LBL-30634

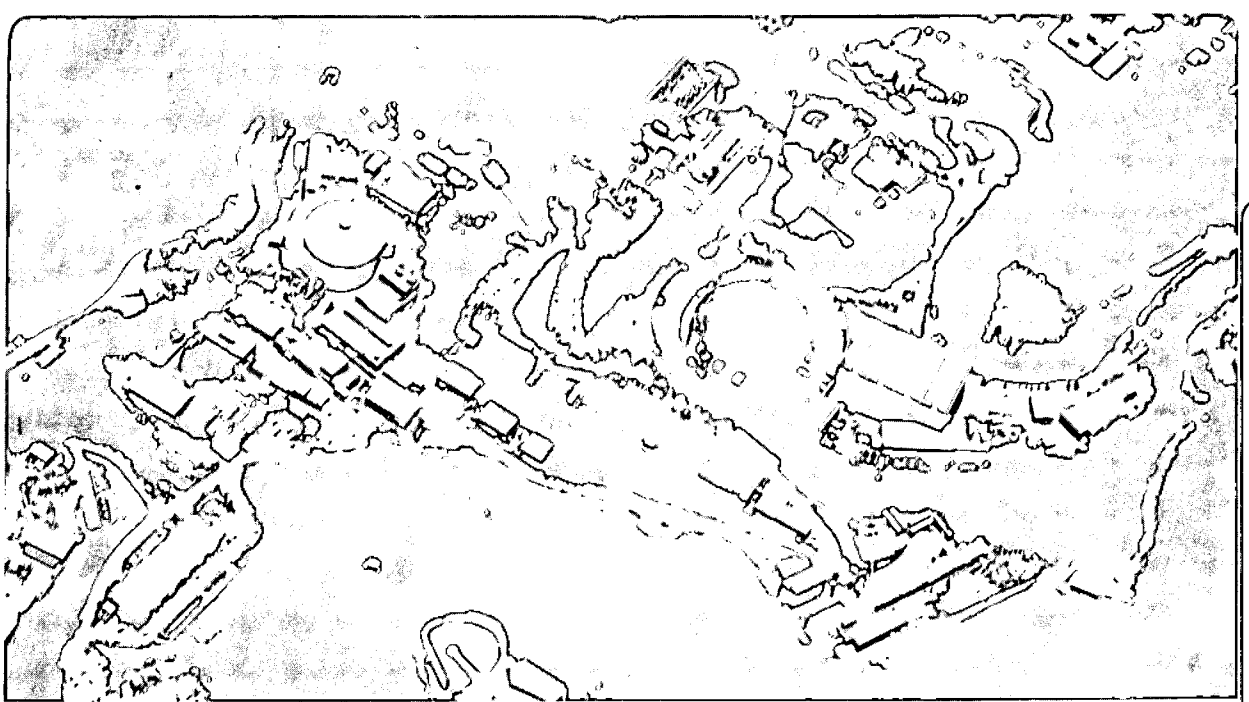
Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

Physics Division

Graph Theory and the Virasoro Master Equation

N.A.J. Obers
(Ph.D. Thesis)

April 1991



LOAN COPY
Circulates
for 4 weeks
Bldg. 50 Library.
LBL-30634
Copy 2

Prepared for the U.S. Department of Energy under Contract Number DE-AC03-76SF00098

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

April, 1991

LBL-30634
UCB-PTH-91/24

Graph Theory and the Virasoro Master Equation *

Niels A.J. Obers

Ph.D. Dissertation

Department of Physics
University of California
and
Theoretical Physics Group
Physics Division
Lawrence Berkeley Laboratory
1 Cyclotron Road
Berkeley, California 94720

*This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY90-21139.

This report has been reproduced directly from the best available copy.

Graph Theory and the Virasoro Master Equation

by

Niels A.J. Obers

Abstract

A brief history of affine Lie algebra, the Virasoro algebra and its culmination in the Virasoro master equation is given. By studying ansätze of the master equation, we obtain exact solutions and gain insight in the structure of large slices of affine-Virasoro space.

We find an isomorphism between the constructions in the ansatz $SO(n)_{diag}$, which is a set of unitary, generically irrational affine-Virasoro constructions on $SO(n)$, and the unlabelled graphs of order n. On the one hand, the conformal constructions, are classified by the graphs, while, conversely, a group-theoretic and conformal field-theoretic identification is obtained for every graph of graph theory.

We also define a class of "magic" Lie group bases in which the Virasoro master equation admits a simple metric ansatz $\{g_{metric}\}$, whose structure is visible in the high-level expansion. When a magic basis is real on compact g, the corresponding g_{metric} is a large system of unitary, generically irrational conformal field theories. Examples in this class include the graph-theory ansatz $SO(n)_{diag}$ in the Cartesian basis of $SO(n)$, and the ansatz $SU(n)_{metric}$ in the Pauli-like basis of $SU(n)$. Finally, we define the "sine-area graphs" of $SU(n)$, which label the conformal field theories of $SU(n)_{metric}$, and we note that, in similar fashion, each magic basis of g defines a generalized graph theory on g which labels the conformal field theories of g_{metric}.

Disclaimer

This document was prepared as an account of work sponsored by the United States Government. Neither the United States Government nor any agency thereof, nor The Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial products process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or The Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof of The Regents of the University of California and shall not be used for advertising or product endorsement purposes.

Lawrence Berkeley Laboratory is an equal opportunity employer.

Acknowledgements

I would like to thank my thesis advisor Professor Martin B. Halpern for his invaluable advice and encouragement during my graduate studies, and for his excellent teaching of physics, both in courses and in private discussions.

I would also like to thank the Physics Department and the Particle Theory Group in LBL for giving me the opportunity to study physics in Berkeley, which has been a very stimulating experience. In particular, I would like to thank Professors O. Alvarez, M.K. Gaillard, L. Hall, M. Suzuki, E. Wichmann and B. Zumino for their outstanding graduate courses, which I enjoyed very much and which taught me a lot.

I should also mention my fellow graduate students of the Particle Theory Group, Greg Anderson, David Brahm, Michael Crescimanno, Scott Hotes, San-Jin Sin and Stephen Hsu, all of which were very pleasant office mates and provided often interesting and motivating discussions. I would also like to thank the postdocs of our group, Elias Kiritsis, Amit Giveon, Massimo Porrati and Dirk-Jan Smit, for many helpful discussions.

The secretaries of the Theory Group in LBL, Luanne Neumann and Betty Moura, who were always willing to answer questions and give any kind of help, certainly deserve a special mention here as well.

I would also like to express my gratitude to my undergraduate thesis advisor Professor J.J. de Swart, who motivated me to continue my studies in the United States.

Finally, I am grateful to my parents for all the support they have given to me, without which this thesis would never have been completed.

This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY85-15857.

Contents

1 Introduction 1

2 Graph Theory, SO(n) Current Algebra and the Virasoro Master Equation 8

2.1 Introduction . . . 8

2.2 General Virasoro Construction on Affine g . . . 11

2.2.1 The Virasoro master equation . . . 11

2.2.2 The diagonal ansatz on SO(n) . . . 16

2.2.3 High-level expansion and unitarity . . . 20

2.3 Graph Theory and $SO(n)_{diag}$ . . . 22

2.3.1 Graph rules . . . 22

2.3.2 Affine-Virasoro constructions as graph functions . . . 22

2.3.3 Automorphisms and isomorphisms . . . 24

2.3.4 Affine-Sugawara nested graphs . . . 25

2.3.5 Affine-Virasoro nested graphs . . . 29

2.3.6 Irreducible graphs . . . 29

2.3.7 Counting old and new constructions . . . 30

2.4 Application to $SO(n \leq 6)_{diag}$ . . . 33

2.5 The Graphs $G_n^{\#}$ of $SO(n)^{\#}_{diag}$ . . . 35

2.5.1 Identity graphs . . . 35

2.5.2 Connected incomplete bipartite graphs . . . 36

2.5.3 Broken N=2 affine-Sugawara nested graphs . . . 37

2.5.4 An edge theorem for $SO(n)^{\#}_{diag}$ . . . 38

2.5.5 Self-K-conjugate constructions . . . 38

2.5.6 Cartesian product graphs . . . 40

2.6 Graph Symmetry and Consistent Subansätze . . . 41

2.7 Exact Solutions in $SO(n)^{\#}_{diag}$ . . . 44

2.7.1 $SO(2n)^{\#}_M \equiv SO(2n)^{\#}[d,3]$ and $SO(2n)^{\#}[d,4]$ . . . 44

2.7.2 The subansatz $SO(2n+1)[d,6]$ . . . 46

2.7.3 The self-K-conjugate constructions on SO(5) . . . 48

2.7.4 $SO(2n+1)^{\#}[d,6]_{1,2}$, $n \geq 3$ . . . 49

2.7.5 $SO(5)^{\#}[d,6]_2$ . . . 50

2.8 The novelty number ν . . . 50

Appendix 2.A: Counting affine-Sugawara nested graphs . . . 52

Appendix 2.B: The deformations $SO(2n+1)^{\#}_2[d,6]$, $n \geq 2$ . . . 56

Appendix 2.C: Decomposition of a graph into irreducible graphs . . . 57

3 Magic Bases, Metric Ansätze and Generalized Graph Theories 58

3.1 Introduction . . . 58

3.2 Magic Bases . . . 60

3.3 Metric Ansätze . . . 60

3.4 High-Level Expansion of g_{metric} . . . 62

3.5 The Ansatz $SU(n)_{metric}$ . . . 63

3.6 $SU(3)_{metric}$ . . . 67

3.6.1 The Self K-Conjugate Construction $SU(3)^{\#}[m,2]$ . . . 67

3.6.2 The Quadratic Deformation $SU(3)^{\#}_3$ . . . 69

3.7 $SU(4)_{metric}$ . . . 71

3.7.1 Solution Methodology . . . 71

3.7.2 $SU(4)^{\#}[m,7]_{1,2}$ . . . 75

3.7.3 $SU(4)^{\#}[m,7]_{3,4}$ . . . 76

3.7.4 $SU(4)_{metric}$ Versus $SO(6)_{diag}$ . . . 77

3.8 $SU(5)_{metric}$ 78

3.8.1 Finite-Order Irrational Central Charge 78

3.8.2 Exact Solutions with Finite-Order Irrationality 80

3.9 Generalized Graph Theories on g 81

3.9.1 The Sine-Area Graphs of SU(n) 81

3.9.2 Generalized Graphs of g 84

3.10 Conclusions 86

Tables 91

Figure Captions 97

Figures 100

Chapter 1

Introduction

The main purpose of this thesis is to find new representations of the Virasoro algebra

$$[L^{(m)}, L^{(n)}] = (m-n)L^{(m+n)} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \quad m,n \in Z \tag{1.1}$$

where the parameter c is called the central charge of the algebra. The Virasoro algebra [1] was first encountered in the study of the open bosonic string, where it was found that the modes of the stress tensor

$$T(z) = \sum_m L^{(m)} z^{-m-2} = -\frac{1}{2}g^{\mu\nu} : J_\mu(z)J_\nu(z) : \tag{1.2}$$

satisfy the Virasoro algebra (1.1) with $c = D$, where D is the number of space-time dimensions. Here :: is a normal ordering and the string momentum

$$J_\mu(z) = \sum_m J_\mu^{(m)} z^{-m-1} \tag{1.3}$$

is a local field, whose modes satisfy the algebra

$$[J_\mu^{(m)}, J_\nu^{(n)}] = g_{\mu\nu} n \delta_{m+n,0}, \quad \mu = 0, \ldots, D-1 \tag{1.4}$$

where $g_{\mu\nu} = (1, -1, \ldots, -1)_{diag}$ is the flat metric.

The algebra (1.4) is in fact a representation of an abelian affine Lie algebra $g = U(1)^D$. More generally, affine Lie algebra, or current algebra on S_1, was discovered independently

in mathematics [2] and physics [3]. In these infinite dimensional algebras the currents of affine (simple) g satisfy

$$[J_a^{(m)}, J_b^{(n)}] = i f_{ab}{}^c J_c^{(m+n)} + mk\eta_{ab}\delta_{m+n,0}, \quad a = 1, \ldots, \dim g \tag{1.5}$$

where $f_{ab}{}^c$ and η_{ab} are respectively the structure constants and Killing metric of g. The level of the affine algebra g_x is defined as $x = 2k/\psi^2$ where ψ is the highest root of g.

The first representations [3] of affine Lie algebras were constructed with world-sheet fermions [3, 4] to implement the proposal of current-algebraic spin and internal symmetry on the string [3]. For example, to construct affine $SU(n)$ at level $x = 1$ take

$$J_a(z) = \sum_m J_a^{(m)} z^{-m-1} = \,{}^\circ_\circ \bar{\psi}^I(z)(T_a)_I{}^J \psi_J(z) \,{}^\circ_\circ \tag{1.6}$$

where T_a are the representation matrices of the n of $SU(n)$, and

$$\psi_I(z) = \sum_r \psi_I^{(r)} z^{-r-1/2}, \quad \bar{\psi}_I(z) = \sum_r \bar{\psi}_I^{(r)} z^{-r-1/2} \tag{1.7a}$$

$$I = 1, \ldots, n, \quad r \in Z + 1/2 \tag{1.7b}$$

are a set of complex world-sheet fermions whose modes satisfy the anticommutation relations

$$\{\psi_I^{(r)}, \psi_J^{(s)}\} = \{\bar{\psi}^{I(r)}, \bar{\psi}^{J(s)}\} = 0 \tag{1.8a}$$

$$\{\psi_I^{(r)}, \bar{\psi}^{J(s)}\} = \delta_I^J \delta_{r+s,0}\,. \tag{1.8b}$$

As another example, affine $SO(n)$ at level $x = 1$ is constructed similarly using the vector representation of $SO(n)$ and real world-sheet fermions.

With the currents of affine g, one obtains the non-abelian generalization of the stress tensor in (1.2), which is the affine-Sugawara construction on g. For simple g the affine-Sugawara construction is

$$T(z) = \frac{1}{2k + Q_g}\eta^{ab} \,{}^\bullet_\bullet J_a(z)J_b(z) \,{}^\bullet_\bullet \tag{1.9}$$

where $\tilde{h}_g = Q_g/\psi^2$ is the dual Coxeter number of g, and $:J_aJ_b: = :J_bJ_a:$ is a symmetric normal ordering. The stress tensor (1.9) satisfies the Virasoro algebra with central charge

$$c = \frac{x \dim g}{x + \tilde{h}_g} \,. \tag{1.10}$$

Examples of affine-Sugawara constructions [3, 5] and coset constructions [3, 5] were also given in the first string era, as well as the vertex operator construction of fermions and $SU(n)_1$ from compactified spatial dimensions [6, 7]. The generalization of these constructions [8, 9, 10] and their applications to the heterotic string [11] mark the beginning of the present era. A general review of affine-Sugawara contructions and coset constructions is given in Section 2.2. See [12, 13, 14] for further historical remarks on affine-Virasoro constructions.

The general Virasoro construction [15-17]

$$T(L) = L^{ab} :J_aJ_b: \tag{1.11}$$

on the currents J_a of affine g [2, 3] systematizes the direct approach used by Bardacki and Halpern [3, 5] to obtain the original affine-Sugawara and coset constructions. Requiring the operator $T(z) = \sum_n L^{(m)} z^{-m-2}$ in (1.11) to satisfy the Virasoro algebra (1.1) results in the Virasoro master equation [15-17]

$$\begin{aligned} L^{ab} &= 2L^{ac}G_{cd}L^{db} - L^{cd}L^{ef} f_{ce}{}^{a} f_{df}{}^{b} - L^{cd} f_{ce}{}^{f} f_{df}{}^{(a} L^{b)e} \\ c &= 2G_{ab}L^{ab} \end{aligned} \tag{1.12}$$

for the inverse inertia tensor $L^{ab} = L^{ba}$. The master equation contains the affine-Sugawara nests * and many new conformal constructions $g^{\#}$ on the currents of affine g.

In particular, broad classes of exact solutions with unitary irrational central charge on compact g have recently been announced [18]. The growing list presently includes the

*The affine-Sugawara nests [18] include the affine-Sugawara constructions [3, 5, 9], the coset constructions [3, 5, 10] and the nested coset constructions [19].

unitary irrational constructions [18,20-23]

$$((\text{ simply-laced } g_x)^q)^{\#}_M \tag{1.13a}$$

$$SU(3)^{\#}_{BASIC} = \begin{cases} SU(3)^{\#}_M \\ SU(3)^{\#}_{D(1)}, SU(3)^{\#}_{D(2)}, SU(3)^{\#}_{D(3)} \\ SU(3)^{\#}_{A(1)}, SU(3)^{\#}_{A(2)} \end{cases} \tag{1.13b}$$

$$SO(n)^{\#}_{diag} = \begin{cases} SO(2n)^{\#}[d,4],\ n \geq 3 \\ SO(5)^{\#}[d,6]_2 \\ SO(2n+1)^{\#}[d,6]_{1,2},\ n \geq 3 \end{cases} \tag{1.13c}$$

$$SU(n)^{\#}_{metric} = \begin{cases} SU(4)^{\#}[m,7]_{1,2} \\ SU(5)^{\#}[m,2] \end{cases} \tag{1.13d}$$

which are obtained in the $BASIC \supset Dynkin \supset Maximal$ sequence of subansätze, in $SO(n)_{diag}$ and in $SU(n)_{metric}$. The exact constructions in (1.13c,d) will be obtained in Chapters 2 and 3 respectively. The value

$$c\left((SU(3)_5)^{\#}_{D(1)}\right) = 2\left(1 - \frac{1}{\sqrt{61}}\right) \simeq 1.7439 \tag{1.14}$$

is the lowest unitary irrational central charge yet observed [22].

A very large number [18]

$$N(g) = 2^{n(g)}, \qquad n(g) = \dim g(\dim g - 1)/2 \tag{1.15}$$

of solutions is expected generically on arbitrary level of any g, e.g. $N(g) \approx \frac{1}{4}$ billion on $SU(3)$, so the exact constructions in eq.(1.13) are only the first glimpse into a generically-irrational affine-Virasoro universe of immense new structure.

A high-level (semi-classical) expansion of the master equation [22] has been developed which marks a bifurcation in the study of new conformal constructions on affine g: The expansion is capable in principle of seeing all solutions whose high-level behavior is $\mathcal{O}(k^{-1})$, which includes all high-k smooth unitary solutions [22]. Following Ref.[22], we refer to

these $\mathcal{O}(k^{-1})$ constructions as the class of *high-k smooth* constructions on g, which we will also call the level-families. As a specific application, the high-level expansion was used in Ref.[22] to see all unitary high-k smooth constructions in the ansatz $SU(3)_{BASIC}$.

In other directions, the master equation has been identified as an Einstein system on the group manifold [16], and a world-sheet action [24] has been obtained for the generic high-level smooth affine-Virasoro construction. Moreover, classical construction of primary fields [24] has been studied in the generic theory. The master equation has also provided an exact C-function and C-theorem [25], so that the associated flow on affine-Virasoro space may be an exact renormalization group equation. Finally, the superconformal master equation [26] has been obtained [1], which collects all the superconformal constructions of the Virasoro master equation on $g_x \times SO(p,q)_1$. This reference also obtains the super C-function and super C-theorem on N=1 affine-Virasoro space, as well as a graph theory of superconformal level-families, and the N=2 superconformal master equation.

As in general relativity, consistent ansätze [18,20,22,23] have played a central role in solving the Virasoro master equation (1.12). Although the conformal field theories of a generic ansatz can be expressed in any basis, there may exist special ansätze whose forms are simple in certain preferred bases. Moreover, the high-level analysis of relatively simple ansätze is a useful tool in seeing the structure of these slices of affine-Virasoro space.

The main topic of this thesis is the study of a family of simple metric ansätze on g

$$g_{metric}: \quad L^{ab} = \frac{1}{2}(\lambda_a + \lambda_b)\eta_{ab} \tag{1.16}$$

in which the inverse inertia tensor is proportional to the Killing metric η_{ab} of g. These metric ansätze are associated to a class of "magic" bases in which any two generators of Lie g commute to no more than a single generator.

A review of the general affine-Virasoro construction (1.11) is given in Section 2.2.1.

[1] In special cases, the question of supersymmetry was addressed earlier in [27, 28].

The remainder of Chapter 2 is devoted to the simplest case of (1.16),

$$SO(n)_{diag}: \quad \text{the diagonal ansatz on } SO(n) \tag{1.17}$$

with diagonal Killing metric η_{ab} in the physicist's standard basis of $SO(n)$. The application of the high-level expansion to this ansatz, shows an isomorphism between this subset of unitary, generically irrational affine-Virasoro constructions and the set of graphs of order n [29, 30]. The graphs label the level-families of $SO(n)_{diag}$, with graph isomorphisms in $Aut\, SO(n)$. Conversely, the isomorphism provides a Lie group and conformal field-theoretic organization of graph theory, which may be interesting in mathematics. The material in Chapter 2 has been published by M.B. Halpern and the author of this thesis in Ref. [23].

In Chapter 3 the generalization of $SO(n)_{diag}$ to the more general class of metric ansätze (1.16) is discussed and it is shown that each of these ansätze is consistent in the Virasoro master equation given a "magic basis" of Lie g. The structure of each g_{metric} is clearly visible in the high-level expansion and, moreover, when a magic basis is real, the corresponding g_{metric} is a large system of unitary, generically irrational conformal field theories.

It is likely that many magic bases can be found, but the two known examples are the Cartesian basis of $SO(n)$, which gives $SO(n)_{diag}$, and the Pauli-like basis of $SU(n)$, which gives the ansatz

$$\bullet\ SU(n)_{metric}: \quad \text{the metric ansatz on } SU(n) \tag{1.18}$$

with a non-diagonal Killing metric. In both cases the sets of of conformal field theories are generically unitary because both magic bases are real. The ansatz $SU(n)_{metric}$ will be discussed in Chapter 3 to illlustrate the features of a general metric ansatz. Moreover, a new phenomenon is observed in the high-level expansion of this ansatz, since irrationalily of the central charge is visible at finite order of the expansion.

We also define the "sine-area graphs" of $SU(n)$ which label the conformal field theories in $SU(n)_{metric}$. The prescription for defining these sine-area graphs for $SU(n)$ can be easily

generalized, so that each magic basis of g defines a generalized graph theory on g which labels the conformal field theories of g_{metric}.

Chapter 2

Graph Theory, SO(n) Current Algebra and the Virasoro Master Equation

2.1 Introduction

The purpose of this chapter is the detailed study, primarily by high-level expansion, of a new ansatz of the Virasoro master equation (1.12)

$$SO(n)_{diag}: \quad \text{the diagonal ansatz on } SO(n)$$

whose set of high-k smooth constructions is generically irrational. High-level analysis provides a strong argument that each of these constructions is unitary down to some finite critical level, in accord with our experience in Refs.[18,20-22] and the additional exact solutions of this chapter.

Our central result is that the physically distinct [20,22] high-k smooth constructions in $SO(n)_{diag}$ are in one-to-one correspondence with the unlabelled graphs of graph theory [29, 30]:

each distinct (high-k smooth) affine-Virasoro construction in $SO(n)_{diag}$

$$\leftrightarrow \text{each unlabelled graph of order } n. \tag{2.1.1}$$

This means, on the one hand, that the high-k smooth constructions in $SO(n)_{diag}$ are classified by the set of all graphs. Conversely, a group-theoretic and conformal field-theoretic identification is obtained for every graph of graph theory, which may be interesting in mathematics.

The isomorphism begins a cross-fertilization of the subjects:

1. Graph theory $\rightarrow$ conformal field theory

   Beyond taxonomy, graph theory is important in counting constructions and the analysis of residual automorphisms [20,22], symmetries, consistent subansätze and exact solutions.

   For example, the asymptotic results

$$N(SO(n)) = \mathcal{O}(e^{n^4(\ln 2)/8}) \tag{2.1.2a}$$

$$N(SO(n)_{diag} = \text{graphs of order } n) = \mathcal{O}(e^{n^2(\ln 2)/2}) \tag{2.1.2b}$$

$$N(\text{affine-Sugawara nests in } SO(n)_{diag}) \leq \mathcal{O}(e^{2n\ln 2}) \tag{2.1.2c}$$

   are seen at large n for the total number of constructions on $SO(n)$, the number of unlabelled graphs, and the number of affine-Sugawara nests in $SO(n)_{diag}$. The asymptotic forms (2.1.2b,c) show a dramatic dominance of new constructions over old constructions, so that

$$\text{the generic graph in } SO(n \gg 1)_{diag} \text{ is a new construction.} \tag{2.1.3}$$

   It also follows from (2.1.2a,b) that the full space of solutions on $SO(n)$ is a structure which is much larger than graph theory.

   Graph theory was particularly helpful in finding the new *self-K-conjugate constructions*, which are the self-complementary graphs [29] of graph theory. These constructions live only on $SO(4n)$ and $SO(4n+1)$ with *half-Sugawara central charge*, whose values raise the question of new rational central charges.

   Graph symmetry also determines a hierarchy of consistent subansätze in $SO(n)_{diag}$. Beginning with the smallest subansätze, we report the following exact unitary irrational constructions,

$$\begin{gathered} SO(2n)^{\#}[d,4]\,, \quad n \geq 3 \\ SO(5)^{\#}[d,6]_2 \\ SO(2n+1)^{\#}[d,6]_{1,2}\,, \quad n \geq 3 \end{gathered} \tag{2.1.4}$$

   and three self K-conjugate constructions

$$SO(4)^{\#}[d,4]\,, \quad SO(5)^{\#}[d,2]\,, \quad SO(5)^{\#}[d,6]_1\,. \tag{2.1.5}$$

   The names of these constructions include the size of the smallest subansatz in which their graphs appear, and we remark that the constructions on $SO(2n+1)$ are the first unitary irrational constructions on non-simply-laced g. The maximal-symmetric construction $SO(2n)^{\#}_M$ [18] also occurs as the most symmetric set of graphs in $SO(2n)^{\#}_{diag}$.

2. Conformal field theory $\rightarrow$ graph theory

   Translating from conformal field theory, we find a number of equivalent categories in graph theory,

   - affine-Sugawara construction = complete graph
   - K-conjugate construction = complement of a graph
   - coset construction = complete N-partite graph

   and a number of categories which are apparently new in graph theory,

   - the affine-Sugawara nested graphs
   - the graphs $G^{\#}_n$ of the new constructions $SO(n)^{\#}_{diag}$
   - the affine-Virasoro nested graphs

- the irreducible and new irreducible graphs
- the broken $N=2$ affine-Sugawara nested graphs.

In general, the names of these graphs are derived from their corresponding conformal constructions. The irreducible graphs are particularly important because every graph can be uniquely constructed from the irreducible graphs by affine-Virasoro nesting [18].

We have also constructed a graph function $\nu(G)$, the novelty number of G, which appears to act as an order parameter for the graphs $G^{\#}$ of new constructions.

The material in this chapter has been published in Ref.[23] and my thesis advisor Prof. M.B. Halpern is the co-author of this paper. This work was a close collaboration with Professor Halpern, since there were a number of earlier papers in which we had learned to work together. I believe that I contributed approximately half the ideas in this paper, and more than half the computational results. In particular, I was solely responsible of the novelty number of Section 2.8.

2.2 General Virasoro Construction on Affine g

2.2.1 The Virasoro master equation

The general affine-Virasoro construction is [15,17]

$$T(L) \equiv L^{ab} :J_a J_b:, \quad [L^{(m)}, L^{(n)}] = (m-n)L^{(m+n)} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{2.2.1}$$

with symmetric normal ordering $T_{ab} = :J_a J_b: = T_{ba}$ [15] on the currents J_a of affine g [2,3]

$$[J_a^{(m)}, J_b^{(n)}] = i f_{ab}{}^c J_c^{(m+n)} + m G_{ab}\delta_{m+n,0} \tag{2.2.2}$$

where $f_{ab}{}^c$ and G_{ab} are respectively the structure constants and general Killing metric of g. Analysis of the system (2.2.1-2) results in the Virasoro master equation and central charge

[15,17]

$$\begin{aligned} L^{ab} &= 2L^{ac}G_{cd}L^{db} - L^{cd}L^{ef} f_{ce}{}^a f_{df}{}^b - L^{cd} f_{ce}{}^f f_{df}{}^{(a} L^{b)e} \\ c &= 2G_{ab}L^{ab} \end{aligned} \tag{2.2.3}$$

for the inverse inertia tensor $L^{ab} = L^{ba}$ of the Virasoro operator (2.2.1). The construction is completely general since g is not necessarily compact or semi-simple. In particular, to obtain level $x_I = 2k_I/\psi_I^2$ of g_I in $g = \oplus_I g_I$ with dual Coxeter number $\tilde{h}_I = Q_I/\psi_I^2$, take

$$G_{ab} = \oplus_I k_I \eta^I_{ab}, \quad f_{ac}{}^d f_{bd}{}^c = -\oplus_I Q_I \eta^I_{ab} \tag{2.2.4}$$

where η^I_{ab} is a Killing metric of g_I. The master equation has been identified in Ref.[16] as an Einstein-like system on the group manifold: The central charge of the general construction is $c = \dim g - 4R$, where R is the curvature scalar.

We remark on some general properties of the master equation which will be useful in the analysis below:

1. The affine-Sugawara construction [3,5,9] L_g is

$$L_g^{ab} = \oplus_I \frac{\eta_I^{ab}}{2k_I + Q_I}, \qquad c_g = \sum_I \frac{x_I \dim g_I}{x_I + \tilde{h}_I} \tag{2.2.5}$$

for arbitrary level of any g, and similarly for L_h when $h \subset g$.

2. K-conjugation covariance [3,5,10,15]. When L is a solution of the master equation on g, then so is the K-conjugate partner $\bar{L}$ of L,

$$\bar{L}^{ab} = L_g^{ab} - L^{ab}, \qquad \bar{c} = c_g - c \tag{2.2.6}$$

while the corresponding constructions $T(L)$ and $T(\bar{L})$ form a commuting pair of Virasoro operators.

3. Affine-Virasoro nests [18]. Repeated embedding by K-conjugation produces the affine-

Virasoro nests. For example, the nests on $g \supset h' \supset h$ are

$$\begin{aligned} L_h \text{ or } L_h^{\#} &: \quad (h \text{ or } h^{\#}) \\ L_{h'} - (L_h \text{ or } L_h^{\#}) &: \quad h'/(h \text{ or } h^{\#}) \\ L_g - L_{h'} + (L_h \text{ or } L_h^{\#}) &: \quad g/h'/(h \text{ or } h^{\#}) \end{aligned} \tag{2.2.7}$$

where L_h is the affine-Sugawara construction on h and $L_h^{\#}$ is any new construction $h^{\#}$ on h. According to eq.(2.2.6), the central charges of these nests are $(c_h \text{ or } c_h^{\#})$, $c_{h'} - (c_h \text{ or } c_h^{\#})$ and $c_g - c_{h'} + (c_h \text{ or } c_h^{\#})$ respectively. The special case of affine-Sugawara nests is realized by restriction to affine-Sugawara constructions at the bottom of the nests. Irreducible constructions [18] are reviewed in Section 2.2.2.

4. Counting. The master equation (2.2.3) is a system of $\dim g(\dim g+1)/2$ coupled quadratic equations on an equal number of unknowns $L^{ab} = L^{ba}$, so that a very large number [18]

$$N(g) = 2^{n(g)}, \qquad n(g) = \dim g(\dim g - 1)/2 \tag{2.2.8}$$

of solutions is expected generically on arbitrary level of affine g, after gauge fixing [22] the inner automorphisms of g. As in general relativity, new solutions of the master equation have generally been obtained with hierarchies of consistent ansätze and subansätze [18,20,22], beginning with the basic ansatz on simply-laced g [18].

5. Radial and angular variables. Unitarity on positive integer level of compact affine g requires [10,18]

$$L^{ab} = \text{real} \tag{2.2.9}$$

in any Cartesian basis, so all unitary solutions are naturally included in the eigenbasis [22]

$$L^{ab} = \sum_c \Omega^{ac}\Omega^{bc}\lambda_c \tag{2.2.10}$$

with λ_a = real the radial variables and $\Omega \in SO(\dim g)$ the angular variables. This eigenbasis is convenient for level x of simple compact g with

$$G_{ab} = k\delta_{ab}, \quad x = 2k/\psi^2 \tag{2.2.11}$$

since the master equation takes the form

$$\lambda_a(1 - 2k\lambda_a) = \sum_{cd} \lambda_c(2\lambda_a - \lambda_d)\hat{f}^2_{cda} \tag{2.2.12a}$$

$$0 = \sum_{cd} \lambda_c(\lambda_a + \lambda_b - \lambda_d)\hat{f}_{cda}\hat{f}_{cdb}, \quad a < b \tag{2.2.12b}$$

$$\hat{f}_{abc} \equiv f_{a'b'c'}\Omega^{a'a}\Omega^{b'b}\Omega^{c'c} \tag{2.2.12c}$$

$$c = 2k\sum_a \lambda_a \tag{2.2.12d}$$

with all Ω dependence in the $SO(\dim g)$-twisted structure constants $\hat{f}_{abc}$ of g.

6. High-level expansion. A high-level (semi-classical) expansion of the system (2.2.12) was developed in Ref.[22], which is capable in principle of seeing all high-k smooth ($\mathcal{O}(k^{-1})$) solutions of the master equation on any manifold. The results at leading order are [22]

$$L^{ab} \simeq \frac{1}{k}L^{ab}_{(0)} = \frac{1}{k}\sum_c \Omega^{ac}_{(0)}\Omega^{bc}_{(0)}\lambda^{(0)}_c, \quad c \simeq c_0 = \sum_a \theta_a \tag{2.2.13a}$$

$$\lambda^{(0)}_a = \frac{\theta_a}{2}, \quad \theta_a = 0 \text{ or } 1,\ a = 1, \ldots, \dim g \tag{2.2.13b}$$

$$0 = \sum_{cd} \theta_c(\theta_a + \theta_b - \theta_d)\hat{f}^{(0)}_{cda}\hat{f}^{(0)}_{cdb}, \quad a < b \tag{2.2.13c}$$

$$\hat{f}^{(0)}_{abc} = f_{a'b'c'}\Omega^{a'a}_{(0)}\Omega^{b'b}_{(0)}\Omega^{c'c}_{(0)} \tag{2.2.13d}$$

so that, in particular, all high-k smooth constructions approach integer central charges c_0 at high level. Values of the high-k twist $\Omega_{(0)}$ are determined by the quantization condition (2.2.13c) (or higher-order analogues) for each choice $\{\theta_a\}$ of the radial variables. The high-level expansion was applied to see all the high-k smooth solutions on $SU(3)$ in the basic ansatz, and the expansion also provided structural clues which were sufficient to obtain the exact form of all the high-k smooth unitary irrational constructions $SU(3)^{\#}_{BASIC}$ in the ansatz [22,21].

The high-level expansion was simple for $SU(3)_{BASIC}$ because the angular variable in this case [22]

$$\Omega^{ab} = \begin{pmatrix} \Omega^{AB}(\phi) & 0 \\ 0 & 1 \end{pmatrix}, \quad \Omega^{AB}(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{2.2.14}$$

is a single angle of rotation ϕ on the Cartan subalgebra, so that the quantization condition (2.2.13c) is a one dimensional problem. More generally, the quantization condition (2.2.13c) on the angular variables will be progressively more difficult to solve on larger groups.

One hope for simplification of the master equation and its high-level expansion is the existence of small consistent ansätze, such as the metric ansatz

$$L^{ab} = \frac{1}{2}(\lambda_a + \lambda_b)\eta^{ab} \tag{2.2.15}$$

where η_{ab} is the Killing metric on compact g. The consistency of a metric ansatz is generally basis dependent, since the form (2.2.15) is not covariant. We restrict our discussion here to the case of Cartesian coordinates, where the metric ansatz becomes a *diagonal ansatz*

$$L^{ab} = \lambda_a \delta_{ab} \tag{2.2.16a}$$

$$\lambda_a(1 - 2k\lambda_a) = \sum_{cd} \lambda_c(2\lambda_a - \lambda_d) f^2_{cda} \tag{2.2.16b}$$

$$f_{cd}{}^{a} f_{cd}{}^{b>a} = 0, \quad \forall c, d \tag{2.2.16c}$$

$$c = 2k \sum_a \lambda_a \tag{2.2.16d}$$

whose consistency condition (2.2.16c) guarantees that the off-diagonal master equation (2.2.12b) is satisfied identically for $\Omega^{ab} = \delta^{ab}$. The consistency condition means that any two generators of g commute to no more than a single generator, which is not true for $SU(3)$ in, say, the Gell-Mann basis. As we note in the following section, however, the consistency condition is satisfied in the physicist's standard Cartesian basis for $SO(n)$. Metric ansätze on other manifolds will be discussed in Chapter 3.

2.2.2 The diagonal ansatz on SO(n)

We label the Cartesian generators J_{ij} of $SO(n \geq 3)$ by the *vector indices* $1 \leq i < j \leq n$, so that $a \equiv (i,j) = 1, \ldots, \dim g = n(n-1)/2$. The Cartesian structure constants and Killing metric are

$$f_{ij;kl}{}^{rs} \equiv \sqrt{\frac{\tau\psi^2}{2}}(\delta_{jk}\delta_i^{[r}\delta_l^{s]} - \delta_{jl}\delta_i^{[r}\delta_k^{s]} - \delta_{ik}\delta_j^{[r}\delta_l^{s]} + \delta_{il}\delta_j^{[r}\delta_k^{s]}) \tag{2.2.17a}$$

$$\eta_{ij;kl} \equiv \delta_{ik}\delta_{jl}, \quad \tau \equiv \begin{cases} 2, & n = 3 \\ 1, & n \geq 4 \end{cases} \tag{2.2.17b}$$

where ψ is the highest root of $SO(n)$ and $A^{[r}B^{s]} \equiv A^rB^s - A^sB^r$. The structure constants in this basis satisfy the consistency condition (2.2.16c) in the form

$$f_{ij;kl}{}^{rs} f_{ij;kl}{}^{tu \neq rs} = 0, \quad \forall\, (i,j) \text{ and } (k,l) \tag{2.2.18}$$

because (r,s) is uniquely determined for each fixed choice (i,j) and (k,l) when $f_{ij;kl}{}^{rs}$ is nonvanishing.

It follows that the diagonal ansatz on $SO(n)$

$$SO(n)_{diag}: \qquad L^{ij;kl} \equiv \frac{L_{ij}}{\psi^2}\delta_{ik}\delta_{jl}, \qquad T(L) = \frac{1}{\psi^2}\sum_{i<j} L_{ij} :(J_{ij})^2: \tag{2.2.19}$$

is a consistent ansatz. The radial variables are $\lambda_{ij} = L_{ij}/\psi^2$, and the simplest form of the ansatz is obtained with the symmetrization convention

$$L_{ij} \equiv L_{ji}, \quad i \neq j; \quad L_{ii} \equiv 0. \tag{2.2.20}$$

The master equation for $SO(n)_{diag}$

$$\begin{aligned} L_{ij}(1 - xL_{ij}) - \tau L_{ij}\sum_{l\neq i,j}^{n}(L_{il} + L_{jl}) + \tau\sum_{l\neq i,j}^{n} L_{il}L_{lj} = 0, \quad i < j \\ c = x\sum_{i<j} L_{ij} \end{aligned} \tag{2.2.21}$$

follows with eqs.(2.2.17) and (2.2.19-20) from the radial equation (2.2.16).

The following properties of $SO(n)_{diag}$ will be useful below:

1. Counting. The master equation (2.2.21) in the diagonal ansatz shows $\dim SO(n) = \binom{n}{2}$ quadratic equations on an equal number of unknowns, so that

$$N(SO(n)_{diag}) = 2^{\binom{n}{2}} \tag{2.2.22}$$

solutions are expected generically for any level of $SO(n)_{diag}$.

2. Unitarity. Unitary solutions on positive integer level of $SO(n)_{diag}$ are recognized when

$$L_{ij} = \text{real} \tag{2.2.23}$$

since the Cartesian currents satisfy $J_{ij}^{(m)\dagger} = J_{ij}^{(-m)}$.

3. K-conjugation covariance. According to eq.(2.2.6), the K-conjugate construction $\tilde{L}_{ij}$

$$\begin{gathered} \tilde{L}_{ij} = L_{ij}(SO(n)) - L_{ij}, \qquad \tilde{c} = \frac{x\,n(n-1)/2}{x + \tau(n-2)} - c \\ L_{ij}(SO(n)) = \frac{1}{x + \tau(n-2)} \end{gathered} \tag{2.2.24}$$

is obtained when L_{ij} is a construction in $SO(n)_{diag}$.

4. Subgroups, cosets and affine-Sugawara nests. The diagonal ansatz on $SO(n)$ contains only those subgroups

$$\begin{gathered} h(SO(n)_{diag}) \equiv SO(m_1) \times SO(m_2) \times \ldots \times SO(m_N) \\ \sum_{i=1}^{N} m_i = n, \quad 2 \le N \le n-1, \quad m_i \ge 1 \end{gathered} \tag{2.2.25}$$

whose generators are a *subset* of the generators J_{ij} of $SO(n)$, and not linear combinations of these generators. Any $SO(1)$ factor in (2.2.25) is the trivial construction $L(SO(1)) = 0$. Moreover, each factor $SO(m_i)$ occurs in its own diagonal subansatz, $SO(m_i)_{diag}$, so further subgroup nesting follows the same pattern * within each $SO(m_i)$. Note that any factor

*For example, $SO(2n)/(SO(n)\times SO(n))/SO(n)_V$, with $SO(n)_V$ the diagonal subgroup of $SO(n)\times SO(n)$, is excluded because this nest requires linear combinations of the generators J_{ij}.

$SO(3)$ is embedded at level $\tau x = 2x$ in $SO(n \ge 4)_x$, while $SO(m_i \ge 4)$ is always a regular embedding.

We define the *fundamental* affine-Sugawara nests $\mathcal{N}_n$ in $SO(n)_{diag}$ as those obtained by subgroup nesting with $SO(n)$ at the top. Moreover, we will say that a fundamental nest $\mathcal{N}_n(d)$ has *depth* d when it contains d layers of subgroup nesting. The first three depths are

$$\begin{aligned} d = 1: &\quad SO(n) \\ d = 2: &\quad \frac{SO(n)}{h(SO(n)_{diag})} \\ d = 3: &\quad \frac{SO(n)}{\frac{SO(m_1)}{h(SO(m_1)_{diag})} \times SO(m_2) \times \ldots \times SO(m_N)} \\ &\quad \frac{SO(n)}{\frac{SO(m_1)}{h(SO(m_1)_{diag})} \times \frac{SO(m_2)}{h(SO(m_2)_{diag})} \times \ldots \times SO(m_N)} \\ &\quad \vdots \\ &\quad \frac{SO(n)}{\frac{SO(m_1)}{h(SO(m_1)_{diag})} \times \frac{SO(m_2)}{h(SO(m_2)_{diag})} \times \ldots \times \frac{SO(m_N)}{h(SO(m_N)_{diag})}} \end{aligned} \tag{2.2.26}$$

and so on for deeper nests. The *bottom* of each nest is the collection of constructions at the bottom of all the nesting columns. The fundamental affine-Sugawara nests $\mathcal{N}_n(d)$ and their K-conjugate nests $\tilde{\mathcal{N}}_n(d)$ on $SO(n)$ † form the set of all affine-Sugawara nests in $SO(n)_{diag}$, which are all known rational constructions in the ansatz.

5. Affine-Virasoro nests [18]. The more general fundamental affine-Virasoro nests on g are those constructed with g at the top, allowing general constructions on smaller manifolds at the bottom of each nest, including new constructions $h^{\#}$, $h \subset g$. Together, the fundamental affine-Virasoro nests and their K-conjugate nests form the set of all affine-Virasoro nests, which contains all affine-Virasoro constructions on g. Examples of fundamental affine-

†The K-conjugate nests of the fundamental affine-Sugawara nests in eq.(2.2.26) are obtained by removing $SO(n)$ from the top of each construction. More generally, the K-conjugate nests $\tilde{\mathcal{N}}_n(d)$ on $SO(n)$ are products of fundamental affine-Sugawara nests on smaller manifolds.

Virasoro nests in $SO(n)_{diag}$ include

$$\frac{SO(n)}{SO(m<n)^{\#}}, \quad \frac{SO(n)}{SO(m)^{\#} \times SO(n-m)}, \quad \frac{SO(n)}{\frac{SO(m<n)}{SO(p<m)^{\#}}} \tag{2.2.27}$$

and the fundamental affine-Sugawara nests in eq.(2.2.26).

6. Irreducible constructions [18]. The reducible constructions on g are the fundamental affine-Virasoro nests of depth $d \geq 2$ and their K-conjugates on g, all of which involve subconstructions on smaller manifolds. The *irreducible constructions* on g are therefore the affine-Sugawara construction on g and any *new irreducible constructions* $g^{\#}, g/g^{\#}$ which contain no subconstructions on smaller manifolds. Note that the new irreducible constructions are such that *both* $g^{\#}$ and $g/g^{\#}$ are non-trivial irreducible constructions, whereas, the single "old" irreducible affine-Sugawara construction is K-conjugate to the trivial construction $L = 0$ on g. The maximal-symmetric constructions [18]

$$SO(2n)^{\#}_{M}, \quad SO(2n)/SO(2n)^{\#}_{M} \tag{2.2.28}$$

are examples of known irreducible constructions which are also found in $SO(2n)_{diag}$.

Irreducible constructions are important because affine-Virasoro space may be organized as the set of fundamental affine-Virasoro nests with irreducible constructions at the bottom of each nest, plus the K-conjugates of these constructions. Moreover, since all irreducible constructions nest identically into larger groups, the irreducible constructions provide a fundamental measure of old versus new constructions, which, loosely speaking, mods out by the affine-Virasoro nesting.

7. $SO(n)$ automorphisms and vector-index relabelling. After gauge-fixing the master equation (or its consistent ansätze), there generally remains a discrete set of residual level-independent automorphisms [20,22] under which the master equation transforms covariantly. The residual automorphisms divide the solutions L into *physically equivalent* sets of solutions called automorphism cycles whose members have the same central charge and conformal weights. We refer below to the automorphism class of any solution L as auto L.

In the case of $SO(n)_{diag}$, any relabelling of the vector indices $\{i\}$ of a solution L_{ij} is also a solution, and it is easily checked that the relabellings are inner automorphic in $SO(n)$. A solution L is said to have a symmetry when one or more inner automorphisms act trivially on L. It follows that

$$\text{auto}\, L = \{\text{non-trivial relabellings of vector indices in } \{L_{ij}\}\} \tag{2.2.29}$$

and $\dim(\text{auto}\, L) \leq n!$, the equality being attained when the solution has no symmetry. A representative of each automorphism cycle is obtained by choosing a particular labelling in each auto L.

8. Conformal weights. The L^{ab}-broken conformal weights of the integrable representation T_a are the eigenvalues of $\Delta = L^{ab}T_aT_b$ [13,18]. The result

$$\Delta_i = \frac{\tau}{2}\sum_{k\neq i}^{n} L_{ki}, \quad 1 \leq i \leq n \tag{2.2.30}$$

is obtained for the n conformal weights Δ_i of the vector representation $(T_{ij})_{IJ} = i(\delta_{iI}\delta_{jJ} - \delta_{jI}\delta_{iJ})\sqrt{\tau\psi^2/2}$ in $SO(n)_{diag}$.

2.2.3 High-level expansion and unitarity

We discuss the high-level expansion [22]

$$L_{ij} = \frac{1}{x}\sum_{p=0}^{\infty} L_{ij}^{(p)} x^{-p}, \quad c = \sum_{p=0}^{\infty} c_p x^{-p} \tag{2.2.31}$$

of the master equation (2.2.21) in the diagonal ansatz. The zeroth order solution is

$$L_{ij}^{(0)} = \theta_{ij}, \quad \theta_{ij} = 0 \text{ or } 1, \quad 1 \leq i \neq j \leq n \tag{2.2.32}$$

and the moments of order $p \geq 1$ are unambiguously computed from the recursion relation

$$L_{ij}^{(p)} = (1 - 2\theta_{ij})\left\{\sum_{q=1}^{p-1} L_{ij}^{(q)} L_{ij}^{(p-q)}\right.$$

$$+\tau \sum_{l\neq i,j}^{n} \left[\sum_{q=0}^{p-1} (L_{ij}^{(q)}(L_{il}^{(p-q-1)} + L_{jl}^{(p-q-1)}) - L_{il}^{(q)} L_{lj}^{(p-q-1)}) \right] \Bigg\} \tag{2.2.33a}$$

$$c_p = \sum_{i<j} L_{ij}^{(p)}. \tag{2.2.33b}$$

The results

$$L_{ij} = \frac{\theta_{ij}}{x} + \frac{L_{ij}^{(1)}}{x^2} + O(x^{-3}), \quad c = \sum_{i<j} \theta_{ij} + \frac{1}{x} \sum_{i<j} L_{ij}^{(1)} + O(x^{-2})$$
$$L_{ij}^{(1)} = -\tau \sum_{l\neq i,j}^{n} [\theta_{ij}(\theta_{il} + \theta_{jl}) + (1 - 2\theta_{ij})\theta_{il}\theta_{lj}] \tag{2.2.34}$$

are obtained through order $p = 1$.

Important features of the high-level expansion in this case are:

1. Each high-k smooth solution in $SO(n)_{diag}$ may be unambiguously labelled by the values $\{\theta_{ij}\}$ of its zeroth order radial variables,

$$L_{ij}(\{\theta_{ij}\}) \leftrightarrow \{\theta_{ij}\}. \tag{2.2.35}$$

   This distinguishes $2^{\binom{n}{2}}$ high-k smooth constructions in $SO(n)_{diag}$, in agreement with the generic counting in (2.2.22). $SO(n)_{diag}$ may also contain sporadic solutions at particular levels, which are inaccessible to high-level analysis (see Appendix 2.B).

2. The moments $L_{ij}^{(p)}$ are real to all orders, so that, according to eq.(2.2.23) each high-k smooth construction in $SO(n)_{diag}$ is "unitary to all orders". More precisely, the reality of $L_{ij}^{(p)}$ guarantees unitarity within the radius of convergence of the high-level expansion. Since there is no reason to suspect a zero radius of convergence [18,20-22], we conjecture that all the high-k smooth solutions in $SO(n)_{diag}$ are unitary down to some finite critical level. The conjecture is true for the exact new constructions in Section 2.7, whose critical levels, in accord with Refs.[18,21,22], are quite low.

2.3 Graph Theory and SO(n)$_{\text{diag}}$

2.3.1 Graph rules

According to eq.(2.2.35), it is natural to represent each high-k smooth construction $L(\{\theta_{ij}\})$ in $SO(n)_{diag}$ by a *labelled graph* [1] G,

$$\text{each high-}k\text{ smooth solution in } SO(n)_{diag} \leftrightarrow \text{each labelled graph } G_n \text{ of order } n$$
$$L(G_n) \leftrightarrow G_n \tag{2.3.1}$$

whose set of points $V(G) \equiv \{i\}$ and (undirected) lines $E(G) \equiv \{(ij)\}$ is obtained by the *graph rules*:

$$SO(n) \text{ vector indices } i \leftrightarrow \text{points } i \text{ in graph } G$$
$$\theta_{ij} = 1 \leftrightarrow \text{line between points } i \text{ and } j \text{ in } G. \tag{2.3.2}$$

An immediate consequence is the high-level form

$$T(L(G)) \sim \frac{1}{x\psi^2} \sum_{(ij)\in E(G)} :(J_{ij})^2: \tag{2.3.3}$$

of the Virasoro operator of each high-k smooth construction $L(G)$ in $SO(n)_{diag}$.

In our discussion of graph theory below, the qualifier "high-k smooth" is implicitly assumed when we refer to constructions in $SO(n)_{diag}$.

2.3.2 Affine-Virasoro constructions as graph functions

Each affine-Virasoro construction L^{ab} in $SO(n)_{diag}$ is computable in principle, through the master equation, as a graph function $L^{ab}(G)$ on its graph G. As an example, we have computed the first two moments of the central charge $c(G)$

$$c_0(G) = \dim E(G) = \frac{1}{2} \sum_i d_i(G) \tag{2.3.4a}$$

[1] A labelled graph of order n is a collection of n labelled points (vertices) and a set of undirected lines (edges) which connect distinct points such that no more than one line connects any two points. The number $2^{\binom{n}{2}}$ of (high-k smooth) solutions in $SO(n)_{diag}$ is equal to the number of labelled graphs of order n [30].

$$c_1(G) = \tau\left(-\frac{3}{2}\sum_i d_i(G)(d_i(G)-1) + 6t_3\right) \leq 0 \qquad (2.3.4b)$$

for any graph G, using eqs.(2.2.31) and (2.2.34). Here $d_i(G)$ is the *degree* [5] of point i in G, and t_3 is the number of triangles in G. The inequality in (2.3.4b) follows from the general result that the asymptotic value c_0 of the central charge is approached from below [22]. Similarly, the high-level $L^{ab}(G)$-broken conformal weights of the vector representation

$$\Delta_i(G) = \frac{\tau\Delta_i^{(0)}(G)}{x} + O(x^{-2}), \quad \Delta_i^{(0)}(G) = \frac{d_i(G)}{2}, \quad i = 1, \ldots, n \qquad (2.3.5)$$

are identified with (2.2.30) as proportional to the degrees of G.

The leading terms of the inverse inertia tensor are

$$L_{ij}(G) = \begin{cases} x^{-1} + x^{-2}\tau(-(d_i(G)+d_j(G)-2) + l(i,j)) + O(x^{-3}), & G \text{ has a line } (ij) \\ 0 - x^{-2}\tau l(i,j) + O(x^{-3}), & G \text{ has no line } (ij) \end{cases} \qquad (2.3.6)$$

where $l(i,j)$ is the number of points $l \neq i,j$ in G which are connected to both of the points i and j. More generally, the exact result

$$L_{ij}(G) = 0 \quad \text{when } G \text{ has no path of any length from } i \text{ to } j \qquad (2.3.7)$$

is obtained to all orders from the recursion relation (2.2.33a).

The result (2.3.7) implies a physical characterization of the disconnected graphs: A graph is *connected* if each distinct pair of points is connected by some path of lines, and disconnected otherwise. Examples are given in Fig. 1. Each disconnected graph is the *union* $G_1 \cup G_2 \cup \ldots \cup G_N$ (see Fig. 1) of some set of connected graphs $\{G_i\}$. It follows from the result (2.3.7) that the disconnected graphs are reducible constructions (see Section 2.2.2) with commuting Virasoro operators $T(L(G_i))$.

[5] The degree $d_i(G) = \sum_{k\neq i} \theta_{ik}$ of point i is the number of lines attached to the point [30].

2.3.3 Automorphisms and isomorphisms

The automorphism cycles of $SO(n)_{diag}$ are easily understood in graph theory. Two graphs are isomorphic when they differ by a relabelling of their points. The particular relabellings of a graph G which preserve the same set of lines $\{\theta_{ij} = 1\}$ form a group auto$G$ of (graph)automorphisms (or trivial isomorphisms) of the graph. Physically, auto$G$ is the symmetry group of the graph G. It follows from our discussion in Section 2.2.2 that

$$SO(n) \text{ automorphisms} = \text{graph isomorphisms} \qquad (2.3.8a)$$

$$\text{auto}\, L(G) = \{\text{non-trivial isomorphisms of } G\} \qquad (2.3.8b)$$

$$1 \text{ representative of auto}\, L(G) \leftrightarrow 1 \text{ unlabelled graph } G \qquad (2.3.8c)$$

and, more physically, that

$$\text{each physically distinct affine-Virasoro construction in } SO(n)_{diag}$$
$$\leftrightarrow \text{each unlabelled graph of order } n. \qquad (2.3.9)$$

This one-to-one correspondence describes an immense structure in $SO(n)_{diag}$, which is itself much smaller than the space of all solutions on $SO(n)$.

It also follows from (2.3.8b) that the dimension of the $SO(n)$ automorphism cycle of a construction $L(G_n)$ in $SO(n)_{diag}$ is equal to the number of non-trivial isomorphisms of G_n, so that

$$\dim(\text{auto}\, L(G_n)) = \frac{n!}{S(G_n)} \qquad (2.3.10)$$

where $S(G) \equiv \dim(\text{auto}\, G)$ is the *symmetry factor* [6] of the graph. The related, but somewhat more technical conclusion

$$\text{symmetry group of } L(G) = \text{auto}\, G \qquad (2.3.11)$$

[6] The basic composition law for symmetry factors is $S(G_1 \cup G_2) = \zeta S(G_1)S(G_2)$ where $\zeta = 2$ when $G_1 = G_2$ and $\zeta = 1$ otherwise.

will be established in Section 2.6.

We employ only *unlabelled graphs* below, unless stated otherwise, as representatives of the physically distinct conformal field theories.

2.3.4 Affine-Sugawara nested graphs

In this section, we identify the graphs of the affine-Sugawara nests in $SO(n)_{diag}$.

1. Affine-Sugawara graphs. These are the graphs of order $n \geq 1$ with all possible lines

$$\begin{aligned} K_n &= \text{complete graph on } n \text{ points} \\ &= \text{affine-Sugawara construction on } SO(n) \end{aligned} \tag{2.3.12}$$

shown for $1 \leq n \leq 6$ in Fig. 2. The affine-Sugawara graphs are the most symmetric connected graphs, with auto $K_n = S_n$, dim(auto K_n) = $n!$ and dim(auto $L(K_n)$)=1. The composition law for affine-Sugawara graphs

$$G(SO(n)) = K_n, \quad G(SO(m) \times SO(n)) = K_m \cup K_n \tag{2.3.13}$$

will be useful below.

2. K-conjugate graphs. The high-level form of K-conjugation in $SO(n)_{diag}$

$$\theta_{ij} \to \tilde{\theta}_{ij} = 1 - \theta_{ij}, \quad c_0 \to \tilde{c}_0 = \binom{n}{2} - c_0 \tag{2.3.14}$$

is obtained from (2.2.24). For each graph G_n of order n, the map (2.3.14) defines a K-conjugate graph * $\tilde{G}_n$ on $SO(n)$

$$\tilde{G}_n : \quad V(\tilde{G}_n) = V(G_n), \quad E(\tilde{G}_n) = E(K_n) - E(G_n) \tag{2.3.15}$$

which represents the K-conjugate theory

$$L(\tilde{G}_n) \equiv \tilde{L}(G_n) = L(K_n) - L(G_n) \tag{2.3.16}$$

*The K-conjugate graph $\tilde{G}$ of a graph G is called $\hat{G} = \tilde{G}$, the *complement* of G, in the literature of graph theory.

of the theory $L(G_n)$. The degrees of $\tilde{G}_n$ satisfy $d_i(\tilde{G}_n) = n - 1 - d_i(G_n)$.

As illustrated in Fig. 3, the K-conjugate graph $\tilde{G}_n$ is obtained on the points of G_n by removing the lines of G_n from the affine-Sugawara graph K_n. It follows that $\tilde{K}_n$ is the totally disconnected graph of order n, such that $L(\tilde{K}_n) = \tilde{L}(K_n) = 0$ is the trivial construction on $SO(n)$, and $\tilde{K}_1 = K_1$ is the trivial graph. It is also clear that

$$\begin{aligned} \text{auto}\, G &= \text{auto}\, \tilde{G} \\ \dim(\text{auto}\, L(G)) &= \dim(\text{auto}\, L(\tilde{G})) \end{aligned} \tag{2.3.17}$$

since K-conjugation is a 1-1 map.

3. Subgroup and fundamental coset graphs. The subgroup graphs in $SO(n)_{diag}$

$$\begin{aligned} G(h(SO(n)_{diag})) &= K_{m_1} \cup K_{m_2} \cup \ldots \cup K_{m_N} \\ \sum_{i=1}^{N} m_i = n, &\quad 2 \leq N \leq n-1, \quad m_i \geq 1 \end{aligned} \tag{2.3.18}$$

are obtained from $h(SO(n)_{diag})$ in (2.2.25) with the composition law (2.3.13). The subgroup graphs are disconnected graphs of order n because of the range restrictions on $\{m_i\}$.

- The fundamental coset graphs of the fundamental coset constructions $SO(n)/h(SO(n)_{diag})$ are obtained by K-conjugation of the subgroup graphs in (2.3.18). A useful identity is

$$G_1 \widetilde{\cup} G_2 = \tilde{G}_1 + \tilde{G}_2 \tag{2.3.19}$$

where the *join* $G_1 + G_2$ of two graphs is defined by connecting every point in G_1 to every point in G_2. It follows that the fundamental coset graphs of $SO(n)_{diag}$ are the connected graphs

$$G(SO(n)/h(SO(n)_{diag})) = \tilde{G}(h(SO(n)_{diag})) = \tilde{K}_{m_1} + \tilde{K}_{m_2} + \ldots + \tilde{K}_{m_N}. \tag{2.3.20}$$

In graph theory, the *complete N-partite graphs* are obtained in this way as the join of $N \geq 2$ totally disconnected graphs. It follows that the affine-Sugawara graphs K_n are the complete

N-partite graphs of order $n = N$, and that

$$\begin{aligned}&\text{fundamental coset graphs in } SO(n)_{diag}\\ &\qquad = \text{complete } N\text{-partite graphs of order } n > N\ .\end{aligned} \tag{2.3.21}$$

Fig. 4 contains a representation of general complete bipartite(2-partite) and 3-partite graphs. In these representations, each circle, called a *lacuna* of the graph, contains one of the totally disconnected graphs $\tilde{K}_{m_i}$ in (2.3.20). The lines of the graphs connect all points in distinct lacunae.

4. Affine-Sugawara nested graphs. The fundamental affine-Sugawara nested graphs $G(\mathcal{N}_n(d))$ of depth d are the graphs of the fundamental affine-Sugawara nests $\mathcal{N}_n(d)$. The affine-Sugawara graphs in Fig. 2 and the fundamental coset graphs in Fig. 4 are the fundamental affine-Sugawara nested graphs of depth 1 and 2 respectively.

Physically, the fundamental coset graphs in Fig. 4 are formed by *removal* ($\ominus$) of subgroup graphs from the complete affine-Sugawara graph. More generally, the fundamental nested graphs at depth d are formed by removal of fundamental nested graphs of depth $d-1$ on smaller manifolds from the affine-Sugawara graph:

$$\begin{aligned}G(\text{nest of depth } d) &= G\left(\frac{\text{Sugawara}}{\text{nest of depth } d-1}\right)\\ &= G(\text{Sugawara}) \ominus G(\text{nest of depth } d-1).\end{aligned} \tag{2.3.22}$$

Alternately, we may think of the nested graphs at depth d as formed by *insertion* ($\ominus\ominus$) of fundamental nested graphs of depth $d-2$ into the fundamental coset graphs

$$\begin{aligned}G(\text{nest of depth } d) &= G\left(\frac{\dfrac{\text{Sugawara}}{\text{subgroups}}}{\text{nest of depth } d-2}\right)\\ &= G(\text{cosets}) \ominus \ominus G(\text{nest of depth } d-2)\end{aligned} \tag{2.3.23}$$

since the nest of depth $d-2$ is itself removed from the subgroups.

A precise definition of this recursive structure is

$$\begin{aligned}&\text{fundamental affine-Sugawara nested graphs of depth } d \geq 3\\ &= \text{fundamental coset graphs with insertion of depth } d-2\\ &\equiv \left\{\begin{array}{l}\text{graphs built by insertion of at least one fundamental affine-}\\ \text{Sugawara nested graph of depth } d-2, \text{ and any number}\\ \text{of affine-Sugawara nested graphs of depth} \leq d-2,\\ \text{in the lacunae of complete } N\text{-partite graphs.}\end{array}\right.\end{aligned} \tag{2.3.24}$$

Insertion of a nested graph in a lacuna of the same order is not allowed. Fig. 5a shows two fundamental affine-Sugawara nested graphs of depth 3, obtained from the coset graphs of Fig. 4 by insertion of depth one affine-Sugawara graphs in their lacunae. Fig. 5b is a fundamental nested graph of depth 4, obtained from a coset graph by inserting another depth 2 coset graph in one of its lacunae [†]. The graphical form of the recursive definition (2.3.24) is given in Fig. 5c.

The last form of the definition (2.3.24) and the schematic representation of the fundamental affine-Sugawara nested graphs in Figs. 4 and 5 are designed to exhibit the N-partite structure of the nests, since we will see below that the N=2 nests play a special role. The complementary representation in Fig. 6 shows the nested graphs as an alternating subtraction (open areas) or addition (shaded areas) of the lines of affine-Sugawara graphs. The bottom of each nest is the set of innermost open and shaded areas. For example, the bottom of the depth-two nest consists of two open areas, which records that two smaller affine-Sugawara graphs have been removed. The open spaces of this representation are not the lacunae of complete N-partite graphs, however, since the spaces do not contain all the

[†] Algebraically, the first four nest depths show the alternating pattern: $G(\mathcal{N}_n(1)) = K_n$, $G(\mathcal{N}_n(2)) = \{+\tilde{K}\}$, $G(\mathcal{N}_n(3)) = \{+\cup K\}$ and $G(\mathcal{N}_n(4)) = \{+\cup+\tilde{K}\}$, where the order of the K's and $\tilde{K}$'s may vary from 1 to $n-1$. More generally, the $d \to d+1$ operations $K \to +\tilde{K}$ (d odd) and $\tilde{K} \to \cup K$ (d even) generate the algebraic form of the fundamental nests at arbitrary depth.

points of the graphs.

The fundamental affine-Sugawara nested graphs $G(\mathcal{N}_n(d))$ are always connected graphs of order n, while the K-conjugate nested graphs $\tilde{G}(\mathcal{N}_n(d))$ are always disconnected [4] graphs of order n. Together, they form the set of *affine-Sugawara nested graphs*, which contains all known rational constructions in $SO(n)_{diag}$.

2.3.5 Affine-Virasoro nested graphs

The more general fundamental affine-Virasoro nested graphs are the graphs of the fundamental affine-Virasoro nests, defined in Section 2.2.2. These graphs retain the subgroup nesting structure in Fig. 6 of the fundamental affine-Sugawara nested graphs, now allowing general graphs at the bottom of the nest. Together, the fundamental affine-Virasoro nested graphs and their K-conjugate graphs form the set of all *affine-Virasoro nested graphs*, which includes all graphs.

2.3.6 Irreducible graphs

A graph G is called *(ir)reducible* if $L(G)$ is an (ir)reducible construction in $SO(n)_{diag}$ (see Section 2.2.2). (Ir)reducible graphs are characterized as follows.

Disconnected graphs are always the unions of graphs on smaller manifolds, so it follows from our discussion in Section 2.2.2 that disconnected graphs are always reducible graphs, and hence that irreducible graphs are always connected. We also know from Section 2.2.2 that a) the affine-Sugawara graph is the only irreducible affine-Sugawara nested graph on each manifold and b) the *new irreducible graphs* G are those for which both G and $\tilde{G}$ are irreducible, and hence connected.

[4] The K-conjugate graphs $\tilde{G}(\mathcal{N}_n(d)) = G(\tilde{\mathcal{N}}_n(d))$ of the fundamental affine-Sugawara nested graphs $G(\mathcal{N}_n(d))$ are unions of fundamental affine-Sugawara nested graphs of lower order.

This establishes the characterization

$$\begin{gathered} G \text{ is a new irreducible graph iff } G \text{ and } \tilde{G} \\ \text{are both non-trivial connected graphs} \end{gathered} \tag{2.3.25}$$

since K_1 is the only irreducible graph on $SO(1)$. The characterization (2.3.25) is a useful tool in the identification of new constructions below. Fig. 7 displays a more complete map of irreducible and reducible graphs in graph space.

Irreducible graphs are important because graph space may be organized as the set of fundamental affine-Virasoro nested graphs with irreducible graphs at the bottom of each nest, plus the K-conjugates of these graphs (see Section 2.2.2). The unique decomposition of a graph into irreducible components is discussed in Appendix 2.C.

2.3.7 Counting old and new constructions

We consider the following basic numbers

$$\begin{aligned} g_n &\equiv \text{number of all graphs of order } n \\ C_n &\equiv \text{number of connected graphs of order } n \\ C(AS)_n &\equiv \left\{ \begin{array}{l} \text{number of connected (fundamental)} \\ \text{affine-Sugawara nested graphs of order } n. \end{array} \right. \end{aligned} \tag{2.3.26}$$

The first two numbers are known in graph theory [29], and the recursion relation

$$C(AS)_n = 2C(AS)_{n-1} + \sum_{\{p(i)\}} \prod_{\substack{i=2 \\ p(i)\neq 0}}^{n-2} \binom{p(i) + C(AS)_i - 1}{p(i)}, \quad c(AS)_2 = 1$$

$$\{p(i) \geq 0\} \text{ are the partitions of } n = \sum_{i=2}^{n-2} ip(i) \tag{2.3.27}$$

is derived in Appendix 2.A. Other numbers of interest [5]

$$D_n \equiv \text{number of disconnected graphs of order } n$$

[5] The connected (fundamental) affine-Sugawara nested graphs $G(\mathcal{N}_n(d))$ are in 1-1 correspondence with the disconnected affine-Sugawara nested graphs $\tilde{G}(\mathcal{N}_n(d))$ by K-conjugation (see Section 2.3.4).

$$= g_n - C_n$$

$$g(AS)_n \equiv \text{number of affine-Sugawara nested graphs of order } n$$

$$= 2C(AS)_n \tag{2.3.28}$$

$$D(AS)_n \equiv \text{number of disconnected affine-Sugawara nested graphs of order } n$$

$$= C(AS)_n$$

$$C_n^{\#} \equiv \text{number of new connected constructions of order } n$$

$$= C_n - C(AS)_n$$

are expressed in terms of the basic numbers (2.3.26). The values of g_n, C_n, $C(AS)_n$ and $C_n^{\#}$ are given for $1 \leq n \leq 10$ in Table 1.

The results of Table 1 show a dramatic dominance of connected new constructions over connected known constructions as n increases. Similar behavior is observed for $g_n^{\#} \equiv g_n - g(AS)_n$ and $g(AS)_n$ when disconnected graphs are included. The asymptotic results [¶]

$$C_n \sim g_n = \mathcal{O}(e^{n^2(\ln 2)/2}), \qquad SO(n \gg 1) \tag{2.3.29a}$$

$$C(AS)_n = g(AS)_n/2 \leq \mathcal{O}(e^{2n\ln 2}), \qquad SO(n \gg 1) \tag{2.3.29b}$$

are a quantitative statement of the dominance of new over old constructions in $SO(n)_{diag}$. The asymptotic bound (2.3.29b) on the number of fundamental affine-Sugawara nests in $SO(n)_{diag}$ is obtained in Appendix 2.A. The corresponding characterization

$$\text{the generic graph in } SO(n \gg 1)_{diag} \text{ is a new connected construction} \tag{2.3.30}$$

follows immediately from (2.3.29).

[¶] It is known in graph theory that the generic large-order graph is connected, and the asymptotic estimate $C_n \sim g_n$ in (2.3.29a) is given in Ref.[29]; The exponential order of g_n is the exponential order of the number $2^{\binom{n}{2}}$ of solutions in $SO(n)_{diag}$, since auto $L(G)$ is combinatoric and hence factorial.

A fundamental measure of new and old constructions is provided by the irreducible graphs, whose definition, loosely speaking, mods out by the affine-Virasoro nesting (see Sections 2.2.2 and 2.3.6). These graphs are counted as follows. At order n, define ir_n, $ir(AS)_n$ and $ir_n^{\#}$ as the total number of irreducible graphs and the number of old and new irreducible graphs respectively. Then we know that

$$ir(AS)_n = 1, \qquad ir_n = ir_n^{\#} + 1 \tag{2.3.31}$$

since the affine-Sugawara graph K_n is the only irreducible affine-Sugawara nested graph on $SO(n)$. It follows from Fig. 7 that

$$ir_n = C_n - C(red)_n \tag{2.3.32a}$$

$$C(red)_n = D_n - 1 = g_n - C_n - 1 \tag{2.3.32b}$$

where $C(red)_n$ is the number of connected reducible graphs in $SO(n)_{diag}$. The last form in (2.3.32b) follows with $D_n = g_n - C_n$. The result for new irreducible graphs [‖]

$$ir_n^{\#} = 2C_n - g_n \tag{2.3.33}$$

is then obtained from eqs.(2.3.31) and (2.3.32).

Numerical values of ir_n, $ir(AS)_n$ and $ir_n^{\#}$ are given for $1 \leq n \leq 10$ in Table 2, which shows that the dominance of new over old constructions in $SO(n)_{diag}$ is even more dramatic after moding out the nests. The asymptotic behavior of the irreducible graphs

$$ir_n^{\#} \sim ir_n \sim C_n \sim g_n = \mathcal{O}(e^{n^2(\ln 2)/2}) \tag{2.3.34}$$

is obtained from eqs.(2.3.29a) and (2.3.33), and, finally, the characterization

$$\text{the generic graph in } SO(n \gg 1)_{diag} \text{ is a new irreducible construction} \tag{2.3.35}$$

follows from this behavior.

[‖] $C_n \geq D_n$ is a consequence of the result (2.3.33), since the number of new irreducible graphs is non-negative.

2.4 Application to $SO(n \leq 6)_{diag}$

Table three lists the unlabelled graphs of order 6, which are the physically distinct constructions in $SO(6)_{diag}$. The table can be used for $SO(n < 6)$ as well, since the constructions with m trivial subgraphs appear first, without the trivial subgraphs, as constructions in $SO(6-m)_{diag}$. The following data are given:

1. The graphs G of the high-level sector numbers $0 \leq c_0 = \dim E(G) \leq 7$.
2. The automorphism group auto G of each graph, e.g. $Z_2 \times S_3$ for $SO(6)/SO(5)/SO(2)$ in sector 6.
3. The dimension of the $SO(6)$ automorphism cycle dim(auto $L(G)$), computed from eq.(2.3.10).
4. The conformal field-theoretic name of each construction. The affine-Sugawara nested graphs are identified from their N-partite characterization in Section 2.3.4. Figs. 8a and 8b show examples of the translation from the symmetrically-drawn graphs of the table to the N-partite forms. The remaining new constructions are assigned an $SO(n)^{\#}$ name which also indicates the size of the subansatz in which the construction is found (see Section 2.6).
5. The $L^{ab}(G)$-broken conformal weights $2x\Delta_i(G) \simeq 2\Delta_i^{(0)} = d_i(G)$ of the vector representation at high level.
6. The K-conjugate graph $\tilde{G}$ of each G. These graphs fill the remaining high-level sectors $8 \leq \tilde{c}_0 = \dim E(\tilde{G}) = 15 - c_0 \leq 15$, with auto $\tilde{G}$ =auto G, dim(auto $L(\tilde{G})$) =dim(auto $L(G)$) and $2\tilde{\Delta}_i^{(0)} = d_i(\tilde{G}) = 5 - d_i(G)$.

In agreement with Table 1, Table 3 shows $g_G = 156$ distinct constructions in $SO(6)_{diag}$, of which $g_n = 1, 2, 4, 11$ and 34 constructions appear first in $SO(n)_{diag}$, n=1,2,3,4 and 5.

The remaining numbers of Table 1 may also be verified for $n = 1, \ldots, 6$ from Table 3, and, in particular, there are 90 new constructions in $SO(6)_{diag}$, of which 79 are connected.

The new irreducible constructions on $SO(n)$ are easily recognized by their name, $SO(n)^{\#}$ or $SO(n)/SO(n)^{\#}$, so that e.g. $SO(4)^{\#}[d,4] \times SO(2)$ in sector 4 is reducible on $SO(6)$ while $SO(4)^{\#}[d,4]$ in sector 3 is irreducible on $SO(4)$.

In agreement with Table 2, Table 3 identifies 9 new irreducible constructions in $SO(4)_{diag}$ and $SO(5)_{diag}$ *

$$\begin{aligned} c_0 = 3: &\quad SO(4)^{\#}[d,4] \\ c_0 = 5: &\quad SO(5)^{\#}[d,2] \;;\; c_0 = 5: \quad SO(5)^{\#}[d,6]_1 \\ c_0 = 4: &\quad SO(5)^{\#}[d,6]_2 \;;\; c_0 = 6: \quad SO(5)/SO(5)^{\#}[d,6]_2 \\ c_0 = 4,5: &\quad SO(5)^{\#}[d,7]_{1,2} \;;\; c_0 = 6,5: \quad SO(5)/SO(5)^{\#}[d,7]_{1,2}. \end{aligned} \tag{2.4.1}$$

The first five constructions of this list are obtained exactly in Section 2.7. Among the 68 new irreducible constructions in $SO(6)_{diag}$, the maximal-symmetric constructions [18]

$$\begin{aligned} c_0 = 6: &\quad SO(6)^{\#}_M \equiv SO(6)^{\#}[d,3] \\ c_0 = 9: &\quad SO(6)/SO(6)^{\#}_M \equiv SO(6)/SO(6)^{\#}[d,3] \end{aligned} \tag{2.4.2}$$

were identified from the high-level behavior of the known solutions. The exact forms of the next most symmetric constructions

$$c_0 = 6: \quad SO(6)^{\#}[d,4] \;;\; c_0 = 9: \quad SO(6)/SO(6)^{\#}[d,4] \tag{2.4.3}$$

are also obtained in Section 2.7.

*The first three irreducible constructions in the list (2.4.1) are examples of self-K-conjugate constructions (see Section 2.5.5).

2.5 The Graphs $G_n^\#$ of $SO(n)^\#_{diag}$

2.5.1 Identity graphs

Intuitively, new constructions are less symmetric than old constructions, which are exceptional points with special inertia tensors, commuting currents and so on. Graph theory provides a more quantitative statement of this expectation.

The affine-Sugawara graphs K_n and their K-conjugate graphs $\tilde{K}_n$ are the most symmetric graphs, with symmetry factors $S(K_n) = S(\tilde{K}_n) = n!$. In fact, the affine-Sugawara nested graphs always have at least a Z_2 symmetry, so that their symmetry factors satisfy

$$S(G(\mathcal{N}_n(d))) = S(G(\tilde{\mathcal{N}}_n(d))) \geq 2 \quad . \tag{2.5.1}$$

This argument goes as follows: By repeated application of auto$G$=auto$\tilde{G}$ and $S(G_1 \cup G_2 \ldots \cup G_N) \geq \prod_{i=1}^N S(G_i)$, the symmetry factor of any affine-Sugawara nest is greater than or equal to the product of symmetry factors of the subgroups at the bottom of the nest, as illustrated in Fig. 10. It follows that, among the affine-Sugawara nests, the chain nest $SO(n)/SO(n-1)/SO(n-2)/\ldots SO(3)/SO(2)$ with $S = S(G(SO(2))) = 2$ has the smallest possible symmetry factor.

In contrast, the *identity graphs* I are completely asymmetric with $S(I)$= dim(auto I)=1, and they are ubiquitous since the generic large-order graph is an identity graph [29]. It follows that

$$\text{the generic new construction in } SO(n \gg 1)^\#_{diag} \text{ is an identity graph} \tag{2.5.2a}$$

$$\text{the generic large-order identity graph is a new construction} \tag{2.5.2b}$$

since the generic large-order graph is also a new construction (see Section 2.3.7). It also follows that constructions with a symmetry are exceptional cases, including those new constructions with $S \geq 2$.

We have already encountered the first 8 non-trivial identity graphs, collected in Fig. 9, which are identified in Table 3 as new constructions in $SO(6)^\#_{diag}$. Moreover, the characterization

$$\text{all identity graphs are new constructions} \tag{2.5.3}$$

follows because the affine-Sugawara nested graphs always have a symmetry.

2.5.2 Connected incomplete bipartite graphs

Connected incomplete bipartite graphs are complete bipartite graphs with one or more lines removed such that the incomplete graph remains connected. Two examples of these graphs are given in Fig. 11a. The K-conjugate graph $\tilde{G}$ of a connected incomplete bipartite graph G is formed by connecting two affine-Sugawara graphs with one or more lines. It follows from the characterization (2.3.25) that

$$\text{connected incomplete bipartite graphs are new irreducible constructions} \tag{2.5.4}$$

since G and $\tilde{G}$ are both connected in this case.

Physically, the connected incomplete bipartite graphs are the *broken N=2 coset graphs* obtained by removing lines from the graphs of the fundamental N=2 cosets $SO(n)/(SO(p)\times SO(n-p))$. The example in Fig. 11b is identified in Table 3 as a new construction in $SO(6)_{diag}$.

An equivalent statement of the result (2.5.4),

$$\text{connected incomplete graphs with } \chi(G) = 2 \text{ are new irreducible constructions} \tag{2.5.5}$$

is obtained in terms of the *chromatic number* [1] $\chi(G)$ of a graph, since a graph is bipartite

[1] A coloring of a graph G is an assignment of a color to every point in G. The chromatic number $\chi(G)$ is the smallest number of colors such that no two points of the same color are connected by a line [30].

iff $\chi(G) = 2$. As examples, the cycle and path graphs

$$C_{2n} = \text{cycle of length } 2n, \quad n \geq 3$$
$$P_n = \text{path of length } n-1, \quad n \geq 4 \tag{2.5.6}$$

are new irreducible constructions, as illustrated with the colors r and w in Fig. 12. The cycle C_6 and the paths P_4, P_5 and P_6 are identified as new irreducible constructions in Table 3.

2.5.3 Broken N=2 affine-Sugawara nested graphs

In this section, we introduce the broken N=2 affine-Sugawara nested graphs, which generalize the broken N=2 coset graphs and which may provide a process which generates all new irreducible graphs from the graphs of the old constructions.

We define the *broken N=2 affine-Sugawara nested graphs* as the connected graphs, shown in Figs. 11 and 13, which are obtained by removing lacunae-connecting lines from the fundamental N=2 affine-Sugawara nested graphs. The K-conjugate graph $\tilde{G}$ of any broken N=2 nest G is also connected since at least one lacunae-connecting line has been removed from G. It follows from the characterization (2.3.25) that [‡]

broken N=2 affine-Sugawara nested graphs are new irreducible constructions. (2.5.7)

An example of this result is given in Fig. 13b, which is identified as a new construction in Table 3.

We have also compiled a list of all broken N=2 affine-Sugawara nests of order $n \leq 6$.

[‡]Broken affine-Sugawara nested graphs are not always new constructions when $N \geq 3$. For example, breaking all the lines between two lacunae of a complete tripartite graph (coset construction) gives a complete bipartite graph (coset construction).

Comparison of this list with the data of Table 3 supports the complementary conjecture,

conjecture 1: At order n, the set of broken N=2 affine-Sugawara nested

graphs contains *all* new irreducible constructions in the lower (2.5.8)

half $1 \leq c_0 \leq \left[\frac{1}{2}\binom{n}{2}\right]$ of the high-level sectors of $SO(n)_{diag}$.

An implication of this conjecture is that all new irreducible constructions in the upper half of the high-level sectors can be obtained by K-conjugation of the broken nests.

2.5.4 An edge theorem for $SO(n)^{\#}_{diag}$

It has been observed empirically for the new constructions (1.13a,b) that [22]

$$\text{rank}\, g < c_0 < \dim g - \text{rank}\, g$$
$$\text{when } g^{\#} \text{ is a new irreducible construction on compact } g \tag{2.5.9}$$

where c_0 is the high-level central charge of $g^{\#}$. The inequalities (2.5.9) are true in $SO(n)^{\#}_{diag}$ as well, since they follow with c_0=dim E from the (stronger) *edge theorem*

$$SO(n)^{\#}_{diag}: \quad n-1 \leq \dim E(G^{\#}_n(irr)) \leq \frac{1}{2}(n-1)(n-2) \tag{2.5.10}$$

where $G^{\#}_n(irr)$ is any new irreducible graph of order n. The proof of the edge theorem is as follows: We know from (2.3.25) that $G^{\#}_n(irr)$ and $\tilde{G}^{\#}_n(irr)$ are both connected, and, moreover, that at least $n-1$ lines are necessary to connect n points. It follows that c_0=dim $E(G^{\#}_n(irr))$ and $\tilde{c}_0$=dim $E(\tilde{G}^{\#}_n(irr))$ are both greater than or equal to $n-1$. The edge theorem (2.5.10) follows since $c_0 + \tilde{c}_0 = n(n-1)/2$ on $SO(n)$.

2.5.5 Self-K-conjugate constructions

An unlabelled graph G is *self-K-conjugate* (or self-complementary [29]) when $\tilde{G} = G$. At the level of labelled graphs, G and $\tilde{G}$ are isomorphic, and the corresponding constructions $L(G)$ and $L(\tilde{G}) = \tilde{L}(G)$ are $SO(n)$ automorphically equivalent, so that $c = \tilde{c} = c_g/2$ for

self-K-conjugate constructions. It follows that a) self-K-conjugate constructions exist only on $SO(4n)$ and $SO(4n+1)$, since $c_0 = \tilde{c}_0 = \dim g/2$ requires that $\dim g$ is even, and b) the *half-Sugawara central charges*

$$\begin{aligned} SO(4n): \quad & c = \frac{xn(4n-1)}{x+4n-2} \\ SO(4n+1): \quad & c = \frac{xn(4n+1)}{x+4n-1} \end{aligned} \tag{2.5.11}$$

are determined for self-K-conjugate constructions before obtaining the exact solutions.

The first six self-K-conjugate constructions are given in Fig. 14, and the first three of these were encountered as new irreducible constructions in $SO(4)^{\#}_{diag}$ and $SO(5)^{\#}_{diag}$. More generally, the number s_n of self-K-conjugate constructions in $SO(n)^{\#}_{diag}$

n	4	5	8	9	12	13	16	17
s_n	1	2	10	36	720	5600	703,760	11,220,000

(2.5.12)

and the asymptotic behavior of s_n

$$\begin{aligned} s_{4n} &= \frac{2^{2n^2-2n}}{n!}(1+O(n^2/2^{4n})) \\ s_{4n+1} &= \frac{2^{2n^2-n}}{n!}(1+O(n^2/2^{4n})) \end{aligned} \tag{2.5.13}$$

are known in graph theory [29].

Although all the self-K-conjugate constructions on a given manifold have the same central charge, each construction is physically distinct (not $SO(n)$ automorphically equivalent to any other), with distinct conformal weights, since each construction is a distinct unlabelled graph. Distinct high-level conformal weights on each manifold is easily verified for the graphs of Fig. 14, and is recorded explicitly in Table 3 for the two self-K-conjugate graphs on $SO(5)$.

The exact form of the first three self-K-conjugate constructions is obtained in Section 2.7. The constructions are generically unitary with generically irrational conformal weights, both of which are expected for generic self-K-conjugate constructions. In this circumstance,

it is possible to imagine that all the self-K-conjugate solutions in a given $SO(n)_{diag}$ are connected by a continuous c-fixed quadratic deformation which is a solution of the full master equation on $SO(n)$.

Except in special cases, we have been unable to construct the half-Sugawara central charges (2.5.11) by affine-Sugawara nesting on any compact g. These central charges, and the values c =13/10, 20/11 and 31/11 reported on $SU(3)$ [21,22], should be investigated carefully since the question of new rational central charges is conceptually important.

2.5.6 Cartesian product graphs

Cartesian product graphs [30] may be defined analytically with our original variables $\{\theta_{ij}, \theta_{ii} = 0\}$, where θ_{ij}=1 is a line from point i to point j. When $\{\theta_{i_1 j_1} = 1\}$ and $\{\theta_{i_2 j_2} = 1\}$ are the lines of two graphs G_{n_1} and G_{n_2}, then the Cartesian product graph $G_{n_1 n_2} = G_{n_1} \times G_{n_2}$ is defined on the product points $[i_1, i_2]$ or $[j_1, j_2]$, with lines

$$\theta_{[i_1,i_2];[j_1,j_2]} = \theta_{i_1 j_1}\delta_{i_2 j_2} + \theta_{i_2 j_2}\delta_{i_1 j_1} \,. \tag{2.5.14}$$

This operation is a direct construction of the high-level inertia tensor $L^{(0)}_{ij} = \theta_{ij}$ of the product graph in terms of the high-level inertia tensors of the component graphs. Pictorially, $G = G_1 \times G_2$ is constructed as shown in Fig. 15: Replace each point in (say) G_2 by copies $G_1', G_1'', \ldots$ of the graph G_1, and each line in G_2 by a set of lines which connect only copied points $i', i'', \ldots$ in the copies of G_1. Since the order $n_1 n_2$ of a product graph $G_{n_1 n_2}$ is multiplicative, it is clear that these graphs are a relatively small subset of all graphs.

It is our intuition that

conjecture 2: Cartesian product graphs are new constructions (except $K_2 \times K_2$ and $K_1 \times G$) (2.5.15)

since the product operation is foreign to the affine-Sugawara nesting operations. It suffices to verify conjecture 2 for products $G_1 \times G_2$ of two graphs, and in fact only for products of

two connected graphs since the identity

$$(G_1 \cup G_2) \times G_3 = (G_1 \times G_3) \cup (G_2 \times G_3) \tag{2.5.16}$$

implies that products of disconnected graphs are new when the products of their connected components are new.

With the characterization (2.3.25) we have checked that conjecture 2 is true for the product graphs through order 8. The conjecture is also true for the maximal-symmetric construction [18]

$$G(SO(2n)/SO(2n)^{\#}_{Kf}) = K_2 \times K_n \tag{2.5.17}$$

whose graphs, given in Fig. 16, were identified from the high-level behavior of the known solutions. More generally, the theorem

$$\text{(connected bipartite graph)} \times \text{(connected bipartite graph)}$$

$$= \text{new irreducible construction (except } K_2 \times K_2) \tag{2.5.18}$$

is established pictorially as follows. When G_1 and G_2 are connected graphs with chromatic number two, then $\chi(G_1 \times G_2)=2$ as well since the two-color scheme of G_1 can be consistently reversed for nearest neighbor copies of G_1 (see Fig. 15). These $\chi = 2$ product graphs are connected and, except for $K_2 \times K_2$, they are incomplete, so (2.5.18) follows from the theorem in (2.5.5). $K_2 \times K_2$ is the complete bipartite graph $SO(4)/(SO(2))^2$. An example of this theorem is the set of n-cubes $Q_n \equiv Q_{n-1} \times K_2$, $Q_1 \equiv K_2$ for $n \geq 3$, shown for $n = 3$ in Fig. 17.

2.6 Graph Symmetry and Consistent Subansätze

In this section, we discuss the *hierarchy of consistent subansätze* in $SO(n)_{diag}$, which may be determined in principle by studying the symmetry groups autoG_n of the graphs of order n.

The subansätze provide the names of the new constructions (see Table 3) and the strategy for exact solutions in the following section.

As a first step, we study the symmetry of the exact solution $L(G)$. Recall from Section 2.3.3 that the lines $\{\theta_{ij} = 1\}$ of G satisfy

$$\theta_{ij} = \theta_{\pi(i)\pi(j)} \tag{2.6.1}$$

when $\pi \in$ auto$G$ is a relabelling in the symmetry group of G. The result (2.6.1) is a high-level symmetry of the construction $L(G)$, expressed as a relation among the high-level components of its inertia tensor $L^{(0)}_{ij} = \theta_{ij}$. In fact, the high-level symmetry (2.6.1) persists to all orders in the high-level expansion, so that the same symmetry

$$L_{ij}(G) = L_{\pi(i)\pi(j)}(G), \quad \forall\, \pi \in \text{auto}G \tag{2.6.2}$$

is obtained for the exact solution $L(G)$. To see this, one needs the iterative lemma

$$L^{(p)}_{ij}(G) = L^{(p)}_{\pi(i)\pi(j)}(G) \quad \text{when} \quad L^{(q<p)}_{ij}(G) = L^{(q<p)}_{\pi(i)\pi(j)}(G). \tag{2.6.3}$$

which is not difficult to check from the recursion relation (2.2.33a). The statement previewed in Section 2.3.3

$$\text{symmetry group of } L(G) = \text{symmetry group of } G = \text{auto}G \tag{2.6.4}$$

follows immediately from the result (2.6.2).

The exact symmetry of $L(G)$ in (2.6.2) determines the smallest consistent subansatz in which the construction is found. As an exercise, we will determine the smallest subansätze of all the new irreducible constructions in $SO(5)^{\#}_{diag}$, whose graphs (up to K-conjugation) are given in Fig. 18: The two graphs of Fig. 18a have auto$G$=Z_2, the non-trivial element being a simultaneous $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ interchange. It follows from (2.6.2) that both graphs occur first in the six-parameter consistent $SO(5)$ subansatz

$$SO(5)[d,6]: \quad L_{12}, \quad L_{34}, \quad L_{13} = L_{24}, \quad L_{14} = L_{23}, \quad L_{15} = L_{25}, \quad L_{35} = L_{45}. \tag{2.6.5}$$

The two graphs of Fig. 18b occur first in the seven-parameter consistent subansatz

$$SO(5)[d,7]: \quad L_{12}, \quad L_{34}, \quad L_{45}, \quad L_{35}, \quad L_{13}=L_{23}, \quad L_{14}=L_{24}, \quad L_{15}=L_{25} \tag{2.6.6}$$

because auto $G=Z_2'$ with non-trivial element $\pi=(12)$. Finally, the self-K-conjugate graph of Fig. 18c occurs first in the two-parameter consistent subansatz

$$SO(5)[d,2]: \quad L_{12}=L_{13}=L_{24}=L_{35}=L_{45}, \quad L_{34}=L_{14}=L_{23}=L_{15}=L_{25} \tag{2.6.7}$$

with auto $G=D_5$ in this case. The hierarchy of subansätze (2.6.5-7) is complete for $SO(5)^{\#}_{diag}$ since each subansatz is K-conjugation covariant.

Note also that $SO(5)[d,2] \subset SO(5)[d,6]$, so the solutions of $SO(5)[d,2]$ will appear as a factorization sector [18,20,22] of higher symmetry in the larger subansatz $SO(5)[d,6]$. More generally, these factorizations are best studied in sum and difference variables, since, according to (2.6.2), the appearance of any smaller subansatz is characterized by the vanishing of a set of difference variables $L_{ij}-L_{\pi(i)\pi(j)}$.

We are now in a position to define our labelling scheme for new constructions, which was employed explicitly in Table 3. The general subansatz in $SO(n)_{diag}$ is named

$$SO(n)[d,s] \tag{2.6.8}$$

where d denotes the diagonal ansatz and s is the size of the subansatz. For example, Table 3 records that all new irreducible constructions in $SO(6)^{\#}_{diag}$ are contained in consistent subansätze of size

$$SO(6)^{\#}_{diag}: \quad s=3,4,5,6,7,7',8,9,9',11 \text{ and } 15 \tag{2.6.9}$$

where s=15 is $SO(6)_{diag}$ itself. Distinct subansätze of the same size are distinguished by primes. The new irreducible solutions are named in their smallest subansatz and numbered according to

$$SO(n)^{\#}[d,s]_i, \quad i=1,2,\ldots \tag{2.6.10}$$

when they fall in the lower half of the high-level sectors of the subansatz. This is a complete labelling of solutions in $SO(n)^{\#}_{diag}$, since the higher sectors are completed by $SO(n)/SO(n)^{\#}[d,s]_i$.

An obvious strategy for new exact solutions is to begin with the smaller subansätze, which contain the graphs of higher symmetry. This is a program which we begin in the following section, but which we cannot finish without further insight: The graphs of lower symmetry occur in larger subansätze, and, in particular, the ubiquitous totally-asymmetric identity graphs occur in no subansatz smaller than $SO(n)_{diag}$ itself.

2.7 Exact Solutions in $SO(n)^{\#}_{diag}$

2.7.1 $SO(2n)^{\#}_M \equiv SO(2n)^{\#}[d,3]$ and $SO(2n)^{\#}[d,4]$

The graphs of the maximal-symmetric construction on $SO(2n)$ [18], shown in Fig. 16, are the most symmetric new irreducible graphs in $SO(n)_{diag}$. The maximal-symmetric construction has auto $G=Z_2\times S_n$ when $n\geq 3$ (and $Z_2\times D_4$ at n=2) and appears first in the three-parameter subansatz

$$SO(2n)_M \equiv SO(2n)[d,3] \begin{cases} L_{ij}=L_{n+i,n+j}=L_h, & 1\leq i<j\leq n \\ L_{i,n+i}=L_c, & 1\leq i\leq n \\ L_{i,n+j}=L_t, & 1\leq i\neq j\leq n \end{cases} \tag{2.7.1}$$

which is the maximal-symmetric subansatz [18] in Cartesian coordinates [§]. The identification

$$SO(2n)^{\#}_M \equiv SO(2n)^{\#}[d,3] \tag{2.7.2}$$

follows for the maximal-symmetric construction in the present taxonomy.

The next most symmetric new irreducible graphs in $SO(2n)_{diag}$, shown in Fig. 19, have

[§] The Cartan-Weyl basis of Ref.[18] is $\lambda=L_c/(2n-2)$ and $L_\pm=(L_t\mp L_h)/2$.

auto $G = S_n$, $n \geq 2$. These constructions appear first in the four-parameter subansatz

$$SO(2n)[d,4] \quad \begin{cases} L_{ij} = L_h, \quad L_{n+i,n+j} = L'_h, \quad 1 \leq i < j \leq n \\ L_{i,n+i} = L_c, \quad 1 \leq i \leq n; \quad L_{i,n+j} = L_t, \quad 1 \leq i \neq j \leq n \end{cases} \tag{2.7.3}$$

which contains the maximal-symmetric subansatz (2.7.1) when $L_h = L'_h$. In sum and difference variables $L_c^{\pm} \equiv L_c \pm L_t$, $L_h^{\pm} \equiv L_h \pm L'_h$, the explicit form of $SO(2n)[d,4]$ is

$$L_c^-(1-(x+n-2)L_c^+ + (n-2)L_c^- - nL_h^+) = 0 \tag{2.7.4a}$$

$$L_c^+(2-(x+2n-2)L_c^+ + 2(n-2)L_c^-) - (x-2)(L_c^-)^2 - 2(n-2)L_c^-L_h^+ = 0 \tag{2.7.4b}$$

$$L_h^-(1-nL_c^+ + (n-2)L_c^- - (x+n-2)L_h^+) = 0 \tag{2.7.4c}$$

$$L_h^+(2-(x+n-2)L_h^+ + 2(n-2)L_c^- - 2nL_c^+) + (n-4)(L_c^-)^2$$
$$+n(L_c^+)^2 - 2(n-2)L_c^-L_c^+ - (x+n-2)(L_h^-)^2 = 0 \tag{2.7.4d}$$

$$c = \frac{xn}{2}(nL_c^+ - (n-2)L_c^- + (n-1)L_h^+) \tag{2.7.4e}$$

which shows the maximal-symmetric subansatz as the factorization sector $L_h^-=0$ in (2.7.4c).

The subansatz (2.7.4) contains 12 known solutions and the following four new solutions

$$L_c^+ = L_h^+ = \frac{1}{x+2n-2}(1+\eta(n-2)R) \tag{2.7.5a}$$

$$L_c^- = \eta R, \quad L_h^- = \eta\sigma R\sqrt{\frac{x+n-6}{x+n-2}} \tag{2.7.5b}$$

$$c = \frac{nx}{2(x+2n-2)}(2n-1-\eta(n-2)(x-1)R) \tag{2.7.5c}$$

$$R \equiv (x^2+2(n-2)x+n^2-8n+8)^{-1/2} \tag{2.7.5d}$$

which are labelled by $\eta = \pm 1$, $\sigma = \pm 1$. The values of η correspond to K-conjugation and the values of σ are $SO(2n)$ automorphically equivalent. For either value of σ, these solutions are identified as

$$\begin{aligned} SO(2n)^{\#}[d,4]: &\quad \eta = +1 \quad (c_0 = n(n+1)/2) \\ SO(2n)/SO(2n)^{\#}[d,4]: &\quad \eta = -1 \quad (c_0 = 3n(n-1)/2) \end{aligned} \tag{2.7.6}$$

by matching the high-level form of the central charge in (2.7.5c) to the number of lines in the graphs of Fig. 19. This family of solutions includes the self-K-conjugate construction [¶] $SO(4)^{\#}[d,4]$ and the K-conjugate pair of constructions $SO(6)^{\#}[d,4]$ and $SO(6)/SO(6)^{\#}[d,4]$, all of which appear in Table 3.

For $x \in N$, these constructions are unitary down to very low level, as expected: A complete list of nonunitary points is $x = 1,2,3$ for $SO(4)$, $x = 1,2$ for $SO(6)$ and $x = 1$ for $SO(8)$ and $SO(10)$. The constructions are generically irrational, with rational subconstructions only for $n = 2$ and levels 1 and 2 of any n [‖].

A special point is the level 3 construction $SO(6)_3^{\#}[d,4]$ which is identical to the maximal-symmetric construction $(SO(6)_3^{\#})_M$ at this level. This phenomenon is an isolated equivalence [*] or accidental crossing of solutions at particular finite levels, since the graphs of the constructions are distinct. The value $c(SO(6)_3^{\#}[d,4]) = c((SO(6)_3^{\#})_M) \simeq 2.9597$ is also the lowest unitary irrational central charge in this family of constructions.

2.7.2 The subansatz SO(2n + 1)[d, 6]

The most symmetric new irreducible graphs in $SO(2n+1)_{diag}$, $n \geq 2$ are the four graph families shown in Fig. 20. All these constructions reside in the six-parameter subansatz

$$SO(2n+1)[d,6] \quad \begin{cases} L_{ij} = L_h, \quad L_{n+i,n+j} = L'_h, \quad 1 \leq i < j \leq n \\ L_{i,n+i} = L_c, \quad 1 \leq i \leq n; \quad L_{i,n+j} = L_t, \quad 1 \leq i \neq j \leq n \\ L_{i,2n+1} = L_r, \quad L_{n+i,2n+1} = L'_r, \quad 1 \leq i \leq n \end{cases} \tag{2.7.7}$$

[¶] The self-K-conjugate construction $SO(4)^{\#}[d,4]$, whose graph is the first in Fig. 19, has the expected central charge $3x/(x+2)$ of $SU(2)$ and irrational conformal weights. Despite the coincidence of central charges, this construction is not a point in the quadratic deformation $(SU(2) \times SU(2))^{\#}$ [18]. $SO(4)^{\#}[d,4]$ should be compared to points in known linear deformations of $SU(2)_x$ [12].

[‖] The central charges at levels 1 and 2 are half integer ($\geq 3/2$) and integer, with irrational conformal weights, so these levels should be compared to particular points of known deformations.

[*] Some of these equivalences are well known. For example, the Sugawara construction $SO(2n)_1$ at level 1 is equivalent to the construction on the maximal torus of $SO(2n)$, although the graphs of these constructions are distinct. The equivalence phenomenon also occurs in irrational constructions at rational points which are affine-Sugawara nests.

with auto $G = S_n$ when $n \geq 3$ [1]. The form of $SO(2n+1)[d,6]$ in sum and difference variables is

$$L_c^-(1-(x+n-2)L_c^+ + (n-2)L_c^- - nL_h^+ - L_r^+) = 0$$

$$L_c^+(2-(x+2n-2)L_c^+ + 2(n-2)L_c^- - 2L_r^+) - (x-2)(L_c^-)^2$$

$$-2(n-2)L_c^-L_h^+ - (L_r^-)^2 + (L_r^+)^2 = 0$$

$$L_h^-(1-nL_c^+ + (n-2)L_c^- - (x+n-2)L_h^+ - L_r^+) + L_r^-(L_r^+ - L_h^+) = 0$$

$$L_h^+(2-(x+n-2)L_h^+ + 2(n-2)L_c^- - 2nL_c^+ - 2L_r^+) + (n-4)(L_c^-)^2$$

$$+n(L_c^+)^2 - 2(n-2)L_c^-L_c^+ - (x+n-2)(L_h^-)^2 + (L_r^+)^2 + (L_r^-)^2 - 2L_h^-L_r^- = 0$$

$$L_r^-(1-nL_c^+ + (n-2)L_c^- - (x+n-1)L_r^+) = 0$$

$$L_r^+(2-(x+2n-1)L_r^+) - (x-1)(L_r^-)^2 = 0$$

$$c = \frac{xn}{2}(nL_c^+ - (n-2)L_c^- + (n-1)L_h^+ + 2L_r^+) \qquad (2.7.8)$$

where $L_c^\pm, L_h^\pm$ are defined in the previous section and $L_r^\pm \equiv L_r \pm L_r'$. This subansatz contains $SO(2n)[d,4]$ when $L_r^- = L_r^+ = 0$, and it also contains the subansatz $SO(5)[d,2]$ when $L_h^+ = L_c^+ = L_r^+$, $L_h^- = L_c^- = -L_r^-$ at $n=2$.

The exact constructions in Sections 2.7.3-5 are solutions of the system (2.7.8). Before the details, here is an overview of the situation.

$SO(2n+1)[d,6]$ contains 48 solutions which are known or were obtained in the previous section, and 64 solutions generically. The generic count of 16 new solutions is accurate except at level 2, where the only new solutions, given in Appendix 2.B, are the new quadratic deformations $SO(2n+1)_2^\#[d,6]$ with $c = n$.

[1] The first column of Fig. 20 shows that the case of $SO(5)$ is special: The graphs at the bottom of the first two graph families are the self-K-conjugate constructions on $SO(5)$, while the third and fourth graph families coincide at order 5. Moreover, the pentagon graph $SO(5)^\#[d,2]$ has the higher symmetry auto $G = D_5$, and occurs first in $SO(5)[d,2]$.

Among the 16 new solutions for $x \neq 2$, we will obtain the following 8 solutions across all n

$$SO(5)^\#[d,6]_1 \text{ and } SO(5)^\#[d,2] \quad \text{(4 copies each)} \qquad (2.7.9a)$$

$$SO(2n+1)^\#[d,6]_{1,2} \text{ and } \frac{SO(2n+1)}{SO(2n+1)^\#[d,6]_{1,2}} \quad (n \geq 3,\ \text{2 copies each}). \qquad (2.7.9b)$$

These constructions are the first two graph families in Fig. 20. We obtain the remaining 8 solutions only for $n = 2$

$$SO(5)^\#[d,6]_2 \text{ and } \frac{SO(5)}{SO(5)^\#[d,6]_2} \quad \text{(4 copies each)}. \qquad (2.7.10)$$

which are the lowest graphs of the third and fourth graph family in Fig. 20.

2.7.3 The self-K-conjugate constructions on SO(5)

The 8 solutions in eq.(2.7.9a) are the self-K-conjugate constructions

$$SO(5)^\#[d,6]_1 \quad \begin{cases} L_c^+ = L_h^+ = L_r^+ = \dfrac{1}{x+3}, & x \neq 2 \\ L_c^- = \sigma L_r^- = \dfrac{\sigma x}{x-2} L_h^- = \dfrac{\eta}{\sqrt{(x-1)(x+3)}}, \end{cases} \qquad (2.7.11a)$$

$$SO(5)^\#[d,2] \quad \begin{cases} L_c^+ = L_h^+ = L_r^+ = \dfrac{1}{x+3}, & x \neq 2 \\ L_c^- = \sigma L_r^- = -\sigma L_h^- = \dfrac{\eta}{\sqrt{(x-1)(x+3)}}, \end{cases} \qquad (2.7.11b)$$

where $\sigma = \pm 1$ are $SO(5)$ automorphically equivalent and $\eta = \pm 1$ is K-conjugation, which is also an $SO(5)$ automorphism in these cases. Both constructions have the half-Sugawara central charge

$$c = \frac{5x}{x+3}, \quad x \neq 2 \qquad (2.7.12)$$

which is characteristic of self-K-conjugate constructions (see Section 2.5.5).

The self-K-conjugate constructions are unitary for integer $x \geq 3$. The conformal weights of the vector representation are irrational for $SO(5)^\#[d,6]$ and rational for $SO(5)^\#[d,2]$, but irrational conformal weights must be expected for higher representations in both cases.

2.7.4 $SO(2n+1)^{\#}[d,6]_{1,2}$, $n \geq 3$

The 8 solutions in eq.(2.7.9b) [‡]

$$
\begin{aligned}
L_c^+ = L_h^+ = L_r^+ &= \frac{1}{x+2n-1}(1+\eta(n-2)S) \\
L_c^- = \eta S, \quad L_h^- &= -\eta\sigma\frac{(1-\epsilon(x+n-3))S}{x+n-2}, \quad L_r^- = \eta\sigma S \\
c &= \frac{nx}{2(x+2n-1)}(2n+1-\eta(n-2)(x-2)S) \\
S &\equiv (x^2+2(n-1)x+n^2-6n+5)^{-1/2}, \quad x \neq 2
\end{aligned}
\tag{2.7.13}
$$

are distinguished by $\eta = \pm 1$, $\sigma = \pm 1$ and $\epsilon = \pm 1$. The values of η correspond to K-conjugation, the values of σ are $SO(2n+1)$ automorphically equivalent, and the values of ϵ label physically distinct solutions with the same central charge but different conformal weights.

For $n \geq 3$, and either value of σ, these solutions are identified as

$$
\begin{aligned}
SO(2n+1)^{\#}[d,6]_1 &: \quad \epsilon = \eta = +1 \quad (c_0 = n(n+3)/2) \\
SO(2n+1)/SO(2n+1)^{\#}[d,6]_1 &: \quad \epsilon = -\eta = +1 \quad (c_0 = n(3n-1)/2) \\
SO(2n+1)^{\#}[d,6]_2 &: \quad \epsilon = -\eta = -1 \quad (c_0 = n(n+3)/2) \\
SO(2n+1)/SO(2n+1)^{\#}[d,6]_2 &: \quad \epsilon = \eta = -1 \quad (c_0 = n(3n-1)/2).
\end{aligned}
\tag{2.7.14}
$$

In this case, the identification requires matching $L_{ij}^{(0)}(G) = \theta_{ij}$ for the appropriate graphs against the high level form of the solutions (2.7.13).

These constructions are unitary for all positive integer level (except $x = 2$). They are also generically irrational, with rational subconstructions only at level 1 [§]. The value

$$
c\left((SO(7)_3^{\#})[d,6]_{1,2}\right) = \frac{9}{16}\left(7 - \frac{1}{\sqrt{17}}\right) \simeq 3.8011 \tag{2.7.15}
$$

[‡] The self-K-conjugate constructions on $SO(5)$ are included in the solutions (2.7.13) when $n = 2$. The identification is $SO(5)^{\#}[d,6]_1$ when $\epsilon = 1$ and $SO(5)^{\#}[d,2]$ when $\epsilon = -1$.

[§] The central charges at level one are half integer ($\geq 3/2$) and integer, with irrational conformal weights, so they should be compared to particular points of known deformations.

is the lowest unitary irrational central charge in this family.

2.7.5 $SO(5)^{\#}[d,6]_2$

The 8 solutions on $SO(5)$ in eq.(2.7.10)

$$
\begin{aligned}
L_c^+ &= \frac{1}{x+3}(1+2\eta(x+1)Q), \quad L_h^+ = \frac{1}{x+3}(1-\eta(x+2)(x-1)Q) \\
L_r^+ &= \frac{1}{x+3}(1-4\eta Q), \quad L_c^- = \frac{-\eta\sigma Q}{x-1}\sqrt{(x^2+2)(x^2-4x-2)} \\
L_h^- &= \frac{-L_r^-}{x} = \frac{-\eta\epsilon Q}{x(x-1)}\sqrt{x(x-4)(x^2+4)} \\
c &= \frac{x}{x+3}(5-\eta(x-1)(x-2)Q) \\
Q &\equiv \sqrt{\frac{x-1}{x^5-x^4-8x^3-4x^2-32x-16}}, \quad x \neq 2
\end{aligned}
\tag{2.7.16}
$$

are distinguished by $\eta = \pm 1$, $\sigma = \pm 1$ and $\epsilon = \pm 1$, where η is K-conjugation, and σ, ϵ label $SO(5)$ automorphically equivalent solutions. By comparison of graphs and solutions, the identification

$$
\begin{aligned}
SO(5)^{\#}[d,6]_2 &: \quad \eta = 1 \quad (c_0 = 4) \\
SO(5)/SO(5)^{\#}[d,6]_2 &: \quad \eta = -1 \quad (c_0 = 6)
\end{aligned}
\tag{2.7.17}
$$

is established for each fixed choice of σ and ϵ.

These constructions are unitary and irrational for all integer $x \geq 5$. The value

$$
c\left((SO(5)_5^{\#})[d,6]_2\right) = \frac{25}{8}\left(1 - \frac{4}{5\sqrt{34}}\right) \simeq 2.6963 \tag{2.7.18}
$$

is the lowest unitary central charge of the family, and this value is also the lowest unitary central charge yet observed on non-simply-laced g.

2.8 The novelty number ν

We have constructed a graph function $\nu(G)$ which we call the *novelty number* of G because it appears to distinguish between the graphs $G(AS)$ of the known rational constructions

and the graphs $G^{\#}$ of the new constructions,

$$\text{conjecture 3:} \qquad \nu(G(AS)) = 0, \qquad \nu(G^{\#}) > 0. \tag{2.8.1}$$

The novelty number is

$$\nu \equiv -\frac{1}{2}\sum_i d_i(G)(2d_i(G)-1) + 9t_3 - 4t_4 + 6t_4' - 12t_4'' + \sum_{(ij)} d_i(G)d_j(G) - 2\sum_{(ijk)} d_i(G) \tag{2.8.2}$$

where

$$\begin{aligned} t_3 &= \text{number of triangles in } G, \quad t_4 = \text{number of squares in } G \\ t_4' &= \text{number of squares with one diagonal in } G \\ t_4'' &= \text{number of } K_4\text{-subgraphs in } G \end{aligned} \tag{2.8.3}$$

and the sums in (2.8.2) are over the lines (ij) of G and the points (ijk) of each triangle in G.

It is sufficient to verify conjecture 3 on connected graphs, since the novelty number is additive $\nu(G_1 \cup G_2) = \nu(G_1) + \nu(G_2)$ on disconnected graphs. The conjecture has been verified for

1. the graphs in the first ten sectors of Table 3,
2. the affine-Sugawara construction K_n and the graphs $\tilde{K}_p + \tilde{K}_{n-p}$ of the fundamental coset constructions $SO(n)/(SO(p) \times SO(n-p))$,
3. the cycle graphs C_{2n} with

$$\nu(C_4) = 0; \quad \nu(C_{2n}) = 2n, \ n \geq 3 \tag{2.8.4}$$

   where C_4 is the graph of $SO(4)/(SO(2) \times SO(2))$,

4. the path graphs P_n with

$$\nu(P_2) = 0; \quad \nu(P_n) = n-3, \ n \geq 3 \tag{2.8.5}$$

   where P_2 and P_3 are the graphs of $SO(2)$ and $SO(3)/SO(2)$,

5. the exact constructions of Ref.[18] and Section 2.7 with

$$\begin{aligned} \nu(SO(2n)/SO(2n)^{\#}_M) &= n(n-1)(n-2) \\ \nu(SO(2n)^{\#}[d,4]) &= n(n-1)/2 \\ \nu(SO(5)^{\#}[d,2]) &= 5 \\ \nu(SO(5)^{\#}[d,6]_1) &= 1 \\ \nu(SO(2n+1)^{\#}[d,6]_1) &= 5n(n-1)/2, \quad n \geq 3 \\ \nu(SO(2n+1)^{\#}[d,6]_2) &= n(n-1)/2, \quad n \geq 3 \\ \nu(SO(5)^{\#}[d,6]_2) &= 2 \end{aligned} \tag{2.8.6}$$

In cases 3, 4 and 5, we emphasize that the novelty number vanishes precisely for those low n members of the graph family which are affine-Sugawara nested graphs.

Appendix 2.A: Counting affine-Sugawara nested graphs

We first obtain the recursion relation (2.3.27) for $C(AS)_n$, the number of connected (fundamental) affine-Sugawara nested graphs. The basic idea is to start from the relation

$$C(AS)_n = D(AS)_n \tag{2.A.1}$$

and express the number of disconnected graphs in terms of $C(AS)_{m<n}$. It is convenient to divide the disconnected graphs into two types I and II,

$$D(AS)_n = D_I(AS)_n + D_{II}(AS)_n \tag{2.A.2a}$$

$$D_I(AS)_n \equiv \left\{ \begin{array}{c} \text{number of disconnected affine-Sugawara nested graphs} \\ \text{of order } n \text{ with at least one trivial subgraph} \end{array} \right. \tag{2.A.2b}$$

$$D_{II}(AS)_n \equiv \left\{ \begin{array}{c} \text{number of disconnected affine-Sugawara nested graphs} \\ \text{of order } n \text{ with no trivial subgraphs} \end{array} \right. \tag{2.A.2c}$$

which are counted separately below [¶].

I. The disconnected graphs of type *I* have the form $K_1 \cup$ (general affine-Sugawara nested graph of order $n-1$), so that

$$D_I(AS)_n = 2C(AS)_{n-1} \tag{2.A.3}$$

follows with eq.(2.A.1).

II. The disconnected graphs of type *II* may be written as

$$\left(\frac{SO(2)}{\vdots}\right)^{p(2)} \times \left(\frac{SO(3)}{\vdots}\right)^{p(3)} \times \ldots\ldots \times \left(\frac{SO(n-2)}{\vdots}\right)^{p(n-2)} \qquad \sum_{i=2}^{n-2} ip(i) = n, \quad p(i) \geq 0 \tag{2.A.4}$$

where the vertical dots indicate arbitrary nesting in each product group $SO(i)$. We emphasize that each of the $p(i)$ factors $(SO(i)/\ldots)$ in the superfactor $(SO(i)/\ldots)^{p(i)}$ is *identical* in unlabelled graph theory.

To count these graphs, we first establish that

$$c(AS)_i^{(p(i))} = \binom{p(i) + c(AS)_i - 1}{p(i)} \tag{2.A.5}$$

is the number of distinct affine-Sugawara nests in each superfactor $(SO(i)/\ldots)^{p(i)}$, where $C(AS)_i$ is the number of fundamental affine-Sugawara nests in each identical factor $SO(i)$. To understand (2.A.5), consider first the example of $(SO(3))^3$. There are two fundamental nests $SO(3)$ and $SO(3)/SO(2)$ in each identical $SO(3)$. The

[¶] Examples of type *I* on $SO(6)$ are $SO(5) \times SO(1) = SO(5)$ and $SO(5)/(SO(4) \times SO(1)) = SO(5)/SO(4)$, while $SO(2) \times SO(4)$ is type *II* because it uses all six points.

distinct nests in $(SO(3))^3$ are

$$\begin{array}{rcl|c} & & SO(3) & \frac{SO(3)}{SO(2)} \\ & & \text{----} & \text{----} \\ SO(3) \times SO(3) \times SO(3) & = & *** & \\ SO(3) \times SO(3) \times \frac{SO(3)}{SO(2)} & = & ** & * \\ SO(3) \times \frac{SO(3)}{SO(2)} \times \frac{SO(3)}{SO(2)} & = & * & ** \\ \frac{SO(3)}{SO(2)} \times \frac{SO(3)}{SO(2)} \times \frac{SO(3)}{SO(2)} & = & & *** \end{array} \tag{2.A.6}$$

where the right-hand side phrases the example as the placement of $p(i) = 3$ identical objects $*$ in $C(AS)_i = 2$ boxes. The result is $C(AS)_i^{(3)} = \binom{3+2-1}{3} = 4$ distinct nests in the superfactor $(SO(3))^3$.

More generally, $C(AS)_i^{(p(i))}$ is computable as the number of ways to place $p(i)$ identical objects in $C(AS)_i$ boxes, which is also the number of ways to partition $p(i)$ identical objects with $C(AS)_i - 1$ walls (the dotted line in eq.(2.A.6)). Equivalently, the result (2.A.5) is the number of ways to place $p(i)$ identical objects on a total number $p(i) + C(AS)_i - 1$ of available sites = objects plus walls.

The total number of nests at fixed $\{p(i)\}$ is a product over the nests of each superfactor, and the result for type *II* nests

$$D_{II}(AS)_n = \sum_{\{p(i)\}} \prod_{\substack{i=2 \\ p(i)\neq 0}}^{n-2} \binom{p(i) + C(AS)_i - 1}{p(i)} \tag{2.A.7}$$

is obtained by summing over all partitions $\{p(i)\}$.

Having computed $D_I(AS)_n$ and $D_{II}(AS)_n$ in terms of $C(AS)_{m<n}$, the recursion relation for $C(AS)_n$, given in eq.(2.3.27) of the text, follows with eqs.(2.A.1), (2.A.2), (2.A.3) and (2.A.7).

We remark in passing that the same argument on the set of *all* disconnected graphs gives the following relation

$$C_n = g_n - D_n = g_n - (D_{I,n} + D_{II,n})$$
$$= g_n - g_{n-1} - \sum_{\{p(i)\}} \prod_{\substack{i=2 \\ p(i)\neq 0}}^{n-2} \binom{p(i)+C_i-1}{p(i)} \tag{2.A.8}$$

which may be used to compute C_n from $C_{m<n}$ and g_n. C_n is computed from g_n by a different route in Ref.[29].

We finally establish an upper bound on $C(AS)_n$ as follows. Any disconnected graph of order n=odd may be decomposed as

$$\text{(connected graph of order } 1 \leq i \leq (n-1)/2) \cup \text{(graph of order } n-i) \tag{2.A.9a}$$

for one or more values of i. Similarly, any disconnected graph of even order may be decomposed either as

$$\text{(connected graph of order } 1 \leq i \leq (n-2)/2) \cup \text{(graph of order } n-i) \tag{2.A.9b}$$

for one or more values of i, or as

$$\text{(connected graph of order } n/2) \cup \text{(connected graph of order } n/2). \tag{2.A.9c}$$

Then, the upper bound on all disconnected graphs

$$D_n \leq \begin{cases} \sum_{i=1}^{(n-1)/2} C_i g_{n-i} & n = \text{odd} \\ \sum_{i=1}^{(n-2)/2} C_i g_{n-i} + (C_{n/2})^2 & n = \text{even} \end{cases} \tag{2.A.10}$$

follows because the decompositions (2.A.9) are not unique.

The bound (2.A.10) may be restricted to affine-Sugawara nested graphs, so that the simple upper bound

$$C(AS)_n = D(AS)_n \leq \sum_{i=1}^{n-1} C(AS)_i C(AS)_{n-i} \tag{2.A.11}$$

is obtained with $D(AS)_n = C(AS)_n$, $g(AS)_{n-i} = 2C(AS)_{n-i}$ and symmetry about $i = n/2$. The right-hand side of (2.A.11) is an upper bound on the right-hand side of the recursion relation (2.3.27) for $C(AS)_n$, so the solution $C(AS)_n^{(\max)}$ of the simple recursion relation

$$C(AS)_n^{(\max)} = \sum_{i=1}^{n-1} C(AS)_i^{(\max)} C(AS)_{n-i}^{(\max)}, \quad C(AS)_1^{(\max)} = 1 \tag{2.A.12}$$

satisfies $C(AS)_n \leq C(AS)_n^{(\max)}$. The solution of (2.A.12) is

$$C(AS)_n^{(\max)} = \frac{1}{2(2n-1)}\binom{2n}{n}, \quad C(AS)_n \leq \frac{1}{2(2n-1)}\binom{2n}{n} \tag{2.A.13}$$

which implies the asymptotic bound in eq.(2.3.29b) of the text.

Appendix 2.B: The deformations $SO(2n+1)_2^{\#}[d,6]$, $n \geq 2$

The only new solutions at level 2 of $SO(2n+1)[d,6]$ are the two-parameter quadratic deformations which we call $SO(2n+1)_2^{\#}[d,6]$,

$$L_c^+ = \frac{1}{n}(1-(n+1)L_r^+ + (n-2)L_c^-), \quad L_h^+ = L_r^+$$
$$L_r^- = \eta R, \quad L_h^- = \frac{\eta}{n}[-R + \epsilon\sqrt{1-4(L_c^-)^2}] \tag{2.B.1}$$
$$R \equiv \sqrt{L_r^+(2-(2n+1)L_r^+)}$$
$$c = n\,.$$

The deformations are labelled by $\epsilon = \pm 1$, $\eta = \pm 1$ and arbitrary values of the deformation parameters L_r^+ and L_c^-. The values of ϵ label two distinct deformations within the construction. The unitary range of the deformation parameters

$$0 \leq L_r^+ \leq \frac{2}{2n+1}, \quad -\frac{1}{2} \leq L_c^- \leq \frac{1}{2} \tag{2.B.2}$$

defines a rectangle in parameter space. At fixed ϵ, the construction is closed under K-conjugation on $SO(2n+1)$

$$\tilde{L}_\eta^{\#}(L_r^+, L_c^-) = L_{-\eta}^{\#}(\frac{2}{2n+1} - L_r^+, -L_c^-) \tag{2.B.3}$$

which shows that K-conjugation in this case is $\eta \to -\eta$ plus a reflection about the center of the unitary rectangle.

The conformal weights of $SO(2n+1)_2^{\#}[d,6]$ are continuous functions of the deformation parameters, and we have checked that the construction is not equivalent to any known quadratic deformation. The physical content of the construction should, as usual, be compared to known linear deformations [12].

Appendix 2.C: Decomposition of a graph into irreducible graphs

In Sections 2.3.5 we noted that the set of all graphs is equal to the set of affine-Virasoro nested graphs. This set consists of the fundamental affine-Virasoro nested graphs and their K-conjugates. Moreover, the decomposition of a given graph as an affine-Virasoro nested graph with irreducible graphs at the bottom of each nest, is unique. As pointed out in Section 2.3.6, this provides a reorganization of graph theory. In this appendix we will describe a procedure for finding the nesting structure of a given graph as well as its irreducible graphs at the bottom.

The procedure for a given graph G is as follows

1 If G is connected go to step 2a; If G is disconnected go to step 2b

2a If $G = K_n$, the complete graph for some n then stop; otherwise compute the K-conjugate graph $\tilde{G}$. If $\tilde{G}$ is connected then G (and $\tilde{G}$) is irreducible and stop; If $\tilde{G}$ is disconnected repeat step 2a for each connected component of G.

2b Go to step 2a for each connected component of G.

By following through the steps described above, one can actually unravel the nesting structure of any graph, and ultimately find the irreducible graphs that are the "basic building blocks" of the graph. More precisely, in this way every graph can be written as a unions and/or joins of irreducible graphs.

Chapter 3

Magic Bases, Metric Ansätze and Generalized Graph Theories

3.1 Introduction

In this chapter, we generalize the graph-theory ansatz $SO(n)_{diag}$ to a family of metric ansätze

$$g_{metric}: \quad L^{ab} = \frac{1}{2}(\lambda_a + \lambda_b)\eta_{ab} \tag{3.1.1}$$

which includes $SO(n)_{diag}$ as a special case with diagonal Killing metric η_{ab} in the Cartesian basis of $SO(n)$. More generally, the metric ansätze are associated to a class of "magic" bases in which any two generators of Lie g commute to no more than a single generator. When a magic basis is real on compact g, the corresponding ansatz g_{metric} is a large system of unitary, generically irrational conformal field theories, whose structure is visible in the high-level expansion.

Moreover, each magic basis of g defines a generalized graph theory on g which, as in $SO(n)_{diag}$, labels the level-families of g_{metric}. The graph isomorphisms of the generalized graphs are permutations in $Aut\, g$, just as the isomorphisms of graph theory live in $Aut\, SO(n)$ - so that isomorphically-equivalent generalized graphs are physically-equivalent level-families of conformal field theories in g_{metric}.

We focus in this chapter on the only other known example of a magic basis, beyond the Cartesian basis of $SO(n)$, which gives $SO(n)_{diag}$ (see Chapter 2), namely the Pauli-like basis of $SU(n)$ [31], which gives the ansatz

$$SU(n)_{metric}: \quad \text{the metric ansatz on } SU(n) \tag{3.1.2}$$

Because the magic basis is real, the set of conformal field theories in the ansatz is generically unitary. Moreover, high-level analysis shows that this set is also generically irrational. In particular, the high-level expansion of $SU(n)_{metric}$ shows a new phenomenon: Due to the trigonometric structure constants of the Pauli-like basis, irrational central charge is clearly visible at finite order of the expansion.

We also obtain a number of exact unitary irrational constructions in $SU(n)_{metric}$

$$SU(4)^{\#}[m,7]_{1,2} \tag{3.1.3a}$$

$$SU(5)^{\#}[m,2] \tag{3.1.3b}$$

and, in particular, the construction in (3.1.3b) shows irrational central charge at finite order of the high-level expansion. Further exact constructions

$$SU(3)^{\#}[m,2] \tag{3.1.4a}$$

$$SU(4)^{\#}[m,7]_{3,4} \tag{3.1.4b}$$

have rational central charge but generically irrational conformal weights, and (3.1.4a) is in fact a self K-conjugate construction (see Sect. 2.5.5) on $SU(3)$.

Moreover, we discuss in some detail the "sine-area graphs" of $SU(n)$, with graph isomorphisms $\in Aut\, SU(n)$, which label the conformal field theories of $SU(n)_{metric}$. Finally, we show that given a magic basis of g one can define, in similar fashion, a generalized graph theory on g, whose generalized graphs label the conformal field theories of the corresponding ansatz g_{metric}.

3.2 Magic Bases

We define a magic basis of Lie g as a basis in which the metric is an involutive automorphisms of g

$$\eta_{ab} = \eta^{ab}, \quad f_{abc} = f^{abc} \tag{3.2.1a}$$

and where the structure constants of g satisfy

$$f_{cd}{}^{a} f_{cd}{}^{b} = (f_{cd}{}^{a})^2 \delta^{ab}, \quad \forall\, ab, c, d \tag{3.2.1b}$$

so that two generators of g commute to no more than a single other generator. A useful identity that follows from (2.2.4) and (3.2.1b) is

$$\psi^{-2} \sum_{cd} (f_{cd}{}^{a})^2 = \tilde{h}_g. \tag{3.2.2}$$

A subset of magic bases is the set of real magic basis which satisfy the stronger conditions

$$\eta_{ab} = \text{real, invertible}, \quad f_{ab}{}^{c} = \text{real} \tag{3.2.3b}$$

$$T_a^{\dagger} = \sum_b \eta_{ab} T_b \tag{3.2.3b}$$

$$f_{cd}{}^{a} f_{cd}{}^{b} = (f_{cd}{}^{a})^2 \delta^{ab}, \quad \forall\, a, b, c, d \tag{3.2.3c}$$

where T_a is a matrix representation of g and dagger is matrix adjoint. The conditions (3.2.3) are sufficient to guarantee those in (3.2.1).

Known examples of real magic bases include the physicist's standard basis choice for $SO(n)$, given in eq.(2.2.17), and the Pauli-like basis of $SU(n)$ which will be given in Section 3.5.

3.3 Metric Ansätze

For any magic basis of g, the metric ansatz (2.2.15) * for the inverse inertia tensor

$$L^{ab} = \frac{L_a}{\psi^2} \eta^{ab} \;\; ; \;\; (L_a - L_b)\eta^{ab} = 0 \tag{3.3.1a}$$

*The two forms (2.2.15) and (3.3.1a) are equivalent, although (3.3.1a) is preferred for computations.

$$T = \frac{1}{\psi^2} \sum_{ab} L_a \eta^{ab} :J_a J_b: \tag{3.3.1b}$$

is a consistent ansatz of the master equation, where the linear constraints in (3.3.1a) require $L_a = L_b$ when $\eta_{ab} \neq 0$, maintaining the symmetry of L^{ab}. We will refer to this class of ansätze as

$$\{g_{metric}\} = \{SO(n)_{diag}, SU(n)_{metric}, \cdots\} . \tag{3.3.2}$$

The reduced master equation in the ansatz (3.3.1)

$$L_a(1 - xL_a) = \sum_{cd} L_c(2L_a - L_d)(f_{cd}{}^a)^2/\psi^2 \tag{3.3.3a}$$

$$(L_a - L_b)\eta^{ab} = 0 \tag{3.3.3b}$$

$$c = x \sum_a L_a \tag{3.3.3c}$$

follows from the master equation (2.2.3), using eqs.(3.2.1) and (3.3.1). Consistency of the quadratic equations (3.3.3a) with the constraints (3.3.3b) is trivially verified for $SO(n)_{diag}$ since the metric is diagonal. More generally, consistency follows from the identity

$$\sum_{cd} L_c(2L_a - L_d)(f_{cd}{}^a)^2\eta_{ab} = \sum_{cd} L_c(2L_b - L_d)(f_{cd}{}^b)^2\eta_{ab} \tag{3.3.4}$$

which may be derived from (3.2.1) and (3.3.3b). It follows that the system (3.3.3) shows an equal number ν of equations and independent unknowns. In particular, for $SO(n)_{diag}$ the number is given in (2.2.22), while the number $\nu(SU(n)_{metric}$ will be given in Section 3.5.

We list some useful properties of the ansatz g_{metric}

1. It follows from (3.2.2) that the affine-Sugawara construction [3,5,9] L_g

$$L^{ab} = \frac{1}{\psi^2}\frac{\eta^{ab}}{x + \tilde{h}_g}, \quad c_g = \frac{x \dim g}{x + \tilde{h}_g} \tag{3.3.5}$$

is always a solution of (3.3.3).

2. K-conjugation [3,5,10,15]. When L_a is a solution of (3.3.3), then its K-conjugate $\tilde{L}_a$

$$\tilde{L}_a = \frac{1}{x + \tilde{h}_g} - L_a, \quad \tilde{c} = c_g - c \tag{3.3.6}$$

is also a solution, and the two corresponding Virasoro operators T and $\tilde{T}$ form a commuting pair.

3. Unitarity [10,18]. In general the unitarity requirement on L_a depends on the explicit form of the metric. However, when the magic basis is real, unitary solutions ($L^{(m)\dagger} = L^{(-m)}$) require

$$L_a = \text{real} \tag{3.3.7}$$

since in this case the hermiticity property of the affine currents is

$$J_a^{(m)\dagger} = \sum_b \eta_{ab} J_b^{(-m)} \tag{3.3.8}$$

which is the affineization of (3.2.3b).

3.4 High-Level Expansion of g_{metric}

The high-level expansion [22] of g_{metric}

$$L_a = \frac{1}{x}\sum_{s=0}^{\infty} L_a^{(s)} x^{-s}, \quad c = \sum_{s=0}^{\infty} c_s x^{-s} \tag{3.4.1}$$

parellels the expansion (2.2.31) of the graph-theory ansatz $SO(n)_{diag}$. The solution of the leading equations $L_a^{(0)}(1 - xL_a^{(0)}) = 0$, $(L_a^{(0)} - L_b^{(0)})\eta_{ab} = 0$ is

$$L_a^{(0)} = \theta_a, \quad c_0 = \sum_a \theta_a \tag{3.4.2a}$$

$$\theta_a \in \{0, 1\}, \quad a = 1, \cdots, \dim g \tag{3.4.2b}$$

$$\theta_a = \theta_b \text{ when } \eta_{ab} \neq 0 \tag{3.4.2c}$$

where any set $\{\theta_a\}$ may be chosen consistent with the constraint in (3.4.2c). In the case of $SO(n)_{diag}$, $\theta_a = \theta_{ij}$ is the adjacency matrix of any graph of order n (see Section 2.3.1).

The higher-order corrections $L_a^{(s\geq 1)}$ for each choice $\{\theta_a\}$ are unambiguously computable from the recursion relation

$$L_a^{(s)} = (1-2\theta_a)\left\{\sum_{t=1}^{s-1} L_a^{(t)} L_a^{(t-s)} + \sum_{cd}\sum_{t=0}^{s-1} L_c^{(t)}(2L_a^{(s-t-1)} - L_d^{(s-t-1)})(f_{cd}{}^a)^2/\psi^2\right\} \tag{3.4.3}$$

so each level-family in g_{metric} may be labelled by its high-level form

$$L(\{\theta_a\}) \leftrightarrow \{\theta_a\}\,. \tag{3.4.4}$$

Moreover, all the coefficients $L_a^{(s)}$ are real when the structure constants are real. It follows for every real magic basis of compact g that each level-family of g_{metric} will be unitary down to some finite critical level (see Section 2.2.3).

For use below, we also give the explicit form of the central charge

$$c = \sum_a \theta_a - \frac{1}{x\psi^2}\sum_{abc}\theta_b[2\theta_a(1-\theta_c)+\theta_c](f_{bc}{}^a)^2 + \mathcal{O}(x^{-2})\,. \tag{3.4.5}$$

through the first non-trivial order.

3.5 The Ansatz $SU(n)_{metric}$

The Pauli-like basis given in Ref.[31] is a real magic basis on $SU(n)$. In this basis, the n^2 generator of the fundamental representation of $U(n)$

$$T_a = T_{\vec{p}} = \sqrt{\frac{\psi^2}{2n}}\,\omega^{p_1p_2/2} g^{p_1} h^{p_2}, \quad \omega = e^{2\pi i/n} \tag{3.5.1a}$$

$$(g)_{IJ} = \omega^{I-1/2}\delta_{IJ}\,, \quad (h)_{IJ} = \delta_{I+1,J} - \delta_{I,n}\delta_{J,1}\,, \quad 1 \leq I,J \leq n \tag{3.5.1b}$$

are labelled on a two-dimensional integer lattice $\vec{p} = (p_1, p_2)$, with periods n. The pullback relation

$$T_{\vec{p}+n(r,s)} = (-)^{(p_1+1)s+(p_2+1)r+nrs} T_{\vec{p}} \tag{3.5.2}$$

can be used to choose the fundamental cell

$$F:\quad 0 \leq p_1 \leq n-1, \quad 0 \leq p_2 \leq n-1, \quad \vec{p} \neq 0 \tag{3.5.3}$$

for the n^2-1 generators of $SU(n)$. We obtain the Killing metric

$$\eta_{\vec{p},\vec{q}} = \frac{2}{\psi^2}\mathrm{Tr}\, T_{\vec{p}} T_{\vec{q}} = \eta^{\vec{p},\vec{q}} = \sigma(\vec{p})\delta_{\vec{p},(-\vec{q})(\mathrm{mod}\, n)} \tag{3.5.4a}$$

$$\sigma(\vec{p}) \equiv \begin{cases} -1 & p_1 = 0 \text{ or } p_2 = 0 \\ (-)^{p_1+p_2+n} & p_1, p_2 \neq 0 \end{cases} \tag{3.5.4b}$$

and the trigonometric structure constants

$$f_{\vec{p},\vec{q}}{}^{\vec{r}} = -\sqrt{\frac{2\psi^2}{n}}\,\sigma(\vec{p},\vec{q})\sin(\pi(\vec{p}\times\vec{q})/n)\delta_{(\vec{p}+\vec{q})(\mathrm{mod}\, n),\vec{r}} \tag{3.5.5a}$$

$$\sigma(\vec{p},\vec{q}) \equiv \begin{cases} 1 & 0 \leq p_1+q_1 \leq n-1,\ 0 \leq p_2+q_2 \leq n-1 \\ (-)^{p_2+q_2+1} & n \leq p_1+q_1 \leq 2n-2,\ 0 \leq p_2+q_2 \leq n-1 \\ (-)^{p_1+q_1+1} & 0 \leq p_1+q_1 \leq n-1,\ n \leq p_2+q_2 \leq 2n-2 \\ (-)^{p_1+q_1+p_2+q_2+n} & n \leq p_1+q_1 \leq 2n-2,\ n \leq p_2+q_2 \leq 2n-2 \end{cases} \tag{3.5.5b}$$

where $\vec{p}\times\vec{q} = p_1q_2 - p_2q_1$ and $\vec{r}(\mathrm{mod}\, n)$, $\forall\vec{r}$ is the representative of $\vec{r}$ in F. Using (3.5.4) and (3.5.5), it is not difficult to verify that the magic basis conditions (3.2.1) and real magic basis conditions (3.2.3) are satisfied in the Pauli-like basis. We note that in this case the condition (3.2.1b) is satisfied as a consequence of the Pauli-like property

$$T_{\vec{p}}T_{\vec{q}} = \sqrt{\frac{\psi^2}{2n}}\,\sigma(\vec{p},\vec{q})\omega^{(\vec{q}\times\vec{p})/2} T_{(\vec{p}+\vec{q})(\mathrm{mod}\, n)} \tag{3.5.6}$$

which is sufficient to guarantee (3.2.1b).

Subsitution of the explicit form of the Killing metric (3.5.4) and structure constants (3.5.5) in the metric ansatz (3.3.3), results in the master equation for $SU(n)_{metric}$

$$L_{\vec{p}}(1-xL_{\vec{p}}) = \frac{2}{n}\sum_{\vec{q}} L_{\vec{q}}(2L_{\vec{p}} - L_{(\vec{p}-\vec{q})(\mathrm{mod}\, n)})\sin^2(\pi(\vec{p}\times\vec{q})/n) \tag{3.5.7a}$$

$$L_{\vec{p}} = L_{(-\vec{p})(\mathrm{mod}\, n)} \tag{3.5.7b}$$

$$c = x \sum_{\vec{p}} L_{\vec{p}} \,. \tag{3.5.7c}$$

The explicit form of the symmetry constraints in (3.5.7b) is

$$L_{(p,0)} = L_{(n-p,0)}, \quad L_{(0,p)} = L_{(0,n-p)} \quad 1 \le p \le [(n-1)/2]$$
$$L_{(p_1,p_2)} = L_{(n-p_1,n-p_2)} \quad \begin{cases} 1 \le p_1 = p_2 \le [(n-1)/2] \\ 1 \le p_1 < p_2 \le n-1 \end{cases} \tag{3.5.8}$$

where $[y]$ is the greatest integer less than or equal to y, so that each variable is paired except for the three unconstrained coefficients

$$L_{(n/2,0)}, \; L_{(0,n/2)}, \; L_{(n/2,n/2)} \tag{3.5.9}$$

on $SU(n)$, n = even. For definiteness, we will choose

$$L_{(p,0)}, \quad L_{(0,p)} \quad 1 \le p \le [n/2] \,, \quad L_{(p_1,p_2)} \quad \begin{cases} 1 \le p_1 = p_2 \le [n/2] \\ 1 \le p_1 < p_2 \le n-1 \end{cases} \tag{3.5.10}$$

as the set of independent variables.

It follows from (3.5.10) that the reduced master equation contains

$$N(SU(n)_{metric}) = 2^{\nu(SU(n)_{metric})}, \quad \nu(SU(n)_{metric}) = \begin{cases} (n^2-1)/2, & n = \text{odd} \\ (n^2+2)/2, & n = \text{even} \end{cases} \tag{3.5.11}$$

solutions generically for any level of $SU(n)$, where $\nu(SU(n)_{metric})$ is the number of independent variables in (3.5.10).

Exactly this number of level-families are seen in the high-level expansion

$$L^{(0)}_{\vec{p}} = \theta_{\vec{p}}, \quad c_0 = \sum_{\vec{p}} \theta_{\vec{p}}, \quad \theta_{\vec{p}} = \theta_{(-\vec{p})(\mathrm{mod}\, n)} \tag{3.5.12}$$

and, since the Pauli-like basis is a real magic basis, each level-family of $SU(n)_{metric}$ is unitary down to some finite critical level (see Section 3.4).

The explicit form of the constraint in (3.5.12) is obtained from (3.5.8) with $L_{\vec{p}} \to \theta_{\vec{p}}$, so that the contribution Δc_0 of an independent $\theta_{\vec{p}} = 1$ to the high-level central charge is

$$\Delta c_0 = \begin{cases} 1 & \vec{p} = \{(n/2,0),(0,n/2),(n/2,n/2)\} \\ 2 & \text{otherwise} \end{cases} \tag{3.5.13}$$

It follows in particular that c_0 is an even integer when n is odd.

As a new phenomenon, we remark that the next order correction to the central charge

$$c = \sum_{\vec{p}} \theta_{\vec{p}} - \frac{2}{xn} \sum_{\vec{p},\vec{q}} \theta_{\vec{q}} [2\theta_{\vec{p}}(1-\theta_{(\vec{p}-\vec{q})(\mathrm{mod}\, n)}) + \theta_{(\vec{p}-\vec{q})(\mathrm{mod}\, n)}] \sin^2(\pi(\vec{p}\times\vec{q})/n) + \mathcal{O}(x^{-2}) \tag{3.5.14}$$

shows generic irrationality [1] already at order x^{-1}, because the squared structure constants $\sin^2(\pi s/n)$ are irrational for $n = 5$ and $n \ge 7$. Beyond the smaller manifolds, rational central charge can occur only sporadically - when the construction has a symmetry, so that certain sum rules are active. For example, the sum rule (3.2.2)

$$\sum_{\vec{q}} \sin^2(\pi(\vec{p}\times\vec{q})/n) = \frac{n^2}{2}, \quad \forall\, \vec{p} \tag{3.5.15}$$

maintains the rational central charge of the affine-Sugawara construction. It is clear that this phenomenon will generate arbitrarily high irrational type [20] as n increases.

For use below, we also give the conformal weights $\Delta = L^{ab}T_aT_b$ [12,18] of the n or $\bar{n}$ of $SU(n)$

$$\Delta[n] = \Delta[\bar{n}] = \frac{1}{2n} \sum_{\vec{p}} L_{\vec{p}} = \frac{c}{2nx} \tag{3.5.16}$$

which are degenerate, and the conformal weights of the adjoint

$$\Delta[n^2-1]_{\vec{p}} = \Delta[n^2-1]_{(-\vec{p})(\mathrm{mod}\, n)} = \frac{2}{n} \sum_{\vec{q}} L_{\vec{q}} \sin^2(\pi(\vec{p}\times\vec{q})/n) \tag{3.5.17}$$

which show a partial degeneracy analogous to the $L_{\vec{p}}$ constraints in (3.5.7b). Generic irrationality of the conformal weights is also visible in the high-level expansion of these results.

[1] The finite-order irrational central charge of these constructions can in principle be seen in other bases, for example the Cartesian basis of Ref.[22]. In this case, the phenomenon arises from the contribution to c_1 of irrational high-level twists $\Omega^{(0)}$.

3.6 $SU(3)_{metric}$

3.6.1 The Self K-Conjugate Construction $SU(3)^{\#}[m,2]$

The simplest case of (3.5.7) beyond $SU(2)_{metric}$, whose solutions coincide with $SO(3)_{diag}$, is $SU(3)_{metric}$

$$L_{01}(1 - xL_{01}) = 2L_{01}(L_{10} + L_{11} + L_{12}) - (L_{10}L_{11} + L_{10}L_{12} + L_{11}L_{12}) \tag{3.6.1a}$$

$$L_{10}(1 - xL_{10}) = 2L_{10}(L_{01} + L_{11} + L_{12}) - (L_{01}L_{12} + L_{01}L_{11} + L_{11}L_{12}) \tag{3.6.1b}$$

$$L_{11}(1 - xL_{11}) = 2L_{11}(L_{10} + L_{01} + L_{12}) - (L_{10}L_{01} + L_{10}L_{12} + L_{01}L_{12}) \tag{3.6.1c}$$

$$L_{12}(1 - xL_{12}) = 2L_{12}(L_{10} + L_{01} + L_{11}) - (L_{10}L_{01} + L_{10}L_{11} + L_{01}L_{11}) \tag{3.6.1d}$$

$$c = 2x(L_{01} + L_{10} + L_{11} + L_{12}) \tag{3.6.1e}$$

showing 4 quadratic equations on an equal number of unknowns, so that $2^4 = 16$ level-families are contained in $SU(3)_{metric}$. According to (3.5.10), we have chosen to write the reduced master equation (3.6.1) in terms of the 4 independent coefficients $L_{01} = L_{02}$, $L_{10} = L_{20}$, $L_{11} = L_{22}$, $L_{12} = L_{21}$. The explicit form of the stress tensor in the ansatz

$$T = \frac{2}{\psi^2} :(-L_{01}J_{01}J_{02} - L_{10}J_{10}J_{20} - L_{11}J_{11}J_{22} + L_{12}J_{12}J_{21}): \tag{3.6.2}$$

is obtained from (3.3.1b) with the metric (3.5.4).

It is easily checked that the ansatz (3.6.1) is covariant under permutations of the coefficients $L_{\vec{p}}$ so that the residual automorphisms are

$$L'_{\vec{p}} = L_{\pi(\vec{p})} \qquad \{\vec{p}\} = \{10, 01, 11, 12\} \tag{3.6.3}$$

where $\pi(\vec{p})$ is any permutation of the independent labels $\{\vec{p}\}$.

Applying our high-level analysis (3.5.12) to the system (3.6.1), we identified the known level-families in the ansatz which are: $L = 0$, the affine-Sugawara construction $L(SU(3)$,

four automorphically equivalent copies of $[U(1)]^2$ and $SU(3)/[U(1)]^2$, accounting for a total of ten level-families. We also find three new K-conjugate pairs

$$\begin{cases} \theta_{01} = \theta_{10} = 1, & \theta_{11} = \theta_{12} = 0 \\ \theta_{01} = \theta_{10} = 0, & \theta_{11} = \theta_{12} = 1 \end{cases} \tag{3.6.4a}$$

$$\begin{cases} \theta_{01} = \theta_{11} = 1, & \theta_{10} = \theta_{12} = 0 \\ \theta_{01} = \theta_{11} = 0, & \theta_{10} = \theta_{12} = 1 \end{cases} \tag{3.6.4b}$$

$$\begin{cases} \theta_{01} = \theta_{12} = 1, & \theta_{10} = \theta_{11} = 0 \\ \theta_{01} = \theta_{12} = 0, & \theta_{10} = \theta_{11} = 1 \end{cases} \tag{3.6.4c}$$

all of which have high-level central charge $c_0 = 4 = \frac{1}{2}\dim SU(3)$. In (3.6.4), the second line of each bracket is the K-conjugate partner

$$\tilde{\theta}_{\vec{p}} = 1 - \theta_{\vec{p}} \tag{3.6.5}$$

of the first line, and, according to the level-independent automorphisms (3.6.3), each bracket is automorphically equivalent

$$\theta'_{\vec{p}} = \theta_{\pi(\vec{p})} \tag{3.6.6}$$

to the other brackets. Moreover, the K-conjugate partners in a given bracket are automorphically equivalent

$$\tilde{\theta}_{\vec{p}} = \theta_{\pi(\vec{p})} \tag{3.6.7}$$

so that each line in (3.6.4) is a copy of a single self K-conjugate construction on $SU(3)$.

The exact form of the new constructions present in the ansatz $SU(3)_{metric}$

$$\begin{cases} L_{01} = L_{10} = \frac{1}{2(x+3)}\left(1 + \sqrt{\frac{x+3}{x-1}}\right), & L_{11} = L_{12} = \frac{1}{2(x+3)}\left(1 - \sqrt{\frac{x+3}{x-1}}\right) \\ L_{01} = L_{10} = \frac{1}{2(x+3)}\left(1 - \sqrt{\frac{x+3}{x-1}}\right), & L_{11} = L_{12} = \frac{1}{2(x+3)}\left(1 + \sqrt{\frac{x+3}{x-1}}\right) \end{cases} \tag{3.6.8a}$$

$$\begin{cases} L_{01} = L_{11} = \frac{1}{2(x+3)}\left(1 + \sqrt{\frac{x+3}{x-1}}\right), & L_{10} = L_{12} = \frac{1}{2(x+3)}\left(1 - \sqrt{\frac{x+3}{x-1}}\right) \\ L_{01} = L_{11} = \frac{1}{2(x+3)}\left(1 - \sqrt{\frac{x+3}{x-1}}\right), & L_{10} = L_{12} = \frac{1}{2(x+3)}\left(1 + \sqrt{\frac{x+3}{x-1}}\right) \end{cases} \tag{3.6.8b}$$

$$\begin{cases} L_{01} = L_{12} = \frac{1}{2(x+3)}\left(1 + \sqrt{\frac{x+3}{x-1}}\right), & L_{10} = L_{11} = \frac{1}{2(x+3)}\left(1 - \sqrt{\frac{x+3}{x-1}}\right) \\ L_{01} = L_{12} = \frac{1}{2(x+3)}\left(1 - \sqrt{\frac{x+3}{x-1}}\right), & L_{10} = L_{11} = \frac{1}{2(x+3)}\left(1 + \sqrt{\frac{x+3}{x-1}}\right) \end{cases} \tag{3.6.8c}$$

$$c = \frac{1}{2}c_{SU(3)} = \frac{4x}{x+3}, \quad x \neq 1 \tag{3.6.8d}$$

can be easily checked from (3.6.1) and shows six automorphic copies of a self K-conjugate construction on $SU(3)$. The organization of the six automorphic copies in (3.6.8) is analagous to the arrangement of the corresponding high-level forms in (3.6.4), and, following Section 2.6, we call the construction

$$SU(3)^{\#}[m,2] \tag{3.6.9}$$

since it occurs in a subansatz of size $\nu = 2$ of the metric (m) ansatz on $SU(3)$. The half-Sugawara central charge (3.6.8d) is characteristic of self K-conjugate construtions, and, as guaranteed by the high-level expansion, the construction is unitary down to a finite critical level, which is $x = 1$ in this case.

We have also computed the L^{ab}-broken conformal weights of this construction for the 3 (or $\bar{3}$) and adjoint representation of $SU(3)$, using (3.5.16) and (3.5.17) respectively. The results

$$\Delta[3] = \Delta[\bar{3}] = \frac{2}{3(x+3)} \tag{3.6.10a}$$

$$\Delta_{\pm}[8] = \frac{1}{2(x+3)}(3 \pm \sqrt{\frac{x+3}{x-1}}) \tag{3.6.10b}$$

show rational, completely degenerate, half-Sugawara conformal weights for the 3 (or $\bar{3}$), but two irrational 4-fold degenerate conformal weights for the adjoint representation.

3.6.2 The Quadratic Deformation $SU(3)^{\#}_3$

Having solved the ansatz (3.6.1) on $SU(3)$ for generic level, we search for sporadic deformations with the predicive method of Ref.[25]: We expect sporadic quadratic deformations for those levels at which two or more constructions, flow-connected at high-level, have coincident central charge. We restrict ourselves to first crossings as the level is lowered.

A plot of the central charges of the five level-families that are present in $SU(3)_{metric}$

is given in Fig. 21, and shows a single first crossing at level three

$$x = 3: \quad SU(3)/[U(1)]^2 \sim SU(3)^{\#}[m,2] \sim [U(1)]^2 \ . \tag{3.6.11}$$

Next, we consider the high-level flow equation [25] of the closed subflow $SU(n)_{metric}$

$$\dot{L}^{(0)}_{\vec{p}} = -L^{(0)}_{\vec{p}}(1 - L^{(0)}_{\vec{p}}), \quad L^{(0)}_{\vec{p}} = L^{(0)}_{(-\vec{p})(\mathrm{mod}\, n)} \tag{3.6.12}$$

which shows that the physical flow between fixed points is $\theta_{\vec{p}} = 1 \to \theta_{\vec{p}} = 0$. The flow on $SU(3)_{metric}$, recorded in Fig. 22, shows that the 4+6+4=14 representatives of the three constructions in (3.6.11) are flow-connected at high-level.

Hence we predict the quadratic deformation

$$SU(3)^{\#}_3, \quad c = 2 \tag{3.6.13}$$

which will contain all 14 constructions that have coalesced at level three. Moreover, we can predict that the deformation $SU(3)^{\#}_3$ will be closed under K-conjugation, since the set of high-level constructions involved in the crossing is closed under K-conjugation.

To find the deformation (3.6.13) explicitly, we set $x = 3$ in the ansatz (3.6.1). The resulting explicit form of $SU(3)^{\#}_3$

$$L_{01} = \frac{1}{12}(1 + \sqrt{3}\cos\theta) + \frac{1}{2\sqrt{6}}\sin\theta\cos\phi \tag{3.6.14a}$$

$$L_{10} = \frac{1}{12}(1 + \sqrt{3}\cos\theta) - \frac{1}{2\sqrt{6}}\sin\theta\cos\phi \tag{3.6.14b}$$

$$L_{11} = \frac{1}{12}(1 - \sqrt{3}\cos\theta) + \frac{1}{2\sqrt{6}}\sin\theta\sin\phi \tag{3.6.14c}$$

$$L_{12} = \frac{1}{12}(1 - \sqrt{3}\cos\theta) - \frac{1}{2\sqrt{6}}\sin\theta\sin\phi \tag{3.6.14d}$$

$$0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi, \quad c = 2 \tag{3.6.14e}$$

shows that the unitary deformation is a two-sphere. Moreover, K-conjugation is reflection through the origin of the sphere

$$\tilde{L}_{\vec{p}}(\theta,\phi) = L_{\vec{p}}(\pi - \theta, \phi + \pi) \tag{3.6.15}$$

and we have checked that all 14 representatives of the constructions in (3.6.11) are particular points in the deformation.

3.7 $SU(4)_{metric}$

3.7.1 Solution Methodology

We have also studied the ansatz $SU(4)_{metric}$

$$L_{01}(1 - xL_{01}) = L_{01}[L_{10} + L_{12} + L_{11} + L_{13} + L_{20} + L_{22} + 2L_{23}] - \frac{1}{2}(L_{10} + L_{12})(L_{11} + L_{13}) - L_{23}(L_{20} + L_{22}) \quad (3.7.1a)$$

$$L_{23}(1 - xL_{23}) = L_{23}[L_{10} + L_{12} + L_{11} + L_{13} + L_{20} + L_{22} + 2L_{01}] - \frac{1}{2}(L_{10} + L_{12})(L_{11} + L_{13}) - L_{01}(L_{20} + L_{22}) \quad (3.7.1b)$$

$$L_{10}(1 - xL_{10}) = L_{10}[L_{01} + L_{23} + L_{11} + L_{13} + L_{02} + L_{22} + 2L_{12}] - \frac{1}{2}(L_{01} + L_{23})(L_{11} + L_{13}) - L_{12}(L_{02} + L_{22}) \quad (3.7.1c)$$

$$L_{12}(1 - xL_{12}) = L_{12}[L_{01} + L_{23} + L_{11} + L_{13} + L_{02} + L_{22} + 2L_{10}] - \frac{1}{2}(L_{01} + L_{23})(L_{11} + L_{13}) - L_{10}(L_{02} + L_{22}) \quad (3.7.1d)$$

$$L_{11}(1 - xL_{11}) = L_{11}[L_{01} + L_{23} + L_{10} + L_{12} + L_{02} + L_{20} + 2L_{13}] - \frac{1}{2}(L_{01} + L_{23})(L_{10} + L_{12}) - L_{13}(L_{02} + L_{20}) \quad (3.7.1e)$$

$$L_{13}(1 - xL_{13}) = L_{13}[L_{01} + L_{23} + L_{10} + L_{12} + L_{02} + L_{20} + 2L_{11}] - \frac{1}{2}(L_{01} + L_{23})(L_{10} + L_{12}) - L_{11}(L_{02} + L_{20}) \quad (3.7.1f)$$

$$L_{02}(1 - xL_{02}) = 2L_{02}(L_{10} + L_{12} + L_{11} + L_{13}) - 2(L_{10}L_{12} + L_{11}L_{13}) \quad (3.7.1g)$$

$$L_{20}(1 - xL_{20}) = 2L_{20}(L_{01} + L_{23} + L_{11} + L_{13}) - 2(L_{01}L_{23} + L_{11}L_{13}) \quad (3.7.1h)$$

$$L_{22}(1 - xL_{22}) = 2L_{22}(L_{01} + L_{23} + L_{10} + L_{12}) - 2(L_{01}L_{23} + L_{10}L_{12}) \quad (3.7.1i)$$

$$c = x(2(L_{01} + L_{23} + L_{10} + L_{12} + L_{11} + L_{13}) + L_{02} + L_{20} + L_{22}) \quad (3.7.1j)$$

showing nine coupled quadratic equations on an equal number of independent variables

$$L_{01} = L_{03},\ L_{23} = L_{21},\ L_{10} = L_{30},\ L_{12} = L_{32},\ L_{11} = L_{33},\ L_{13} = L_{31} \quad (3.7.2a)$$

$$L_{02},\ L_{20},\ L_{22} \quad (3.7.2b)$$

so that $2^9 = 512$ level-families will be contained in the ansatz $SU(4)_{metric}$.

It is easily checked that the ansatz (3.7.1) is covariant under

$$L'_{\vec{p}} = L_{\pi(\vec{p})}\ ,\quad \{\vec{p}\} = \{01, 10, 11, 12, 23, 13, 02, 20, 22\} \quad (3.7.3)$$

when π is any of the following permutations (of the independent labels $\{\vec{p}\}$)

$$\begin{aligned} \pi_1 &: \ 01 \leftrightarrow 23 \\ \pi_2 &: \ 10 \leftrightarrow 12 \\ \pi_3 &: \ 11 \leftrightarrow 13 \\ \pi_4 &: \ 01 \leftrightarrow 10,\ 23 \leftrightarrow 12,\ 02 \leftrightarrow 20 \\ \pi_5 &: \ 01 \leftrightarrow 11,\ 23 \leftrightarrow 13,\ 02 \leftrightarrow 22 \end{aligned} \quad (3.7.4)$$

or any combination of these five permuations. For simplicity we have omitted inert labels in the permutations of eq.(3.7.4).

To obtain explicit solutions of the system (3.7.1), it is useful to introduce the sum and difference variables

$$L^{\pm}_{01} = L_{01} \pm L_{23} \quad (3.7.5a)$$

$$L^{\pm}_{10} = L_{10} \pm L_{12} \quad (3.7.5b)$$

$$L^{\pm}_{11} = L_{11} \pm L_{13} \quad (3.7.5c)$$

in which case eq.(3.7.1) takes the form

$$L^{-}_{01}[-1 + 2(L_{20} + L_{22}) + L^{+}_{11} + L^{+}_{10} + xL^{+}_{01}] = 0 \quad (3.7.6a)$$

$$L_{10}^{-}[-1+2(L_{02}+L_{22})+L_{11}^{+}+L_{01}^{+}+xL_{10}^{+})=0 \tag{3.7.6b}$$

$$L_{11}^{-}[-1+2(L_{02}+L_{20})+L_{01}^{+}+L_{10}^{+}+xL_{11}^{+})=0 \tag{3.7.6c}$$

$$L_{01}^{+}[2-(L_{10}^{+}+L_{11}^{+}-(x+2)L_{01}^{+}]-(x-2)(L_{01}^{-})^2+2L_{10}^{+}L_{11}^{+}=0 \tag{3.7.6d}$$

$$L_{10}^{+}[2-(L_{01}^{+}+L_{11}^{+}-(x+2)L_{10}^{+}]-(x-2)(L_{10}^{-})^2+2L_{01}^{+}L_{11}^{+}=0 \tag{3.7.6e}$$

$$L_{11}^{+}[2-(L_{01}^{+}+L_{10}^{+}-(x+2)L_{11}^{+}]-(x-2)(L_{11}^{-})^2+2L_{01}^{+}L_{10}^{+}=0 \tag{3.7.6f}$$

$$2L_{02}[1-2(L_{10}^{+}+L_{11}^{+}-xL_{02}]+(L_{10}^{+})^2-(L_{10}^{-})^2+(L_{11}^{+})^2-(L_{11}^{-})^2=0 \tag{3.7.6g}$$

$$2L_{20}[1-2(L_{01}^{+}+L_{11}^{+}-xL_{20}]+(L_{01}^{+})^2-(L_{01}^{-})^2+(L_{11}^{+})^2-(L_{11}^{-})^2=0 \tag{3.7.6h}$$

$$2L_{22}[1-2(L_{01}^{+}+L_{10}^{+}-xL_{22}]+(L_{01}^{+})^2-(L_{01}^{-})^2+(L_{10}^{+})^2-(L_{10}^{-})^2=0 \tag{3.7.6i}$$

$$c=x[2(L_{01}^{+}+L_{10}^{+}+L_{11}^{+})+L_{02}+L_{20}+L_{22}] \tag{3.7.6j}$$

which shows a factorization in the first three equations.

We may thus consider eight different sectors of the equations

$$I.\quad L_{10}^{-}=0, L_{01}^{-}=0, L_{11}^{-}=0: \quad 64 \text{ solutions} \tag{3.7.7a}$$

$$II.\quad \begin{cases} a.\ L_{10}^{-}\neq 0, L_{01}^{-}=0, L_{11}^{-}=0: & 64 \text{ solutions} \\ b.\ L_{10}^{-}=0, L_{01}^{-}\neq 0, L_{11}^{-}=0: & 64 \text{ solutions} \\ c.\ L_{10}^{-}=0, L_{01}^{-}=0, L_{11}^{-}\neq 0: & 64 \text{ solutions} \end{cases} \tag{3.7.7b}$$

$$III.\quad \begin{cases} a.\ L_{10}^{-}\neq 0, L_{01}^{-}\neq 0, L_{11}^{-}=0: & 64 \text{ solutions} \\ b.\ L_{10}^{-}\neq 0, L_{01}^{-}=0, L_{11}^{-}\neq 0: & 64 \text{ solutions} \\ c.\ L_{10}^{-}=0, L_{01}^{-}\neq 0, L_{11}^{-}\neq 0: & 64 \text{ solutions} \end{cases} \tag{3.7.7c}$$

$$IV.\quad L_{10}^{-}\neq 0, L_{01}^{-}\neq 0, L_{11}^{-}\neq 0: \quad 64 \text{ solutions} \tag{3.7.7d}$$

and, using the permutation automorphisms $\pi_{4,5}$ in (3.7.4), it is not difficult to show that the 3 sectors a,b,c within II are automorphic copies of each other, as well as the 3 sectors within III.

The sectors in eq.(3.7.7) may be understood as consistent subansätze, which, as in $SO(n)_{diag}$, can be obtained by restricting to those solutions which posses a symmetry

$$L_{\vec{p}}=L_{\pi(\vec{p})}, \quad \pi\in Aut\,SU(n)\,. \tag{3.7.8}$$

In particular, requiring invariance under the three permuations π_1,π_2,π_3 gives a six parameter subansatz $SU(4)[m,6]$ which contains the $2^6=64$ solutions in sector I. Requiring, invariance under any two permutations in the set $\{\pi_1,\pi_2,\pi_3\}$ defines three automorphically equivalent seven parameter subansätze, each of which contains the $2^7=128$ solutions in sector I and IIa,IIb or IIc. For example, the particular subansatz

$$SU(4)[m,7]: \quad L_{01}=L_{23}, \quad L_{10}=L_{12} \tag{3.7.9}$$

follows from (3.7.8) using the permutations π_1 and π_2 given in (3.7.4), and contains the solutions in sectors I and IIa. We have solved this π_1 and π_2 symmetric subansatz completely, and an overview of the 128 solutions in the subansatz is given below.

All the 64 solutions in sector I are known constructions. The complete list is

$$\begin{gathered} L=0\,(1), \quad U(1)\,(3), \quad [U(1)]^2\,(3), \\ [U(1)]^3\,(1), \quad \frac{[SU(2)]^2}{[U(1)]^2}\,(3), \quad \frac{[SU(2)]^2}{U(1)}\,(6), \\ \frac{[SU(2)]^2}{[U(1)]^2}\times U(1)\,(3), \quad [SU(2)]^2\,(3), \\ \frac{[SU(2)]^2}{U(1)}\times U(1)\,(6), \quad [SU(2)]^2\times U(1)\,(3) \end{gathered} \tag{3.7.10}$$

and their K-conjugates on $SU(4)$, where the number of automorphic copies of each solution is given in parentheses.

Among the 64 solutions in sector IIa, we find 32 known solutions, which are

$$[U(1)]^2\,(2), \quad [U(1)]^3\,(2), \quad SO(4)^{\#}[d,4]\,(4),$$

$$\frac{[SU(2)]^2}{[U(1)]^2}(2), \quad SO(4)^{\#}[d,4]\times U(1)(4), \quad \frac{[SU(2)]^2}{[U(1)]^2}\times U(1)(2) \tag{3.7.11}$$

and their K-conjugates on $SU(4)$. The remaining 32 solutions are 4 copies each of 4 physicallly distinct K-conjugate pairs

$$SU(4)^{\#}[m,7]_{1,2}, \quad \frac{SU(4)}{SU(4)^{\#}[m,7]_{1,2}} \tag{3.7.12a}$$

$$SU(4)^{\#}[m,7]_{3,4}, \quad \frac{SU(4)}{SU(4)^{\#}[m,7]_{3,4}} \tag{3.7.12b}$$

whose explicit forms will be given in Sects. 3.7.2 and 3.7.3. We remark at this point that the level-families in (3.7.12a) have generically unitary irrational central charge, while those in (3.7.12b) have rational central charge but generically irrational conformal weights.

3.7.2 $SU(4)^{\#}[m,7]_{1,2}$

The explicit form of the solutions in (3.7.12a) is

$$L_{11}^{+} = \frac{1}{x+4}(1+\eta 2(x+2)R), \quad L_{11}^{-} = \sigma\sqrt{\frac{x^2-4x-4}{x^4-16x^2+16}} \tag{3.7.13a}$$

$$L_{10}^{+} = \frac{1}{x+4}\left(1-\eta R(4-\epsilon(x+4)\sqrt{x(x-4)}\right), \quad L_{10}^{-}=0 \tag{3.7.13b}$$

$$L_{01}^{+} = \frac{1}{x+4}\left(1-\eta R(4+\epsilon(x+4)\sqrt{x(x-4)}\right), \quad L_{01}^{-}=0 \tag{3.7.13c}$$

$$L_{02} = \frac{1}{2(x+4)}\left(1-\eta R(x^2+2x-4+2\epsilon(x+4)\sqrt{(x-4)/x})\right) \tag{3.7.13d}$$

$$L_{20} = \frac{1}{2(x+4)}\left(1-\eta R(x^2+2x-4-2\epsilon(x+4)\sqrt{(x-4)/x})\right) \tag{3.7.13e}$$

$$L_{22} = \frac{1}{2(x+4)}\left(1+\eta(16R-\xi\frac{x+4}{x})\right) \tag{3.7.13f}$$

$$c = \frac{1}{2(x+4)}(15x-\eta(x-2)(x^2+4)R) - \frac{(1+\xi)\eta}{2} \tag{3.7.13g}$$

$$R \equiv (x^4-16x^2+16)^{-1/2} \tag{3.7.13h}$$

where $\eta=\pm 1$ is K-conjugation, $\sigma,\epsilon=\pm 1$ labels automorphic copies at fixed η and ξ, and ξ labels physically distinct level-families according to

$$SU(4)^{\#}[m,7]_1: \quad \eta=\xi=1 \tag{3.7.14a}$$

$$SU(4)^{\#}[m,7]_2: \quad \eta=-\xi=1\ . \tag{3.7.14b}$$

In fact, the two constructions in (3.7.14) are related by a #-coset construction [22]

$$SU(4)^{\#}[m,7]_2 = SU(4)^{\#}[m,7]_1\times U(1) \tag{3.7.15}$$

with $U(1)$-current J_{22}.

The construction is unitary on positive integer levels $x\geq 5$ and generically irrational. The lowest unitary central charge of the family in (3.7.13) is at level 5

$$c\left(SU(4)_5^{\#}[m,7]_1\right) = \frac{1}{18}\left(57-\frac{87}{\sqrt{241}}\right) \simeq 2.8553 \tag{3.7.16}$$

and, more generally, the central charge approaches integer values at high level.

3.7.3 $SU(4)^{\#}[m,7]_{3,4}$

The explicit form of the solutions in (3.7.12b) is

$$L_{11}^{+} = \frac{1}{x+4}, \quad L_{11}^{-}\frac{\sigma\eta}{\sqrt{x(x+4}} \tag{3.7.17a}$$

$$L_{01}^{+} = \frac{1}{x+4}\left(1-\eta\epsilon\sqrt{\frac{x+4}{x}}\right), \quad L_{01}^{-}=0 \tag{3.7.17b}$$

$$L_{10}^{+} = \frac{1}{x+4}\left(1+\eta\epsilon\sqrt{\frac{x+4}{x}}\right), \quad L_{10}^{-}=0 \tag{3.7.17c}$$

$$L_{02} = \frac{1}{2(x+4)}\left(1-\frac{2\eta\epsilon}{x}\sqrt{\frac{x+4}{x}}+\frac{\eta\epsilon\xi}{x}\sqrt{\frac{(x+4)(x^2+4)}{x}}\right) \tag{3.7.17d}$$

$$L_{20} = \frac{1}{2(x+4)}\left(1+\frac{2\eta\epsilon}{x}\sqrt{\frac{x+4}{x\xi}}-\frac{\eta\epsilon\xi}{x}\sqrt{\frac{(x+4)(x^2+4)}{x}}\right) \tag{3.7.17e}$$

$$L_{22} = \frac{1}{2(x+4)}\left(1-\eta\frac{x+4}{x}\right) \tag{3.7.17f}$$

$$c = \frac{15x}{2(x+4)}-\frac{\eta}{2} \tag{3.7.17g}$$

where $\eta = \pm 1$ is K-conjugation, $\sigma, \epsilon = \pm 1$ labels automorphic copies and $\xi = \pm 1$ labels physically distinct K-conjugate pairs with equal central charge but different conformal weights. The constructions are named according to

$$SU(4)^{\#}[m,7]_3: \quad \eta = \xi = 1 \tag{3.7.18a}$$

$$SU(4)^{\#}[m,7]_4: \quad \eta = -\xi = 1 \ . \tag{3.7.18b}$$

The constructions are unitary on $x \in N$, have rational central charge and generically irrational conformal weights. It is interesting to note that a special type of #-coset construction can be observed in this family

$$SU(4)/SU(4)^{\#}[m,7]_{3,4} = SU(4)^{\#}[m,7]_{3,4} \times U(1) \tag{3.7.19}$$

where the $U(1)$-curent is J_{22}. This is an example of a # coset construction, which involves both L and its K-conjugate construction $\tilde{L}$

$$L + L(U(1)) = \omega \tilde{L} \omega, \quad \omega \in Aut\, SU(4)\,; \quad \tilde{L} = L_g - L\,. \tag{3.7.20}$$

It folows that the central charge of the constructions L and $\tilde{L}$ can be computed before obtaining their exact form since

$$c + 1 = \tilde{c} = c_g - c \tag{3.7.21}$$

so that

$$c = \frac{c_g - 1}{2}, \quad \tilde{c} = \frac{c_g + 1}{2}\ . \tag{3.7.22}$$

The results (3.7.22) with $g = SU(4)$ are in agreement with the rational central charges $c(SU(4)^{\#}[m,7]_{3,4}) = (7x-2)/(x+4)$ and $c(SU(4)/SU(4)^{\#}[m,7]_{3,4}) = (8x+2)/(x+4)$.

3.7.4 SU(4)$_{\text{metric}}$ Versus SO(6)$_{\text{diag}}$

Since $SO(6) = SU(4)$, we have determined the overlap between the two metric ansätze $SO(6)_{diag}$ and $SU(4)_{metric}$. It is not difficult to establish that the two subansätze

$$SO(6)[d,7'] \sim SU(4)[m,7] \tag{3.7.23}$$

are physically equivalent. A labelled representative of the subansatz $SO(6)[d,7']$ is

$$SO(6)[d,7']: \begin{cases} L_{12} = L_{34}, \ L_{13} = L_{24}, \ L_{14}, \ L_{23}, \ L_{56} \\ L_{15} = L_{16} = L_{45} = L_{46}, \ L_{25} = L_{26} = L_{35} = L_{36} \end{cases} \tag{3.7.24}$$

and the exact relation with $SU(4)[m,7]$ is

$$L_{12} = L_{11}, \quad L_{13} = L_{13}, \quad L_{15} = L_{01}, \quad L_{25} = L_{10}$$

$$L_{14} = L_{20}, \quad L_{23} = L_{02}, \quad L_{56} = L_{22} \tag{3.7.25}$$

which is written in the form $L_{ij} = L_{\vec{\beta}}$ for the seven independent coefficients of the ansatz. Moreover, the identifications

$$SO(6)]^{\#}[d,7']_{1,2,3,4} = SU(4)]^{\#}[m,7]_{1,2,3,4} \tag{3.7.26}$$

are established by comparing high-level central charges and conformal weights of the new #-constructions in $SO(6)[d,7']$ (which can be found in the high-level sectors 6,7 and 8 of Table 3) and those in $SU(4)[m,7]$.

3.8 SU(5)$_{\text{metric}}$

3.8.1 Finite-Order Irrational Central Charge

In Section 3.5. we pointed out that irrational central charge at finite order of the high-level expansion is seen for the generic level-family in $SU(n)_{metric}$. To illustrate this new phenomenon, we have investigated $SU(5)_{metric}$, which is the smallest ansatz in $SU(n)_{metric}$ that shows finite order irrational central charge, since $n = 5$ is the lowest integer for which $\sin^2(\pi/n)$ is irrational. From (3.5.7), it is straightforward to express the 12 equations of the ansatz in terms of the 12 independent variables

$$L_{01}, L_{02}, L_{10}, L_{20}, L_{11}, L_{22}, L_{12}, L_{24}, L_{13}, L_{34}, L_{14}, L_{23} \tag{3.8.1}$$

and, with (3.5.14), we have analyzed the high-level central charges of the $2^{12} = 4096$ level-families in the ansatz. The results below are given for the lower half ($c_0 \leq 12$) of the high-level sectors, since the rest of the constructions can be obtained by K-conjugation.

1. There are three exact rational central charges

$$c = 0, 2 \text{ and } 4 \tag{3.8.2}$$

which correspond to the known constructions $L = 0$, $[U(1)]^2$ and $[U(1)]^4$.

2. There are eight distinct potentially-irrational central charges

$$\begin{aligned} c = \{ & 6 - \frac{12}{x}, \\ & 8 - \frac{1}{x}(24, 36), \\ & 10 - \frac{1}{x}(36, 48, 60), \\ & 12 - \frac{1}{x}(48, 60)\} + \mathcal{O}(x^{-2}) \end{aligned} \tag{3.8.3}$$

which may show irrationality at higher finite order, or may be the high-level forms of exact irrational constructions.

3. There are fifteen distinct central charges

$$\begin{aligned} c = \{ & 4 - \frac{48}{5x}(s_1^2, s_2^2), \\ & 6 - \frac{48}{5x}(2s_1^2, 2s_2^2, 2s_1^2 + s_2^2, s_1^2 + 2s_2^2), \\ & 8 - \frac{48}{5x}(3s_1^2 + s_2^2, s_2^2 + 3s_1^2, 3s_1^2 + 2s_2^2, 2s_1^2 + 3s_2^2), \\ & 10 - \frac{48}{5x}(4s_1^2 + 3s_2^2, 3s_1^2 + 4s_2^2), \\ & 12 - \frac{48}{5x}(5s_1^2 + 4s_2^2, 4s_1^2 + 5s_2^2, 6s_1^2 + 4s_2^2)\} + \mathcal{O}(x^{-2}) \end{aligned} \tag{3.8.4a}$$

$$s_1^2 \equiv \sin^2(\pi/5) = (5 - \sqrt{5})/8, \quad s_2^2 \equiv \sin^2(2\pi/5) = (5 + \sqrt{5})/8 \tag{3.8.4b}$$

which are already irrational at this order.

Each central charge in (3.8.3) and (3.8.4) collects a large number of constructions which are degenerate at this order, and we have determined that the total number of irrational

constructions represented in (3.8.4) already dominates the total number of potentially-irrational constructions represented in (3.8.3). Moreover, most of the potentially-irrational constructions show irrational conformal weights at this order.

The observation of unitary irrational central charge at $c_0 = 4$ in (3.8.4) (and $c_0 = 20$ by K-conjugation) requires an update of the phenomenological inequalities of Ref.[22]: So far, all known unitary irrational irreducible constructions satisfy the high-level central charge inequalities

$$\operatorname{rank} g \leq c_0 \leq \dim g - \operatorname{rank} g \tag{3.8.5}$$

which now include $c_0 = \operatorname{rank} g$ and $\dim g - \operatorname{rank} g$.

We finally remark that at least some of the constructions with $c = 12 - 60/x + \mathcal{O}(x^{-2})$ are self K-conjugate constructions on $SU(5)$. For example, the candidate

$$\begin{cases} \theta_{01} = \theta_{10} = \theta_{11} = \theta_{12} = \theta_{13} = \theta_{14} = 1 \\ \theta_{02} = \theta_{20} = \theta_{22} = \theta_{24} = \theta_{34} = \theta_{23} = 0 \end{cases} \tag{3.8.6a}$$

$$c = \frac{1}{2}c_{SU(5)} = \frac{12x}{x+5} = 12 - \frac{60}{x} + \mathcal{O}(x^{-2}) \tag{3.8.6b}$$

is the high-level form of a self K-conjugate construction, because

$$\tilde{\theta}_{\vec{p}} = 1 - \theta_{\vec{p}} = \theta_{\pi(\vec{p})} \tag{3.8.7a}$$

$$\pi(\vec{p}): \quad 01 \leftrightarrow 02, 10 \leftrightarrow 20, 11 \leftrightarrow 22, 12 \leftrightarrow 24, 13 \leftrightarrow 34, 14 \leftrightarrow 23 \tag{3.8.7b}$$

is the high-level form of the level-independent automorphism $L'_{\vec{p}} = L_{\pi(\vec{p})}$.

3.8.2 Exact Solutions with Finite-Order Irrationality

We have obtained the exact form of one of the K-conjugate pairs in $SU(5)_{metric}$

$$L_{01} = L_{02} = L_{10} = L_{20} = L_{11} = L_{22} = \frac{1}{2(x+5)}\left(1 + \eta\frac{(x+5)\sqrt{5} - 1}{\sqrt{5x^2 + 20x - 24}}\right) \tag{3.8.8a}$$

$$L_{12} = L_{24} = L_{13} = L_{34} = L_{14} = L_{23} = \frac{1}{2(x+5)}\left(1 - \eta\frac{(x+5)\sqrt{5} + 1}{\sqrt{5x^2 + 20x - 24}}\right) \tag{3.8.8b}$$

$$c = \frac{12x}{x+5}\left(1 - \eta\frac{1}{\sqrt{5x^2+20x-24}}\right) \tag{3.8.8c}$$

($\eta = \pm 1$ is K-conjugation) whose irrational central charge is visible at finite order. The solutions were obtained from a consistent two-parameter subansatz whose form is recorded in the result, so we call the constructions

$$SU(5)^{\#}[m,2],\ \eta = 1\ ;\quad SU(5)/SU(5)^{\#}[m,2],\ \eta = -1\,. \tag{3.8.9}$$

The construction $SU(5)^{\#}[m,2]$ with $c = 12 - 12(5+1/\sqrt{5})/x + O(x^{-2})$ is included in (3.8.4). Both constructions are unitary on $x \in N$, with rational points at $x = 1, 2$ and 3, so that the value

$$c\left(SU(5)^{\#}_{4}[m,2]\right) = \frac{48}{9}\left(1 - \frac{1}{2\sqrt{34}}\right) \simeq 4.8760 \tag{3.8.10}$$

is the lowest unitary irrational central charge in the K-conjugate pair.

3.9 Generalized Graph Theories on g

3.9.1 The Sine-Area Graphs of SU(n)

The signal that the conformal field theories of $SO(n)_{diag}$ are isomorphic to graph theory is that the high-level form of the ansatz $L^{(0)}_{ij} = \theta_{ij} \in \{0,1\}$, $1 \le i < j \le n$ is the adjacency matrix of any graph of order n (see Section 2.3.1). This correspondence also underlies the group- and conformal field-theoretic organization of graph theory described in Chapter 2, which begins by defining the graphs on $SO(n)$ with graph isomorphisms in $Aut\,SO(n)$.

In analogy with the graph theory ansatz $SO(n)_{diag}$, we will define in this subsection a generalized graph theory on $SU(n)$ interpreting the high-level form of the ansatz, $\{\theta_{\vec{p}}\}$ as an "edge-list". The resulting graphs will label the conformal field theories in $SU(n)_{metric}$.

To obtain a generalized graph theory on $SU(n)$, imagine first that the $\theta_{\vec{p}}$'s of $SU(n)_{metric}$ are unconstrained in a larger ansatz. We define the "sine-area graphs" $\{G\}$ of $SU(n)$ as the 2^{n^2-1} sets of vectors $\{\vec{p}\}$ for which $\theta_{\vec{p}} = 1$, e.g. the 8 sine-area graphs of $SU(2)$ in Fig.

23. The name "sine-area" is motivated below. As usual, complete sine-area graphs satisfy $\theta_{\vec{p}} = 1$, $\forall\,\vec{p}$ and complementary sine-area graphs $\bar{G}$ are generated by $\bar{\theta}_{\vec{p}} = 1 - \theta_{\vec{p}}$.

To complete the definition of a generalized graph theory, we specify the graph isomorphisms,

$$G' \simeq G \tag{3.9.1}$$

which define equivalence classes of sine-area graphs analogous to the unlabelled graphs of graph theory. Guided by the master equation (3.5.7), we choose the graph isomorphisms

$$\theta'_{\vec{p}} = \theta_{\pi(\vec{p})},\quad \vec{p} \in F \tag{3.9.2a}$$

as those permutations π of $\{\vec{p}\}$ which satisfy

$$\rho_\pi(\vec{p})\rho_\pi(\vec{q})\,f_{\pi(\vec{p}),\pi(\vec{q})}{}^{\pi(\vec{r})} = f_{\vec{p},\vec{q}}{}^{\vec{r}}\,\rho_\pi(\vec{r}) \tag{3.9.2b}$$

$$\rho_\pi(\vec{p})\rho_\pi(\vec{q})\eta_{\pi(\vec{p}),\pi(\vec{q})} = \eta_{\vec{p},\vec{q}} \tag{3.9.2c}$$

where $\{\rho_\pi(\vec{p})\}$ is a set of phase factors $|\rho_\pi(\vec{p})| = 1$, $\vec{p} \in F$ for each allowed permutation.

Using the metric and structure constants of $SU(n)$ in (3.5.4,5) we obtain the explicit form of the graph isomorphisms in (3.9.2)

$$\pi[(\vec{p}+\vec{q})(\mathrm{mod}\,n)] = (\pi(\vec{p}) + \pi(\vec{q}))(\mathrm{mod}\,n) \tag{3.9.3a}$$

$$s_{\vec{p},\vec{q}} = \frac{\rho_\pi(\vec{p})\rho_\pi(\vec{q})}{\rho_\pi((\vec{p}+\vec{q})(\mathrm{mod}\,n))}\frac{\sigma(\pi(\vec{p}),\pi(\vec{q}))}{\sigma(\vec{p},\vec{q})}s_{\pi(\vec{p}),\pi(\vec{q})} \tag{3.9.3b}$$

$$\pi[(-\vec{p})(\mathrm{mod}\,n)] = (-\pi(\vec{p}))(\mathrm{mod}\,n) \tag{3.9.3c}$$

$$\sigma(\vec{p}) = \rho_\pi(\vec{p})\rho_\pi((-\vec{p})(\mathrm{mod}\,n))\sigma(\pi(\vec{p}))\,. \tag{3.9.3d}$$

where $\pi(\vec{p})$ is a permutation of $\vec{p} \in F$, $\{\rho_\pi(\vec{p})\}$ is a set of phase factors and $s_{\vec{p},\vec{q}} \equiv \sin(\pi(\vec{p}\times\vec{q})/n)$. The relations (3.9.3a) and (3.9.3b) need only be satisfied for those $\vec{p},\vec{q}$ pairs with non-zero $s_{\vec{p},\vec{q}}$.

The graph isomorphisms (3.9.2) are real magic-basis-preserving automorphisms of $SU(n)$ because the automorphisms

$$T'_{\vec{p}} = \rho_\pi(\vec{p}) T_{\pi(\vec{p})} \tag{3.9.4}$$

satisfy $T'^{\dagger}_{\vec{p}} = \sum_{\vec{q}} \eta_{\vec{p},\vec{q}} T'_{\vec{q}}$. It also follows from (3.9.3b) that the graph isomorphisms (3.9.2) are sine-area preserving

$$\{\text{sine-area}(\pi(\vec{p}), \pi(\vec{q}))\} = \{\text{sine-area}(\vec{p}, \vec{q})\} \tag{3.9.5}$$

where

$$\text{sine-area}(\vec{p}, \vec{q}) \equiv |\sin(\pi(\vec{p} \times \vec{q})/n)| \tag{3.9.6}$$

is the sine-area of any vector pair $(\vec{p}, \vec{q}\,|\theta_{\vec{p}} = \theta_{\vec{q}} = 1)$ in a sine-area graph. In particular, all the graphs in an equivalence class have the same set of sine-areas. All permutations $\pi(\vec{p})$ on $SU(2)$ are graph isomorphisms [1], and the equivalence classes of the sine-area graphs of $SU(2)$ are arranged horizontally in Fig. 23.

Following graph theory, a self-complementary sine-area graph

$$\bar{G} \simeq G \tag{3.9.7}$$

is recognized when it is isomorphic to its complement (see Section 2.5.5).

The high-level constraints $\theta_{\vec{p}} = \theta_{(-\vec{p})(\text{mod}\, n)}$ of $SU(n)_{metric}$ select a subset of sine-area graphs with dimension $2^{\nu(SU(n)_{metric})}$, which, according to eq.(3.4.4), are in one-to-one correspondence with the level-families of $SU(n)_{metric}$. As in $SO(n)_{diag}$, the complete sine-area graph on $SU(n)$ is the affine-Sugawara construction [3,5,9] on this manifold, complementary sine-area graphs are K-conjugate partners [3,5,10,15], and self-complementary sine-area graphs are self K-conjugate constructions.

[1] The 6 permutations of (10), (01) and (11) are generated by $\pi_1(01 \leftrightarrow 10)$ and $\pi_2(01 \leftrightarrow 11)$, which satisfy (3.9.4) with $\rho_{\pi_1}(10) = \rho_{\pi_1}(11) = -\rho_{\pi_1}(01) = 1$ and $\rho_{\pi_2}(01) = \rho_{\pi_2}(11) = i\rho_{\pi_2}(10) = i$.

In further detail, the action of any graph isomorphism π on a solution $L_{\vec{p}}$ in $SU(n)_{metric}$ is

$$L'_{\vec{p}} = L_{\pi(\vec{p})}, \quad \vec{p} \in F \tag{3.9.8}$$

which mirrors the high-level form in (3.9.2a). We have checked from (3.5.7), (3.9.2) and (3.9.8) that the constraints of $SU(n)_{metric}$ are invariant under any graph isomorphism

$$L_{\vec{p}} = L_{(-\vec{p})(\text{mod}\, n)} \quad \rightarrow \quad L'_{\vec{p}} = L'_{(-\vec{p})(\text{mod}\, n)}\,. \tag{3.9.9}$$

and that $L'_{\vec{p}}$ is automorphically equivalent to $L_{\vec{p}}$ in $SU(n)_{metric}$. The point is that, as in $SO(n)_{diag}$, the graph isomorphisms are residual automorphisms of the ansatz, so that isomorphically-equivalent sine-area graphs represent physically-equivalent level-families of conformal field theories in the ansatz. There are no constraints in $SU(2)_{metric}$, so all the sine-area graphs of $SU(2)$ are in $SU(2)_{metric}$, and the conformal- field theoretic names of the equivalence classes of $SU(2)_{metric}$ are included in Fig. 23.

As another example, the $2^4 = 16$ sine-area graphs of $SU(3)_{metric}$ are shown in Fig. 24, along with their equivalence classes and the corresponding conformal-field theoretic name of each class. In this case, the equivalence classes of the figure can be verified from eq.(3.6.3), which is the action of the graph isomorphisms on the set of independent variables. We remark in particular that each vertical pair in the self K-conjugate construction $SU(3)^{\#}[m, 2]$ is a pair of self-complementary sine-area graphs.

The enumeration of equivalence classes on $SU(n)$ is the next step in the study of sine-area graphs.

3.9.2 Generalized Graphs of g

It is clear that the high-level form of each g_{metric}

$$L_a^{(0)} = \theta_a \in \{0, 1\}, \quad a = 1, \cdots, \dim g \tag{3.9.10a}$$

$$\theta_a = \theta_b \text{ when } \eta_{ab} \neq 0 \tag{3.9.10b}$$

defines, as in $SU(n)_{metric}$, an "edge-list" of a generalized graph theory on g, although a subset of the generalized graphs will be selected when the constraints in (3.9.10b) are non-trivial.

The discussion in Section 3.9.1 can be generalized for any g, giving a prescription which defines a generalized graph theory on g, given a magic basis and the corresponding edge-list $\{\theta_a, a = 1, \cdots, \dim g\}$ of the corresponding ansatz g_{metric}. Without further information, we may think of the generalized graphs as the $2^{\dim g}$ sets of index-edges $\{a\}$ for which $\theta_a = 1$. The isomorphisms of the generalized graphs are those index-edge permutations π which satisfy

$$\theta'_a = \theta_{\pi(a)} \tag{3.9.11a}$$

$$\rho_\pi(a)\rho_\pi(b) f_{\pi(a)\pi(b)}{}^{\pi(c)} = f_{ab}{}^c \rho_\pi(c) \tag{3.9.11b}$$

$$\rho_\pi(a)\rho_\pi(b)\eta_{\pi(a)\pi(b)} = \eta_{ab} \tag{3.9.11c}$$

so that $\pi \in Aut\, g$. We have also checked that, when the magic basis is real and $\{\rho_\pi(a)\}$ is a set of phase factors, then π is a real magic-basis-preserving automorphism of g and

$$L'_a = L_{\pi(a)} \tag{3.9.12}$$

is automorphically equivalent to L_a in g_{metric}. It follows that isomorphically-equivalent generalized graphs are physically-equivalent level-families of conformal field theories in g_{metric}.

We finally note that the possibility of a simpler description exists when the edge-index a has a composite structure. In the case of the graphs of $SO(n)_{diag}$, for example, we have

$$a = ij, \quad 1 \leq i < j \leq n \tag{3.9.13}$$

and the set of allowed permutations $\in Aut\, SO(n)$ in (3.9.11) is the set of all permutations of the graph points $\{i\}$.

3.10 Conclusions

The general class of metric ansätze developed in this chapter, enables us to study large slices of affine-Virasoro space.

We have shown that, given any magic basis of simple g, the Virasoro master equation admits a metric ansatz g_{metric}

$$\text{magic basis of } g \rightarrow \text{conformal field theories of } g_{metric} \tag{3.10.1}$$

whose level-families are classified by their high-level form. The 1-1 correspondence between the high-level form of each solution and the exact level-family is the essence of the simplicity of the ansatz.

Each g_{metric} is a large system of generically irrational conformal field theories. Moreover, when a magic basis is real on compact g, high-level analysis shows that the level-families of the corresponding g_{metric} are generically unitary as well. Each of the two examples of a real magic basis, has resulted in a corresponding metric ansatz with remarkable structure: The level-families of $SO(n)_{diag}$ are isomorphic to the graphs of order n [29] and $SU(n)_{metric}$ shows irrational central charge at finite order of the high-level expansion. Both ansätze contain self K-conjugate constructions.

Guided by the master equation, we also defined the sine-area graphs of $SU(n)$, which label the conformal field theories of $SU(n)_{metric}$. In similar fashion, each magic basis of g defines a generalized graph theory on g, with graph isomorphisms in $Aut\, g$, whose generalized graphs label the conformal field theories of g_{metric}. The overall picture is

$$\begin{array}{ccc} & \text{conformal field theories of } g_{metric} & \\ \text{magic basis of } g \begin{array}{c}\nearrow \\ \searrow\end{array} & \uparrow\downarrow & \\ & \text{generalized graph theory on } g & \end{array} \tag{3.10.2}$$

and we expect that many other magic bases can be found, with their corresponding metric ansätze and generalized graph theories.

The master equation seems to define graph theories in order to classify its own solutions. This connection between the Virasoro master equation and graph theories has so far been the best organizing principle, classifying large subsets of the bewildering number of solutions of the master equation. In particular, each magic basis will give a corresponding ansatz g_{metric}, whose size is as large as one graph-theory unit. The mathematical structure that the master equation imposes on its own (generalized) graph theories, associating conformal field theories to (generalized) graphs, might be interesting both in physics and mathematics.

Bibliography

[1] M.A. Virasoro, Phys. Rev. D1 (1970) 2933.

[2] V.G. Kač, Funct. Anal. App. 1 (1967) 328; R.V. Moody, Bull. Am. Math. Soc. 73 (1967) 217.

[3] K. Bardakci and M.B. Halpern, Phys. Rev. D3 (1971) 2493.

[4] P. Ramond, Phys. Rev. D3 (1971) 2415.

[5] M.B. Halpern, Phys. Rev. D4 (1971) 2398.

[6] M.B. Halpern, Phys. Rev. D12 (1975) 1684.

[7] T. Banks, D. Horn and H. Neuberger, Nucl. Phys. B108 (1976) 119.

[8] J. Lepowsky and R.L. Wilson, Comm. Math. Phys. 62 (1978) 43; I.B. Frenkel and V.G. Kač, Inv. Math. 62 (1980) 23; G. Segal, Comm. Math. Phys. 80 (1981) 301.

[9] E. Witten, Comm. Math. Phys. 92 (1984) 455; G. Segal, unpublished; V.G. Knizhnik and A.M. Zamolodchikov, Nucl. Phys. B247 (1984) 83.

[10] P. Goddard, A. Kent and D. Olive, Phys. Lett. B152 (1985) 88.

[11] D. Gross, J.A. Harvey, E. Martinec and R. Rohm, Phys. Rev. Lett. 54 (1985) 502.

[12] J.K. Freericks and M.B. Halpern, Ann. of Phys. 188 (1988) 258; Erratum, ibid. 190 (1989) 212.

[13] M.B. Halpern, Ann. of Phys. 194 (1989) 247.

[14] P. Goddard and D. Olive, Int. J. Mod. Phys. A1 (1986) 303.

[15] M.B. Halpern and E. Kiritsis, Mod. Phys. Lett. A4 (1989) 1373; Erratum ibid. A4 (1989) 1797.

[16] M.B. Halpern and J.P. Yamron, Nucl. Phys. B332 (1990) 411.

[17] A.Yu Morozov, A.M. Perelomov, A.A. Rosly, M.A. Shifman and A.V. Turbiner, Int. J. Mod. Phys. A5 (1990) 803.

[18] M.B. Halpern, E. Kiritsis, N.A. Obers, M. Porrati and J.P. Yamron, Int. J. Mod. Phys. A5 (1990) 2275.

[19] E. Witten, in Memorial Volume for V. Knizhnik, ed. L. Brink et al., World Scientific, 1990.

[20] M.B. Halpern and N.A. Obers, Int. J. Mod. Phys. A6 (1991) 1835.

[21] S. Schrans and W. Troost, Nucl. Phys. B345 (1990) 584.

[22] M.B. Halpern and N.A. Obers, Nucl. Phys. B345 (1990) 607.

[23] M.B. Halpern and N.A. Obers, "*Graph Theory, SO(n) Current Algebra and the Virasoro Master Equation*", Berkeley preprint, UCB-PTH-90/31, (1990). To appear in Comm. Math. Phys.

[24] M.B. Halpern and J.P. Yamron, Nucl. Phys. B351 (1991) 333.

[25] A. Giveon, M.B. Halpern, E.B. Kiritsis and N.A. Obers, "*Exact C-Function and C-Theorem on Affine-Virasoro Space*", Berkeley preprint, UCB-PTH-90/48, (1990). To appear in Nucl. Phys. B

[26] A. Giveon, M.B. Halpern, E.B. Kiritsis and N.A. Obers, "*The Superconformal Master Equation*", Berkeley preprint, UCB-PTH-91/2, (1991). To appear in Int. J. Mod. Phys. A

[27] N. Mohammedi, "*General Super-Virasoro Construction on Affine g*", Trieste preprint, PRINT-90-0645 (1990).

[28] E. Ragoucy and P. Sorba, Phys. Lett. B256 (1991) 30.

[29] F. Harary and E.M. Palmer, "*Graphical enumeration*", New York, London, Academic Press 1973.

[30] A.T. White, "*Graphs, groups and surfaces*", Amsterdam, London, North-Holland/American Elsevier 1973.

[31] G. 't Hooft, Nucl. Phys. B138 (1978) 1; A. Belavin, JETP Lett. 32 (1980) 169; A. Gonzalez-Arroyo and M. Okawa, Phys. Rev. D27 (1983) 2397; T. Eguchi and R. Nakayama, Phys. Lett. B122 (1983) 59; J. Patera and H. Zassenhaus, J. Math. Phys. 29 (1988) 85; D.B. Fairlie, P. Fletcher and C.K. Zachos, Phys. Lett. B218 (1989) 203, J. Math. Phys. 31 (1990) 1088; I. Bars, Phys. Lett. B245 (1990) 35.

Tables

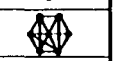
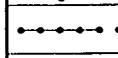
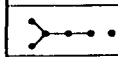
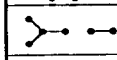
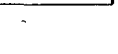
Table 1. Connected constructions in $SO(n)_{diag}$

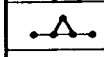
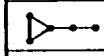
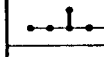
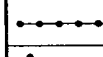
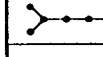
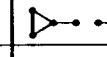
manifold $SO(n)_{diag}$	all constructions g_n	connected constructions C_n	fundamental affine-Sugawara nests $C(AS)_n$	new connected constructions $C_n^{\#}$
$SO(1)$	1	1	1	0
$SO(2)$	2	1	1	0
$SO(3)$	4	2	2	0
$SO(4)$	11	6	5	1
$SO(5)$	34	21	12	9
$SO(6)$	156	112	33	79
$SO(7)$	1,044	853	90	763
$SO(8)$	12,346	11,117	261	10,856
$SO(9)$	274,668	261,080	766	260,314
$SO(10)$	12,005,168	11,716,571	2,312	11,714,259

Table 2. Irreducible constructions in $SO(n)_{diag}$

manifold $SO(n)_{diag}$	total irreducible constructions ir_n	irreducible affine-Sugawara nests $ir(AS)_n$	new irreducible constructions $ir_n^{\#}$
$SO(1)$	1	1	0
$SO(2)$	1	1	0
$SO(3)$	1	1	0
$SO(4)$	2	1	1
$SO(5)$	9	1	8
$SO(6)$	69	1	68
$SO(7)$	663	1	662
$SO(8)$	9,889	1	9,888
$SO(9)$	247,493	1	247,492
$SO(10)$	11,427,975	1	11,427,974

Table 3. The graphs of $SO(6)_{diag}$

c_0	G	auto G	dim auto L(G)	conformal construction L(G)	$\{2\Delta_1^{(0)}\}$	$\tilde{G}$
0		S_6	1	L=0	(0,0,0,0,0,0)	
1		$Z_2 \times S_4$	15	SO(2)	(0,0,0,0,1,1)	
2		$Z_2 \times S_3$	60	SO(3)/SO(2)	(0,0,0,1,1,2)	
		$(Z_2)^4$	45	$(SO(2))^2$	(0,0,1,1,1,1)	
3		$S_3 \times S_3$	20	SO(3)	(0,0,0,2,2,2)	
		$S_3 \times Z_2$	60	SO(4)/SO(3)	(0,0,1,1,1,3)	
		$Z_2 \times Z_2$	180	$SO(4)^{\#}[d,4]$	(0,0,1,1,2,2)	
		$Z_2 \times Z_2$	180	$(SO(3)/SO(2)) \times SO(2)$	(0,1,1,1,1,2)	
		$(Z_2)^3 \times S_3$	15	$(SO(2))^3$	(1,1,1,1,1,1)	
4		$D_4 \times Z_2$	45	$SO(4)/(SO(2))^2$	(0,0,2,2,2,2)	
		$Z_2 \times Z_2$	180	SO(4)/SO(3)/SO(2)	(0,0,1,2,2,3)	
		S_4	30	SO(5)/SO(4)	(0,1,1,1,1,4)	
		Z_2	360	$SO(5)^{\#}[d,6]_2$	(0,1,1,2,2,2)	
		Z_2	360	$SO(5)^{\#}[d,7]_1$	(0,1,1,1,2,3)	
		$S_3 \times Z_2$	60	$SO(3) \times SO(2)$	(0,1,1,2,2,2)	
		$S_3 \times Z_2$	60	$(SO(4)/SO(3)) \times SO(2)$	(1,1,1,1,1,3)	
		$(Z_2)^3$	90	$(SO(3)/SO(2))^2$	(1,1,1,1,2,2)	
		$Z_2 \times Z_2$	180	$SO(4)^{\#}[d,4] \times SO(2)$	(1,1,1,1,2,2)	

c_0	G	auto G	dim auto L(G)	conformal construction L(G)	$\{2\Delta_1^{(0)}\}$	$\tilde{G}$
5		$(Z_2)^3$	90	SO(4)/SO(2)	(0,0,2,2,3,3)	
		$Z_2 \times Z_2$	180	SO(5)/SO(4)/SO(2)	(0,1,1,2,2,4)	
		S_5	6	SO(6)/SO(5)	(1,1,1,1,1,5)	
		D_5	72	$SO(5)^{\#}[d,2]$	(0,2,2,2,2,2)	
		Z_2	360	$SO(5)^{\#}[d,6]_1$	(0,1,1,2,3,3)	
		Z_2	360	$SO(5)^{\#}[d,7]_2$	(0,1,2,2,2,3)	
		Z_2	360	$SO(5)/SO(5)^{\#}[d,7]_2$	(0,1,2,2,2,3)	
		$(Z_2)^3$	90	$SO(6)^{\#}[d,5]_1$	(1,1,1,1,3,3)	
		S_3	120	$SO(6)^{\#}[d,7]_1$	(1,1,1,1,2,4)	
		Z_2	360	$SO(6)^{\#}[d,9]_1$	(1,1,1,2,2,3)	
		Z_2	360	$SO(6)^{\#}[d,9']_1$	(1,1,2,2,2,2)	
		Z_2	360	$SO(6)^{\#}[d,11]_1$	(1,1,1,2,2,3)	
		$D_4 \times Z_2$	45	$(SO(4)/(SO(2))^2) \times SO(2)$	(1,1,2,2,2,2)	
		$S_3 \times Z_2$	60	$(SO(3)/SO(2)) \times SO(3)$	(1,1,2,2,2,2)	
		$Z_2 \times Z_2$	180	$(SO(4)/SO(3)/SO(2)) \times SO(2)$	(1,1,1,2,2,3)	
6		$S_4 \times Z_2$	15	SO(4)	(0,0,3,3,3,3)	
		$Z_2 \times S_3$	60	$SO(5)/(SO(3) \times SO(2))$	(0,2,2,2,3,3)	
		$(Z_2)^3$	90	$SO(5)/SO(4)/(SO(2))^2$	(0,2,2,2,2,4)	
		Z_2	360	SO(5)/SO(4)/SO(3)/(SO(2))	(0,1,2,2,3,4)	
		$Z_2 \times S_3$	60	SO(6)/SO(5)/SO(2)	(1,1,1,2,2,5)	

c_0	G	auto G	dim auto L(G)	conformal construction L(G)	$\{2\Delta_1^{(0)}\}$	$\tilde{G}$
6		Z_2	360	$SO(5)/SO(5)^{\#}[d,6]_2$	(0,2,2,2,3,3)	
		Z_2	360	$SO(5)/SO(5)^{\#}[d,7]_1$	(0,1,2,3,3,3)	
		D_6	60	$SO(6)^{\#}_{M} \equiv SO(6)^{\#}[d,3]$	(2,2,2,2,2,2)	
		S_3	120	$SO(6)^{\#}[d,4]$	(1,1,1,3,3,3)	
		Z_2xZ_2	180	$SO(6)^{\#}[d,7]_1$	(1,1,2,2,3,3)	
		Z_2xZ_2	180	$SO(6)^{\#}[d,8]_1$	(1,1,2,2,3,3)	
		Z_2xZ_2	180	$SO(6)^{\#}[d,8]_2$	(1,1,2,2,2,4)	
		Z_2	360	$SO(6)^{\#}[d,9]_2$	(1,2,2,2,2,3)	
		Z_2	360	$SO(6)^{\#}[d,9]_2$	(1,1,2,2,3,3)	
		Z_2	360	$SO(6)^{\#}[d,11]_2$	(1,2,2,2,2,3)	
		Z_2	360	$SO(6)^{\#}[d,11]_3$	(1,1,1,2,3,4)	
		Z_2	360	$SO(6)^{\#}[d,11]_4$	(1,2,2,2,2,3)	
		Z_2	360	$SO(6)^{\#}[d,11]_5$	(1,1,2,2,2,4)	
		I	720	$SO(6)^{\#}[d,15]_1$	(1,1,2,2,3,3)	
		$(S_3)^2xZ_2$	10	$(SO(3))^2$	(2,2,2,2,2,2)	
		$(Z_2)^3$	90	$(SO(4)/SO(2))xSO(2)$	(1,1,2,2,3,3)	
7		Z_2xS_3	60	$SO(5)/SO(3)$	(0,2,2,2,4,4)	
		S_3	120	$SO(5)/SO(4)/SO(3)$	(0,1,3,3,3,4)	
		Z_2xZ_2	180	$SO(5)/((SO(3)/SO(2))x(SO(2))$	(0,2,3,3,3,3)	
		$(Z_2)^3$	90	$SO(6)/SO(5)/(SO(2))^2$	(1,2,2,2,2,5)	

c_0	G	auto G	dim auto L(G)	conformal construction L(G)	$\{2\Delta_1^{(0)}\}$	$\tilde{G}$
7		Z_2xZ_2	180	$SO(6)/SO(5)/SO(3)/SO(2)$	(1,1,2,2,3,5)	
		Z_2	360	$SO(5)/SO(4)^{\#}[d,4]$	(0,2,2,3,3,4)	
		$(Z_2)^3$	90	$SO(6)^{\#}[d,5]_2$	(2,2,2,2,3,3)	
		Z_2xZ_2	180	$SO(6)^{\#}[d,6]$	(2,2,2,2,3,3)	
		S_3	120	$SO(6)^{\#}[d,7]_2$	(1,2,2,2,3,4)	
		Z_2xZ_2	180	$SO(6)^{\#}[d,7]_2$	(1,1,3,3,3,3)	
		Z_2xZ_2	180	$SO(6)^{\#}[d,7]_3$	(1,1,2,2,4,4)	
		Z_2xZ_2	180	$SO(6)^{\#}[d,7]_4$	(2,2,2,2,3,3)	
		Z_2xZ_2	180	$SO(6)^{\#}[d,8]_3$	(1,1,2,3,3,4)	
		Z_2xZ_2	180	$SO(6)^{\#}[d,8]_4$	(2,2,2,2,2,4)	
		Z_2xZ_2	180	$SO(6)^{\#}[d,8]_5$	(1,2,2,3,3,3)	
		Z_2	360	$SO(6)^{\#}[d,9]_3$	(2,2,2,2,3,3)	
		Z_2	360	$SO(6)^{\#}[d,9]_4$	(1,2,2,3,3,3)	
		Z_2	360	$SO(6)^{\#}[d,11]_6$	(1,2,2,3,3,3)	
		Z_2	360	$SO(6)^{\#}[d,11]_7$	(1,2,2,2,3,4)	
		Z_2	360	$SO(6)^{\#}[d,11]_8$	(1,2,2,2,3,4)	
		I	720	$SO(6)^{\#}[d,15]_2$	(1,2,2,3,3,3)	
		I	720	$SO(6)^{\#}[d,15]_3$	(1,2,2,2,3,4)	
		I	720	$SO(6)^{\#}[d,15]_4$	(1,1,2,3,3,4)	
		S_4xZ_2	15	$SO(4)xSO(2)$	(1,1,3,3,3,3)	

Figure Captions

Fig.1 a) Connected graphs;

b) Disconnected graphs.

Fig.2 complete graphs = affine-Sugawara constructions = fundamental affine-Sugawara nests of depth 1.

Fig.3 K-conjugate graphs on $SO(3)$ and $SO(4)$.

Fig.4 complete N-partite graphs = fundamental coset constructions = fundamental affine-Sugawara nests of depth 2.

Fig.5 a) fundamental coset graphs with depth 1 insertion = fundamental affine-Sugawara nests of depth 3;

b) fundamental coset graphs with depth 2 insertion = fundamental affine-Sugawara nests of depth 4;

c) fundamental coset graphs with depth $d-2$ insertion = fundamental affine-Sugawara nests of depth d.

Fig.6 Complementary representation of fundamental affine-Sugawara nested graphs.

Fig.7 Irreducible and reducible graphs in graph space. The dashed lines indicate the action of K-conjugation on the graphs of each category.

Fig.8 a) The complete bipartite graph $SO(6)/(SO(5)\times SO(1)) = SO(6)/SO(5)$;

b) The fundamental affine-Sugawara nested graph $SO(6)/((SO(3)/SO(2))\times(SO(1))^3)$ $= SO(6)/SO(3)/SO(2)$.

Fig.9 The first eight identity graphs are new constructions in $SO(6)_{diag}$.

Fig.10 The symmetry factor of this nests is $S(G(SO(2)\times SO(1)))\cdot S(G(SO(1))\cdot S(G(SO(1)) = 2$.

Fig.11 a) Connected incomplete bipartite graphs, or broken N=2 coset graphs, are new irreducible constructions.

b) The broken N=2 coset graph $SO(6)/SO(6)^{\#}[d,5]_2$.

Fig.12 The cycles $C_{2n}, n \geq 3$ and paths $P_n, n \geq 4$ are broken N=2 coset graphs and hence new irreducible constructions.

Fig.13 a) Broken N=2 affine-Sugawara nested graphs are new irreducible constructions.

b) The broken N=2 affine-Sugawara nested graph $SO(6)^{\#}[d,4]$.

Fig.14 The first six self-K-conjugate constructions.

Fig.15 The Cartesian product graphs $G_1 \times (K_2 \times K_2)$.

Fig.16 The maximal-symmetric construction $SO(2n)^{\#}_M$.

Fig.17 The n-cubes $Q_n = K_2 \times Q_{n-1}, n \geq 3$ are broken N=2 coset graphs and hence new irreducible constructions.

Fig.18 The labelling is used to obtain the subansatz of the graph.

Fig.19 $SO(2n)^{\#}[d,4]$.

Fig.20 $SO(2n+1)^{\#}[d,6]$.

Fig.21 Central charges in $SU(3)_{metric}$: The crossing at level three is the deformation $SU(3)^{\#}_3$.

Fig.22 High-level flow of the level-families in $SU(3)_{metric}$. The brackets show the set $\{\vec{p}\}$ for which $\theta_{\vec{p}} = 1$.

Fig.23 The sine-area graphs of $SU(2)$.

Fig.24 The sine-area graphs of $SU(3)_{metric}$.

Figures

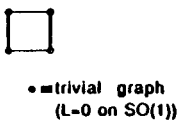


Fig. 1a

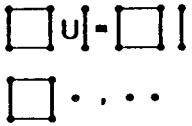


Fig. 1b

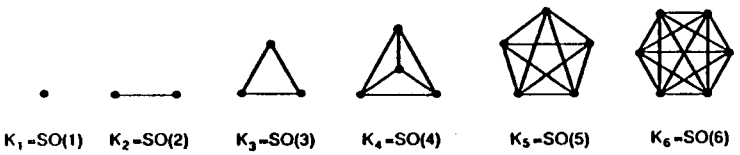


Fig. 2

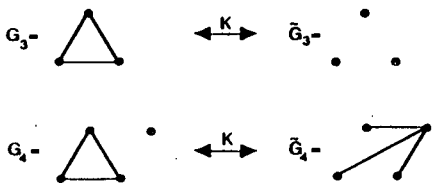
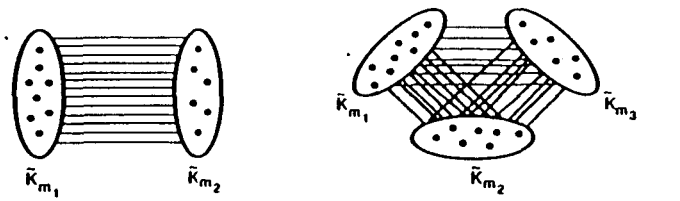


Fig. 3



N=2: $SO(m_1+m_2)/(SO(m_1)xSO(m_2))$ N=3: $SO(m_1+m_2+m_3)/(SO(m_1)xSO(m_2)xSO(m_3))$

Fig. 4

Fig. 5a

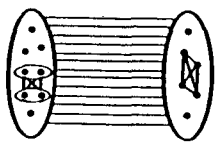


Fig. 5b

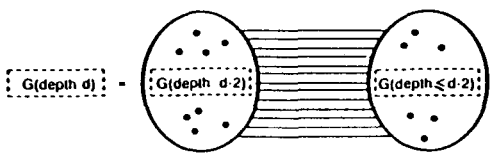


Fig. 5c

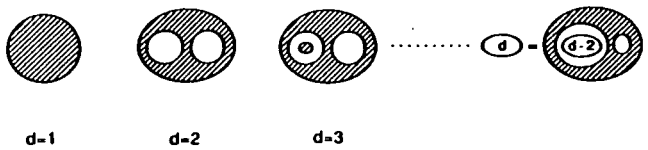


Fig. 6

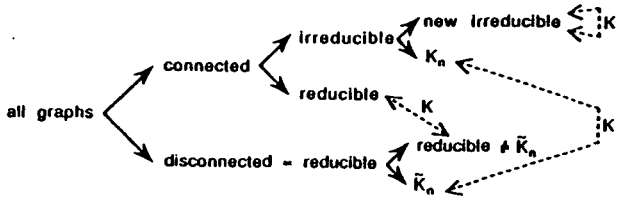


Fig. 7

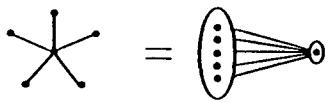


Fig. 8a

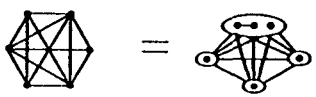


Fig. 8b

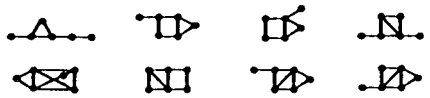


Fig. 9

$$\left(\frac{\frac{SO(5)}{SO(4)}}{\left(\frac{SO(3)}{SO(2)\times SO(1)}\right)\times SO(1)}\right)\times SO(1) =$$

Fig. 10

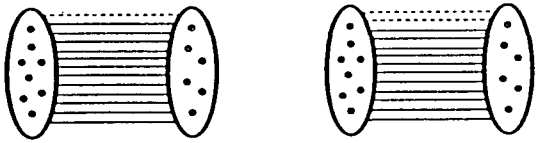


Fig. 11a

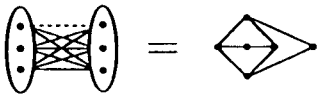


Fig. 11b

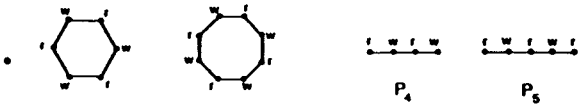


Fig. 12

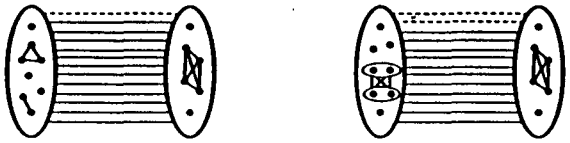


Fig. 13a

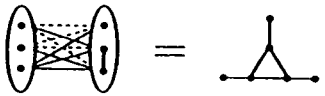


Fig. 13b

Fig. 14

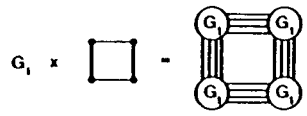


Fig. 15

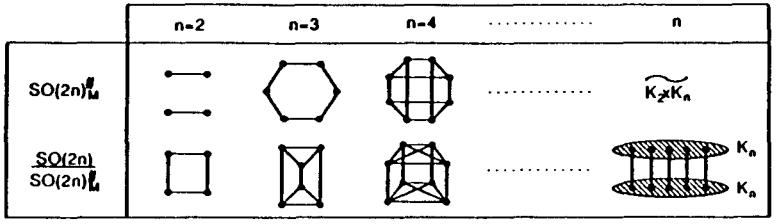


Fig. 16

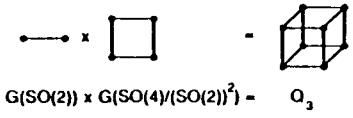


Fig. 17

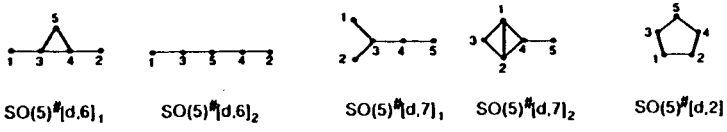


Fig. 18a Fig. 18b Fig. 18c

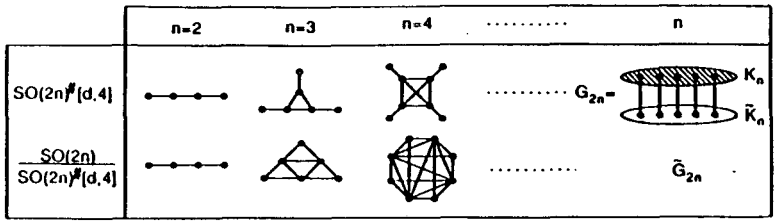


Fig. 19

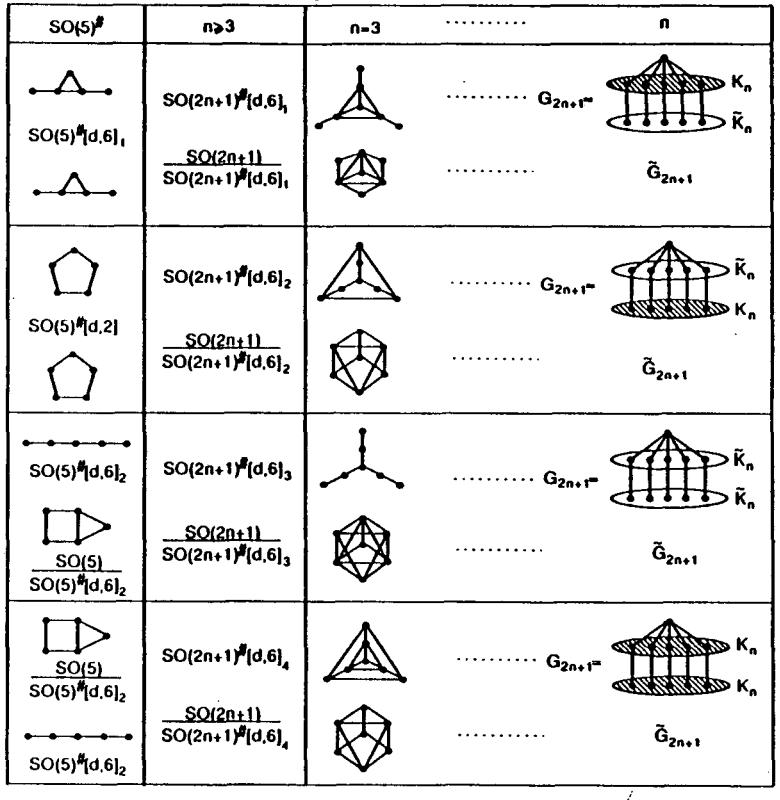


Fig. 20

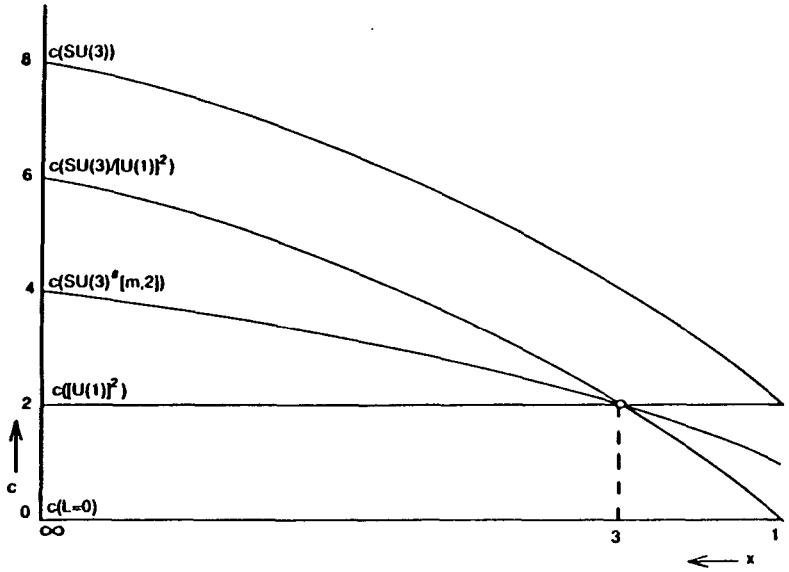


Fig. 21

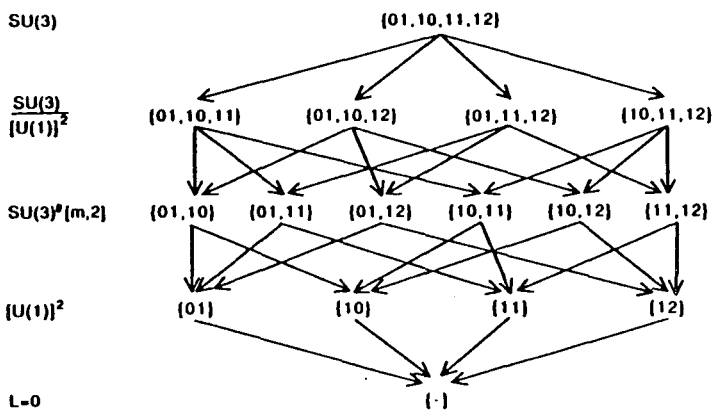


Fig. 22

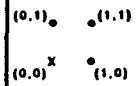
conformal field theory	sine-areas	G
L=0	(·)	
U(1)	(·)	
SU(2)/U(1)	(1)	
SU(2)	(1,1,1)	

Fig. 23

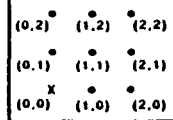
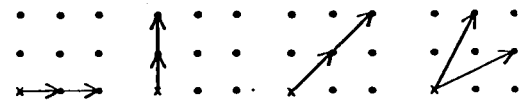
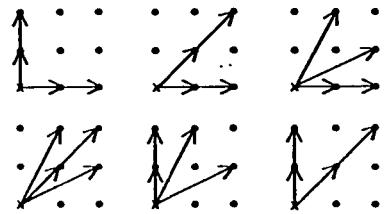
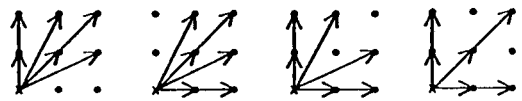
conformal field theory	sine-areas	G
L=0	(-)	
$[U(1)]^2$	(0)	
$SU(3)^{\#}[m,2]$	$([0]^2,[\frac{\sqrt{3}}{2}]^4)$	
$\frac{SU(3)}{[U(1)]^2}$	$([0]^3,[\frac{\sqrt{3}}{2}]^{12})$	
SU(3)	$([0]^4,[\frac{\sqrt{3}}{2}]^{24})$	

Fig. 24

LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
INFORMATION RESOURCES DEPARTMENT
BERKELEY, CALIFORNIA 94720