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E.M. Schlueter, R.W. Zimmerman, N.G. W. Cook,
and P.A. Witherspoon

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*E. M. Schlueter,*[*,†] *R. W. Zimmerman,*[†] *N. G. W. Cook,*[*,†] *and P. A. Witherspoon*[*,†]

[*]Department of Materials Science and Mineral Engineering
University of California

[†]Earth Sciences Division
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

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*E. M. Schlueter, R. W. Zimmerman, N. G. W. Cook, and P. A. Witherspoon*

ABSTRACT

Fractal dimensions of pores in five sedimentary rocks are estimated from scanning electron micrographs of thin sections. It is found that the area-perimeter relationship of the pores follows the law derived by Mandelbrot for islands whose boundaries are fractal: $P \simeq A^{D/2}$, where $D$ is the fractal dimension of the pore perimeter. The fractal dimensions of the pores of four sandstones were found to lie between 1.31 and 1.40, while that of an Indiana limestone was found to be 1.20. A brief discussion is given of how the fractal dimension, along with a pore-size distribution, can be used to estimate the permeability.

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When the microstructure of a typical sedimentary rock is examined under an optical microscope at low resolution, the pore-grain interfaces appear to be smooth. However, when the scanning electron microscope with higher resolution is used instead (see Fig. 1), the pore-grain interface commonly appears rough, and fractal surface behavior has often been found to span all length scales below the pore size.[1,2] Possible diagenetic mechanisms by which the pore surfaces may become fractal have been discussed by several researchers.[3,4] In this letter, we show that the pore cross-sections of various reservoir-type sedimentary rocks satisfy the perimeter-area relation for fractal islands derived by Mandelbrot,[5] in which $P \approx A^{D/2}$. Since the fractal dimension is a parameter that quantifies the roughness of the pore surfaces, it is reasonable to expect that it has some influence on the permeability of the rock, which is a physical property of extreme importance in many areas of the earth sciences. In particular, since the permeability measures the viscous resistance of the rock to fluid flowing through its pores, permeability would be expected to correlate with the amount of surface area of the pore system. We will (below) construct a model that allows reasonable predictions of the permeability, based on the fractal dimension of the pores and the pore-size distribution. In contrast to previous models that have attempted to relate permeability to fractal dimension,[6] all of the parameters in our model have an unambiguous physical meaning, and are readily measurable from scanning electron micrographs of rock thin sections.

For each family of standard planar shapes, geometrically similar but of different sizes, one can define a characteristic length as[7]

$$\varepsilon = Perimeter / Area^{1/2} . \tag{1}$$

The ratio $\varepsilon$ is the same for each of the similarly-shaped features, and is independent of their size. For example, since the perimeter of a circle of radius $r$ is equal to $2\pi r$,

and the area of the disc bounded by such a circle is $\pi r^2$, it follows that

$$\varepsilon(circle) = P/A^{1/2} \tag{2}$$

Similarly, the ratios $\varepsilon = 4$ and $\varepsilon = 6/3^{1/4}$ can be found for squares and equilateral triangles, respectively. Hence, each of these shapes obeys a relationship of the form $P = \varepsilon A^{1/2}$, where $\varepsilon$ is some constant.

Consider now a collection of similar "islands" with *fractal* "coastlines" of length $P(\alpha)$ and area $A(\alpha)$. Mandelbrot[5] has shown that for each island whose boundary is a fractal curve, there exists an analogue of $\varepsilon$ that can be defined by

$$\varepsilon = P^{1/D}/A^{1/2}, \tag{3}$$

where $D$ is the fractal dimension of the coastlines of the similarly-shaped islands. The area and length of each of the islands is measured by using an area-dependent yardstick $\alpha^* = \rho[A_i(\alpha)]^{1/2}$ for the $i-th$ island, with $\rho$ being an arbitrary but fixed small parameter, and $\alpha$ a fixed yardstick. The length of the coastline of the $i-th$ island is $P_i(\alpha^*) = N_\rho \alpha^*$, where $N_\rho$ is the number of segments of length $\alpha^*$ needed to traverse the perimeter. For similarly-shaped islands, $N_\rho$ is independent of the size of the island. From the definition of Hausdorff-Besicovitch fractal dimension[7] in the limit of small $\alpha$,

$$P_i(\alpha) = P_i^0 \alpha^{(1-D)} = P_i(\alpha^*)[\alpha/\alpha^*]^{(1-D)}. \tag{4}$$

Thus,

$$P_i(\alpha) = N_\rho \alpha^{(1-D)} \alpha^{*D} = N_\rho \rho^D \alpha^{(1-D)} [A_i(\alpha)]^{D/2} . \tag{5}$$

Therefore, islands that are similar in form will satisfy the following perimeter-area relation:[5,8]

$$P(\alpha) = \varepsilon^D [A(\alpha)]^{D/2} , \tag{6}$$

where the parameter $\varepsilon$ depends on the length of the measuring yardstick, $\alpha$. This equation holds for any given yardstick $\alpha$ that is small enough to measure the smallest island accurately.

In order to study the pore structure of the rock samples, specimens of cylindrical shape were first impregnated with a blue-dyed epoxy. Thin sections were then prepared and imaged (Fig. 1) with a scanning electron microscope (SEM). The basic method involves counting size and perimeter pixel units for every feature in a standard SEM micrograph of some fixed magnification. The field imaged by the micrograph must contain a large enough number of pores to assure a statistically representative sample; we have found that 30-40 pores suffice for this purpose. The analysis was carried out using both a manual and an automated image analysis procedure. The manual technique involved overlaying a square grid, with grid size of 2.54 mm, and visually counting the number of grid blocks occupied by the area of each pore, as well as the number of grid blocks that the perimeter passes through. Digital images with typical image sizes of $482 \times 640$ pixels, and 8 bits per pixel to quantify the darkness level, were used for comparison with the manual technique. The size of each pixel was about 3 μm on a side. The image analysis program sets a threshold level of darkness to distinguish between pore space and mineral grains. The digitized thin section (Fig. 2) then shows mineral grains in black, and pore space in white. This method was used to estimate the area-perimeter statistics for groups of pores in a thin-section.

Individual pores were also studied by changing the magnification of the SEM to cover feature sizes from approximately 10 μm to sizes larger than the typical grain size.

According to Eq. (6), the fractal dimension $D$ can be estimated from the slope of a plot of $\ln P$ *vs.* $\ln A$, since $d\ln P/d\ln A = D/2$. Analysis of the perimeter - area data from several reservoir rocks confirms their fractal nature (see Figs. 2 and 3). The constants $D$ and $\varepsilon$ appearing in Eq. (6) were found by performing a linear regression on the log perimeter - log area data. From this analysis we find pore fractal dimensions ranging from 1.31 to 1.40 for the four sandstones examined, and 1.20 for Indiana limestone (Table 1). However, it is important to realize that existence of a power-law relationship between area and perimeter, with a non-integer slope, is necessary but not sufficient for the shapes to be fractals. For example, imagine a section in which there was a large circular pore, followed by a somewhat smaller hexagonal pore, and then a still smaller pentagonal pore, etc. Such a section would also exhibit a $\ln P$ *vs.* $\ln A$ slope that differed from 1/2, but these shapes are obviously not fractals.

The mathematical analysis described above that led to the relationship between the fractal dimension $D$ and the slope $d\ln A/d\ln P$ was predicated on the assumption that the different features were of different sizes, but that the boundary of each feature was a fractal of the same dimension $D$. One way to test this assumption is to examine a single feature under different magnifications. As an example, consider the Berea sandstone pore shown in Fig. 4 under various levels of magnification from 54X - 120X. A plot of the relationship between $\ln P$ and $\ln A$ for this pore under the different magnifications (see Fig. 5) yields a fractal dimension of 1.33, which is very close to the value of 1.31 that was estimated from the plot of $\ln A$ *vs.* $\ln P$ for *different* features. This correspondence verifies the fractal-like quality of the pore boundaries, at least over a certain range of scales.

The permeability of a rock is a physical property of extreme importance in many areas of geophysics and the earth sciences. Since the permeability is controlled by the

geometry and topology of the pore space, it is not unreasonable to expect that it will be somehow related to the fractal dimension of the pore space. We now show how the fractal dimension of the pore space of a rock can be used, in conjunction with a classical model for permeability, to yield reasonable estimates of the permeability. The Kozeny-Carman model[10] for transport through a porous medium is based on the idealization of the pore space as consisting of a bundle of parallel tubes, the total conductance of which is merely the sum of the individual conductances. It is traditional to then divide this result by a tortuosity factor $\tau=3$, to account for the fact that in an hydraulically isotropic rock, we would expect only one-third of the total number of tubes to be oriented in each of the three orthogonal directions. If $n(A)$ is the number-distribution function for pores of cross-sectional area $A$ in an area of rock having total cross-section of $A_{total}$, and $C(A)$ is the conductance of each pore of area $A$, then the total conductance can be expressed as

$$C_{total} = \int_0^\infty n(A)C(A)dA \ . \tag{7}$$

In practice, of course, the distribution function $n(A)$ will vanish for all $A$ greater than some $A_{max}$, although it is often convenient to represent $n(A)$ by a function that drops off, say, exponentially as $A \to \infty$.

If the pore tubes were all of circular cross-section, their individual conductances would be given by the exact Hagen-Poiseuille law.[11] The Hagen-Poiseuille solution can be modified to account for irregular cross-sections by using the "hydraulic radius" approximation, which predicts[11] a conductance of $A^3/2P^2$ for a tube of cross-sectional area $A$ and perimeter $P$. Invoking the fractal power-law relationship $P=\varepsilon^D A^{D/2}$, the hydraulic conductance can be expressed as $C(A) = A^{3-D}/2\varepsilon^{2D}$. Combining this with the general expression (7) for the total conductance yields

$$C_{total} = \frac{1}{\tau}\int_0^\infty \frac{n(A)A^{3-D}}{2\varepsilon^{2D}}dA \ . \tag{8}$$

We now define a normalized distribution function $\beta(A) = n(A)A/\phi A_{total}$, where the total porosity $\phi$ is defined as $A_{pores}/A_{total}$. This distribution function has the property that $\int\beta(A)dA = 1$. The total conductance can now be expressed as

$$C_{total} = \frac{\phi A_{total}}{2\varepsilon^{2D}\tau}\int_0^\infty A^{2-D}\beta(A)dA \ . \tag{9}$$

For example, assume that the pore-size distribution can be fit to a lognormal distribution:

$$\hat{\beta}(u) = (2\pi\sigma_u^2)^{-1/2}\exp[-(u-u_m)^2/2\sigma_u^2] \ , \tag{10}$$

where $u = \ln A$, $u_m$ is the mean value of $\ln A$, and $\sigma_u$ is the variance of $\ln A$. The permeability coefficient $k$ can then be estimated as

$$k = \frac{C_{total}}{A_{total}} = \frac{\phi}{2\varepsilon^{2D}\tau}\int_0^\infty A^{2-D}\beta(A)dA$$

$$= \frac{\phi}{2\sqrt{\pi}\tau\varepsilon^{2D}\sigma_u}\int_{-\infty}^{\infty} e^{(2-D)u}\exp[-(u-u_m)^2/2\sigma_u^2]du$$

$$= \frac{\phi}{2\tau\varepsilon^{2D}}e^{(2-D)u_m}e^{\sigma_u^2(2-D)^2/2} \tag{11}$$

For a lognormally-distributed random variable, one can show that the expected value of $A^n$ is related to the mean value of $A$ by $\overline{A^n} = (\bar{A})^{n^2} e^{n(1-n)}$. Denoting the mean and variance of $A$ by $A_m$ and $\sigma_A$, we can derive relationships between mean and variance of $A$ and the mean and variance of $\ln A$, which then allows us to rewrite Eq. (11) in the form

$$k = \frac{\phi}{2\tau\varepsilon^{2D}} (A_m)^{2-D} [1 + (\sigma_A^2 / A_m)^2]^{(2-D)^2} . \tag{12}$$

As with the standard Kozeny-Carman model, if the pore *sizes* are held constant, the predicted permeability scales with the porosity, which is to say it is proportional to the *number* of pores. Since the parameter $\varepsilon$ quantifies the perimeter of the pore cross-sections, when the pores are projected back into three-dimensions $\varepsilon$ will in some sense represent the pore surface area; hence $k$ is a decreasing function of $\varepsilon$. Finally, Eq. (11) shows that $k$ is an increasing function of both the mean pore size and the variance of the pore size, as would be expected for an essentially parallel arrangement of conductors, as is assumed in our model. Our result also bears a resemblance to that derived by Hansen and Skjeltorp,[6] whose expression for $k$ included some length scale raised to the $2-D$ power. Our result is more explicit in that our length scale is clearly identified in terms of the pore-size distribution.

Image analysis of the pore system of the Berea sandstone yields the values $A_m = 77.9\,\mu\text{m}^2$, $\sigma_A = 60.9\,\mu\text{m}^2$, $\varepsilon = 0.731\,\mu\text{m}^{-1/2}$, and $D = 1.31$. Equation (12) then predicts a permeability of $1.51 \times 10^{-12}$ m$^2$ (about 1.5 darcys), which is of the same order of magnitude as our experimentally measured value of $0.46 \times 10^{-12}$ m$^2$. Since the permeability of rocks can range over many orders of magnitude, from about $10^{-11}$ m$^2$ down to about $10^{-20}$ m$^2$, this prediction is not trivial. Of course, more accurate estimates of the permeability will require more sophisticated models[9,12] than that

of parallel tubes, which somehow account for factors such as the interconnectedness of the pore tube network and the converging-diverging nature of the pore channels; the above example was intended to be a plausible demonstration of the use of fractal information for making quantitative predictions of the permeability.

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[a] Also at the Department of Materials Science and Mineral Engineering, University of California, Berkeley.

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[11] See, for example, F. M. White, *Viscous Fluid Flow* (McGraw-Hill, New York, 1974). The error incurred by using the hydraulic radius approximation for irregular cross-sections is typically less than 30%, which is acceptable for our purposes.

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Table 1. Fractal dimension $D$ and correlation coefficient $r$ measured from perimeter-area data of five sedimentary rocks.

| Rock | $D$ | $r$ |
|---|---|---|
| Berea sandstone | 1.31 | 0.99 |
| Boise sandstone | 1.40 | 0.98 |
| Massilon sandstone[9] | 1.40 | 0.98 |
| Saint-Gilles sandstone | 1.34 | 0.98 |
| Indiana limestone | 1.20 | 0.99 |

## Figure Captions

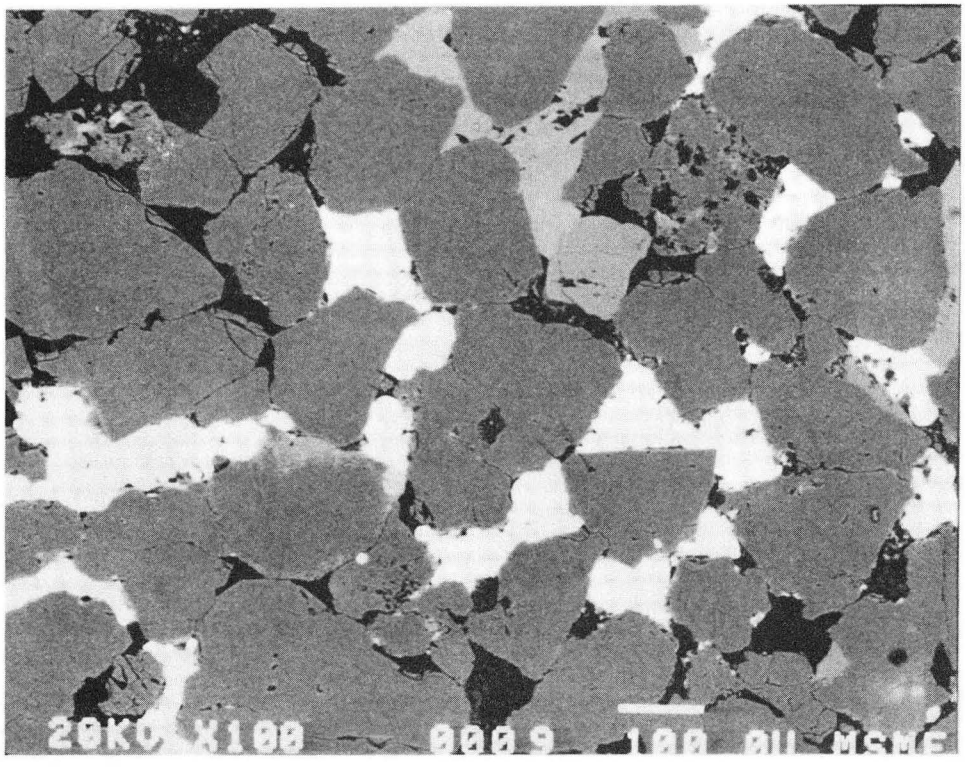
Fig. 1. Typical thin section of Berea sandstone. The rock is composed mainly of quartz grains (dark-gray phase), feldspar grains (medium-gray phase), and products of grain dissolution (light-grey phase). The pore space is impregnated with Wood's metal alloy (white phase) and epoxy (black phase).

Fig. 2. Pore-space contours obtained from computerized image analysis of a Berea sandstone thin section.

Fig. 3. Fractal area-perimeter relationship for Berea sandstone. The fractal dimension $D$ is equal to twice the slope $d\ln P/d\ln A$ [see Eq. (6)].

Fig. 4. Pore-space contours from a Berea sandstone pore at different magnifications.

Fig. 5. Fractal area-perimeter relationship for a Berea sandstone pore, at different magnifications. The inferred fractal dimension of 1.33 agrees closely with the value 1.31 determined from the plot in Fig. 3.

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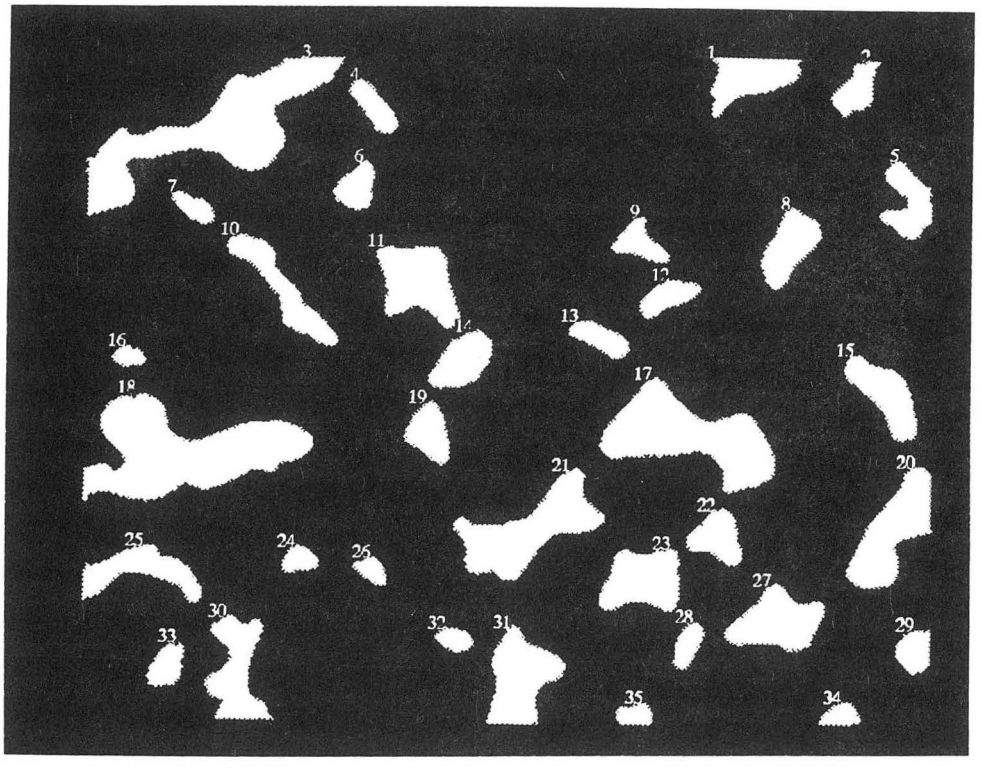


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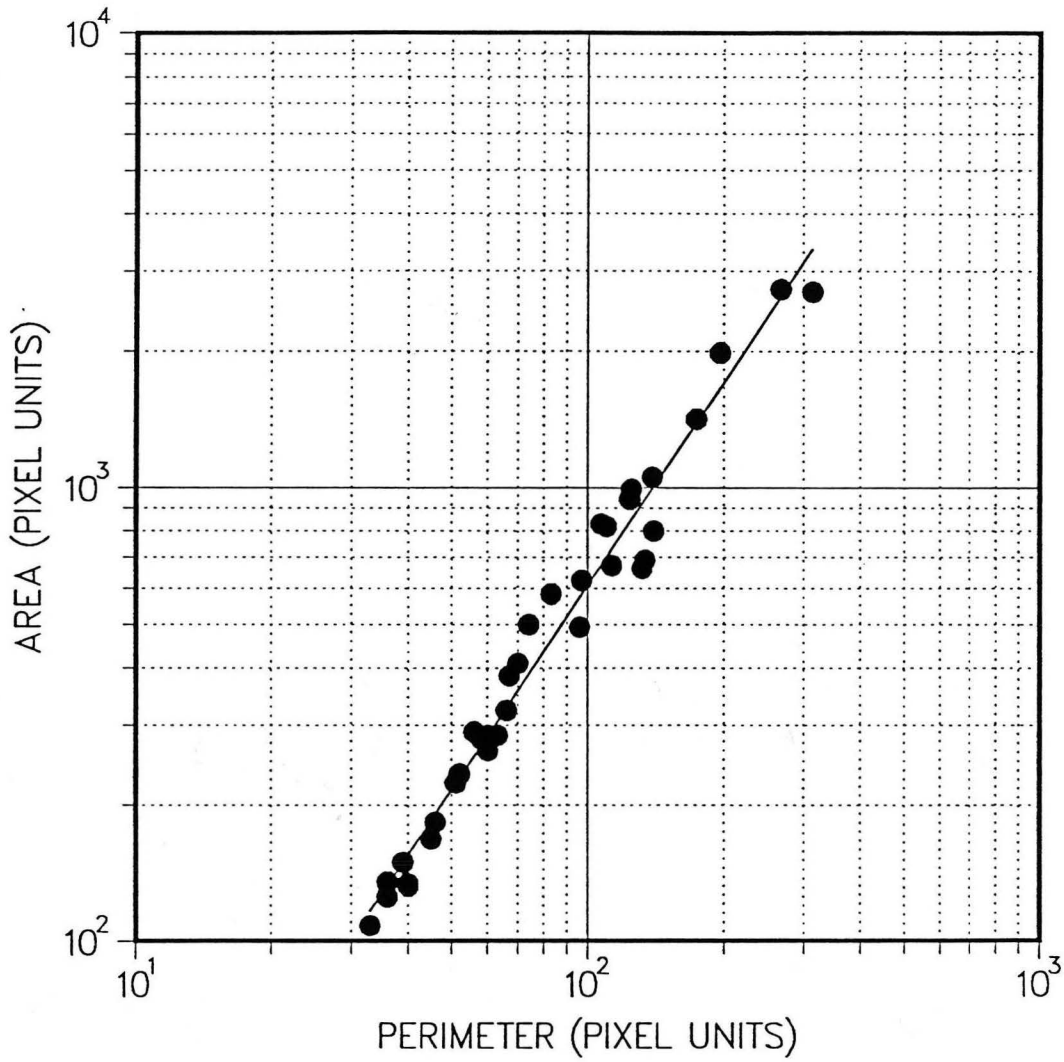


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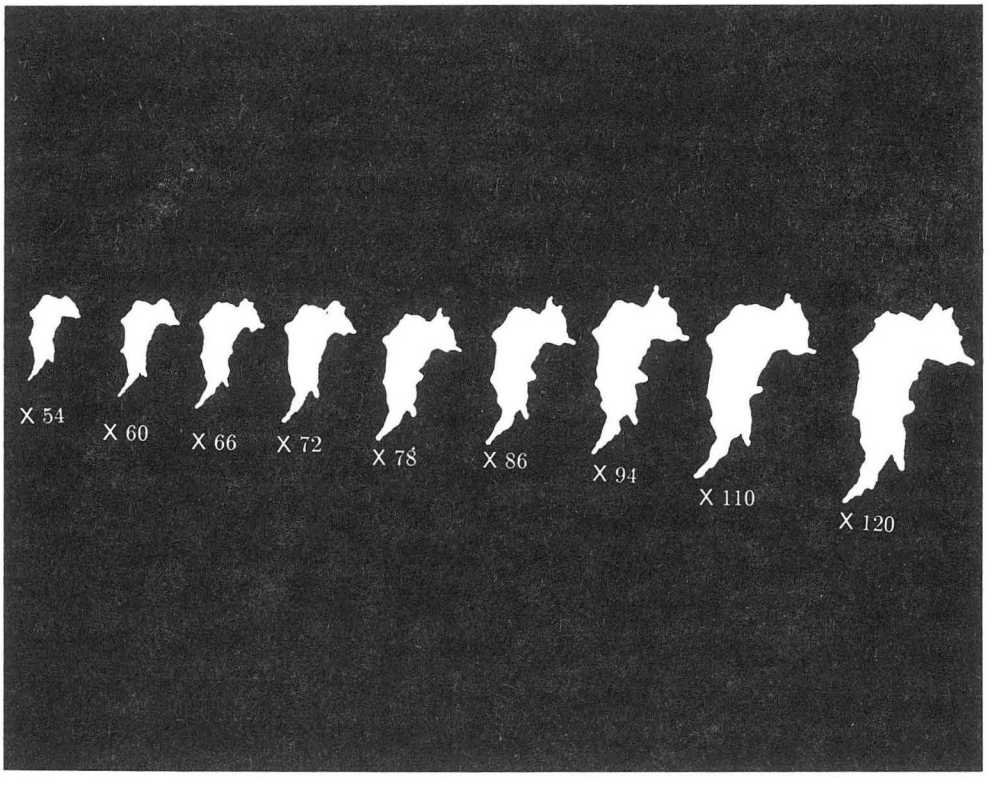
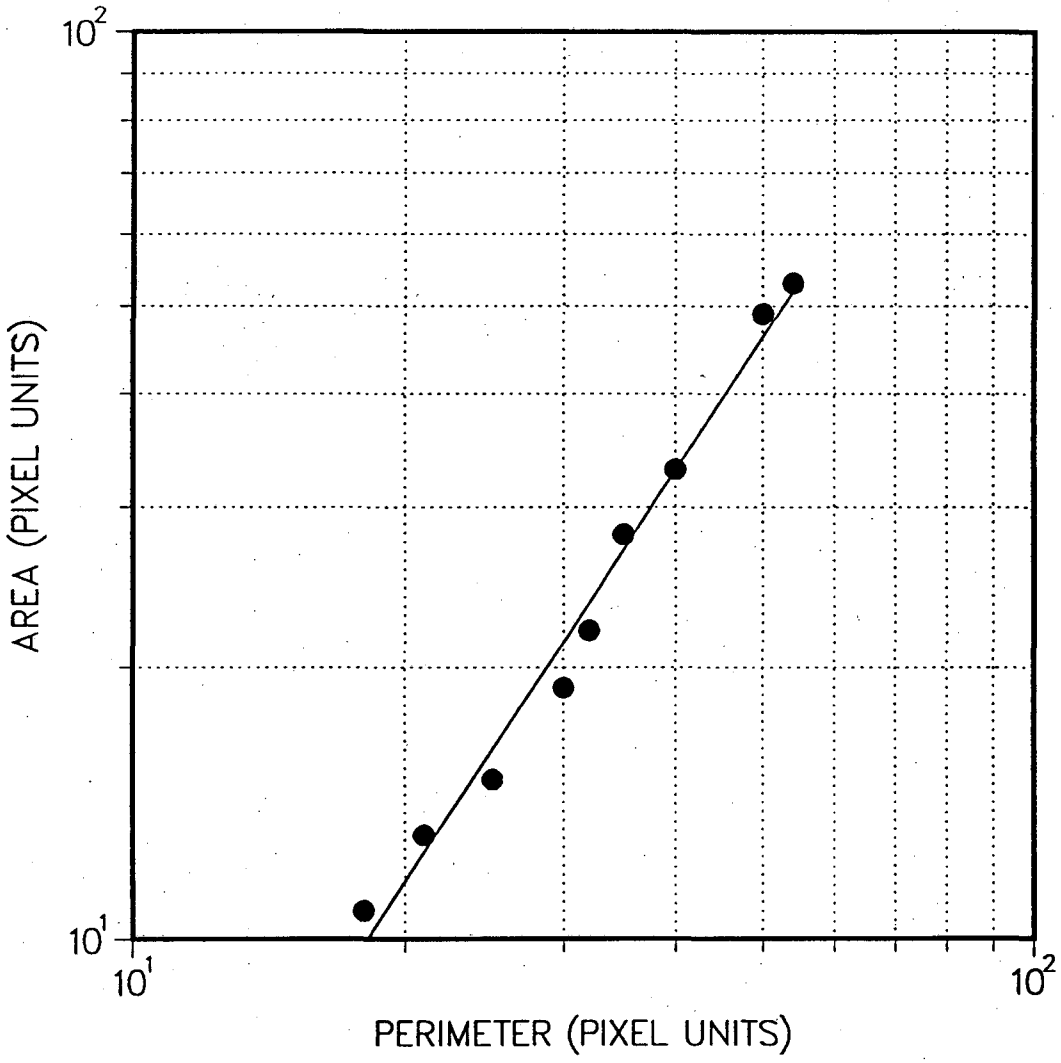



Fig. 4. Pore-space contours from a Berea sandstone pore at different magnifications.

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LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
INFORMATION RESOURCES DEPARTMENT
BERKELEY, CALIFORNIA 94720