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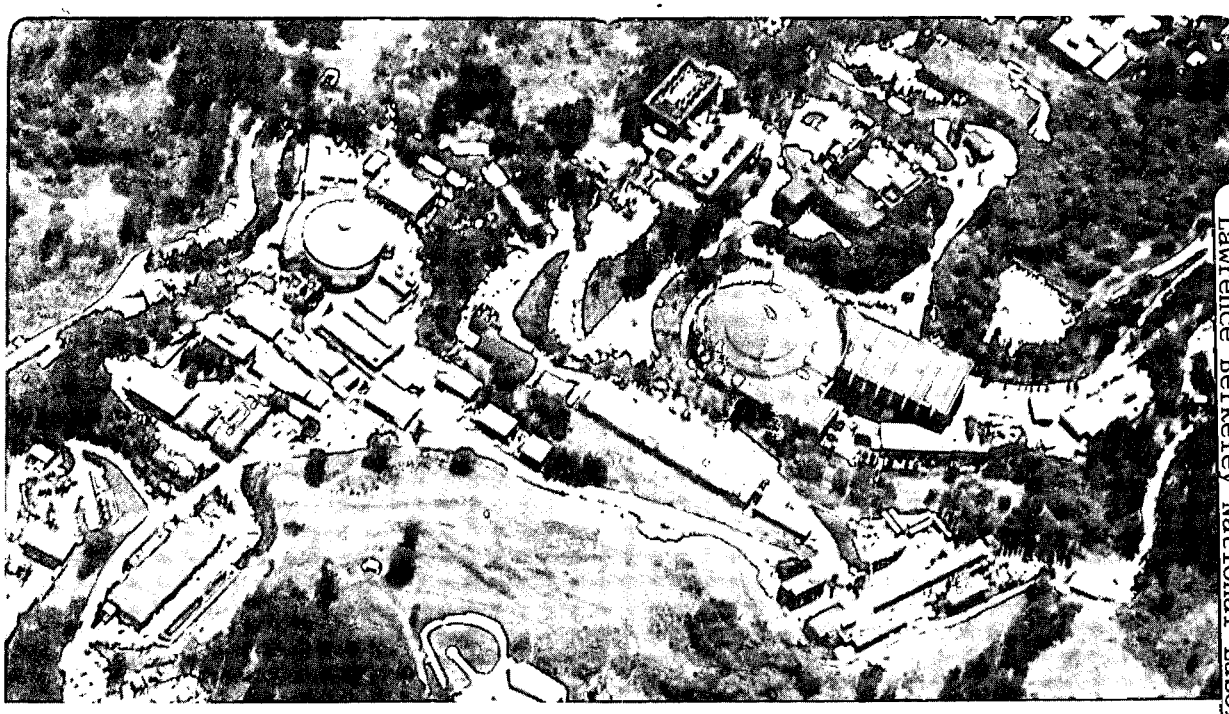

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RAPIDLY-CONVERGENT QUADRATURES FOR INTEGRAL OPERATORS WITH SINGULAR KERNELS*

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Rapidly-Convergent Quadratures for Integral Operators with Singular Kernels

Bradley K. Alpert

Abstract

A numerical integration method that has high-order convergence for singular integrands is presented. Based on endpoint corrections to the trapezoidal rule, the quadratures are well suited to integral equations encountered in problems of potential theory. Several numerical examples, as well as actual quadrature weights, are included.

Key Words. numerical integration, singular kernels, quadrature rules, corrected trapezoidal rules

AMS(MOS) subject classifications. 65D30, 65B15, 65R20

An integral equation to be solved numerically must be reduced to a finite-dimensional problem. The Nyström method achieves this reduction by approximating the integral with a quadrature (a weighted average of values of the integrand at selected points). The quadrature's rate of convergence to the integral, as the number of points increases, affects the amount of computation needed to achieve a given accuracy in the solution to the integral equation. The convergence rate depends on the behavior of the integrand and in particular on whether the integrand is singular in the interval of integration. In this paper we develop rapidly-convergent quadratures for several types of singularities encountered in physical problems.

The quadratures are based on the trapezoidal rule, with modifications to the points and weights at the ends of the interval of integration. It is designed to give high-order convergence for integrands which are several times differentiable except at isolated singularities of known type. The method arises from a technique developed by Rokhlin [4]. Changes to his method have been made with the aim of solving non-periodic integral equations; in addition, one improvement permits higher order quadratures in practice. These differences from [4] are noted as they are presented. The dissertation [1] of the present author gives an earlier report of the present work.

In §1 the Nyström method is introduced. In §2 we construct quadrature rules for differentiable and for singular integrands. In §3 we analyze their convergence properties, and several numerical examples are given in §4. An appendix provides actual quadrature weights for various parameter values.

1 Nyström Method

A linear Fredholm integral equation of the second kind is an expression of the form

$$f(x) - \int_a^b K(x,t)\, f(t)\, dt = g(x), \tag{1}$$

where the kernel K is in $\mathcal{L}^2[a,b]^2$ and the unknown f and right-hand-side g are in $\mathcal{L}^2[a,b]$. We use the symbol $\mathcal{K}$ to denote the integral operator of Eq. (1), which is given by the formula

$$(\mathcal{K}f)(x) = \int_a^b K(x,t)\, f(t)\, dt,$$

for all $f \in \mathcal{L}^2[a,b]$ and $x \in [a,b]$. Then Eq. (1) written in operator form is

$$(I - \mathcal{K})f = g. \tag{2}$$

The Nyström, or *quadrature*, method for the numerical solution of integral equations approximates the integral operator $\mathcal{K}$ by the finite-dimensional operator R, characterized by points $x_1, x_2, \ldots, x_n \in [a,b]$ and weights $w_1, w_2, \ldots, w_n \in \mathcal{R}$, and given by the formula

$$(Rf)(x) = \sum_{j=1}^{n} w_j\, K(x, x_j)\, f(x_j),$$

for all $f \in \mathcal{L}^2[a,b]$ and $x \in [a,b]$. Substitution of R for $\mathcal{K}$ in Eq. (2), combined with the requirement that the resulting equation hold for $x = x_1, x_2, \ldots, x_n$, yields the following system of n equations in the n unknowns $f_1, f_2, \ldots, f_n$:

$$f_i - \sum_{j=1}^{n} w_j\, K(x_i, x_j)\, f_j = g(x_i), \qquad i = 1, \ldots, n. \tag{3}$$

This system is typically solved by standard methods of numerical linear algebra, such as Gaussian elimination.

The approximation $\langle f_1, \ldots, f_n \rangle$ to the solution f of Eq. (1) may be extended to all $x \in [a,b]$ by the natural formula

$$f_R(x) = g(x) + \sum_{i=1}^{n} w_j\, K(x, x_j)\, f_j, \tag{4}$$

which satisfies $f_R(x_i) = f_i$ for $i = 1, \ldots, n$. How large is the error $e_R = f - f_R$ of the approximate solution? We follow the derivation by Delves and Mohamed in [2]. Rewriting Eqs. (3) in operator form, we have

$$(I - R)f_R = g, \tag{5}$$

and combining Eqs. (2) and (5) yields

$$(I - \mathcal{K})e_R = (\mathcal{K} - R)f_R.$$

Provided that $(I - \mathcal{K})^{-1}$ exists, we obtain the error bound

$$\|e_R\| \leq \|(I - \mathcal{K})^{-1}\| \cdot \|(\mathcal{K} - R)f_R\|. \tag{6}$$

The error depends, therefore, on the conditioning of the original integral equation, as is apparent from the term $\|(I - \mathcal{K})^{-1}\|$, and on the fidelity of the quadrature R to the integral operator $\mathcal{K}$. It is not necessary that $\|\mathcal{K} - R\|$ be small, rather merely that R approximate $\mathcal{K}$ well near the solution f. In this paper we develop quadrature rules that have this property; nevertheless, they have a somewhat different form than R does and are defined only on the mesh points $x_1, \ldots, x_n$ rather than the whole interval $[a, b]$.

2 Corrected Trapezoidal Rules

It is well known that the trapezoidal rule for integration can be modified at the ends via the Euler-Maclaurin summation formula to a rapidly convergent rule, provided the integrand is sufficiently differentiable. We will suppose, instead, that the integrand is singular at one end of the interval and the form of the singularity is known. In this case a modification *at that end* may be determined so that the corrected trapezoidal rule is rapidly convergent.

2.1 Differentiable Integrands

We begin with the assumption that the integrand is differentiable throughout the interval of integration. For positive integers l, n, and $m = 2l + 2$, and a function $f \in C^m[a, b]$, the Euler-Maclaurin summation formula is given by the equation (see, *e.g.*, [5])

$$\int_a^b f(x)\,dx = T_n(f) + D_n^m(f, a) - D_n^m(f, b) + E_n^m(f), \tag{7}$$

where

$$
\begin{aligned}
T_n(f) &= h\left(\frac{1}{2}f(a) + f(a+h) + \cdots + f(b-h) + \frac{1}{2}f(b)\right) \\
D_n^m(f,x) &= \sum_{i=1}^{m/2-1} h^{2i}\frac{B_{2i}}{(2i)!}f^{(2i-1)}(x) \\
E_n^m(f) &= h^m \sum_{i=0}^{n-1} \int_{a+hi}^{a+h(i+1)} \frac{B_m((x-a)/h - i) - B_m}{m!}\, f^{(m)}(x)\, dx \\
&= -h^m \frac{B_m}{m!}(b-a)\, f^{(m)}(\xi)
\end{aligned}
$$

and $h = (b-a)/n$ and $a < \xi < b$. Here B_i are the Bernoulli numbers,

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \ldots.$$

The derivatives which appear in $D_n^m(f,a)$ and $D_n^m(f,b)$ may be approximated by finite differences, to obtain high-order quadrature rules which depend only on the values of f at equispaced points. Using the Taylor expansion for f about the point a we have, for $i = 1, 2, \ldots, m-2$,

$$f(a + i\,h) = f(a) + \sum_{j=1}^{m-2} f^{(j)}(a)\,\frac{(i\,h)^j}{j!} + f^{(m-1)}(\nu_i)\,\frac{(i\,h)^{m-1}}{(m-1)!} \tag{8}$$

with $a < \nu_i < a + ih$. Eqs. (8) can be considered to be a system of equations in the unknowns $hf^{(1)}(a), h^2 f^{(2)}(a), \ldots, h^{m-2}f^{(m-2)}(a)$, and the matrix A^m of this system, given by the formula

$$A^m = \begin{pmatrix} 1 & \frac{1}{2!} & \cdots & \frac{1}{(m-2)!} \\ 2 & \frac{2^2}{2!} & \cdots & \frac{2^{m-2}}{(m-2)!} \\ \vdots & & & \vdots \\ m-2 & \frac{(m-2)^2}{2!} & \cdots & \frac{(m-2)^{m-2}}{(m-2)!} \end{pmatrix},$$

is non-singular, since the functions $x^j/j!$ for $j = 1, 2, \ldots$ form a Chebyshev system. We define the vector $v_h^m = \langle v_{h,1}^m, \ldots, v_{h,m-2}^m \rangle^T$ of finite differences by the expression

$$v_h^m = (A^m)^{-1} \begin{pmatrix} f(a+h) - f(a) \\ f(a+2h) - f(a) \\ \vdots \\ f(a+(m-2)h) - f(a) \end{pmatrix}$$

and from Eqs. (8) we obtain, for $j = 1, \ldots, m-2$, the error bound

$$\left| v_{h,j}^m - h^j f^{(j)}(a) \right| < e_m\, h^{m-1} \sup_{\xi \in [a, a+(m-2)h]} \left| f^{(m-1)}(\xi) \right|, \tag{9}$$

where e_m is independent of $[a,b]$, h, and the function f.

The expressions $v^m_{h,j}$ for the derivatives are used to define the left-end correction ${}_1D^m_n(f,a)$ by the formula

$$ {}_1D^m_n(f,a) = \sum_{i=1}^{m/2-1} h \frac{B_{2i}}{(2i)!} v^m_{h,2i-1}. \tag{10} $$

Similarly, in Eqs. (8) we replace a by b and i by $-i$ to obtain finite-difference expressions $w^m_h = \langle w^m_{h,1}, \ldots, w^m_{h,m-2}\rangle^T$ for the derivatives at the right endpoint $\langle hf'(b), h^2 f^{(2)}(b), \ldots, h^{m-2} f^{(m-2)}(b)\rangle^T$, and we define the right-end correction ${}_1D^m_n(f,b)$ by the formula

$$ {}_1D^m_n(f,b) = \sum_{i=1}^{m/2-1} h \frac{B_{2i}}{(2i)!} w^m_{h,2i-1}. \tag{11} $$

We thus obtain the corrected trapezoidal rule

$$ {}_1T^m_n(f) = T_n(f) + {}_1D^m_n(f,a) - {}_1D^m_n(f,b) \tag{12} $$

that depends only on the values of f at the equispaced points a, $a+h, \ldots,$ $b-h$, b. Combination of the Euler-Maclaurin formula (Eq. 7) with the finite differences error bound (9) yields the bound

$$ \left| {}_1T^m_n(f) - \int_a^b f(x)\,dx \right| < {}_1e_m\,(b-a)\,h^m \sup_{\xi\in[a,b]} \left| f^{(m)}(\xi) \right|, $$

where ${}_1e_m$ is independent of $[a,b]$, h, and the function f.

Quadrature rules equivalent to ${}_1T^m_n$ are derived by Weinberger [6] using a different technique.

Alternatively, the corrections to the trapezoidal rule may be concentrated in the subintervals $[a, a+h)$ and $(b-h, b]$ by replacing h by $h' = h/(m-1)$ in Eqs. (8) to define revised endpoint corrections ${}_2D^m_n(f,a)$ and ${}_2D^m_n(f,b)$ by the formulae

$$ {}_2D^m_n(f,a) = \sum_{i=1}^{m/2-1} h \frac{B_{2i}}{(2i)!} v^m_{h',2i-1} \tag{13} $$

$$ {}_2D^m_n(f,b) = \sum_{i=1}^{m/2-1} h \frac{B_{2i}}{(2i)!} w^m_{h',2i-1}. \tag{14} $$

These endpoint corrections give us the "crowded" corrected trapezoidal rule

$$ {}_2T^m_n(f) = T_n(f) + {}_2D^m_n(f,a) - {}_2D^m_n(f,b). \tag{15} $$

For the quadrature ${}_2T_n^m$ we obtain the error bound (as for ${}_1T_n^m$)

$$\left| {}_2T_n^m(f) - \int_a^b f(x)\,dx \right| < {}_2e_m\,(b-a)\,h^m \sup_{\xi\in[a,b]} \left| f^{(m)}(\xi) \right|,$$

where ${}_2e_m$ is independent of $[a,b]$, h, and the function f. This rule has the advantage over the equispaced rule that the constant of the error term is smaller; it has the disadvantage that the coefficients are larger and hence produce larger round-off errors. Independently of these two characteristics, the crowded rule is a suitable starting point for construction of rules for integral equations with singular kernels, as we shall see below.

Coefficients of finite-difference expressions for the derivatives, equispaced corrected trapezoidal rules, and crowded corrected trapezoidal rules are given in an appendix.

2.2 Singular Integrands

We now consider integrands of the form

$$f(x) = \phi(x)\,s(x) + \psi(x) \tag{16}$$

for all $x \in (0,b]$, where $\phi, \psi \in C^m[0,b]$ and the function s contains the singular part of f. In particular, we construct quadratures for f, where s is given by the formula $s(x) = \log(x)$ or the formula $s(x) = x^\alpha$ with $0 < |\alpha| < 1$.

We define T_n' to be the trapezoidal rule on $[0,b]$ minus the left-end, by the formula

$$T_n'(f,b) = h\left(f(h) + \cdots + f(b-h) + \frac{1}{2}f(b)\right),$$

where $h = b/n$, and define the right-end corrected rule T_n'' by the formula

$$T_n''(f,b) = T_n'(f,b) - {}_2D_n^m(f,b), \tag{17}$$

where the term ${}_2D_n^m(f,b)$ is the finite-difference approximation to the Euler-Maclaurin correction, as defined in Eq. (14). Given a positive integer l and two finite sequences $\beta = \langle \beta_1, \beta_2, \ldots, \beta_l \rangle$ and $\chi = \langle \chi_1, \chi_2, \ldots, \chi_l \rangle$, with $0 < \chi_1 < \chi_2 < \cdots < \chi_l \le 1$, we define the left-end correction L_h^β by the formula

$$L_h^\beta(f) = \sum_{i=1}^{l} \beta_i\, f(\chi_i\, h).$$

We now define the linear mapping $T_n^\beta : \mathcal{L}^1[0,b] \to \mathcal{R}$ by the formula

$$T_n^\beta(f,b) = T_n''(f,b) + L_h^\beta(f). \tag{18}$$

Although L_h^β and T_n^β depend on χ, this sequence is generally implied (see Remark 2.1 below) and we omit the symbol χ from our notation. The mapping T_n^β with suitable choice of β and χ will be used as a quadrature formula for functions f of the form given in Eq. (16). We now show how β may be chosen, given χ.

For positive integers k and n we consider the following system of linear equations with respect to the unknowns $\beta^n = \langle \beta_1^n, \ldots, \beta_{2k}^n \rangle$:

$$\begin{aligned} \sum_{j=1}^{2k} \beta_j^n \, (\chi_j \, h)^i &= \int_0^b x^i \, dx - T_n''(x^i, b) \\ \sum_{j=1}^{2k} \beta_j^n \, (\chi_j \, h)^i \, s(\chi_j \, h) &= \int_0^b x^i \, s(x) \, dx - T_n''(x^i \, s(x), b), \end{aligned} \tag{19}$$

for $i = 0, 1, \ldots, k-1$. We will see below that this system has a unique solution β^n and for functions f of the form given in Eq. (16) the rule $T_n^{\beta^n}$ is a quadrature with convergence of order at least $k-1$.

Remark 2.1 In Rokhlin [4], the points $\chi_1, \ldots, \chi_{2k}$ are chosen to be equispaced, namely $\chi_i = i/(2k)$. While this choice leads to satisfactory quadratures for $k \leq 3$, larger values of k correspond to coefficients $\beta_1^n, \ldots, \beta_{2k}^n$ which are large enough to introduce substantial round-off errors to fixed-precision computations. We have found that alternative spacing of the points can be chosen so as to delay the growth in the coefficients. Letting $\chi_1, \ldots, \chi_{2k}$ be *half-Chebyshev* points

$$\chi_i = 1 - \cos\left(\frac{2i-1}{4k} \cdot \frac{\pi}{2}\right) \tag{20}$$

results in $\beta_1^n, \ldots, \beta_{2k}^n$ which are satisfactory for $k \leq 5$. Coefficients for both equispaced points and half-Chebyshev points are given in the appendix.

Our primary purpose in this paper is the development of quadratures for integral operators with diagonally-singular kernels. We consider kernels of the form

$$K(x,t) = \phi(x,t) \, s(|x-t|) + \psi(x,t),$$

where $\phi, \psi \in C^m[a,b]^2$ and s is as introduced above. To compute the value

$$(\mathcal{K}f)(c) = \int_a^b K(c,t) \, f(t) \, dt, \tag{21}$$

the interval $[a,b]$ is divided into $[a,c]$ and $[c,b]$. The integrand is singular on one end of each of these intervals, so the quadrature $T_n^{\beta^n}$ is applicable. We augment

our notation by defining the quadrature ${}_LT_n^{\beta^n}$ for an interval $[c, b]$ with a left-end singularity by the formula

$$ {}_LT_n^{\beta^n}(f(x), c, b) = T_n^{\beta^n}(f(x+c), b-c). $$

Similarly, we define the quadrature ${}_RT_n^{\beta^n}$ for an interval $[a, c]$ with a right-end singularity by the formula

$$ {}_RT_n^{\beta^n}(f(x), a, c) = T_n^{\beta^n}(f(c-x), c-a). $$

Now we define a quadrature for the integral in Eq. (21) by defining the mapping $T_{ni}^k : \mathcal{L}^1[a,b]^2 \to \mathcal{R}$ by the formula

$$ T_{ni}^k(G) = {}_RT_i^{\beta^i}(G_i, a, x_i) + {}_LT_{n-i}^{\beta^{n-i}}(G_i, x_i, b), \tag{22} $$

and further define $T_{n0}^k(G) = {}_LT_n^{\beta^n}(G_0, a, b)$ and $T_{nn}^k(G) = {}_RT_n^{\beta^n}(G_n, a, b)$. Here $G : [a,b]^2 \to \mathcal{R}$ and $G_i : [a,b] \to \mathcal{R}$ is the restriction of G defined by the formula $G_i(t) = G(x_i, t)$ where $x_i = a + i\,h$ for $i = 0, 1, \ldots, n$, and $h = (b-a)/n$. The quadrature T_{ni}^k has convergence of order $k-1$, uniformly in i, as we will show in the next section.

3 Analytical Properties of the Corrected Rules

The following lemma is a restatement of a classical result (see, *e.g.*, [3]).

Lemma 3.1 *Suppose that the function* $s : (0, b] \to \mathcal{R}$ *is given by the formula* $s(x) = \log(x)$ *or the formula* $s(x) = x^\alpha$ *with* $0 < |\alpha| < 1$. *Then the system of Eqs. (19) has a unique solution* $\beta^n = \langle \beta_1^n, \ldots, \beta_{2k}^n \rangle$.

The next two lemmas are proven in [4]. Lemma 3.2 states that the coefficients $\beta_1^n, \ldots, \beta_{2k}^n$ have limiting values as $n \to \infty$ and that the rate of convergence to these values depends on the difference between the orders of correction k on the left end and m on the right end. Lemmas 3.2, 3.3, and 3.4 will be used in the proof of Theorem 3.5, below.

Lemma 3.2 (Rokhlin) *Suppose that* k *and* m *are positive integers with* $k < m$, *the function* s *is as specified in Lemma 3.1, and for each positive integer* n, *the coefficients* $\beta^n = \langle \beta_1^n, \ldots, \beta_{2k}^n \rangle$ *are the solution to the linear system of Eqs. (19). Then there exist coefficients* $\beta = \langle \beta_1, \ldots, \beta_{2k} \rangle$ *and a constant* $c > 0$ *such that*

$$ |\beta_i^n - \beta_i| < \frac{c}{n^{m-k}} $$

for $i = 1, \ldots, 2k$ *and all* n.

Lemma 3.3 (Rokhlin) *Suppose that the function s is one of the singular functions specified in Lemma 3.1, k is a positive integer, and $\varphi \in C^k[0,b]$ satisfies*

$$0 = \varphi(0) = \varphi'(0) = \cdots = \varphi^{(k)}(0).$$

Then the function $w = \varphi \cdot s$ is defined on the closed interval $[0,b]$ and

$$0 = w(0) = w'(0) = \cdots = w^{(k-1)}(0).$$

For the function $w = \varphi \cdot s$, whose kth derivative is unbounded near 0, the simplified Euler-Maclaurin error expression

$$E_n^k(w) = -h^k \frac{B_k}{k!} (b-a)\, w^{(k)}(\xi),$$

with $0 < \xi < b$, is not useful. In this case we substitute the following bound.

Lemma 3.4 *Suppose that k and n are positive integers and the function $w : [0,b] \to \mathcal{R}$ is as specified in Lemma 3.3, above. Then the error $E_n^k(w)$ of the Euler-Maclaurin formula (Eq. 7) has the bound*

$$E_n^k(w) < h^k \sup_{x\in[0,1]} \left| \frac{B_k(x) - B_k}{k!} \right| \cdot \int_0^b \left| w^{(k)}(x) \right| dx. \tag{23}$$

Thus, there is a constant $c_k > 0$, independent of n, b, and the function w, such that

$$\left| T_n''(w,b) - \int_0^b w(x)\, dx \right| < h^k\, c_k \int_0^b \left| w^{(k)}(x) \right| dx, \tag{24}$$

where $h = b/n$.

Proof. The bound (23) is immediate from the definition of E_n^k given in Eq. (7). The error bound (24) then follows from the combination of the Euler-Maclaurin formula, the observation that $D_n^k(w,0) = 0$, the definition of T_k'' (Eq. 17), and bound (9). □

The following theorem is the foundation for the corrected trapezoidal rules for singular functions. It is a slight generalization of a theorem found in Rokhlin [4], in that the error bound is established for integrals taken on subintervals $[0, \frac{i}{n}b]$, for $i = 1, \ldots, n$, of the interval $[0,b]$, provided that the trapezoidal points' spacing $h = b/n$ is preserved.

Theorem 3.5 *Suppose that k and m are positive integers with $2 \le k < m$ and the function $f : (0,b] \to \mathcal{R}$ is given by the formula $f = \phi \cdot s + \psi$, where $\phi, \psi \in C^m[0,b]$ and the function s is as specified in Lemma 3.1. Further suppose that for each positive integer n, β^n is the solution of Eqs. (19) and $T_n^{\beta^n}$ is defined*

by Eq. (18). Then there exists $c > 0$ independent of b and the function f such that

$$\left|T_i^{\beta^i}(f, i/n \cdot b) - \int_0^{\frac{i}{n}b} f(x)\,dx\right| < c\,b\,h^{k-1} \sup_{\xi\in[0,b]} \left(\left|\phi^{(k)}(\xi)\right| + \left|\psi^{(k)}(\xi)\right|\right) \tag{25}$$

for all n and $i = 1, \ldots, n$.

Proof. We write the function f as a sum of two functions; one is integrated exactly by the quadrature and the other has several zero derivatives at 0.

Let $P(\varphi)$ denote the k-term Taylor expansion of a function $\varphi \in C^m[0, b]$ about $x = 0$:

$$P(\varphi)(x) = \sum_{j=0}^{k-1} \frac{\varphi^{(j)}(0)}{j!}\, x^j.$$

Now we define $\phi_r = \phi - P(\phi)$ and $\psi_r = \psi - P(\psi)$ so we have

$$0 = \phi_r(0) = \psi_r(0) = \ldots = \phi_r^{(k-1)}(0) = \psi_r^{(k-1)}(0). \tag{26}$$

We further define $f_p = P(\phi) \cdot s + P(\psi)$ and $f_r = \phi_r \cdot s + \psi_r$, so that $f = f_p + f_r$, and we let $b_i = \frac{i}{n}b$. Now we bound the error by the inequality

$$\begin{aligned}\left|T_i^{\beta^i}(f, b_i) - \int_0^{b_i} f(x)\,dx\right| \;<\; & \left|T_i^{\beta^i}(f_p, b_i) - \int_0^{b_i} f_p(x)\,dx\right| \\ & + \left|T_i^{\beta^i}(f_r, b_i) - \int_0^{b_i} f_r(x)\,dx\right|,\end{aligned}$$

where by Eqs. (19) the first term on the right vanishes. By the definition of $T_i^{\beta^i}$ (Eq. 18), the second term satisfies

$$\left|T_i^{\beta^i}(f_r, b_i) - \int_0^{b_i} f_r(x)\,dx\right| < \left|T_i''(f_r, b_i) - \int_0^{b_i} f_r(x)\,dx\right| + \left|\sum_{j=1}^{2k} \beta_j^i\, f_r(\chi_j h)\right|.$$

By Eq. (26) and Lemma 3.3 the function $f_r = \phi_r \cdot s + \psi_r$ vanishes at 0 and has vanishing derivatives,

$$0 = f_r(0) = f_r'(0) = \ldots = f_r^{(k-2)}(0),$$

from which, in combination with error bound (24), we obtain

$$\left|T_i''(f_r, b_i) - \int_0^{b_i} f_r(x)\,dx\right| < h^{k-1}\, c_k \int_0^b \left|f_r^{(k-1)}(x)\right| dx \tag{27}$$

for some constant c_k independent of n, b, i, and the function f_r. For the remaining term, we define $M_\phi = k! \sup\left|\phi^{(k)}(x)\right|$ and $M_\psi = k! \sup\left|\psi^{(k)}(x)\right|$, both suprema

taken for $x \in [0, b]$, therefore finite, and $M_\beta = \sup \beta_j^i$, taken over positive integers i and $j = 1, \ldots, 2k$. M_β is finite by Lemma 3.2, and we obtain

$$\begin{aligned}
\left| \sum_{j=1}^{2k} \beta_j^i \, f_r(\chi_j \, h) \right| &\leq M_\beta \sum_{j=1}^{2k} |f_r(\chi_j \, h)| \\
&\leq M_\beta \sum_{j=1}^{2k} \left(M_\phi \, (\chi_j \, h)^k \, |s(\chi_j \, h)| + M_\psi \, (\chi_j \, h)^k \right) \\
&< M_\beta \, (2k) \, (M_\phi + M_\psi) \, h^{k-1}, \qquad (28)
\end{aligned}$$

for sufficiently large n. Combining bounds (27) and (28) yields (25). □

Note that Theorem 3.5 establishes that the quadrature $T_n^{\beta^n}(f, b)$ converges to the integral of f on the fixed interval $[0, b]$, with order of convergence at least $k - 1$. Additionally, however, it establishes the same convergence on subintervals $[0, \frac{i}{n} b]$ of $[0, b]$ with correspondingly fewer quadrature points. This characteristic is essential for proper treatment of non-periodic integral operators; in fact, it assures uniform convergence of the quadratures T_{ni}^k defined in Eq. (22).

Corollary 3.6 *Suppose that k and m are positive integers with $k < m$ and that the kernel $K : [a, b] \times [a, b] \to \mathcal{R}$ is given by the formula*

$$K(x, t) = \phi(x, t) \, s(|x - t|) + \psi(x, t)$$

where $\phi, \psi \in C^m([a, b] \times [a, b])$ and s is as specified in Lemma 3.1. We further suppose $f \in C^m[a, b]$ and define the function $G : [a, b] \times [a, b] \to \mathcal{R}$ by the formula $G(x, t) = K(x, t) \, f(t)$. There exists $c_k > 0$ such that

$$\left| T_{ni}^k(G) - \int_{x_i}^b G(x, t) \, f(t) \, dt \right| < \frac{c_k}{n^{k-1}} \qquad (29)$$

for all n and $i = 0, 1, \ldots, n$.

Proof. We observe that M_ϕ and M_ψ defined by the formulae

$$\begin{aligned}
M_\phi &= \sup_{x,t \in [a,b]} \frac{\partial^k \phi(x, t)}{\partial t^k} \\
M_\psi &= \sup_{x,t \in [a,b]} \frac{\partial^k \psi(x, t)}{\partial t^k}
\end{aligned}$$

are finite, then apply Theorem 3.5. □

The quadratures T_{ni}^k achieve uniform convergence and thus, combined with the kernel K and the coefficient p, represent an operator R which approximates

the integral operator $\mathcal{K}$ defined above. An issue that arises in computing Rf for a function f, however, is that T_{ni}^k requires the values of f at non-equispaced points in the interval $[a, b]$ in addition to the points $a, a+h, \ldots, b-h, b$. This issue can be handled by using k-point interpolation, for $f \in C^m[a, b]$. We have the estimate

$$\left| f(\xi) - \sum_{i=1}^{k} c_i(\xi)\, f(a + (i + j_\xi)\, h) \right| < \frac{c}{n^k},$$

where $c_1(\xi), \ldots, c_k(\xi)$ are Lagrange interpolation coefficients and j_ξ is chosen such that $\xi \in [a + (1 + j_\xi)\, h, a + (k + j_\xi)\, h]$. Replacing $f(\xi)$ in $T_{ni}^k(Kf)$ by the interpolation preserves our error estimates.

4 Numerical Examples

In this section we present numerical examples of the corrected quadrature rules applied to differentiable and singular integrands. FORTRAN routines were written which incorporate the corrections developed in §2, and the quadratures were computed in double-precision arithmetic on a Sun Sparcstation 1. The correction weights themselves, which are given in the appendix, were computed exactly with Maple (for differentiable integrands), and in quadruple-precision arithmetic in FORTRAN running on a DEC microVAX (for singular integrands).

Table 1 shows the errors in using quadratures of order 4, 8, and 12 to approximate the integral of a smooth function, comparing the equispaced corrected trapezoidal rule to the "crowded" corrected trapezoidal rule. Errors from the uncorrected trapezoidal rule are also shown for comparison. We make several observations:

1. Full single or double precision accuracy is easily achievable with the higher-order rules.

2. The observed rate of convergence matches that expected quite closely, as can be seen by comparing the errors for various n.

3. The "crowded" rules get a substantial jump on the equispaced rules and show small errors almost as soon as the integrand is resolved by the quadrature points.

4. The crowded rules ultimately achieve somewhat less precision than the equispaced rules, due to roundoff error resulting from the large correction weights, but this effect is important only if full double-precision accuracy is required.

Table 1: *The quadrature rules* ${}_1T_n^m$ *of Eq. (12) and* ${}_2T_n^m$ *of Eq. (15) are used to compute* $\int_0^1[\cos(21x)+\sin(22x)]\,dx$ *and relative errors are shown for various values of* n, m.

	Trapez.	Equispaced rule ${}_1T_n^m$		
n	Rule	$m=4$	$m=8$	$m=12$
10	−0.397E+00	−0.235E+00	0.128E+01	−0.499E+01
20	−0.936E−01	0.840E−02	−0.119E−01	0.385E−02
40	−0.231E−01	0.137E−02	−0.181E−04	−0.226E−05
80	−0.575E−02	0.108E−03	0.863E−07	−0.847E−10
160	−0.144E−02	0.733E−05	0.617E−09	0.676E−13
320	−0.359E−03	0.474E−06	0.286E−11	−0.189E−15
640	−0.897E−04	0.301E−07	0.966E−14	−0.227E−14
1280	−0.224E−04	0.190E−08	−0.170E−14	−0.151E−14

	"Crowded" rule ${}_2T_n^m$		
n	$m=4$	$m=8$	$m=12$
10	0.425E−02	0.103E−04	0.119E−06
20	0.133E−02	0.866E−06	0.653E−09
40	0.109E−03	0.462E−08	−0.420E−11
80	0.748E−05	0.201E−10	0.121E−11
160	0.486E−06	0.793E−13	0.810E−12
320	0.309E−07	0.000E+00	0.863E−13
640	0.195E−08	−0.227E−14	−0.172E−12
1280	0.122E−09	−0.170E−14	0.130E−12

In summary, the corrected rules should be preferred to the trapezoidal rule whenever high accuracy is desirable. The "crowded" corrected trapezoidal rule of order 12 performs very well.

Table 2 shows the errors from using the corrected trapezoidal rule for singular functions to compute the integral of an integrand with logarithm singularity. The rule was applied for various numbers of quadrature points n and various k, and for equispaced and half-Chebyshev $\chi_1, \ldots, \chi_{2k}$. In each case the correction weights $\beta = \langle \beta_1, \ldots, \beta_{2k} \rangle$ were chosen to be the limiting values (of Lemma 3.2), given in the appendix. Note that Theorem 3.5 assures us of convergence of order at least $k-1$. We observe:

1. Although we have proven that the order of convergence is at least $k-1$, the actual order of convergence appears to lie between k and $k+1$ and the convergence pattern is somewhat irregular.

2. The correction weights from equispaced points $\chi_1, \ldots, \chi_{2k}$ are much larger than those from half-Chebyshev points; nevertheless, both versions of the

Table 2: *The quadrature rule T_n^β of Eq. (18) is used to compute the integral $\int_0^1[\cos(21x)+\sin(22x)+\log(x)(\cos(23x)+\sin(24x))]\,dx$ and relative errors are shown for various values of n, k. The right end is corrected to eighth order ($m = 8$).*

	Equispaced points $\chi_1, \ldots, \chi_{2k}$			
n	$k = 2$	$k = 3$	$k = 4$	$k = 5$
10	−0.548E+00	−0.219E−01	0.429E−01	−0.608E−02
20	0.386E−02	−0.657E−02	0.506E−03	0.394E−04
40	0.403E−02	−0.242E−03	−0.169E−05	0.614E−06
80	0.663E−03	−0.341E−05	−0.236E−06	0.482E−08
160	0.964E−04	0.381E−06	−0.971E−08	0.809E−11
320	0.139E−04	0.545E−07	−0.363E−09	−0.593E−12
640	0.202E−05	0.509E−08	−0.135E−10	0.258E−14
1280	0.292E−06	0.414E−09	−0.497E−12	0.861E−15

	Half-Chebyshev points $\chi_1, \ldots, \chi_{2k}$			
n	$k = 2$	$k = 3$	$k = 4$	$k = 5$
10	−0.510E−01	−0.667E−02	−0.503E−02	−0.110E−02
20	0.143E−02	0.599E−04	−0.166E−04	0.553E−05
40	0.484E−03	0.615E−05	0.732E−06	0.877E−07
80	0.750E−04	0.349E−06	0.351E−07	0.653E−09
160	0.108E−04	0.203E−07	0.132E−08	−0.312E−12
320	0.156E−05	0.125E−08	0.492E−10	−0.148E−12
640	0.227E−06	0.796E−10	0.183E−11	−0.445E−14
1280	0.327E−07	0.519E−11	0.670E−13	−0.100E−14

Table 3: *The quadrature rule T_{ni}^k of Eq. (22) is used to compute $F(x) = \int_0^1[\cos(21xt)+\sin(22xt)+s(|x-t|)(\cos(23xt)+\sin(24xt))]\,dt$, with $x = i/n$ for $i = 0, 1, \ldots, n$. The function s is singular and has one of three forms given below. The relative $\mathcal{L}^2$ errors for the quadratures with $k = 4$ are shown.*

	Three Choices of Integrand $f(x,t)$		
n	$s(x) = \log(x)$	$s(x) = x^{-1/2}$	$s(x) = x^{1/2}$
10	0.302E−03	0.134E−03	0.482E−04
20	0.527E−05	0.187E−05	0.579E−06
40	0.681E−07	0.487E−07	0.206E−07
80	0.573E−08	0.266E−08	0.611E−09
160	0.283E−09	0.132E−09	0.181E−10
320	0.118E−10	0.613E−11	0.547E−12
640	0.454E−12	0.300E−12	0.169E−13
1280	0.216E−13	0.155E−12	0.271E−14

quadrature rule perform well and lead to nearly full precision accuracy in practice.

The uncorrected trapezoidal rule, with the left end omitted, gives very slow convergence for this problem and is not practical for achieving high accuracy.

Our final examples of the corrected trapezoidal rules, in Table 3, demonstrate the uniform convergence of the quadratures for various integral operators. In these examples, we approximate the integral $F_i = \int_0^1 f(x_i, t)\, dt$ for $x_i = i/n$ and $i = 0, 1, \ldots, n$, by the quadrature $T_{ni}^k(f)$, defined in Eq. (22). The table shows the relative $\mathcal{L}^2$ error of the approximations, defined by the formula

$$e_{\mathcal{L}^2} = \left(\frac{\sum_{i=0}^n [T_{ni}^k(f) - F_i]^2}{\sum_{i=0}^n F_i^2} \right)^{1/2} . \tag{30}$$

By taking ratios of the errors for different n, it can be seen that (for $k = 4$) the rate of convergence is uniformly of order at least 4. For the logarithm singularity, the order of convergence is about 4.5, and for the square root singularity, it is nearly 5. In each case, $n = 20$ produces roughly single-precision accuracy.

A Quadrature Weights

This appendix contains various weights and correction coefficients used in the quadratures of the paper. The coefficients in the finite difference expressions for odd-numbered derivatives, which appear in the Euler-Maclaurin formula, as well as the corresponding correction coefficients are tabulated (Tables 4 and 5 and 6). The limiting values of the correction weights used for the singularities $s(x) = \log(x)$ and $s(x) = x^\alpha$ for $\alpha = \pm\frac{1}{2}$ are shown for equispaced correction points (Table 7) and for half-Chebyshev correction points. The correction weights for all finite n (which are infinite in number) are obtained by interpolation from the weights for a few values of n. The coefficients of the interpolating polynomials are quite numerous, however, and are not shown. They are available from the author.

Table 4: *Finite-difference coefficients for odd-numbered derivatives to various orders of approximation are shown. The table entries are coefficients from the formula* $h^j f^{(j)}(x) = (1/d) \sum_{i=0}^{m-2} c_i\, f(x + i\,h) + \mathrm{O}(h^m)$, *for* $j = 1, 3, 5, 7, 9$ *and* $m = 4, 6, 8, 10, 12$.

	$hf^{(1)}(x)$					$h^5 f^{(5)}(x)$		
	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$
d	2	12	60	840	2520	2	6	288
c_0	−3	−25	−147	−2283	−7381	−7	−81	−8591
c_1	4	48	360	6720	25200	40	575	72492
c_2	−1	−36	−450	−11760	−56700	−95	−1790	−278313
c_3		16	400	15680	100800	120	3195	640752
c_4		−3	−225	−14700	−132300	−85	−3580	−979878
c_5			72	9408	127008	32	2581	1039656
c_6			−10	−3920	−88200	−5	−1170	−774402
c_7				960	43200		305	399408
c_8				−105	−14175		−35	−136347
c_9					2800			27788
c_{10}					−252			−2565

	$h^3 f^{(3)}(x)$				$h^7 f^{(7)}(x)$		$h^9 f^{(9)}(x)$
	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$	$O(h^{10})$	$O(h^{12})$	$O(h^{12})$
d	2	8	240	30240	2	24	2
c_0	−5	−49	−2403	−420475	−9	−605	−11
c_1	18	232	13960	2876868	70	5628	108
c_2	−24	−461	−36706	−9389763	−238	−23583	−477
c_3	14	496	57384	19227792	462	58632	1248
c_4	−3	−307	−58280	−27098442	−560	−95802	−2142
c_5		104	39128	27147960	434	107520	2520
c_6		−15	−16830	−19395138	−210	−83958	−2058
c_7			4216	9693648	58	45048	1152
c_8			−469	−3229227	−7	−15897	−423
c_9				645412		3332	92
c_{10}				−58635		−315	−9

Table 5: *Equispaced endpoint corrections transform the familiar trapezoidal rule into a high-order quadrature for functions with several continuous derivatives. The quadrature rules are given by the formula* $\int_a^b f(x)\,dx = T_n(f) + (h/d)\sum_{i=0}^{m-2} c_i\,[f(a+i\,h)+f(b-i\,h)] + \mathrm{O}(h^m)$, *where* $h=(b-a)/n$ *and* $T_n(f) = h\,[\frac{1}{2}f(a)+f(a+h)+\cdots+f(b-h)+\frac{1}{2}f(b)]$.

	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$
d	1	24	1440	120960	7257600	958003200
c_0	0	−3	−245	−23681	−1546047	−216254335
c_1		4	462	55688	4274870	679543284
c_2		−1	−336	−66109	−6996434	−1412947389
c_3			146	57024	9005886	2415881496
c_4			−27	−31523	−8277760	−3103579086
c_5				9976	5232322	2939942400
c_6				−1375	−2161710	−2023224114
c_7					526154	984515304
c_8					−57281	−321455811
c_9						63253516
c_{10}						−5675265

Table 6: *Corrections crowded into the intervals* $[a, a+h)$ *and* $(b-h, b]$ *convert the trapezoidal rule into a high-order quadrature for functions with several continuous derivatives. The quadrature rules have the form* $\int_a^b f(x)\,dx = T_n(f) + (h/d)\sum_{i=0}^{m-2} c_i\,[f(a+\frac{i}{m-1}h)+f(b-\frac{i}{m-1}h)] + \mathrm{O}(h^m)$, *where* $h=(b-a)/n$ *and* $T_n(f) = h\,[\frac{1}{2}f(a)+f(a+h)+\cdots+f(b-h)+\frac{1}{2}f(b)]$. *These rules have smaller error constants, but larger coefficients, than the equispaced corrected rules.*

	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$
d	1	8	288	17280	89600	87091200
c_0	0	−3	−125	−7889	−41943	−41374135
c_1		4	30	13832	−372570	4717178004
c_2		−1	240	−57421	2898654	−43825028709
c_3			−190	133056	−9112466	180245487576
c_4			45	−130067	15663360	−428859839166
c_5				58744	−15657582	649926182400
c_6				−10255	9131810	−649910688834
c_7					−2897574	428850243624
c_8					388311	−180220260891
c_9						43821791596
c_{10}						−4703691465

Table 7: *The limiting values of the correction weights, at equispaced points, for the corrected trapezoidal rules for singular functions are tabulated. The corrected quadrature rules are given by the formula* $\int_a^b f(x)\,dx \doteq T_n'(f) + h\sum_{i=1}^{2k}\beta_i f(a+\frac{i}{2k}h) + (h/d)\sum_{i=0}^{m-2} c_i f(b-\frac{i}{m-1}h)$, *where* $h = (b-a)/n$, $T_n'(f) = h\,[f(a+h)+f(a+2h)+\cdots+f(b-h)+\frac{1}{2}f(b)]$, *and the coefficients* d *and* $c_0,\ldots,c_{m-2}$ *are given by Table 6.*

Singularity $s(x) = \log(x-a)$

	$k=2$	$k=3$
β_1	0.1601298415357170E+01	0.2228766018460087E+01
β_2	−0.3382558521919485E+01	−0.1231212070062612E+02
β_3	0.3627888464434128E+01	0.3157965997308673E+02
β_4	−0.1346628357871812E+01	−0.3840391590010434E+02
β_5		0.2267350459115253E+02
β_6		−0.5265893981968890E+01

	$k=4$	$k=5$
β_1	0.3093483401777122E+01	0.4335882754006495E+01
β_2	−0.3101788376740780E+02	−0.6931374375731333E+02
β_3	0.1362059155903270E+03	0.4590379369965628E+03
β_4	−0.3147474808724213E+03	−0.1630223863113166E+04
β_5	0.4215054127612634E+03	0.3509093316535128E+04
β_6	−0.3287854038787327E+03	−0.4819809483739626E+04
β_7	0.1388011671370668E+03	0.4271587513841620E+04
β_8	−0.2455521037187227E+02	−0.2374001155274844E+04
β_9		0.7547424129498612E+03
β_{10}		−0.1049488171922296E+03

Singularity $s(x) = (x-a)^{-1/2}$

	$k=2$	$k=3$
β_1	0.3338954623777353E+01	0.5156862384200115E+01
β_2	−0.1036918555513964E+02	−0.3718025418572306E+02
β_3	0.1238817390561390E+02	0.1050278047880726E+03
β_4	−0.4857942974251604E+01	−0.1388205130918806E+03
β_5		0.8799431664143500E+02
β_6		−0.2167821653610409E+02

Table 7: (continued)

Singularity $s(x) = (x-a)^{-1/2}$ (continued)

	$k=4$	$k=5$
β_1	0.7889576157976986E+01	0.1191640221424121E+02
β_2	−0.1014839102693306E+03	−0.2431211398143441E+03
β_3	0.4982052353339497E+03	0.1823927314310889E+04
β_4	−0.1241778604543411E+04	−0.7083666085060807E+04
β_5	0.1751093993580452E+04	0.1635686640624526E+05
β_6	−0.1419085152097947E+04	−0.2381679753707451E+05
β_7	0.6179863268019096E+03	0.2217597560756524E+05
β_8	−0.1123274649636003E+03	−0.1285030321889555E+05
β_9		0.4231931207608981E+04
β_{10}		−0.6062289570994023E+03

Singularity $s(x) = (x-a)^{1/2}$

	$k=2$	$k=3$
β_1	0.1076226369733505E+01	0.1403733895743621E+01
β_2	−0.1472473730142097E+01	−0.6106269754659710E+01
β_3	0.1382935017750346E+01	0.1458214411894674E+02
β_4	−0.4866876573417537E+00	−0.1639615278631818E+02
β_5		0.8952282755716858E+01
β_6		−0.1935738229429337E+01

	$k=4$	$k=5$
β_1	0.1761384695584808E+01	0.2301960768321394E+01
β_2	−0.1382118344852977E+02	−0.2856244752621036E+02
β_3	0.5459150117813370E+02	0.1642204330046712E+03
β_4	−0.1173574845498706E+03	−0.5199121092937581E+03
β_5	0.1507790199321616E+03	0.1011593120489479E+04
β_6	−0.1147784911579322E+03	−0.1264688737067177E+04
β_7	0.4762309598361213E+02	0.1027260908792144E+04
β_8	−0.8297842633159577E+01	−0.5280676861522366E+03
β_9		0.1570202164406574E+03
β_{10}		−0.2066565945589198E+02

Table 8: *The limiting values of the correction weights, at half-Chebyshev points, for the corrected trapezoidal rules for singular functions are tabulated. The quadratures are given by* $\int_a^b f(x)\,dx \doteq T_n'(f) + h\sum_{i=1}^{2k}\beta_i\, f(a+\chi_i\, h) + (h/d)\sum_{i=0}^{m-2} c_i\, f(b - \frac{i}{m-1}\, h)$, *where* $h = (b-a)/n$, $\chi_i = 1 - \cos((2i-1)\pi/(8k))$, $T_n'(f) = h\,[f(a+h) + f(a+2h) + \cdots + f(b-h) + \frac{1}{2}f(b)]$, *and the coefficients* d *and* $c_0, \ldots, c_{m-2}$ *are given by Table 7.*

Singularity $s(x) = \log(x-a)$

	$k=2$	$k=3$
β_1	0.5603992216960789E−01	0.2395413023703419E−01
β_2	0.2870630583396921E+00	0.1743092878178258E+00
β_3	0.2563533589170907E+00	−0.1631724711422584E+00
β_4	−0.9945633942639063E−01	0.9387069242218954E+00
β_5		−0.5880276430135620E+00
β_6		0.1142297718790650E+00

	$k=4$	$k=5$
β_1	0.3338537800974016E−01	0.1916264301337224E−02
β_2	−0.1737066212781738E+00	0.1567268574189051E+00
β_3	0.1198836762516400E+01	−0.4334786391079948E+00
β_4	−0.2591374963545573E+01	0.7215818540402889E+00
β_5	0.3883697343902998E+01	0.1482966328206821E+01
β_6	−0.2430220427598129E+01	−0.6595938685246877E+01
β_7	0.6270755476376530E+00	0.1169258889232862E+02
β_8	−0.4769301964491491E−01	−0.9958531304701892E+01
β_9		0.4103424553775595E+01
β_{10}		−0.6712561210148048E+00

Singularity $s(x) = (x-a)^{-1/2}$

	$k=2$	$k=3$
β_1	0.8353164416920675E−01	0.3888639530245085E−01
β_2	0.1911747004918826E+00	0.8640672385266692E−01
β_3	0.3657110693973134E+00	0.3880300543933000E−01
β_4	−0.1404174140584028E+00	0.7069014304227469E+00
β_5		−0.4537263116382184E+00
β_6		0.8272875662102377E−01

Table 8: (continued)

Singularity $s(x) = (x-a)^{-1/2}$ (continued)

	$k=4$	$k=5$
β_1	0.2823063071751857E−01	0.1604517803043481E−01
β_2	−0.1016030472473191E+00	−0.2672763418512212E−01
β_3	0.8957498450168363E+00	0.4952499866825667E+00
β_4	−0.1944630011805141E+01	−0.1916835699721412E+01
β_5	0.3079540564910779E+01	0.6247645777407226E+01
β_6	−0.1835281366554477E+01	−0.1233093115169777E+02
β_7	0.3822511312525950E+00	0.1631924962150504E+02
β_8	−0.4257746290791726E−02	−0.1237865585994330E+02
β_9		0.4849695950284900E+01
β_{10}		−0.7747361683625560E+00

Singularity $s(x) = (x-a)^{1/2}$

	$k=2$	$k=3$
β_1	0.3685550676579945E−01	0.1729631990952830E−01
β_2	0.3411680628277778E+00	0.2024441114244492E+00
β_3	0.2026521530973060E+00	−0.2155157764926719E+00
β_4	−0.8067572269088319E−01	0.9910326020047757E+00
β_5		−0.6154646588009935E+00
β_6		0.1202074019549121E+00

	$k=4$	$k=5$
β_1	0.7512196681423955E−01	−0.6317627640571476E−01
β_2	−0.4538891994788910E+00	0.7370068868212822E+00
β_3	0.2050275204711021E+01	−0.2828700163634487E+01
β_4	−0.4084723895152133E+01	0.6657003574957570E+01
β_5	0.5503263764388441E+01	−0.8199869710972154E+01
β_6	−0.3512451256619964E+01	0.4165258132856930E+01
β_7	0.1038747521068656E+01	0.3554455470298683E+01
β_8	−0.1163441057313696E+00	−0.5923097194990581E+01
β_9		0.2913478385611404E+01
β_{10}		−0.5123591045429318E+00

References

[1] B. Alpert. *Sparse Representation of Smooth Linear Operators.* PhD thesis, Yale University, December, 1990.

[2] L. M. Delves and J. L. Mohamed. *Computational Methods for Integral Equations.* Cambridge University Press, 1985.

[3] S. Karlin and W. Studden. *Tchebycheff Systems with Applications in Analysis and Statistics.* Wiley and Sons, 1966.

[4] V. Rokhlin. End-point corrected trapezoidal quadrature rules for singular functions. *Computers and Mathematics with Applications*, 20:51–62, 1990.

[5] J. Stoer and R. Bulirsch. *Introduction to Numerical Analysis.* Springer Verlag, 1980.

[6] H. F. Weinberger. Some remarks on good, simple, and optimal quadrature formulas. In Carl de Boor and Gene H. Golub, editors, *Recent Advances in Numerical Analysis*, pages 207–229. Academic Press, 1978.

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