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VOLUME ELEMENT FOR LOOP DIAGRAM IN THE STRING PICTURE OF DUAL MODEL*

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ABSTRACT

Using perturbation theory and the vertex function for the joining and separation of strings, we calculate momentum independent part of the orientable loop diagram in the string picture of the orbital dual model. The calculation is done for the case where space-time has the critical dimensionality.

I. INTRODUCTION

The quantum theory of a relativistic string [1] provides a space-time picture underlying dual models. With the development of the path integral formulation [2,3], one can apply quantum-mechanical methods to systems of interacting strings to calculate scattering amplitude in higher orders of perturbation theory.

The functional integral formulation gives us the momentum dependent part of the amplitude in a straightforward manner [2,3], but the momentum independent part (the so-called volume element) is difficult to derive from it. As with ordinary quantum-mechanical systems, alternative forms of perturbation theory can be used to calculate the volume element. Using perturbation theory with the

vertex function for three strings, calculated by Mandelstam [2], we find the volume element for the loop diagram.

Loop calculations in the context of the operator formalism of dual models have been performed by several authors [4]. Hsue, Sakita, and Virasoro [5] have represented the one-loop amplitude as a functional integral over an annulus. Recently Brink and Fairlie [6] have studied the loop amplitude purely within the functional formalism. Their calculation is very illuminating about the origin of the partition function in the volume element, but it is not completely rigorous and they make assumptions concerning one factor in the volume element.

We have used the fact that the volume element is independent of the external momenta and have chosen a special configuration in which all incoming and outgoing strings are infinitesimally short and have zero $(d-2)$-momentum. In that case it is straightforward to calculate the loop amplitude directly. Comparing the amplitude with momentum dependent part, which has been calculated for any general graph by Mandelstam [2], we read off the volume element.

In Sec. II the problem and the method is discussed. In Sec. III the perturbation theory calculation is performed and the volume element for the loop is found. In the appendix the neglect of certain integration regions in the loop diagram is justified.

II. LOOP IN THE STRING PICTURE

An orientable loop diagram in the string picture corresponds to the splitting and recombination of one of the intermediate strings. This is represented as a slit in the (σ,τ) plane of the two

dimensional surface representing the motion of the string in space time (fig. 1). The momentum-dependent part for a configuration like that of fig. 1 has been calculated [2] and is equal to $\mathrm{Exp}\left\{-\frac{1}{2}\sum_{r\neq s} P_r P_s N(Z_r, Z_s)\right\}$ where Z_r is the projection of the string r at $\tilde{\tau}_i$ (or $\tilde{\tau}_f$) on any region conformally equivalent to the region of fig. 1 *, N is the Neumann function for that region and P_r is the d-momentum of the string r.

R1 can be conformally transformed onto a semiannulus in the lower (or upper) half plane, with the strings at $\tilde{\tau}_i$ ($\tilde{\tau}_f$) transforming onto the points Z_i's on the positive real axis. The region R1 can also be conformally transformed to the region shown in fig. 3 (R3), which can itself be transformed to region of fig. 4 (R4) by cutting the cylinder in R3 along the dotted line and opening it.

Since the amplitude [2] is invariant under conformal transformations, the amplitudes calculated from any of the diagrams in figs. 1, 2, 3, and 4 are equal. We shall apply perturbation theory to region R4 but we can transform the result to the variables corresponding to the semiannular configuration. So we have

$$A^1 = \sum \int d\tau_2 \cdots d\tau_n \, V_1 \, e^{-(P^-_{12}-P^-_{ini})(\tau_2-\tau_1)} \times V_2 \, e^{-(P^-_{23}-P^-_{ini+1})} \cdots \otimes e^{+\sum P^-_i \tilde{\tau}_i} \tag{1}$$

---

* We refer to this region as R1, and to regions figs. 2, 3, and 4 as R2, R3, and R4.

where P^-_{ini} refers to the incoming particles, P^-_{ini+1} to the intermediate states between vertices i and i+1 and $\tilde{\tau}_i$ is the initial or final time for the string i. We do not have factor of i in the exponents because we are using a Wick rotated τ. The symbol $\sum\!\!\!\!\!\!\int$ stands for summation over all possible intermediate strings. Since the states at τ_0 and τ_{n+1} for the intermediate strings correspond to the same state in the cylinder diagram, we take the trace over these states when working with the region R4.

In the expression (1) we have an integration over (n-1) interaction times. One of the interaction times (e.g., τ_1) can be kept fixed. Since the translation in τ only changes the integrand just by a phase $e^{-i\tau\sum_{r=1}^{n} P_r^{(-)}}$, the integration over τ_1 would lead to a δ-function in the variable $P^{(-)}$.

Vertex function

The vertex function for three strings has been calculated in ref. 2. For the case where an infinitely short unexcited string with $p^i = 0$ joins a string with level number n and any momentum, the vertex vanishes unless the third string is in the same excited state as the second. In this case the vertex is simply:

$$V_{123} = \left\{ \alpha_1^{\alpha_1} \alpha_2^{\alpha_2} (-\alpha_3)^{\alpha_3} \right\}^{P_1^- + P_2^- - P_3^-} \qquad \alpha_3 = -(\alpha_2 + \alpha_1)$$

$$= \alpha_1^{\alpha_1(P_1^- + P_2^- - P_3^-)} \; \alpha_2^{-\alpha_1(P_1^- + P_2^- - P_3^-)} .$$

But

$$P_1^- = \frac{P_i^{\,2} - 1}{2P_i^+} = \frac{-1}{\alpha_1}$$

and (2)

$$P_3^- = \frac{\left(P_i^{(2)} + P_i^{(1)}\right)^2 + n - 1}{2P_2^+} = P_2^-$$

so

$$V_{123} = \alpha_1^{-1}\,\alpha_2 = e^{-\ell n\alpha_1/\alpha_2} \quad . \tag{3}$$

We see that the vertex is very simple in this case. Making use of this fact we take all external particles in (fig. 4) to be infinitely short unexcited strings with zero d-2 momentum.

<u>Neumann function and momentum dependent part of the integrand</u>

Since the points $r_1 e^{i\theta}$ and $r_2 e^{i\theta}$ of fig. 2 transform onto the same points of figs. 1 or 4 we may require that $N(r_1 e^{i\theta}, Z') = N(r_2 e^{i\theta}, Z')$. The analysis of ref. 2 goes through if $N(r_1 e^{i\theta}, Z') = N(r_2 e^{i\theta}, Z')$ + (a function independent of Z') and leads to the same result. The boundary conditions are the same as before; $\partial N(Z,Z')/\partial N_Z$ must be independent of Z' on the real Z-axis. The function satisfying these conditions is as follows:

$$N(Z,Z') = \ell n\left|\left(1 - \frac{Z'}{Z}\right)\left(1 - \frac{Z'^*}{Z}\right)\prod_{n=1}^{\infty}\left(1 - \frac{Z}{Z'}\,\omega^n\right)\left(1 - \frac{Z'}{Z}\,\omega^n\right)\right.$$

$$\left.\times\left(1 - \frac{Z}{Z'^*}\,\omega^n\right)\left(1 - \frac{Z'^*}{Z}\,\omega^n\right)\right| - \frac{\ell n\, Z/\omega\;\ell n(Z'Z'^*/\omega^2)}{\ell n\,\omega} - 2\ell n\, Z/\omega$$

(modulo functions of Z) (4)

where $\omega = r_2/r_1$. For the rest of the discussion we fix the outer radius of the semiannulus and set $r_1 = 1$, so $\omega = r_2$.

Using this Neumann function we find, after some algebra that the momentum-dependent part [2] for unexcited strings with $P_i = 0$ is

$$\sum_{i<j(Z_j<Z_i)} (\alpha_i/\alpha_j + \alpha_j/\alpha_i) \left[\ln\left(1 - \frac{Z_j}{Z_i}\right) \prod_{n=1}^{\infty} \left(1 - \frac{Z_i}{Z_j}\omega^n\right) \right.$$

$$\left. \times \left(1 - \frac{Z_j}{Z_i}\omega^n\right) - \frac{\ln \frac{Z_i}{\omega} \ln \frac{Z_j}{\omega}}{\ln \omega} - \ln \frac{Z_j}{\omega} \right] . \qquad (5)$$

Interaction times in terms of Z_r's

If all α's were strictly zero the transformation from region R4 to R2 would be $\rho' = -\ln Z$. In this case the point $\tilde{Z}_r$ on the semiannulus corresponding to an interaction time τ_r would coincide with Z_r. When α_r is small but not zero $\bar{Z}_r - Z_r$ is of order of α_r and we need a transformation from R2 to R4 which is correct to order α, since the energy P^- is proportional to α^1. This transformation is as follows $\rho' = \sum \frac{1}{2}\alpha_s \mathcal{N}(Z,Z_s) - \alpha_0 \ln Z$, where $\mathcal{N}$ is a complex function whose real part is the Neumann function and α_0 is the value of α for the intermediate string. τ_r corresponds to a point $\bar{Z}_r$ (close to Z_r) where the derivative of ρ' with respect to Z vanishes

$$\partial\rho/\partial Z = \sum_s \frac{1}{2}\alpha_s \, \partial\mathcal{N}/\partial Z - \frac{\alpha_0}{Z}\Bigg|_{Z_r} = 0 .$$

Taking only dominant terms (to zeroth order in the α's) we get

$$\alpha_r \frac{1}{\bar{Z}_r - Z_r} - \frac{\alpha_0}{\bar{Z}_r} \cong 0 \rightarrow \bar{Z}_r = \tilde{Z}_r\left(1 + \frac{\alpha_r}{\alpha_0}\right) \quad . \tag{6}$$

Therefore $\tau_r = -\alpha_0 \ln \bar{Z}_r + \sum_s \frac{1}{2} \alpha_s N(\bar{Z}_r, Z_s) = - \ln \bar{Z}_r +$

$$+ \sum_{s \neq r} \frac{1}{2} \alpha_s N(Z_s, Z) + \alpha_r \left\{ \ln\left[\frac{\bar{Z}_r - Z_r}{\bar{Z}_r^{\,2}} \omega \prod_{n=1}^{\infty} (1 - \omega^n)^2\right] - \right.$$

$$\left. - (\ln Z_r/\omega)^2/\ln \omega \right\} .$$

Using (6) we get:

$$\bar{Z}_r - Z_r = \frac{\alpha_r Z_r}{a_0}$$

and (7)

$$\tau_r = - \alpha_0 \ln \bar{Z}_r + \sum_{s \neq r} \frac{1}{2} \alpha_s N(Z_s, Z_r) +$$

$$+ \alpha_r \left[\ln \frac{\alpha_r}{\alpha_0} + \ln\left[\frac{\omega}{Z_r} \prod_{n=1}^{\infty} (1 - \omega^n)^2 - \frac{\left(\ln \frac{Z_r}{\omega}\right)^2}{\ln \omega} \right]\right] .$$

III. LOOP AMPLITUDE

In this section we use eq. (1) to calculate the loop amplitude for scattering of short strings. In the appendix we show that the contribution from the region where $\tau_{r+1} < \tau_r$ (fig. 7) tends to zero when the α's $\rightarrow 0$ (like a power of α). Hence, in the expression (1), we only have to integrate over the regions where $0 = \tau_1 < \tau_2 < \ldots < \tau_n$ (fig. 5). If the momentum of the long string

is initially $P_0^{\,i}$, $P_0^{(-)}$, the value of p^- for the long string in all the intermediate states will be p_0^-, from (2). Therefore, between the interaction times τ_r and τ_{r+1},

$$P_{ini}^{(-)} - P_{inter}^{(-)} = \sum_{i=1}^{m} -\frac{1}{\alpha_i} - \left(\sum_{i=r+1}^{m} -\frac{1}{\alpha_i} + P_0^{(-)} \right)$$

$$= -\left(\sum_{i=1}^{r} \frac{1}{\alpha_i} + P_0^{(-)} \right)$$

where m is the number of incoming particles. Since the level number and momentum of the intermediate string is unchanged, the summation over all possible states of the intermediate reduces to one term. Inserting the vertex factors, we find for the integrand of (1)

$$I = \left(\frac{\alpha_1}{\alpha_0}\right)^{-1}\left(\frac{\alpha_2}{\alpha_0}\right)^{-1}\cdots\left(\frac{\alpha_n}{\alpha_0}\right)^{-1} \mathrm{Exp}\left\{ -\sum_{r=1}^{n-1}\left(P_0^{(-)} + \sum_{i=1}^{r}\frac{1}{\alpha_i}\right) \times (\tau_{r+1} - \tau_r) \right\} e^{+\sum_i \tilde{\tau}_i P_i^{(-)}}$$

$$= \left(\frac{\alpha_1}{\alpha_0}\right)^{-1}\left(\frac{\alpha_2}{\alpha_0}\right)^{-1}\cdots\left(\frac{\alpha_n}{\alpha_0}\right)^{-1} \mathrm{Exp} - \left\{ P_0^{(-)}\,\tau_{n+1} - \sum_{r=1}^{n} \tau_r/d_r \right\}$$

$$\otimes\ \mathrm{Exp}\left(+ \sum_{i=1}^{n} \tilde{\tau}_i P_i^{(-)} \right)$$

Equation (8) Continued on Next Page

Equation (8) Continued

$$I = \mathrm{Exp}\left\{\left(\sum_{r=1}^{n} \frac{\tau_r}{\alpha_r} - \ln \frac{\alpha_r}{\alpha_0}\right) - P_0^- \tau_{n+1}\right\} \exp\left(+ \sum_{i=1}^{n} \tilde{\tau}_i P_i^{(-)}\right) \quad . \tag{8}$$

In the last step overall energy conservation; $\sum_{r=1}^{n} \frac{1}{\alpha_r} = 0$ is used.

From (8) it is clear why one needs the τ_r in terms of the Z_r's to first order in the α's.

The only part in the integrand which depends on n and $p_0{}^i$ is $\mathrm{Exp}\left[-P_0^{(-)} \tau_{n+1}\right]$ where $P_0^{(-)} = \frac{P^{i2} + n - 1}{2P_+}$. The sum over intermediate states is then:

$$\int dP_0^+ \, dP_0^- \, d^{d-2} P_{0^i} \sum_{\substack{\text{all} \\ \text{excitations}}} \delta\left(2P_0^+P_0^- - P_i{}^2 - (n-1)\right)$$

$$\times \mathrm{Exp}(-P_0^- \tau_{n+1})$$

$$= \int \frac{dP_0^+}{2P_0^+} \, d^{d-2} P_{0^i} \sum_{\substack{\text{all} \\ \text{excitations}}} \mathrm{Exp}\left[-\frac{P^{i^2} - 1}{2P_0^+} \tau_{n+1}\right]$$

$$\times \mathrm{Exp}\left(-n \frac{\tau_{n+1}}{2P_0^+}\right) \quad . \tag{9}$$

From the transformation $\rho' = -\alpha_0 \ln Z$ we have $Z = e^{-\rho'/\alpha_0}$ and $\omega = e^{-\tau_{n+1}/\alpha_0} = e^{-\tau_{n+1}/2P_0^+}$. Equation (9) then becomes:

$$\frac{1}{2}\int \frac{d\alpha_0}{\alpha_0}\, d^{d-2}\, p_i \;\mathrm{Exp}\left[-P_i^{\,2}\,\frac{\tau_{n+1}}{\alpha}\right]\;\mathrm{Exp}\left(\frac{\tau_{n+1}}{\alpha}\right)\prod_{n=1}^{\infty}\left[\frac{1}{1-\omega^n}\right]^{d-2}$$

$$= \frac{1}{2}\int_0^{\infty} \frac{d\alpha_0}{\alpha_0}\,\frac{1}{\omega}\prod_{n=1}^{\infty}\left(\frac{1}{1-\omega^n}\right)^{d-2}\left(\sqrt{\frac{\pi}{\tau_{n+1}/\alpha_0}}\right)^{d-2} .$$

We wish to change the integration from α_0 to ω. Using the relations $\omega = e^{-\tau_{n+1}/\alpha_0}$ we get $d\alpha_0/\alpha_0 = -d\omega/\omega\ln\omega$, and (9) is equal to

$$+\frac{1}{2}\int_0^1 d\omega\,\frac{1}{\pi\omega^2}\left(\frac{\pi}{-\ln\,\omega}\right)^{d/2}\prod_{n=1}^{\infty}\left(\frac{\pi}{1-\omega^n}\right)^{d-2} .$$

Collecting the above results and inserting τ_r in terms of Z_r from (7) into (8) and changing variables from τ_r's to Z_r's, we obtain the result:

$$A^1 = \frac{1}{2}\int\prod_{i=2}^{n}\frac{dZ_i}{-Z_i}\,\frac{d\omega}{\pi\omega^2}\left(\frac{\omega}{-\ln\,\omega}\right)^{d/2}\left(\prod_{n=1}^{\infty}\frac{1}{1-\omega^n}\right)^{d-2}$$

$$\times\;\mathrm{Exp}\left[\sum_{s\neq r}\frac{1}{2}\,\frac{\alpha_r}{\alpha_s}\,N(Z_r,Z_s)\right]\otimes\;\mathrm{Exp}\left[\ln\frac{\omega}{Z_r}\prod_{n=1}^{\infty}(1-\omega^n)^2 - \right.$$

$$\left. -\frac{(\ln\,Z_r/\omega)^2}{\ln\,\omega}\right] .$$

The Z_i's are ordered so that $1 = Z_1 \geq Z_2 \geq Z_3 \cdots \geq Z_n \geq \omega$.

We may isolate the momentum-dependent part and rewrite the expression as:

$$\frac{1}{2}\int \prod_{i=2}^{n} \frac{dZ_i}{-Z_i}\, d\omega\, \frac{1}{\pi\omega^2} \left(\frac{\pi}{-\ln \omega}\right)^{d/2} \left(\prod_{n=1}^{\infty} \frac{1}{1-\omega^n}\right)^{d-2}$$

$$\times \operatorname{Exp} \sum_r \left\{ \ln\left[\frac{\omega}{Z_r} \prod_{n=1}^{\infty} (1-\omega^n)^2\right] - \frac{(\ln Z_r/\omega)^2}{\ln \omega} \right\}$$

$$\otimes \operatorname{Exp} \sum \frac{1}{2} \frac{\alpha_s}{\alpha_r} N(Z_r, Z_s) \quad .$$

Comparing with (5) we can read off the volume element to be

$$\frac{1}{2}\left(\prod_{i=2}^{n} \frac{1}{-Z_i}\right) \frac{1}{\pi\omega^2} \left(\frac{\pi}{-\ln \omega}\right)^{d/2} \left(\prod_{n=1}^{\infty} \frac{1}{1-\omega^n}\right)^{d-2}$$

$$\times \operatorname{Exp}\left\{ \sum_r \left(\ln\left[\frac{\omega}{Z_r} \prod_{n=1}^{\infty} (1-\omega^n)^2\right] - \frac{(\ln Z_r/\omega)^2}{\ln \omega} \right)\right\}$$

which agrees with the operatorial method calculation. Since we have assumed that the intermediate state has transverse degrees of freedom only, we have restricted ourselves to 26 dimensional space. One can treat a nonplanar orientable diagram in exactly the same way by considering the configuration in (fig. 6).

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APPENDIX

Here we wish to justify the neglect of diagrams with $\tau_r < \tau_s$, $r > s$. Consider vertices as in (fig. 7) where τ_{r+1} precedes τ_r.

We want to show that, in the limit where the length of the strings goes to zero the contribution of diagrams as fig. 7 is infinitely small compared to the diagram that we have been considering in the text (the graph with $\tau_r < \tau_s$ if $r < s$). If the length of the strings are of order ε, then the vertex A is proportional to $\left(\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}(\alpha_1+\alpha_2)^{-\alpha_1-\alpha_2}\right)^{1/\alpha_1+1/\alpha_2-1/\alpha_1+\alpha_2} \simeq O(1)$ and B is proportional to ε^{-1}. When $\tau_r < \tau_{r+1}$ both vertices are of order ε^{-1}. We can also consider the integral corresponding to the diagrams like fig. 7. In this situation the shaded region transforms to a small neighborhood of Z_r and Z_{r+1}, and these two points are very close to each other too (close means $(Z_{r+1} - Z_r)$ is of order of ε). The integrand involves a factor as

$$\exp\left[\left(\frac{1}{\alpha_1+\alpha_2} - \frac{1}{\alpha_1} - \frac{1}{\alpha_2}\right)\left(\tau_r - \tau_{r+1}\right)\right] \simeq \exp\left(\frac{3}{2\varepsilon}\tau_{r+1}\right) \times (\text{term}$$

which is function of τ_r). τ_{r+1} integration will give a term of order $\varepsilon.\left(\int_{-\infty}^{\tau_r} \exp\frac{3}{2\varepsilon}\tau_{r+1} \simeq \frac{2}{3}\varepsilon \exp\frac{3}{2\varepsilon}\tau_r\right)$. Therefore the the ratio of a diagram like (fig. 7) over the graph that we have been considering is $\varepsilon^{-1}\cdot\varepsilon/\varepsilon^{-2} = \varepsilon^2$. This justifies the neglect of (fig. 7) in our calculation of volume element.

FOOTNOTES AND REFERENCES

* This work was supported in part by the U. S. Atomic Energy Commission.

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FIGURE CAPTIONS

Fig. 1. Space-time surface corresponding to a loop.

Fig. 2. Semiannulus in the lower half plane.

Fig. 3. Cylinder diagram

(a) long strings

(b) short strings.

Fig. 4. Cut cylinder.

Fig. 5. Loop for short strings.

Fig. 6. Diagrams for nonplanar string loop.

Fig. 7. Configuration in which $\tau_{r+1} < \tau_r$.

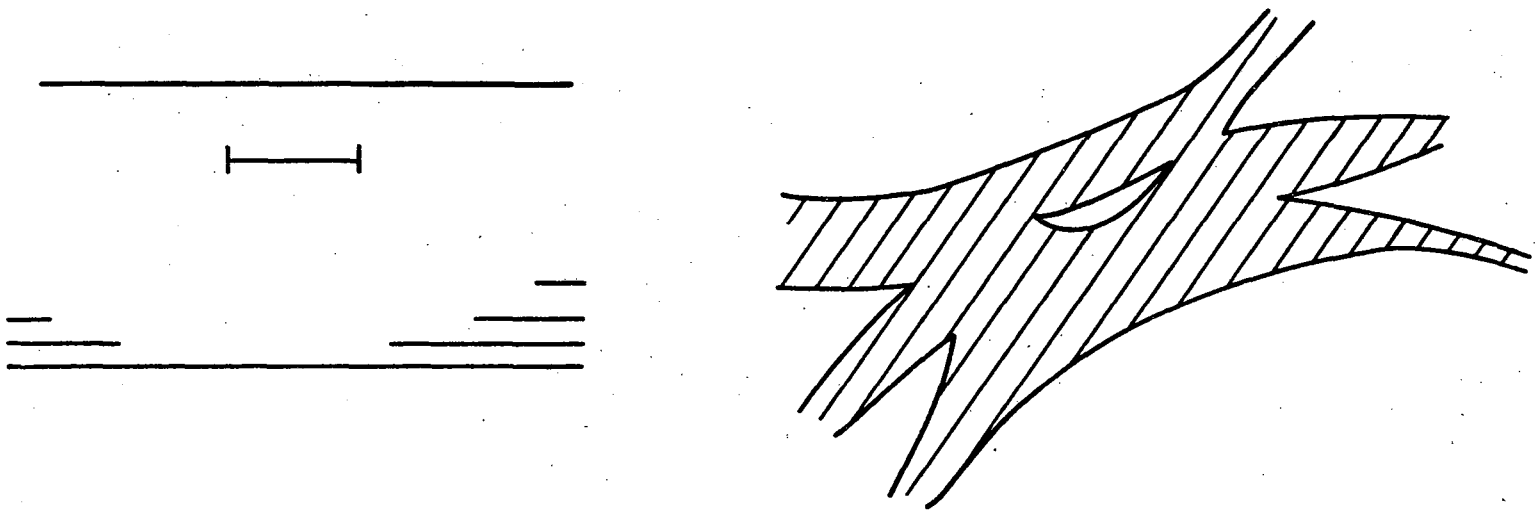

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Fig. 1.

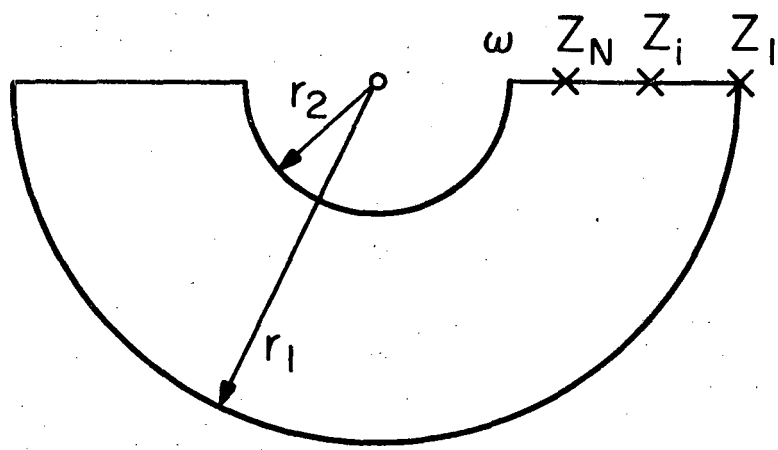

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Fig. 2.

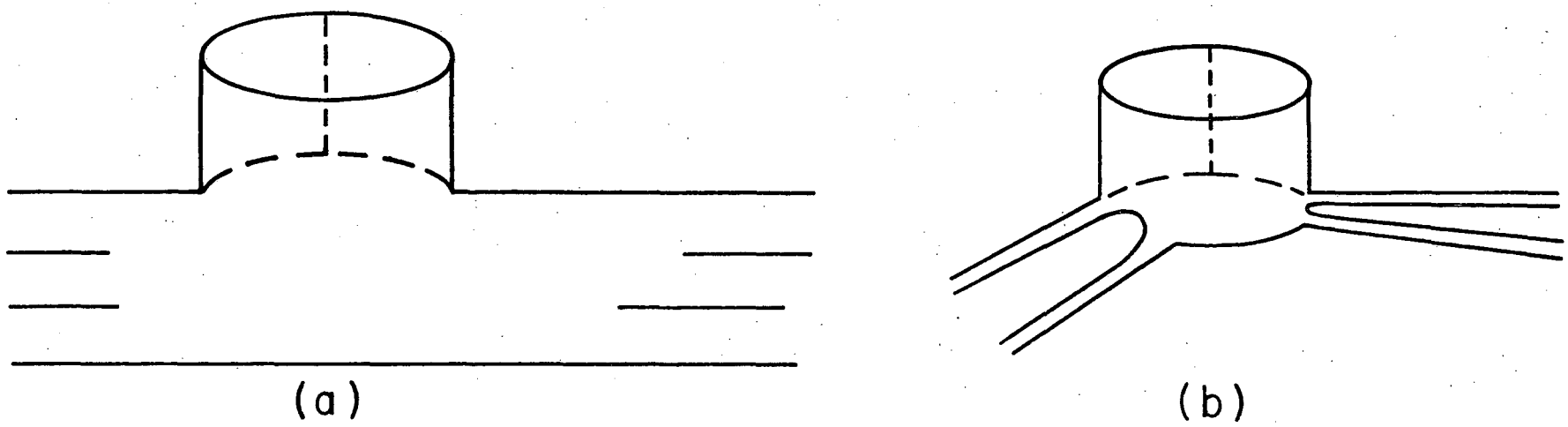

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Fig. 3.

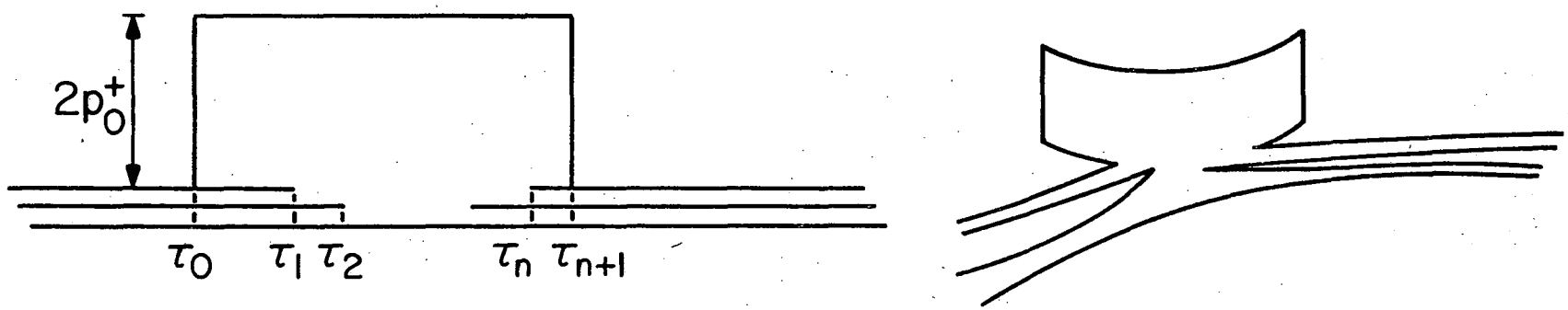

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Fig. 4.

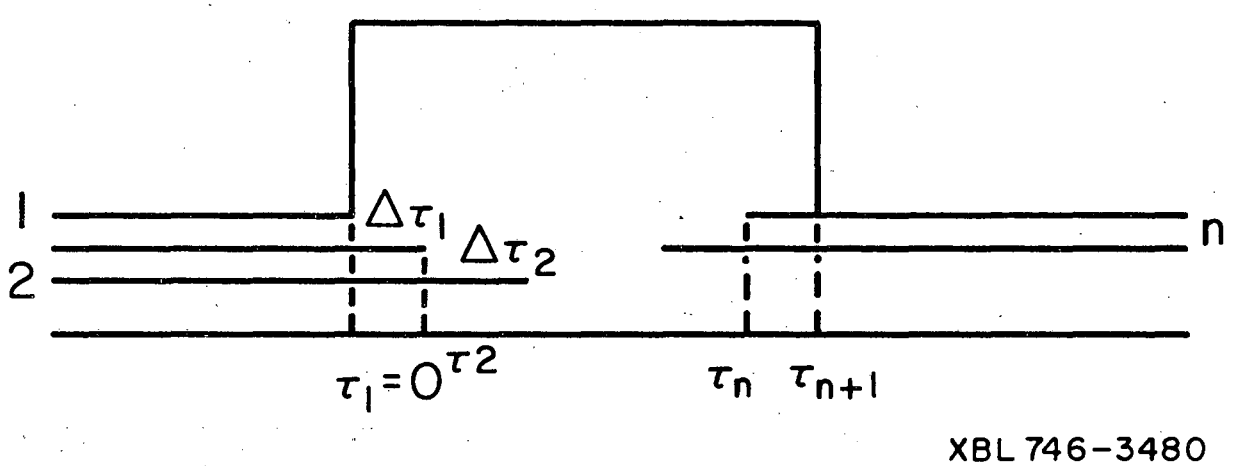

Fig. 5.

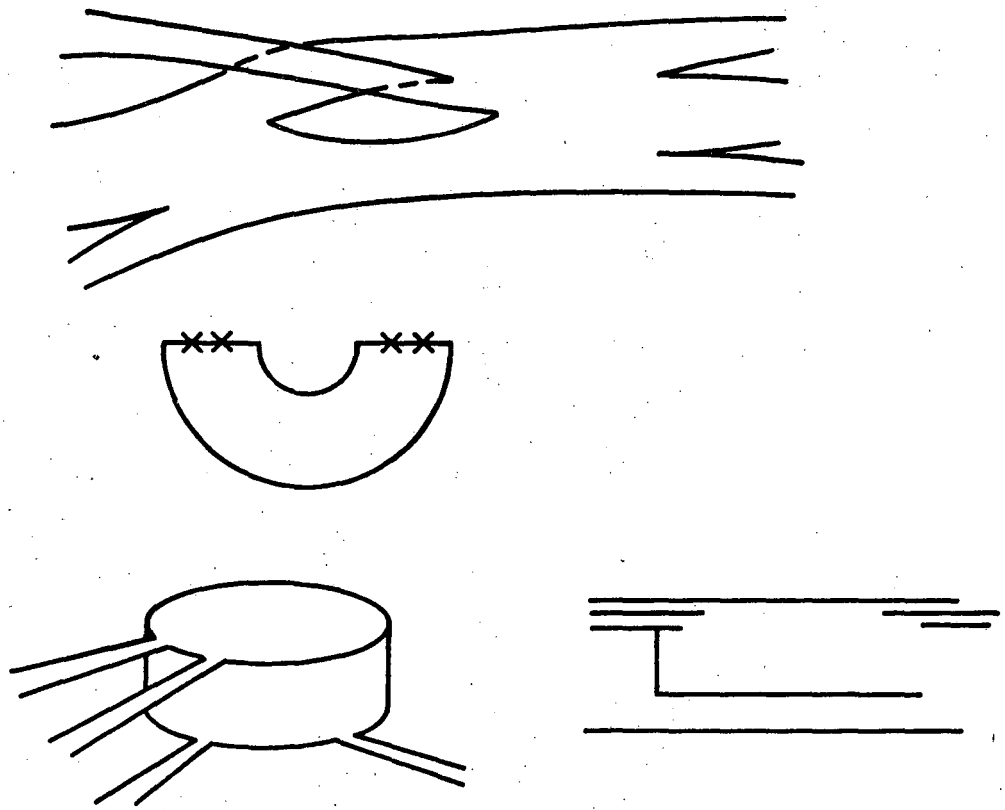

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Fig. 6.

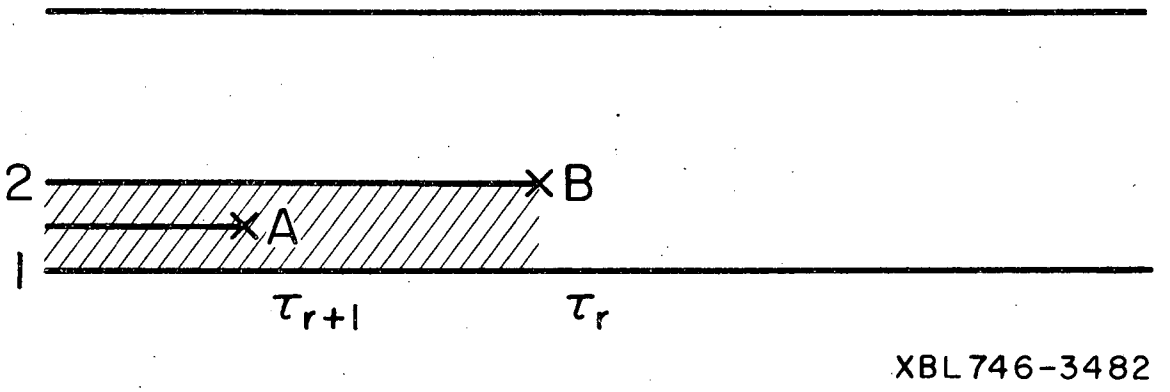

Fig. 7.

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