamma(x,y+s(z-y))\,ds.$$

Let

$$\Psi(x,y) = \Gamma_\lambda(x,y) - \Gamma(x,y) = \int R(x,y,z)\phi_\lambda(y-z)\,dz.$$

Let ∂_l denote the partial derivative with respect to y_l, $l = 1,\ 2,\ 3$. We have

$$\begin{aligned}\partial_l\partial_m\Gamma(x,y) &= -\sum_{n=1}^{3}[\partial_l\partial_m u_n(y,t)]\partial_n H_\lambda(x-y) \\ &\quad -\sum_{n=1}^{3}[\partial_l u_n(y,t)\partial_m + \partial_m u_n(y,t)\partial_l]\partial_n H_\lambda(x-y) \\ &\quad +\sum_{n=1}^{3}[(u_n(x,t) - u_n(y,t)]\partial_l\partial_m\partial_n H_\lambda(x-y).\end{aligned}$$

Since H_λ and its derivatives satisfy Lemma 4.4 with δ replaced by λ, we have

$$\begin{aligned}|\partial_l\partial_m\Gamma(x,y)| &\le C\lambda^{-4}, \quad \forall x,\ y \\ |\partial_l\partial_m\Gamma(x,y)| &\le C|x-y|^{-4}, \quad |x-y| \ge \lambda.\end{aligned}$$

Therefore

$$|\Psi(x,y)| \le \int |R(y,z)|\phi_\lambda(y-z)\,dz \le C\lambda^{-2}, \quad \forall x,\ y$$

and for $|x-y| \ge 2\lambda$,

$$\begin{aligned}|\Psi(x,y)| &\le 9\lambda^2 \max_{|\beta|=2}\max_{|z-y|\le\lambda}|\partial^\beta\Gamma(x,z)|\int \phi_\lambda(y-z)\,dz \\ &\le C_1\lambda^2(|x-y|-\lambda)^{-4} \\ &\le C\lambda^2|x-y|^{-4}\end{aligned}$$

since $|x-z| \geq |x-y| - |z-y| \geq |x-y|/2 \geq \lambda$. The result in Lemma 6.1 with $|\beta| = 2$ and δ replaced by λ is applicable and we have

$$\|\sigma^{(22)}\|_{0,p} \preceq C\lambda \|\varepsilon_i\|_{0,p,h}.$$

This finishes the proof of (7.8).

The diffusion term $\sigma^{(3)}$ has negative contribution since

$$\begin{aligned}
&\int_{\mathbf{R}^3} |\sigma(x,t)|^{p-2}\sigma(x,t)\nabla^2\sigma(x,t)\,dx \\
&= -\int \nabla[|\sigma(x,t)|^{p-2}\sigma(x,t)] \cdot \nabla\sigma(x,t)\,dx \\
&= -(p-1)\int |\sigma(x,t)|^{p-2}|\nabla\sigma(x,t)|^2\,dx \\
&\leq 0
\end{aligned}$$

with

$$\begin{aligned}
\nabla[|\sigma|^{p-2}\sigma] &= |\sigma|^{p-2}\nabla\sigma + \sigma\nabla|\sigma|^{p-2} \\
&= |\sigma|^{p-2}\nabla\sigma + \sigma[(p-2)|\sigma|^{p-4}\sigma\nabla\sigma] \\
&= (p-1)|\sigma|^{p-2}\nabla\sigma.
\end{aligned}$$

For the statistical error term involving $\sigma^{(4)}$, we consider the stochastic integral

$$\begin{aligned}
\xi(t) &= \sum_i \xi_i(t) \\
&= -\sqrt{2\nu}\sum_i \int_0^t \|\sigma\|_{0,p}^{1-p} f_i(s)\cdot\varepsilon_i(s)\,h^3 \qquad (7.11)
\end{aligned}$$

where the (vector) stochastic differential

$$f_i(t) = \int_{\mathbf{R}^3} |\sigma(x,t)|^{p-2}\sigma(x,t)[\nabla H_\lambda(x - X_i(t))\cdot dW_i]\,dx.$$

We will show that

$$\max_{0\leq t\leq T} |\xi(t)| \preceq Ch(h/\delta)^{1/2}|\ln h|. \qquad (7.12)$$

ξ is a continuous martingale since each summand ξ_i is. The quadratic variation of $\xi(t)$ is

$$\langle\xi\rangle(t) = 2\nu h^3 \int_0^t \|\sigma\|_{0,p}^{2(1-p)}\{\sum_i g_i^{(2)}(s)\cdot\varepsilon_i^2(s)\,h^3\}\,ds$$

with $g_i^{(2)}(t) = \sum_{l=1,m}^{3} [g_{i;l,m}(t)]^2$ where

$$g_{i;l,m}(t) = \int |\sigma(x,t)|^{p-2}\sigma_l(x,t)\partial_m H_\lambda(x - X_i(t))\, dx.$$

It has been shown that any continuous martingale can be transformed into a Brownian motion by a (random) change of time according to its quadratic variation. More specifically, we have

$$\xi(t) = W(\langle\xi\rangle(t)) \tag{7.13}$$

almost surely for a suitably defined Brownian motion W. See Theorem 4.6 in [18] for more details and a proof. By utilizing (7.13), one can prove the next lemma which generalizes Lemma 6.2 to continuous martingales.

Lemma 7.2 *Let $M(t)$ be a continuous martingale. If the quadratic variation $\langle M\rangle(t) \le at$ for certain constant $a > 0$, then for fixed $T > 0$,*

$$P\{\max_{0\le t\le T} |M(t)| \ge b\} \le c_1(\sqrt{aT}/b)\exp(-c_2 b^2/aT).$$

See p. 232 in [18] for a proof. We need to estimate the quadratic variation $\langle\xi\rangle(t)$. By Hölder's inequality,

$$\sum_i g_i^{(2)} \cdot \varepsilon_i^2\, h^3 \le \|g_i^{(2)}\|_{0,\frac{p}{p-2},h} \cdot \|\varepsilon_i^2\|_{0,\frac{p}{2},h} = \|g_i^{(2)}\|_{0,\frac{p}{p-2},h} \cdot \|\varepsilon_i\|_{0,p,h}^2.$$

Moreover, it follows from the generalized Young's inequality (iii) in Lemma 4.6 that

$$\begin{aligned}
\|g_{i;l,m}^2\|_{0,\frac{p}{p-2},h} &= \|g_{i;l,m}\|_{0,\frac{2p}{p-2},h}^2 \\
&\le |||\nabla H_\lambda(x - X_i)|||_2^2 \cdot \left\||\sigma|^{p-2}\sigma\right\|_{0,\frac{p}{p-1}}^2 \\
&= |||\nabla H_\lambda(x - X_i)|||_2^2 \cdot \|\sigma\|_{0,p}^{2(p-1)}.
\end{aligned}$$

Since $\|\nabla H_\lambda\|_{0,2}^2 \le C_1\lambda^{-3}$ and $\sum_i |\nabla H_\lambda(x - X_i)|^2\, h^3 \preceq C_2\lambda^{-3}$,

$$\|g_i^{(2)}\|_{0,\frac{p}{p-2},h} \preceq C\lambda^{-3}\|\sigma\|_{0,p}^{2(p-1)}.$$

Therefore by (7.1)

$$\langle\xi\rangle(t) \preceq C_1\nu h^3\lambda^{-3}\int_0^t \|\varepsilon_i(s)\|_{0,p,h}^2\, ds \preceq C\nu h^3\lambda^2\delta^{-3}t \le C\nu h^3\delta^{-1}t. \tag{7.14}$$

Notice that the event of (7.14) being violated has extremly small but positive probability since it is possible that

$$|\varepsilon_i(t)| > \lambda(\lambda/\delta)^{1/2} \tag{7.15}$$

for some i and t. In Lemma 7.2 the quadratic variation is assumed to be bounded by a constant times t. Therefore we need to modify the process $\xi(t)$ before applying Lemma 7.2. To remove the undesired feature (7.15), we use the stopping time

$$\tau_i = \inf\{t \geq 0; |\varepsilon_i(t)| = \lambda(\lambda/\delta)^{1/2}\}$$

to define the truncated process

$$\tilde{\varepsilon}_i(t) = \begin{cases} \varepsilon_i(t); & 0 \leq t \leq \tau_i, \\ \varepsilon_i(\tau_i); & \tau_i < t \leq T \end{cases}$$

and the martingale

$$\tilde{\xi}(t) = -\sqrt{2\nu} \sum_i \int_0^t \|\sigma\|_{0,p}^{1-p} f_i(s) \cdot \tilde{\varepsilon}_i(s)\, h^3.$$

The quadratic variation $\langle\tilde{\xi}\rangle(t)$ of the martingale $\tilde{\xi}(t)$ satisfies

$$\langle\tilde{\xi}\rangle(t) \leq C\nu h^3 \delta^{-1} t.$$

Let $b = Ch(h/\delta)^{1/2}|\ln h|$. By Lemma 7.2,

$$\begin{aligned} & P\{\max_{0\leq t\leq T} |\tilde{\xi}(t)| \geq b\} \\ & \leq C_1(C_2 h^3 \delta^{-1} T)^{1/2} b^{-1} \exp[-C_3 b^2 h^{-3} \delta T^{-1}] \\ & \leq C_1(C_2 T)^{1/2} \exp[-C_3 T^{-1} C^2 |\ln h|^2] \end{aligned} \tag{7.16}$$

which goes to zero faster than any polynomial rate by choosing C large enough or h small enough. Furthermore,

$$\begin{aligned} & P\{\max_{0\leq t\leq T} |\xi(t)| \geq Ch(h/\delta)^{1/2}|\ln h|\} \\ & \leq P\{\max_{0\leq t\leq T} |\tilde{\xi}(t)| \geq Ch(h/\delta)^{1/2}|\ln h|\} + \\ & \quad P\{|\varepsilon_i(t)| > \lambda(\lambda/\delta)^{1/2}, \text{ for some } i \text{ and } t\}. \end{aligned}$$

Hence (7.12) follows from (7.16) and (7.1).

For $\theta^{(2)}$ it follows Hölder's inequality and generalized Young's inequality that

$$\begin{aligned}
&\frac{1}{2}(p-1)\|\sigma\|_{0,p}^{1-p}\int|\sigma|^{p-2}(d\sigma)^2\,dx \\
&= \nu(p-1)h^3\|\sigma\|_{0,p}^{1-p}\Big\{\int|\sigma|^{p-2}\Big\{\sum_i[\nabla H_\lambda(x-X_i)]^2\varepsilon_i^2\,h^3\Big\}\,dx\Big\}dt \\
&\le \nu(p-1)h^3\|\sigma\|_{0,p}^{1-p}\cdot\|\sigma\|_{0,p}^{p-2}\cdot|||\nabla H_\lambda(x-X_i)|||_1\cdot\|\varepsilon_i^2\|_{0,\frac{p}{2},h}dt \\
&\preceq Ch^3\lambda^{-3}\|\sigma\|_{0,p}^{-1}\cdot\|\varepsilon_i\|_{0,p,h}^2dt \\
&\le C\|\sigma\|_{0,p}^{-1}\cdot\lambda^2\|\varepsilon_i\|_{0,p,h}^2dt
\end{aligned}$$

since $\lambda = h^{q'}$ with $q' < 3/5$. By combining (7.6), (7.7), (7.8), and (7.12), we obtain

$$\begin{aligned}
\|\sigma(\cdot,t)\|_{0,p} \preceq\ & C\Big\{\delta^m + h(h/\delta)^{1/2}|\ln h| \\
&+\int_0^t[\eta(s)+\lambda^2\|\varepsilon_i(s)\|_{0,p,h}^2\|\sigma(\cdot,s)\|_{0,p}^{-1}]\,ds\Big\}. \quad (7.17)
\end{aligned}$$

There is an undesirable term $\|\sigma\|_{0,p}^{-1}$ in (7.17). It is quite plausible that

$$\lambda\|\varepsilon_i\|_{0,p,h} \le C\|\sigma\|_{0,p}^{-1}. \quad (7.18)$$

However, (7.18) is difficult to prove even if it is true. A simple way to get around this difficulty is to consider the sum of the squares:

$$\zeta(t) = \|e_i(t)\|_{0,p,h}^2 + \|\varepsilon(x,t)\|_{-1,p}^2 + \lambda^2\|\varepsilon_i(t)\|_{0,p,h}^2.$$

Let $\kappa = \delta^m + h(h/\delta)^{1/2}|\ln h|$. We have

$$\frac{d}{dt}\|e_i(t)\|_{0,p,h}^2 \le 2\|e_i(t)\|_{0,p,h}\|de_i/dt\|_{0,p,h} \preceq C[\kappa^2+\zeta(t)] \quad (7.19)$$

and

$$\lambda^2\frac{d}{dt}\|\varepsilon_i(t)\|_{0,p,h}^2 \le 2\lambda^2\|\varepsilon_i(t)\|_{0,p,h}\|d\varepsilon_i/dt\|_{0,p,h} \preceq C[\kappa^2+\zeta(t)] \quad (7.20)$$

by (7.3) and (7.4). It follows from Itô's formula and (7.6) that

$$\begin{aligned}
d\|\varepsilon(x,t)\|_{-1,p}^2 &= d\|\sigma(x,t)\|_{0,p}^2 \\
&= 2\|\sigma\|_{0,p}d\|\sigma\|_{0,p} + [d\|\sigma\|_{0,p}]^2 \\
&= 2\|\sigma\|_{0,p}(\theta^{(1)}+\theta^{(2)}) + (2-p)(d\|\sigma\|_{0,p})^2. \quad (7.21)
\end{aligned}$$

We assume that $p \geq 2$ so that the last term in (7.21) can be ignored. By integrating (7.21) we obtain

$$\|\varepsilon(x,t)\|_{-1,p}^2 \preceq C\Big\{\kappa^2 + \int_0^t \zeta(s)\,ds\Big\}. \tag{7.22}$$

The only thing that requires some justification in obtaining (7.22) is about the statistical error term involving $\sigma^{(4)}$. There is an extra factor $\|\sigma\|_{0,p}$ in the stochastic integral (7.11). We follow the same argument as before except that the quadratic variation is now bounded by

$$C_1\nu h^3\lambda^{-3}\int_0^t \|\sigma(x,s)\|_{0,p}^2\|\varepsilon_i(s)\|_{0,p,h}^2\,ds \preceq C\nu h^6\delta^{-2}t$$

by the assumptions (7.1) and (7.2). Combining (7.19), (7.20), and (7.22), we have

$$\zeta(t) \preceq C\Big\{\kappa^2 + \int_0^t \zeta(s)\,ds\Big\}. \tag{7.23}$$

By Gronwall's inequality, $\zeta(t) \preceq \bar{C}\kappa^2$. Hence

$$\eta(t) \preceq C[\delta^m + h(h/\delta)^{1/2}|\ln h|] \tag{7.24}$$

for $0 \leq t \leq T$.

To complete the proof, we need to justify the assumptions (3.15), (7.1), and (7.2) for $0 \leq t \leq T$. (7.2) is an immediate consequnce of (7.24). Since $h^3 \cdot \max_i |e_i|^p \leq \|e_i\|_{0,p,h}^p$, we have

$$\max_i |e_i| \leq h^{-3/p}\|e_i\|_{0,p,h} \preceq Ch^{\frac{3}{2}-\frac{q}{2}-\frac{3}{p}}|\ln h| \leq \lambda^2$$

provided that $m > 2q'/q$ with q' sufficiently close to q. By (C2) and (6.10),

$$\begin{aligned}\|d\varepsilon_i/dt\|_{0,p,h} &\preceq C_1[\delta^{-1}\|e_i(t)\|_{0,p,h} + \|\varepsilon_i(t)\|_{0,p,h} + (h/\delta)^{3/2}|\ln h|] \\ &\preceq C[\|\varepsilon_i(t)\|_{0,p,h} + \delta^{m-1} + (h/\delta)^{3/2}|\ln h|].\end{aligned}$$

Hence by Gronwall's inequality,

$$\|\varepsilon_i(t)\|_{0,p,h} \preceq \bar{C}[\delta^{m-1} + (h/\delta)^{3/2}|\ln h|]$$

and

$$\max_i |\varepsilon_i| \leq h^{-3/p}\|\varepsilon_i\|_{0,p,h} \preceq Ch^{-\frac{3}{p}}(h/\delta)^{3/2}|\ln h| \leq \lambda^{5/2}\delta^{-3/2}$$

for $t < T$ by choosing p, m large enough and h small enough. This completes the proof for the convergence of particle paths.

Remark The constraint $1/3 < q$ in Theorem 1 follows from the condition $2\epsilon < 3q-1$ in the stability lemma 2.2 since the proof in [5] used the estimate

$$\|\varepsilon_i\|_{0,p,h} \leq 2h^{-1}\|\varepsilon_i\|_{-1,p,h}$$

which loses a factor of h. Here we estimate $\|\varepsilon_i\|_{0,p,h}$ directly from the consistency and the stability lemmas so that the correct factor δ^{-1} is obtained. This removes the constraint $1/3 < q$.

For the convergence of discrete velocity we have

$$\begin{aligned}
&\|\widetilde{u}_i^h(t) - u(X_i(t),t)\|_{0,p,h} \\
&\quad\leq \|\widetilde{u}_i^h(t) - u_i^h(t)\|_{0,p,h} + \|u_i^h(t) - u(X_i(t),t)\|_{0,p,h} \\
&\quad\preceq C[\eta(t) + \delta^m + h(h/\delta)^{1/2}|\ln h|] \\
&\quad\preceq C[\delta^m + h(h/\delta)^{1/2}|\ln h|]
\end{aligned}$$

by (C1), (S1), and (7.24). For the convergence of the continuous velocity, we again consider the lattice points of spacing h^2 inside the ball $B(R_0)$ and write

$$\begin{aligned}
\widetilde{u}^h(x,t) - u(x,t) &= [\widetilde{u}^h(x,t) - \widetilde{u}^h(z_k,t)] + [\widetilde{u}^h(z_k,t) - u^h(z_k,t)] \\
&\quad + [u^h(z_k,t) - u(z_k,t)] + [u(z_k,t) - u(x,t)] \\
&= u^{(1)} + u^{(2)} + u^{(3)} + u^{(4)}
\end{aligned}$$

where $z_k = h^2 \cdot k$ is the lattice point closest to x. The set of all points x closest to z_k is the square centered at z_k with side length h^2. We have $|u^{(4)}| \leq Ch^2$ and $|u^{(3)}| \preceq C[\delta^m + h(h/\delta)^{1/2}|\ln h|]$. For $u^{(1)}$ we write

$$\begin{aligned}
\widetilde{u}^h(x,t) - \widetilde{u}^h(z_k,t) &= \sum_i [K_\delta(x - \widetilde{X}_i) - K_\delta(z_k - \widetilde{X}_i)] \cdot \omega_i h^3 \\
&= (x - z_k) \cdot \sum_i \nabla K_\delta(z_k - \widetilde{X}_i + Y_{ik}) \cdot \omega_i h^3 \\
&= (x - z_k) \cdot \sum_i \nabla K_\delta(z_k - X_i + Z_{ik}) \cdot \omega_i h^3
\end{aligned}$$

with $|Z_{ik}| \leq |\widetilde{X}_i - X_i| + |Y_{ik}| \preceq \delta$. Hence $|u^{(1)}| \preceq Ch^2$ by (6.20). For $u^{(2)}$ we write

$$\begin{aligned}
\widetilde{u}^h(z_k,t) - u^h(z_k,t) &= \sum_i [K_\delta(z_k - \widetilde{X}_i) - K_\delta(z_k - X_i)] \cdot \omega_i h^3 \\
&= \sum_i \nabla K_\delta(z_k - X_i + Y_{ik}) \cdot e_i \omega_i h^3.
\end{aligned}$$

It follows from the argument for $v^{(11)}$ in (S1) that

$$\Big\| \sum_i \nabla K_\delta(z_k - X_i + Y_{ik}) \cdot e_i \omega_i h^3 \Big\|_{0,p,h^2} \preceq C \|e_i\|_{0,p,h}.$$

Notice that $\nabla K_\delta(z_k - X_i + Y_{ik})$ is approximated by $\nabla K_\delta(z_k - Z_i)$ where Z_i is the closest lattice point in $\lambda \cdot \mathbf{Z}^3$, not in $h^2 \cdot \mathbf{Z}^3$. This justifies the convergence of continuous velocity.

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