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Topics in Particle Physics and Cosmology

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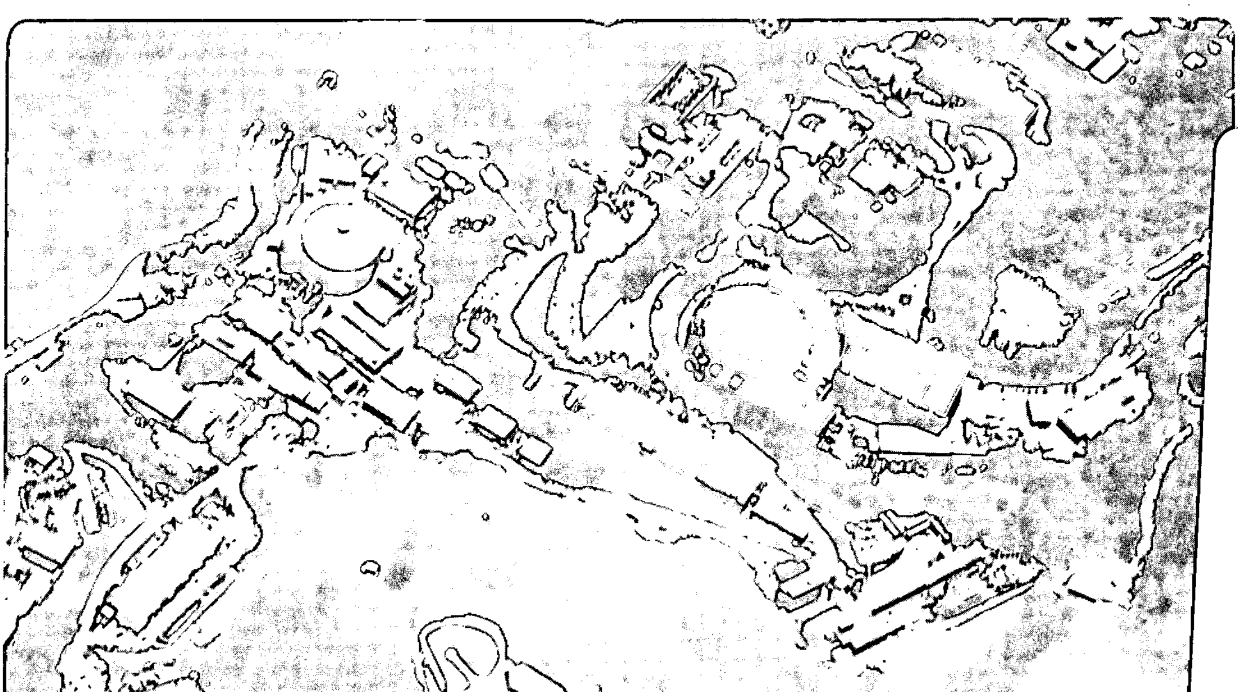

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Topics in Particle Physics and Cosmology *

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Ph.D. Dissertation

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Abstract

The Standard Model of particle physics, together with the Big Bang model of the early universe, constitute a framework which encompasses our current understanding of fundamental laws and beginning of our universe. Despite recent speculative trends, quantum field theory remains the theoretical tool of choice for investigating new physics either at high energy colliders, or in the early universe. In this dissertation, several field theoretic phenomena relevant to cosmology or particle physics are explored.

A common theme in these explorations is the structure of the vacuum state in quantum field theory. First, we discuss first-order phase transitions in the early universe, in which the effective vacuum state of the universe shifts discontinuously as the temperature drops below some critical point. We find that the dynamics of a certain type of first-order phase transitions can lead to production of primordial black holes, which could constitute the dark matter of our universe. Alternatively, supercooled first-order phase transitions may be the cause of an extended inflationary epoch in the early universe, which is generally regarded as necessary to solve several cosmological puzzles. We derive limits on such scenarios based on nearly model-independent percolation properties of the transition.

We also study some nonperturbative aspects of the field theory vacuum. We show that non-topological solitons of a single fermion and Higgs fields can only exist in strongly coupled theories. In particular, we find that at the lowest energy fermionic excitations in the Standard Model are single fermions, and not bound states of fermion plus Higgs. Finally, we investigate the intriguing behavior of instanton-induced cross sections. We discover Higgs-Higgs cross sections which increase exponentially with center of mass energy due to the presence of instanton solutions related to vacuum instability.

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Topics in Particle Physics and Cosmology

By

Stephen Dao-Hui Hsu

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For my parents, who deserve all of the credit.

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Contents

Chapter I

The Big Picture - an overview of topics

A Introduction

Theorists of this generation have inherited a dazzling array of ideas, models and theories with which to describe the world around us. The Seventies and Eighties have provided confirmation after confirmation that quantum field theories, and in particular gauge theories, offer correct descriptions of the (thus far discovered) fundamental interactions. We even have at our disposal the $SU(3)\otimes SU(2)\otimes U(1)$ Standard Model of particle physics, which thus far accounts extremely well for all experimental high energy data.

Faced with the phenomenological successes of the Standard Model, and with our seemingly satisfactory understanding (at least at the perturbative level) of its theoretical underpinnings, theoretical physicists have opted to pursue many different lines of investigation. Some look to higher, or lower, dimensionalities in search of "interesting" mathematical structures. Others attempt to extend, or generalize the Standard Model, hoping to account for some of its shortcomings. Still others look to cosmology or astrophysics for clues as to new physics. The attitude adopted in this thesis is that there are many interesting physical phenomena which occur quite generically in quantum field theories in four dimensions. We will investigate the relevance of some of these phenomena to long standing puzzles of cosmology and astrophysics, as well as to new high energy physics.

The common theme (if one exists) of the investigations in this thesis is the subject of the field theory vacuum, or ground state. This subject is extremely important in cosmology, as dramatic effects often result from phase transitions at finite temperature. These effects include inflation, baryogenesis, creation of topological and non-topological solitons and many other non-equilibrium phenomena. It is crucial that we have a first-principles understanding of the dynamics of these transitions in order to conduct future investigations of the early universe.

The relevance of the ground state to particle physics is of course obvious. The symmetries of the vacuum may not coincide with those of the Lagrangian, and in that case the theory is said to exhibit spontaneous symmetry breaking. This is precisely what is observed in both QCD and the electroweak sector. In the case of QCD, approximate global symmetries are broken by strong interactions of which we have little first-principles understanding. In the case of electroweak interactions it is gauge symmetries which are broken, and due to our present ignorance we can at least pretend that they are broken due to perturbative dynamics (i.e., the Higgs sector). Future experimental efforts at the Superconducting Super Collider and Large Hadron Collider will focus on determining the dynamics of electroweak symmetry breaking, whether it be due to perturbative scalar dynamics, nonperturbative dynamical symmetry breaking, or something yet undreamt of.

The models we investigate in this thesis are all of the first type. That is, the vacuum state will always be determined by a perturbatively calculable scalar effective potential. These models allow a large range of interesting behavior, including various types of phase transitions at finite temperature, all of which can be studied using perturbative and semiclassical techniques. They also allow the possibility of multiple vacua, and in the case of weak coupling the resulting nonperturbative effects can be studied using semiclassical techniques such as instantons. (See, however, chapters IV and V for examples where semiclassical techniques can break down.)

Of course, purely scalar theories have naturalness problems, as no symmetry can protect scalar masses from quadratic divergences. These quadratic divergences tend to force the scalar mass to be roughly the cut-off of the theory. For the Standard Model, this cut-off is typically thought to be the Planck or Grand Unification scale, and the naturalness problem found in the Higgs sector is referred to as the Hierarchy problem. There are two well known solutions to the Hierarchy problem. In supersymmetric theories, extra scalars and fermions are added in just the proper representations to cancel all quadratic divergences. An alternative is to rid the theory of scalar fields entirely, and rely on nonperturbative dynamics of strongly coupled gauge theories to break symmetries and give fermion masses. This alternative is generically referred to as technicolor or dynamical symmetry breaking. In the supersymmetric case the effects of the new weakly coupled fields can be integrated out, leaving us again with a scalar effective potential. The second case, however, involves strong couplings and therefore is much less well understood, even with QCD as a model.

It is certainly possible in principle that strongly coupled gauge theories may exhibit dynamics and finite temperature behavior which is qualitatively different from that found in scalar models. However, the range of phenomena that can be studied in scalar models is quite broad, and the techniques (both perturbative and nonperturbative) are well developed, so we will adopt the attitude that they provide a good guide for the types of phenomena likely to be found in more complicated theories.

In the remainder of this introduction we will review the theoretical foundations for the investigations that are to follow in subsequent chapters.

B The effective potential

The effective potential is a much studied and extremely useful object [1]. It is the quantum analog of the classical potential, and therefore determines the lowest energy state of a quantum system. Consider for the moment a classical scalar field theory given by the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi) \tag{I.1}$$

The lowest energy translationally invariant configuration of this theory is obviously just $\phi_c(x) = constant = \phi_{min}$, where ϕ_{min} is the minimum of $U(\phi)$.

When we quantize the theory $U(\phi)$ loses its interpretation as an energy density, and instead gives the various interaction vertices for the quantized field ϕ. However, one's intuition suggests that, at least when quantum effects are "small" (we will quantify this in a moment), the vacuum state of the full theory is not so different from ϕ_c. In other words, there should at least be some limit in which our classical intuition holds true.

Now consider the construction of the effective potential, $V_{eff}(\phi)$. Let $Z[J]$ be the generating functional of n-point functions $G(x_1, ..., x_n) \equiv \langle\phi(x_1), ..., \phi(x_n)\rangle$. Then connected Green's functions are generated by $iW[J] = lnZ[J]$, and one-particle-irreducible (1PI) truncated Green's functions $i\Gamma^n$ (vertex functions for $n > 2$) are generated by $\Gamma[\overline{\phi}]$, which is just the Legendre transform of W[J]:

$$W[J] = \Gamma[\overline{\phi}] + \int d^4x J(x)\overline{\phi}(x), \tag{I.2}$$

where

$$\overline{\phi}(x) = \langle 0^+|\hat{\phi}(x)|0^-\rangle = \frac{\delta W[J]}{\delta J(x)} = \frac{\int D\phi\ \phi(x)\ e^{iS[\phi,J]}}{\int D\phi\ e^{iS[\phi,J]}} \tag{I.3}$$

is the expectation value of the field operator $\hat{\phi}$ in the presence of source J. Γ can be formally expanded in terms of the 1PI Green's functions:

$$\Gamma[\overline{\phi}] = \sum_{n=2}^{\infty} \frac{1}{n!} d^4x_1...d^4x_n\overline{\phi}(x_1)...\overline{\phi}(x_n)\Gamma^n(x_1...x_n). \tag{I.4}$$

We note that any connected Green's function generated by $W[J]$ can now be written as the sum of tree graphs in a theory whose "effective action" is given by $\Gamma[\phi]$. That is, the quantum theory described by $W[J]$ is equivalent to a classical (i.e., involving only tree graphs, and no loops) theory described by the effective action $\Gamma[\overline{\phi}]$.

Defining $\tilde{\Gamma}(p_1...p_n)$ as the Fourier transform of $\Gamma(x_1...x_n)$, and taking $\overline{\phi}(x) = \overline{\phi}$ to be a homogeneous, stationary state yields

$$\Gamma[\overline{\phi}] = \sum_{n+2}^{\infty} \frac{\overline{\phi}^{\,n}}{n!} \int d^4x\tilde{\Gamma}(0, ..., 0). \tag{I.5}$$

Since the effective action $\Gamma[\overline{\phi}]$ can be written in terms of a derivative expansion,

$$\Gamma[\overline{\phi}] = \int d^4x[-V(\overline{\phi}) + A(\partial_\mu\overline{\phi})^2 + ...]. \tag{I.6}$$

We see that for constant $\phi(x) = \overline{\phi}$, (I.5) can be identified with $V(\phi) \equiv V_{eff}(\phi)$, which we will henceforth refer to as the effective potential. For constant configurations a solution to $dV_{eff}/d\overline{\phi} = 0$ is also a solution to $\delta\Gamma/\delta\overline{\phi} = 0$. From (I.2) and (I.3) we see that

$$\frac{\delta\Gamma}{\delta\overline{\phi}} = -J. \tag{I.7}$$

Therefore, if $dV_{eff}/d\overline{\phi} = 0$ is satisfied for some nonzero $\overline{\phi}$, we will have found a nonzero stationary point for the effective action, even when the source $J(x)$ vanishes. This is precisely the signal for spontaneous symmetry breaking. To study the properties of the broken theory, we can define a shifted field with zero expectation value,

$$\phi' = \phi - \overline{\phi}, \tag{I.8}$$

and determine the spectrum of the broken theory by expanding perturbatively about the new vacuum state.

At this point it is enlightening to stop for a moment and consider the relationship of this formal machinery to the intuitive classical expectations expressed early on. In particular, how is the new effective potential related to the classical potential $U(\phi)$? By examining in greater detail the expression (I.5), we see that each term $\Gamma^n(x_1...x_n)$ will generally have contributions from tree graphs (i.e., from $U(\phi)$ itself) and from higher order processes (i.e., loops) that are suppressed by the assumed small couplings of the interactions. Specifically, for the scalar theory at hand, V_{eff} can be computed exactly to one loop, yielding

$$V_{eff} = U + i\int \frac{d^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{U''(\phi)}{k^2+i\epsilon}\right) \tag{I.9}$$

$$= U + \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} ln(k^2 + U''(\phi) - i\epsilon). \tag{I.10}$$

We note in passing that the second expression can only be obtained by analytic continuation of the first expression from where it is well defined (i.e., $k^2 > U''(\phi)$). If $U(\phi)$ contains only renormalizable interactions (dimension four or less), then the divergences in (I.10) can be absorbed into counterterms of the form already appearing in the theory.

It is well known that the loop expansion in field theory is equivalent to an expansion in powers of Planck's constant $\hbar$. Therefore, it is obvious that our expression for V_{eff} has the correct classical limit $U(\phi)$ as we take $\hbar \to 0$. Furthermore, in the limit of weak coupling we expect the effective potential to closely resemble the classical potential. In fact, one can show that the effective potential is equal to the stationary expectation value of the Hamiltonian

$$V_{eff}(\overline{\phi}) = \langle a|H|a\rangle \tag{I.11}$$

(i.e., for states $|a\rangle$ such that $\delta\langle a|H|a\rangle = 0$) under the constraints $\langle a|a\rangle = 1$ and $\langle a|\hat{\phi}|a\rangle = \overline{\phi}$. That is, it gives the minimum energy of all normalized states which yield the specified vacuum expectation value.

This particular definition is problematic if one allows states $|a\rangle$ which are not localized in ϕ. By localized, we mean states which are smeared versions of eigenstates of ϕ. If we include nonlocalized states the set of $|a\rangle$'s is too general, and the effective potential so defined is always convex, contrary to our classical intuition. If, however, we restrict ourselves to localized $|a\rangle$'s, V_{eff} can be nonconvex, and even complex in the region where it is nonconvex [2]. The restriction to localized states does not commute with the Hamiltonian, and the complex part is related to an instability for the localized state to decay into a superposition of states.

The restriction to localized states is relevant to our purpose of studying ground states and phase transitions, and the definition of the effective potential given in (I.2) in fact coincides with that restriction. We will continue referring to this object as the effective potential, although others may prefer to call it the one-particle irreducible potential, since it is computed by summing 1PI graphs at a particular order in the loop expansion.

We note briefly here that interesting effects can arise due to quantum corrections even in weak coupling. Consider the standard model in the limit of a heavy top quark (i.e., $m_t >> m_Z, m_{Higgs}$) [3]. The largest radiative corrections to the Higgs potential in this case come from the top itself, and lead to an instability. That is,

$$V_{eff} = -(2A+B)\sigma^2\phi^2 + A\phi^4 + B\phi^4 ln(\phi^2/\sigma^2), \tag{I.12}$$

where $m^2_{Higgs} = 4\sigma^2(2A+3B)$, $\langle\phi\rangle = \sigma = 246\,\mathrm{GeV}$, and

$$B = \frac{1}{64\pi^2}(6m_W^4 + 3m_Z^4 + m_{Higgs}^4 - 12m_t^4). \tag{I.13}$$

Note that a heavy top quark causes B to be negative, and hence V_{eff} to become unbounded from below for large values of ϕ. It can be shown that higher loop effects restore the boundedness of the potential from below, but a new lower minimum does appear at large values of ϕ. While it may be disturbing that the vacuum we observe (i.e., the one with SU(2) violating vacuum expectation value equal to 250 GeV) is not the true minimum, it is certainly consistent with observational constraints as long as our vacuum is metastable and has a very long lifetime. There will, however, be nonperturbative effects due to the nontrivial structure of the effective potential at large field values. Some of these effects, particularly their relation to Green's functions and cross sections calculated in the metastable vacuum, will be discussed in chapter V.

We now turn to finite temperature effects on the effective potential. The finite temperature effective potential can be interpreted as the free energy of a system in equilibrium statistical mechanics, and therefore V_{eff} is useful for determining the equilibrium state of the system. To study field theory at finite temperature, we imagine placing the system in contact with a thermal bath at temperature T. Then the background in which we study a particular scattering process is no longer an empty vacuum, but rather a thermal distribution of particles. We can now define finite-temperature Green's functions in terms of ensemble averages,

$$G^{\beta}(x_1 ... x_2) = N\sum e^{-\beta E(a)}\langle a|\phi(x_1)...\phi(x_2)|a\rangle \tag{I.14}$$

$$= \frac{Tr\,[e^{-\beta H}\phi(x_1)...\phi(x_2)]}{Tr\ e^{-\beta H}}. \tag{I.15}$$

The sum runs over a complete set of states $|a\rangle$ with energies $E(a)$, and can be written as a trace in the space of states.

In analogy with the zero temperature case we can define a generating functional $Z^{\beta}[J]$, from which the above Green's functions can be obtained. Then

$$Z^{\beta}[J] = \frac{Tr\,[e^{-\beta H} exp(i\int d^4x J(x)\phi(x))]}{Tr\ e^{-\beta H}}, \tag{I.16}$$

and $G^\beta(x_1...x_n)$ can be obtained by varying $Z^\beta[J]$ with respect to $J(x_1)...J(x_n)$ in the usual way. The important point is that $Z^\beta[J]$ can be written in terms of a Euclidean path integral over scalar fields satisfying periodic (antiperiodic in the case of fermions) boundary conditions in Euclidean time $\beta\hbar$. In particular [4],

$$Z^\beta[J] = N\int D\phi_\beta\, exp(-\int_0^\beta dx_0 \int d^3x\mathcal{L}_E - JA) \tag{I.17}$$

where the measure $D\phi_\beta$ is restricted to periodic paths $\phi(0,\vec{x}) = \phi(\beta,\vec{x})$, $\mathcal{L}_E$ is the Euclidean Lagrangian and N is chosen to make $Z^\beta[0] = 1$.

We now see that, at least formally, the only difference between zero and finite temperature field theory is in the boundary conditions imposed on the fields. Having made this modification in boundary conditions, we can bring to bear all of the techniques (i.e., diagrammatic expansion or functional methods) originally developed for the zero temperature case. In particular, we can see quite directly what becomes of the effective potential when we go to finite temperature. Consider equation (I.10). Due to the periodicity in t_0, integrals over k_0 become sums over discrete energies given by $\omega_m^2 = \beta^{-2} + \vec{k}^2$. This gives

$$V_{eff} = U + \frac{1}{2\beta}\int\sum_m \frac{d^3k}{(2\pi)^4} ln(\omega_k^2 + \vec{k}^{\,2} + U''(\phi) - i\epsilon). \tag{I.18}$$

Simplifying this expression, and subtracting off the zero temperature effective potential, yields the temperature dependent correction,

$$\Delta_\beta V_{eff}(\phi) = \frac{1}{2\pi^2\beta}\int_{-\infty}^{\infty} dk\; k^2 ln(1 - e^{-\beta E_k}), \tag{I.19}$$

where $E_k^2 = \vec{k}^2 + U''(\overline{\phi})$. Although we have said little about renormalization up to this point, we note that $\Delta_\beta V_{eff}(\phi)$ is finite, and hence the zero temperature counterterms necessary to ensure finiteness also suffice for the finite temperature potential.

In the high temperature limit ($\overline{\phi}\beta << 1$) the correction becomes

$$\Delta_\beta V_{eff}(\phi) = \frac{U''(\overline{\phi})}{24\beta^2} - \frac{\pi^2}{90\beta^4} + O(\overline{\phi}^4). \tag{I.20}$$

If, for example, the Lagrangian contains a ϕ^4 interaction, the first term has the form of a temperature dependent mass term ($\overline{\phi}\,^2T^2$), and therefore tends to increase the free energy of any state with nonzero vacuum expectation value $\overline{\phi}$. This term ensures that at sufficiently high temperature the state of lowest free energy (the finite temperature equivalent of the ground state, henceforth referred to as the thermal ground state) is the maximally symmetric state with zero vacuum expectation value. We therefore see that symmetries which are spontaneously broken at zero temperature

may be "restored" at finite temperature. The exact details of how the system adjusts its thermal ground state as the temperature is continuously decreased (the case of interest in the early universe) will be discussed in what follows.

C The early universe

In this section we will review the basic assumptions that make up the Hot Big Bang model of the early universe. This model has several notable successes [28], including its prediction of the microwave background and of primordial light element abundances produced in nucleosynthesis. It also provides possible mechanisms which can account for various observations such as the overabundance of matter versus antimatter (the problem of baryogenesis) and the preponderance of dark versus luminous matter (the dark matter problem) in our universe.

To specify a model of the early universe we need two ingredients: a physical theory which fixes the dynamics and interactions, and a set of initial conditions. We will take for our theory classical general relativity plus a quantum field theory that is (hopefully) consistent with current high energy experiments. Formally, then, the theory can be written in terms of the following action:

$$S = \int d^4x\sqrt{-g}\left[\frac{-1}{16\pi G}R + \mathcal{L}_{matter}\right] \tag{I.21}$$

where G is Newton's constant, R is the Einstein curvature scalar and $\mathcal{L}_{matter}$ is the particle physics Lagrangian.

This still leaves us a great deal of freedom, as there are countless proposed models of particle physics beyond the standard model, each of which has its own cosmological implications. As for initial conditions, we will take the universe to be homogeneous and isotropic and at some initial temperature T_i.

The metric of a spacetime with homogeneous and isotropic three dimensional subspace can always be written in the Robertson-Walker form,

$$ds^2 = dt^2 - R^2(t)(\frac{dr^2}{1-kr^2} - r^2d\Omega^2). \tag{I.22}$$

Here $R(t)$ is the Robertson-Walker scale factor, r is a dimensionless spatial coordinate, and the parameter k describes the geometrical curvature of the three dimensional spatial subspace:

$$^3R = \frac{6k}{R(t)^2}. \tag{I.23}$$

It is always possible to rescale $r \to r/|k|^{1/2}, R(t) \to R(t)/|k|^{1/2}$, so that $k = +1$, -1 or 0.

If the contents of the universe are treated as a fluid, then homogeneity and isotropy ensure that the matter stress tensor is given in terms of the energy density ρ and pressure p by

$$T^{\mu\nu} = (p+\rho)g^{\mu\nu} - pu^{\mu}u^{\nu}, \tag{1.24}$$

with $u^{\mu} = (1,0,0,0)$. The equation of state for the fluid then determines the dependence of p and ρ on the temperature T. For example, a fluid consisting of relativistic degrees of freedom (radiation dominated) would have

$$\rho(T) = \sum_i \frac{g_i}{2\pi^2}\frac{\pi^4}{15}T^4 \tag{1.25}$$

$$p(T) = \rho(T)/3, \tag{1.26}$$

where the sum is over particle species, and the weight factor g_i is the number of spin degrees of freedom, times one for bosons and 7/8 for fermions.

The equations of motion resulting from the action defined above, under the assumptions of isotropy and homogeneity, can be reduced to the Friedmann equation and the equation for energy conservation:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{R^2} \tag{1.27}$$

$$\frac{d}{dt}(\rho R^3) = -p\frac{dR^3}{dt}. \tag{1.28}$$

These equations, when combined with an equation of state, determine the evolution of the scale factor $R(t)$, pressure p and energy density ρ with time. This also allows us to solve for the temperature T(t) as a function of time. For the relativistic fluid described above, ignoring for the moment the curvature term k/R^2, we find $R(t) \propto t^{1/2}$ and $T(t) \propto t^{-1/2}$.

We note that if the curvature term is negligible the expansion rate $\dot{R}/R$ and cooling rate $\dot{T}/T$ of the universe are just dependent on the quantity $\sqrt{G\rho} \sim T^2/M_{Planck}$. The inverse of this quantity is often referred to as the Hubble timescale. For energy densities much less than the Planck density $\sim M^4_{Planck}$, this expansion or cooling rate is much less than the inverse of the timescale over which typical particle reactions or thermal fluctuations take place (e.g., $\delta t^{-1} \sim T$). This relatively slow expansion and cooling of the universe allows us to make the approximation (used implicitly above) that the system is able to track its thermodynamic equilibrium as the temperature decreases. Obviously this assumption depends critically on the particular rates for particle or thermal processes under consideration. For instance, it is possible for weakly coupled particles to interact (annihilate or scatter) so seldom that their

abundance fails to track equilibrium. This situation is known as "freeze-out", and can be used to calculate the relic abundance of stable, weakly coupled particles such as the neutrino [5]. Later we will consider the case of a first order phase transition, in which the vacuum state of the universe is unable (at least temporarily) to track its free energy minimum. This is due to the extremely low rate for fluctuations which interpolate between the metastable vacuum and the true minimum of the free energy.

Before continuing in our discussion of the thermodynamics of the early universe, we digress for a moment to reconsider the question of initial conditions. In particular, although we specified the initial values T_i, ρ_i, p_i, we said nothing about the initial value of the scale factor R_i. Since there is only one dimensionful constant in the gravitational sector of the theory, we might expect that $R_i \sim M^{-1}_{Planck}$. However, unless k is extremely small, this would be a disaster as the curvature term would then dominate the Friedmann equation, and lead either to a curvature dominated universe (for $k < 0$) or a universe that quickly recollapses ($k > 0$). If we work in the rescaled basis, where $k = +1$, -1 or 0, it is clear that only $k = 0$ or an $R_i >> M^{-1}_{Planck}$ would allow the universe to have lived as long as it has and to be as flat as it is. This generic problem, of fine-tuned initial conditions, is known as the flatness problem.

Another problem of initial conditions for Robertson-Walker universes is understanding how regions separated by large distances today ended up at the same temperature. This is known as the horizon problem. A horizon in a Robertson-Walker universe denotes a region that is in causal contact. Consider a photon whose geodesic satisfies $ds^2 = 0$ emitted at coordinates $(t = 0, r = 0, \theta = 0, \phi = 0)$ along a radial path satisfying $\theta = \phi = 0$. This hypothetical photon would reach coordinate r_H at time t_0, where

$$\int_0^{r_H} \frac{dr}{\sqrt{1-kr^2}} = \int_0^{t_0} \frac{dt}{R(t)}. \tag{1.29}$$

The horizon size at time t_0 is the physical distance between $r = 0$ and $r = r_H$ at time t_0:

$$D_H(t_0) = R(t_0)\int_0^{r_H} \frac{dr}{\sqrt{1-kr^2}} = R(t_0)\int_0^{t_0} \frac{dt}{R(t)}, \tag{1.30}$$

where the physical distance between two points A and B is $D_{AB} = \int_A^B \sqrt{-ds^2}$.

For power law expansion ($R(t) = R_i(t/t_i)^n$), the horizon size increases as

$$D_H(t_0) = t_0/(1-n) \qquad (n<1) \tag{1.31}$$

$$= t_0 ln(t_0/t_i) \qquad (n=1) \tag{1.32}$$

$$= t_0/(n-1)(t_0/t_i)^{n-1} \qquad (n>1). \tag{1.33}$$

Now, consider the physical volume at some past time t that would correspond to a horizon volume today at t_0. The size of this region, relative to a horizon size at that time, would be

$$\frac{d(t)}{d_H(t)} = \frac{R(t)}{R(t_0)}\frac{d_H(t_0)}{d_H(t)} = \left(\frac{t_0}{t}\right)^{1-n} \qquad (n < 1). \tag{I.34}$$

From the last equality, we see under either radiation dominated ($n = 1/2$) or matter dominated ($n = 2/3$) expansion our present horizon volume must correspond to a volume which at early times consisted of many causally disconnected volumes. It is then difficult to understand why the microwave background (which decoupled from equilibrium at temperatures of about $1eV$), is so uniform today. Such uniformity would require that the initial temperature distribution (or, at least the temperature distribution at $1eV$) of the universe was uniform over distances which were acausally separated at the time.

We see now that to accommodate a universe like the one we observe in a Robertson-Walker framework requires extremely unnatural initial conditions, both in the initial scale factor and in the distribution of energy density. An attractive possible solution to these problems is the idea of inflation [26, 6]. Inflation postulates an era when the equation of state of the quantum fields in the theory was such that the universe expanded exponentially, $R(t) \sim e^{Ht}$, while maintaining a constant energy density, ρ. Sufficient exponential expansion solves both the flatness and horizon problems. The rapid growth of the scale factor suppresses any curvature, leading to a flat universe with negligible curvature, hence the generic prediction from inflation that $\Omega \equiv \rho/\rho_c = 1$ (ρ_c is defined as the critical energy density necessary for a flat universe). Secondly, one can check that during the era of exponential growth, the horizon size increases dramatically, and it is possible for our universe to have originated from a relatively small initial volume that was causally connected before inflation.

The state necessary to ensure exponential growth of the scale factor is one in which vacuum energy dominates the matter field stress tensor $T^{\mu\nu}$. The stress tensor for a scalar theory such as that described by (2.1) is

$$T^{\mu\nu} = 1/2(\partial^\mu\phi\partial^\nu\phi) + V_\beta(\phi) \tag{I.35}$$

where $V_\beta(\phi)$ is the full temperature dependent effective potential. If the ground state of the theory is one in which the derivatives vanish (translationally invariant), then the stress tensor is simply given by the (possibly nonzero) potential energy $V_\beta(\phi)$ times $g_{\mu\nu}$. In this case the solution to the Robertson-Walker equations of motion, neglecting the curvature term, which quickly becomes irrelevant, is $R(t) = R(t_0)e^{Ht}$, where $H^2 = \frac{8\pi G}{3}T^{00}$. Here $T^{00} = \rho$ is simply the energy density of the "false vacuum"

state. We note that the energy conservation equation (I.28) is automatically satisfied by the $\rho = -p$ equation of state of the false vacuum.

The spacetime of the exponentially inflating universe is actually a De Sitter spacetime. It exhibits an $O(4)$ symmetry which is larger than the symmetry of the general class of Robertson-Walker spacetimes we have been considering. There are a number of interesting aspects of De Sitter space, specifically the presence of quantum fluctuations in the various fields due to background gravitational curvature. These quantum fluctuations can be the source of density perturbations necessary to seed subsequent structure formation in the early universe. We will not discuss either topic here, but they are investigated in [7].

From the above discussion it is clear that there is an intricate interplay between the ground state of the quantum fields in the theory, and the large-scale evolution of the metric. This brings us back to the discussion of the statistical, or thermodynamical, properties of the early universe. We now turn to the details of how the universe could find itself in a "false vacuum" state, and indeed, how it might manage to extricate itself. One simple scenario involves the quantum fields undergoing a first-order phase transition as the universe cools. By first-order, we mean a phase transition in which the minimum of the effective potential (or free energy) shifts discontinuously as the temperature is decreased. This type of phase transition occurs generically in a large class of perturbative theories with scalar potentials. In figure 1 we sketch the type of finite temperature behavior of the effective potential which would lead to such a transition. We note that at zero temperature, the true minimum of the potential occurs at nonzero vacuum expectation of the field ϕ. At zero temperature the local minimum at $\bar{\phi} = 0$ is metastable, and represents a state of much larger energy density than the true minimum. However, at finite temperature the corrections are such that above some critical temperature T_c, the minimum of the potential (i.e., free energy) occurs at zero vacuum expectation value.

Now let us follow the evolution of the ground state of the theory, beginning at temperatures $T > T_c$. We will assume that the system is in thermal equilibrium for these large temperatures, and therefore occupies a state with $\bar{\phi} = 0$. At temperatures less than T_c, the system suddenly has available states of much lower free energy. However, because of the barrier in the effective potential, the system cannot homogeneously shift to the lower minimum. The classical equation of motion for the scalar field is satisfied by the field remaining constant at the local minimum.

Because of the lower minimum there are field configurations which are energetically favored over the false vacuum state, and which can be excited by thermal fluctuations or by quantum tunneling. Quantum tunneling proceeds by classically forbidden paths, which nonetheless contribute to the path integral for the system. We leave a discussion of quantum tunneling until later, when we discuss semiclassical techniques in general. For now we will merely mention that tunneling typically results in the nucleation of some sort of critical bubble which then evolves according to its classical equations of motion [45].

Thermal fluctuations represent excursions from the equilibrium state, and occur at a rate [8]

$$\Gamma \sim T^4 exp(-\beta F), \tag{1.36}$$

where F is the free energy of the fluctuation. The subsequent evolution of a fluctuation, if it remains in thermal equilibrium, is governed by the equations of motion gotten from the finite temperature effective potential. Alternatively, it is possible that the timescale over which a configuration evolves is insufficient to ensure thermal equilibrium. In that case the evolution of the configuration cannot be computed using the free energy, but rather should be studied using the zero-temperature potential, taking into account interactions with the background thermal spectrum of particles.

The fluctuations we have in mind are typically some sort of bubble, in which the field ϕ probes the true minimum of its potential at the core before patching on continuously to the false vacuum. This description is of course in a semiclassical spirit, where the bubble is treated to a good approximation as a classical field configuration. Bubbles of sufficiently large size are typically free energetically favored to expand, and will be termed critical. Subcritical bubbles will collapse soon after appearing, and will not interest us here.

Now consider the total (thermal plus quantum) nucleation rate for critical bubbles. If this rate is large compared to the inverse of the Hubble timescale at temperatures of order the critical temperature T_c, a large number of critical bubbles will appear within each horizon volume. These bubbles will typically grow and percolate before the universe cools appreciably below T_c. In that event, we will avoid the situation in which the false vacuum energy density dominates the thermal energy density and the universe begins inflating. This is the type of non-supercooled first-order phase transition we will study in chapter II.

If, however, the total nucleation rate is small, the universe will begin expanding at an exponential rate, and the temperature will drop correspondingly. In this case we end up with an extremely cold De Sitter universe, which is trapped in a metastable vacuum state. Critical bubbles will continue to nucleate, but now primarily through quantum tunneling. Detailed studies of the subsequent evolution of these bubbles [26] shows that they will never percolate, and that the majority of the universe remains in De Sitter space.

The scenario of inflation we have described is known as "old inflation", and is unfortunately not successful. Later attempts to devise successful scenarios generally rely on fine tuning of scalar potentials to achieve "slow-rollover" second-order transitions [28], in which the scalar field evolves continuously to its zero temperature minimum, but which also provide for vacuum energy dominance and sufficient expansion. These fine-tunings of the scalar potential are unnecessary to achieve the type of phase transition required for the original scenario, and it would be very appealing if the first-order type of transition could be made to work.

A recent proposal, dubbed "extended inflation" [29], makes use of modifications to Einstein gravity in order to make a first-order scenario successful. In chapter III we will study the percolation of the vacuum in extended inflation, and determine that the scenario leads to the production of a large number of primordial black holes. This result will allow us to place some fairly model independent constraints on such scenarios.

Before turning to other subjects, we mention briefly the relation of the so-called "cosmological constant" problem [9] to the idea of inflation. In quantum field theory the zero point or vacuum energy of a system is arbitrary in the sense that one can always add to or subtract from it without affecting the physics. Therefore, it is always possible in principle to define the theory so that the vacuum energy density is zero by adding the appropriate constant to the potential. We say in principle because the constant to be added must cancel order by order in perturbation theory any higher loop corrections to the zero point energy. (This also applies to any nonperturbative contributions.)

Now, we know that to good accuracy the zero point energy of the theory of our universe is zero because gravity couples directly to energy density. The aforementioned invariance of the quantum theory is not an invariance of classical gravity, and we can place a constraint on the vacuum energy density simply because the universe is not vacuum energy dominated today. This constraint gives (roughly) $\rho_{vac} < 10^{-29} g/cm^3 \simeq 10^{-47}\,\mathrm{GeV}^4$. Since the quantum corrections to the vacuum energy density are expected to be roughly Λ^4, where Λ is the cutoff for the theory, it is obvious that the cancellation between these corrections and the arbitrary constant must be exceedingly fine.

In studying inflation we must, in a sense, assume a solution to the cosmological constant problem. That is, we assume that there is some mechanism that fixes the actual zero point energy of the theory, and that what appears as the stress tensor in the Friedmann equations is measured relative to this zero point.

D Saddlepoints

We now return from the early universe to the subject of saddlepoint expansions and their use in evaluating field theoretic quantities via the Feynman path integral. We have in fact already encountered saddlepoint approximations, as the one loop expression given in (I.10) for the effective potential is equivalent to the result of a saddlepoint expansion of the effective action about $\phi = \overline{\phi}$.

The saddlepoint approximation is useful in general for the approximation of integrals of the type

$$I = \int dx\, e^{-f(x)} \tag{I.37}$$

where the function $f(x)$ has a stationary point about which it can be expanded. In other words, $f(x) \simeq f(x_0) + 1/2 f''(x_0)(x-x_0)^2 + ...$, where $f'(x_0) = 0$ and $f''(x_0) > 0$. This expansion allows the integral to be approximated as

$$I \simeq e^{-f(x_0)} \int dx\, e^{-1/2\, f''(x_0)(x-x_0)^2} + ... \tag{I.38}$$

The last integral is simply a Gaussian, and can be directly evaluated. The approximation is known as a saddlepoint approximation because if the function $f(x)$ is taken to be the real part of some analytic function $w(z = x + iy)$, then by the Cauchy-Riemann equations $f(z)$ is harmonic and satisfies $\partial^2 f(z)/\partial x^2 = -\partial^2 f(z)/\partial y^2$. This implies that the critical point is actually a saddlepoint of $f(z)$.

These techniques can be used to approximate Euclidean path integrals if one is somewhat cavalier about extending the results to infinite dimensions. In particular, the Euclidean action can be expanded about a stationary point as

$$S_E[\phi] \simeq S_E[\phi_0] + 1/2 \int dx\, dy\, \frac{\delta^2 S_E}{\delta\phi(x)\delta\phi(y)}|_{\phi_0} (\phi(x) - \phi_0)(\phi(y) - \phi_0) + ..., \tag{I.39}$$

where by functional differentiation

$$\frac{\delta^2 S_E}{\delta\phi(x)\delta\phi(y)}|_{\phi_0} = -(\Box + U''(\phi_0))\delta(x-y). \tag{I.40}$$

Here, a stationary point is nothing more than a solution to the Euclidean equations of motion. Keeping track of $\hbar$'s tells us that the quadratic piece is actually $\mathcal{O}(\hbar)$, and therefore corresponds to the first loop correction to the saddlepoint result. The saddlepoint expansion used here is actually equivalent to the semiclassical, or loop, expansion as we will see below.

For example, we can use this type expansion to compute the saddlepoint result for the Euclidean analytic continuation of W[J], the generating functional for n-point functions. We have

$$e^{-W_E[J]} = \int D\phi\, e^{-S_E[\phi, J]}, \tag{I.41}$$

which gives

$$\begin{aligned} e^{-W_E[J]} &\simeq e^{-S_E[\phi_0, J]} \int D\phi\, e^{1/2 \int dx \phi(x)(\Box + U''(\phi_0))\phi(x)} & \text{(I.42)} \\ &= e^{-S_E[\phi_0, J]} \int D\phi\, det[\Box + U''(\phi_0)]^{-1/2}. & \text{(I.43)} \end{aligned}$$

Using the result $det A = e^{Tr\, ln A}$ and rotating back from Euclidean space yields

$$W[J] = S[\phi_0, J] + 1/2 Tr\, ln\, [\Box + U''(\phi_0)] + ... \tag{I.44}$$

From the result for $W[J]$, we can quickly get to the effective potential by noting that to the order $\mathcal{O}(\hbar)$ in which we are working, the above equation implies $\delta W/\delta J = \phi_0$. Using the definition of W[J] given in (I.2), $\delta W/\delta J = \overline{\phi}$, we have

$$\Gamma[\overline{\phi}] = S[\overline{\phi}] + 1/2 Tr\, ln\, [\Box + U''(\overline{\phi})] + ... \tag{I.45}$$

Now, for constant configurations $\overline{\phi}$, the above result becomes an expansion of the effective potential V_{eff} in terms of the classical potential $U(\overline{\phi})$ and the first quantum correction. The $\mathcal{O}(\hbar)$ correction found here is precisely equal to the one found earlier by diagrammatic means, since

$$Tr\, ln[\Box + U''(\overline{\phi})] = \int d^4k\, ln(k^2 + U''(\overline{\phi})). \tag{I.46}$$

The saddlepoint found above is an almost trivial one: it corresponds to a uniform value of the field ϕ throughout spacetime. Below we will consider two more interesting types of saddlepoints, namely solitons and instantons. Here by soliton we mean a stable, time independent solution of the classical equations of motion. It is the last property that makes the configuration a saddlepoint. The first two properties are generally true of topological solitons (i.e., kinks, monopoles) [41], but are not necessarily true of a more complicated class of solutions known as nontopological solitons (Q-balls, fermion-scalar bound states, etc.) [39, 40]. Solitons, because of their stability (or near-stability, in the nontopological case), can be treated as extended particle-like excitations. On the other hand, instantons are solutions of the classical equations which are localised in time as well as space. Rather than representing particle-like states in the theory, they will provide a means for saddlepoint expansion of various nonperturbative amplitudes in the path integral.

The prototypical (topological) soliton we will briefly discuss here is the (1+1) dimensional kink, which is trivially related to "domain wall" solutions in higher dimensions. The kink is a solution to the classical equations derived from the following Lagrangian,

$$\mathcal{L}/\hbar = \frac{1}{g^2\hbar} \int dx \left[\frac{1}{2}(\partial_\mu\phi')^2 - \frac{\mu^2}{2}(1 - \phi'^2)^2\right], \tag{I.47}$$

where we have rescaled the fields by the small coupling, $\phi' = g\phi$. The point we wish to emphasize here is that in weak coupling g, where we expect the semiclassical expansion to be valid, the ϕ field configuration corresponding to the soliton is of magnitude $1/g\hbar$. This can be seen by dimensional analysis: as the ϕ' field Lagrangian has no g dependence, we expect the solutions to scale as $\phi' \sim \mu$, which implies $\phi \sim \mu/g$.

The quantum fluctuations about the soliton solutions are higher order in the loop expansion, and therefore are $\mathcal{O}(\hbar g)$. Hence, it makes perfect sense in the full quantum theory to retain the soliton solutions, and to imagine that their classical properties will still be approximately valid. In chapter IV, we will discuss putative new nontopological solitons which can exist in weakly coupled scalar-fermion theories. (One such theory is the standard model with two Higgs doublets). Careful examination of these "would-be" solitons shows that the quantum fluctuations are actually larger than the solutions themselves. In contrast to the case discussed above, the saddlepoint expansion about these solutions is untrustworthy.

Finally, we come to the subject of instantons [45, 10], and their use in computing nonperturbative Green's functions. In the brief discussion presented here, we will refer to a generic class of instanton solutions $\phi_I(x,\xi)$ which satisfy the relevant Euclidean equations of motion and have finite action $S_E[\phi_I]$. The parameter ξ represents the zero modes of the solution, typically given by translations or internal space rotations. Instanton solutions of this type occur in non-Abelian gauge theories such as QCD and the electroweak gauge theory, as well as in scalar theories with nontrivial vacuum structure (eg multiple vacua).

The instanton configurations can be used to compute contributions to n-point Green's functions in the saddlepoint approximation. In the case of the electroweak sector of the standard model these one instanton Green's functions lead to the violation of (B+L) number, which is conserved classically but violated by anomalies. An index theorem relates the change in winding number of the $SU(2)$ gauge fields in the instanton background to the change in (B+L) number [11]. Alternatively, iteration of instanton-antiinstanton configurations can give contributions to perturbatively allowed processes, such as (B+L) conserving processes in the electroweak theory.

Consider the following Euclidean n-point function,

$$\begin{aligned} G(x_1,...x_n) &= \langle \phi(x_1),...\phi(x_n) \rangle & (1.48)\\ &= Z[0]^{-1} \int D\phi\ \phi(x_1)...\phi(x_n) e^{-S_E[\phi]}. & (1.49) \end{aligned}$$

We can compute $G(x_1,...x_n)$ in the instanton background as follows:

$$G(x_1,...x_n) = B \int d\xi\ \phi_I(x_1,\xi)...\phi_I(x_n,\xi) e^{-S_E[\phi_I]}, \qquad (1.50)$$

where $B = (det[\Box + U''(\phi_I)])^{-1/2}$ is a function of the determinant evaluated in the instanton background. We see that in the one instanton approximation the full functional integral has been reduced to an integral over zero modes.

If we Fourier transform, the Green's function becomes

$$\begin{aligned} G(x_1,...x_n) &= B \int dx_1...dx_n\ e^{i\sum p_i x_i} \int d\xi\ \phi_I(x_1,\xi)...\phi_I(x_n,\xi)\ e^{-S_E[\phi_I]} & (1.51)\\ &= B \int d(x_1-\xi)...d(x_n-\xi)\ e^{i\sum p_i\cdot(x_i-\xi)} & (1.52)\\ &\quad \cdot \int d\xi\ e^{i\xi\cdot\sum p_i}\ \phi_I(x_1-\xi)...\phi_I(x_n-\xi)\ e^{-S_E[\phi_I]}, & (1.53) \end{aligned}$$

which yields

$$G(x_1,...x_n) = B\ \delta(\textstyle\sum p_i)\ \tilde{\phi}_I(p_1)...\tilde{\phi}_I(p_n)\ e^{-S_E[\phi_I]}. \qquad (1.54)$$

The n-point functions we have obtained have a form as if they were obtained from a local operator of the form $B\ \phi\phi...\phi\ e^{-S_E[\phi_I]}$. For $n > 4$ these are local, higher dimension operators, and hence naively lead to unitarity violation when used to calculate scattering at sufficiently high energies.

Of course, we have to check that these particular Green's functions actually contribute to scattering between on-shell modes. That is, we must check to see that $G(p_1,...p_n)$ has poles at $p_i^2 = -m^2$ (recall, we are in Euclidean space). However, this will be a general property of instanton solutions with finite action. In order to have finite action, the solutions $\phi_I(x-\xi)$ must asymptotically approach the vacuum state $\langle\hat{\phi}\rangle \equiv \sigma$ upon which the spectrum of states with masses m^2 is built. In other words, in the asymptotic limit $x - \xi >> 0$ the solutions satisfy

$$(\Box + m^2)\,(\phi_I(x-\xi) - \sigma) = 0, \qquad (1.55)$$

whereas for $x \sim \xi$, the configuration has some nonzero value of order one over the small coupling in the theory. It is easy to see that a configuration with the above properties always has the desired pole structure. Given that fact, it is a straightforward application of the LSZ procedure to arrive at an S-matrix element by truncating the on-shell poles.

The above arguments, when repeated for the case of $SU(2)$ instantons of the electroweak sector, lead to the prediction that (B+L) violating cross sections, although extremely small at low energy, grow exponentially with center of mass energy and may even violate unitarity at sufficiently high energies. This is a subject of ongoing interest [46, 50, 48] which will be discussed in greater detail in chapter V. There we study the Green's functions induced by scalar instantons present in a theory with non-degenerate minima (i.e., Higgs sector with heavy top quark, as previously discussed). More sophisticated saddlepoint computations allow us to compute to leading order the exact rate of exponential growth of the cross section, and also to show that the saddlepoint approximation actually breaks down at large energies, before any conclusion can be drawn concerning unitarity violation, or observability of the relevant process.

Figure 1: The finite temperature effective potential for a first-order phase transition. At temperature T_c the two phases have equal free energy.

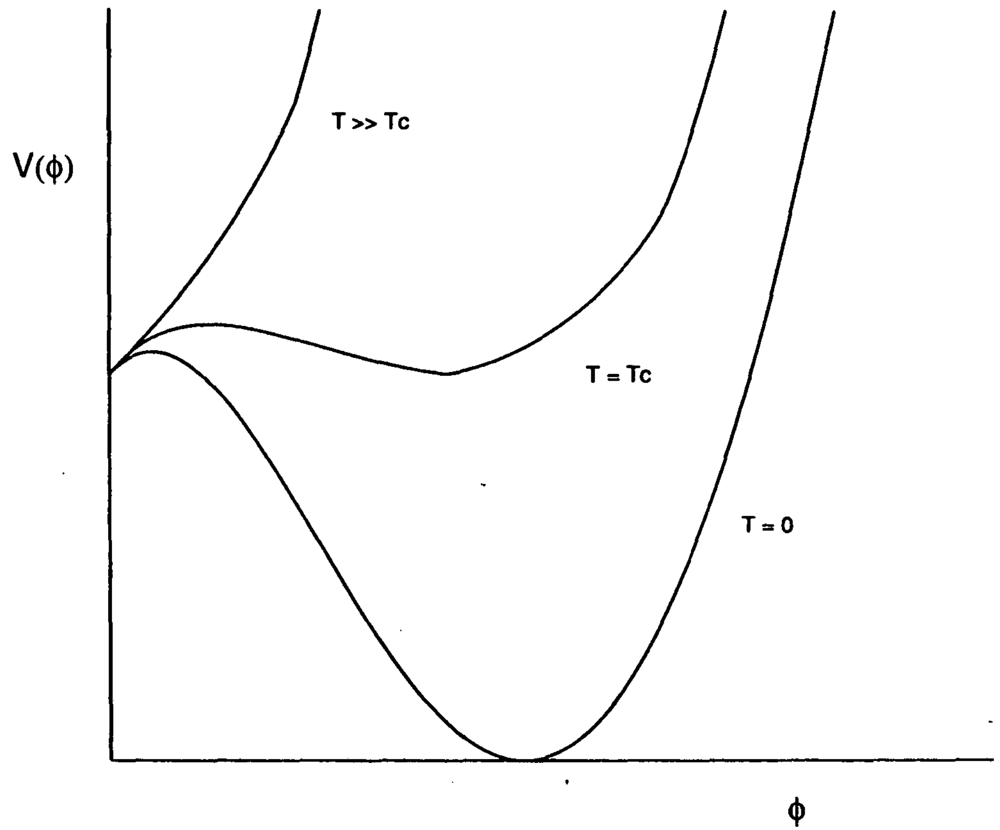


fig 1

Chapter II

Cosmological production of black holes

A Introduction

The framework of the standard Friedmann–Robertson–Walker big bang, together with a theory of particle physics, allows us to calculate, at least in principle, the present contents of the universe. However, there are several open problems: we do not know the particle physics of baryon number violation, nor the particle physics corresponding to dark matter. Furthermore, while accurate calculations of relic particle abundances can be made, it is much harder to compute abundances of extended, non-pointlike relics produced at phase transitions.

It is critical to consider any mechanism which at high temperatures transforms energy from radiation to a form which redshifts more slowly. For example, suppose that at temperature T_c, during the radiation dominated era, a fraction f_M of the radiation energy is converted to a form which redshifts like matter. Today this matter will contribute to $\Omega = \rho/\rho_c$ an amount

$$\Omega_M \simeq 10^8 \left(\frac{T_c}{GeV}\right) f_M, \tag{II.1}$$

where ρ is energy density, and ρ_c is the value which makes the universe critical. At high T_c only a very weak conversion mechanism is required to produce significant Ω_M. Alternatively, if the conversion mechanism at high T_c is too strong, the universe will be overclosed. Examples include the cosmological monopole problem, which constrains certain phase transitions, and a limit of about 1000 TeV on the mass of point particle candidates for dark matter. In this chapter we give a new mechanism for primordial black hole (PBH) production at a first order phase transition. It produces PBHs only

rarely, but this may suffice to give the dark matter.

It is an unproven, but widely believed, conjecture in general relativity (the "hoop conjecture" [12]) that if a mass M is located inside a region of radius $R_s \leq 2GM$, it forms a black hole. Such a black hole emits radiation [13] with a spectrum similar to a thermal one of temperature $T = 1/(8\pi GM)$ [14]. The black hole lifetime is therefore $\tau = 0(G^2M^3) \simeq 10^{10}$ year $(M/10^{39}GeV)^3$. PBHs produced with masses less than 10^{39} GeV will reach a temperature of the Planck mass before today. The final state of such an exploding PBH is unclear; we will not consider the case of PBH remnants as dark matter [15].

Could the dark matter be dominantly in PBHs of mass $0(10^{39}GeV)$? No, they have a temperature today of about 10 MeV, and the diffuse background radiation in this region implies $\Omega < 10^{-9}$ for such masses [16]. Furthermore, if the temperature of the PBHs today is larger than m_e, they will emit significant numbers of e^- and e^+. If our galactic halo were made of these PBHs, the positrons would slow down in the interstellar medium and annihilate to produce 512 keV line radiation. We find that this implies that the halo should be dominated by PBHs of mass greater than 10^{41} GeV. If $\Omega = 1$ in PBHs of mass $O(10^{41}$ GeV), they will lead to a diffuse background radiation peaking at energies of 0.25 MeV with a flux of 0.1 $cm^{-2}s^{-1}$. This is just below the observed background, and apparently is barely unable to explain the observed feature in the MeV region.

Many dark matter candidates, monopoles and stable point particles for example, are more dangerous in overclosing the universe the earlier they are produced. This is not the case for PBHs. At temperatures larger than 10^8 GeV the horizon mass is less than 10^{41} GeV. Hence, prior PBH production is unimportant for dark matter, unless PBH's can be made to collide and "cannibalise" before evaporating. We argue later that significant cannibalism does not occur. Consequently, we find that if PBHs are the dark matter today, the scale of the particle physics responsible for the phase transition is less (for our mechanism, much less) than 10^8 GeV.

Cosmological production mechanisms of PBHs are not new. Carr showed that the Harrison–Zeldovich scale–invariant density perturbation spectrum, which is commonly taken for the origin of large scale structure, leads to PBH formation as the perturbations enter the horizon [17]. The resulting spectrum of PBHs is a steeply falling power law, implying an Ω in PBHs today which is much too small to be of interest for the dark matter. Gross, Perry and Yaffe calculated PBH production rates via quantum tunneling [18]. The tunneling rate becomes exponentially suppressed at temperatures less than the Planck mass, and hence this mechanism does not lead to PBHs massive enough to survive until today. More interesting from the viewpoint of dark matter is the mechanism of Hawking, Moss and Stewart [19]. They produce horizon size PBHs at a first order phase transition by the unlikely event that many neighboring bubbles grow to horizon size before percolating. Providing the phase transition occurs at $T_c < 10^8$ GeV, the PBH masses will be large enough to survive until today. Critically closing the universe with these PBHs requires a choice for the bubble nucleation rate. In this chapter we provide an alternative production mechanism, in which the PBH masses are much less than the horizon mass at formation, and where the density of PBHs is largely independent of the bubble nucleation rate.

B The mechanism

We illustrate our mechanism in a simple field theory with a real scalar field ϕ coupled to a Dirac fermion ψ:

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + i\bar{\psi}(\not{\partial} - m')\psi + \mu^2\phi^2 + \gamma\phi^3 - \lambda\phi^4 + \alpha\bar{\psi}\psi\phi, \tag{II.2}$$

where all the parameters are chosen to be real and positive. At high T the vacuum is at $\langle\phi\rangle = 0$, while at lower temperatures this becomes a false vacuum with the true vacuum appearing for positive $\langle\phi\rangle = v(T)$, as shown in Figure 1. At the critical temperature T_c, the two phases coexist and are separated by a transition region in which ϕ interpolates between the two vacua. The energy per unit area of the boundary region is the surface tension $\sigma \equiv \Delta^3$. The scale Δ is a function of the parameters of the scalar potential, and is typically, but not always, of order T_c.

At T_c, there will be a non-zero difference in vacuum energy*: $B(T_c) = V(\phi = 0) - V(\phi = v(T_c)) \neq 0$. This is because the vacuum, or bag, pressure tending to collapse the regions with $\phi = 0$ is countered by the pressure due to the fermions being transmitted and reflected from the boundary. For convenience we take $m' << T_c$ and ignore m' from now on. The fermions have a mass difference $m = \alpha v(T_c)$ across the phase boundary and the fermion pressure arises from the change in momentum of fermions which are reflected or transmitted at the boundary.

Suppose that bubble nucleation is appreciable at some temperature T_N which is not very much less than T_c. As the bubbles grow and convert false to true vacuum, we assume that the released energy reheats the bulk of the universe and does not accumulate as domain wall kinetic energy. Throughout this analysis we assume a

*Here we use a different definition for T_c than in the introduction. While in thermal equilibrium, the evolution of regions of different phases is governed by the free energy or finite temperature effective potential. However, when the evolution is on timescales so short that equilibrium is not achieved, there can be effects such as those discussed here.

"thermalon" mechanism which always maintains the universe at a single homogeneous temperature. Finally, we assume that the temperature reaches T_c while the fraction of the universe in the true vacuum, f_T, is much less than 1/2. This will prevent further bubble nucleation, establishing an era of quasi-static equilibrium [20] (QSE). Such a phase transition is of great interest: the transition proceeds very slowly at a rate governed by the rate of expansion of the universe:

$$\dot{f}_T = \frac{3}{B(T_c)}H(p+\rho), \tag{II.3}$$

where H is the Hubble parameter during this era, p the pressure and ρ the energy density. The rate at which vacuum energy is released is equal to the rate of doing work of expansion together with the rate at which energy density must be created to fill the increased physical volume. For $p \simeq \rho \simeq T_c^4, H = t_H^{-1}$, the era of quasi-static equilibrium lasts for a time

$$\frac{\tau_{QSE}}{t_H} \simeq \frac{B(T_c)}{T_c^4}. \tag{II.4}$$

We consider the case that $\tau_{QSE}/t_H \lesssim O(1)$ since in this case no significant inflation takes place, and for many simple estimates the expansion of the universe can be ignored. In the case that the mass acquired by the fermion crossing the boundary, $m = \alpha v(T_c)$, is less than T_c, we find $B(T_c)/T_c^4 = m^2/T_c^2$, so that this condition is straightforward to satisfy. The controlled nature of the slow-burn phase transition allows us to make simple calculations.

When f_T approaches 1/2, bubble collisions become frequent. We assume that at T_N the nucleation rate is fast enough so that when the bubbles collide they are sufficiently small that a period of bubble coalescence takes place. The original scale of the bubbles is erased and the final scale of the bubbles depends on the dominant dynamics of bubble coalescing. During coalescence of two bubbles of size r, bubble walls must move a distance r in time $\simeq \tau_{QSE}$. Bubble wall speeds are limited by frictional drag: the transmitted fermions impart a momentum of order $(m/T_c)^2 T_c$ to the wall. This turns out to be a severe limitation, so that bubbles coalesce by setting up bulk fluid flows rather than having walls moving rapidly with respect to the fluid. Bulk motion of fluids is set up because fermions crossing the phase boundary have their momentum changed. We find that in time τ_{QSE} bubbles coalesce by fluid flow up to an average scale $r_c = \bar{\epsilon} t_H$, where

$$\bar{\epsilon} \simeq \left(\frac{\Delta}{T_c}\right)\left(\frac{m}{T_c}\right)^2\left(\frac{T_c}{M_{pl}}\right)^{1/3}. \tag{II.5}$$

After the era of coalescence, further expansion of the bubbles causes f_T to become greater than 1/2. The true vacuum percolates to produce a connected region of true vacuum containing shrinking bubbles of false vacuum. We will argue that eventually a few of these shrinking bubbles will form black holes.

Consider first the idealized situation of a density $n_B = r_c^{-3}$ of identical spherical, collapsing bubbles of radius $r(t)$. Since $1 - f_T = n_B r^3$,

$$\dot{f}_T \simeq \frac{r^2 \dot{r}}{\bar{\epsilon}^3 t_H^3}. \tag{II.6}$$

As the bubbles get smaller, $\dot{r}$ must increase in order to satisfy the condition of quasi-static equilibrium, equation (II.3). When the bubble walls reach the speed of light, quasi-static equilibrium will be lost and the bubble radius will be

$$\hat{r} \simeq \bar{\epsilon}^{3/2} \left(\frac{T_c}{m}\right) t_H. \tag{II.7}$$

Next we consider the case of a distribution of initial bubble radii about the average $\bar{r} = \bar{\epsilon} t_H$. If the temperature in the true vacuum during QSE is uniform throughout the universe, the rate of collapse of bubbles will be approximately independent of their size. This, of course, neglects surface tension which causes smaller bubbles to contract slightly faster than larger bubbles. Therefore, at the end of QSE, while $\bar{r}$ has decreased to $\bar{\epsilon}^{1/2}(T_c/m)\bar{r}$ (and smaller bubbles have already disappeared), a bubble with initial radius $r_i > \bar{r}$ has only decreased to $r_i - \bar{r}$. Thus, as QSE ends, we are left with the bubbles on the large end of the size distribution, most of which have barely decreased in radius.

After QSE is lost, work is done on the bubbles as they undergo relativistic collapse

$$W = P_{net}\Delta V = (B(T) - P_{matter})\Delta V. \tag{II.8}$$

There are two separate effects which contribute to P_{net}. One is the increase in $B(T)$ as the temperature falls beneath T_c: $B(T) \simeq B(T_c) + B'(T_c)(T - T_c)$. The time δt necessary for relativistic collapse of a bubble is at least $\delta\tau \geq \bar{\epsilon} t_H$. The corresponding temperature drop will be $\delta T \sim (M_{Pl})^{1/2}(t_H^{-3/2})\delta t \sim T_c(\frac{\delta t}{t_H}) \sim \epsilon T_c$, so $\delta B(T) \sim |B'(T_c)|\epsilon T_c$ which we will approximate as $\sim (\Delta^3 T_c)\bar{\epsilon}$. The second contribution comes from the fact that the bubble walls become increasingly porous to fermions as they become highly relativistic. Consider the scattering of fermions from a bubble wall, which we have idealized as a step-function potential. Since the thickness of the bubble wall $\sim \Delta^{-1}$, and the de Broglie wavelength of fermions is $\sim T^{-1}$, this is a relatively good approximation for nonrelativistic scattering with $\Delta > T_c$. In this case we have the following results from elementary quantum mechanics:

$$E_\psi < m \quad \rightarrow \quad \text{reflection, } \Delta p \simeq 2E_\psi$$

$$E_\psi > m \quad \rightarrow \quad \begin{cases} \text{probability } (\frac{m}{E_\psi})^4 \text{ of reflection} \\ \text{probability } (1 - (\frac{m}{E_\psi})^4) \text{ of transmission, (with } \Delta p \simeq \frac{m^2}{2E_\psi}). \end{cases}$$

Note that these results already indicate that very energetic particles ($E_\psi >> m$) barely interact with the interface between true and false vacuum. For the extremely relativistic case, in which the domain wall has a velocity $\beta_{DW} \sim 1$, we can still apply the results above if we calculate the scattering process in the frame of the domain wall and then Lorentz transform back to the cosmological frame. In the case where $\gamma_{DW} >> 1$, we find that virtually all fermions penetrate the wall, and that the momentum change is $\Delta p \simeq \frac{m^2}{2E_\psi(1+\beta_{DW})}$. The above considerations imply that P_{matter} in equation (II.8) decreases as the collapse becomes relativistic. This means that we can approximate P_{net} in equation (II.8) by

$$P_{net} \simeq (\Delta^3 T_c)\bar{\epsilon} + T_c^4(m/T_c)^2. \tag{II.9}$$

The dominant term for small $\bar{\epsilon}$ is the latter, which yields the following mass formula:

$$M_{PBH} \simeq P_{net}\Delta V \simeq T_c^4(\epsilon t_H)^3(m/T_c)^2 \tag{II.10}$$

$$\simeq \left(\frac{\Delta}{T_c}\right)^3 \left(\frac{m}{T_c}\right)^8 \frac{M_{pl}^2}{T_c}. \tag{II.11}$$

This of course applies only for bubbles with $\epsilon > \bar{\epsilon}$. Bubbles smaller than or only slightly larger than $\bar{r} = \bar{\epsilon} t_H$ will collapse without forming black holes. Bubbles with $\epsilon >> \bar{\epsilon}$ produce heavier PBH. In general, the mass distribution of black holes formed can be given in terms of the initial distribution of bubble sizes after percolation. Let $N(\epsilon - \bar{\epsilon})d\epsilon$ be the number of bubbles in the $\epsilon, \epsilon + d\epsilon$ range. Presumably $N(\epsilon - \bar{\epsilon})$ is peaked about $\epsilon = \bar{\epsilon}$, with some power law or exponential fall off. Then the number of black holes formed with mass in the $m, m + dm$ range is $\propto (\epsilon - \bar{\epsilon})^3 N(\epsilon - \bar{\epsilon})d\epsilon$, which applies for $\epsilon > \epsilon_{min} > \bar{\epsilon}$. ($\epsilon_{min}$ represents the smallest bubble which can collapse to form a black hole.)

Given the mass estimate (II.11), we can now estimate how far a bubble must collapse before falling into its Schwarzschild radius r_s. Using equations (II.5) and (II.11):

$$\frac{r_s}{r_i} = \frac{r_s}{\epsilon r_H} \simeq \left(\frac{\Delta}{T_c}\right)^2 \left(\frac{T_c}{M_{pl}}\right)^{2/3} \tag{II.12}$$

where we drop factors of m/T_c from now on. We see that for high T_c (i.e., GUT scales or higher), this factor is not extremely small. However, to produce PBHs which are interesting as dark matter, equation (II.11) requires a much lower T_c. This leads to $r_s/r_i << 1$, which can only be achieved for bubbles which are extremely spherical at collapse.

Bubbles which have fractional asphericities greater than (r_s/r_i) at the point of collapse will probably self collide, rather than form PBHs. In considering coalescence, we calculated the maximum size of bubbles that could become roughly spherical in a Hubble time. To understand exactly how spherical such a bubble can become, it is necessary to include the effects of damping on the motion of bubble walls. During the QSE era, we can treat an individual bubble as a stationary, nearly spherical membrane with surface tension $\sigma \sim \Delta^3$ and radius $r = \epsilon r_H$. We can now study the damping of an arbitrary perturbation on our membrane by examining the excited normal modes. We can characterize a given mode by its amplitude δr and wavelength λ. A bubble minimizes its surface energy by becoming spherical. For fixed δr, it is clearly the longest wavelength modes ($\lambda \sim r$) that are hardest to eliminate. The equation of motion for such a mode is

$$(\sigma + T_c^4\delta r)\dot{\delta r} + \gamma\ddot{\delta r} + \frac{\sigma}{r}\left(\frac{\delta r}{r}\right) = 0 \tag{II.13}$$

where γ is the damping factor and $(\sigma + T_c^4\delta r)$ is the effective mass per unit area (surface tension plus mass per unit area of bulk fluid) to be moved.

The damping factor represents the rate of energy dissipation into the bulk fluid. Each bounce of the membrane produces a sound wave in the fluid which carries off an energy per unit area of $\sim T_c^4\delta r\beta^2$, leaving the membrane with kinetic energy $\sim \sigma\beta^2$. Here β is the velocity of the membrane when $\delta r = 0$. Since initially, $T_c^4\delta r >> \sigma$, we expect each succeeding bounce to have a drastically smaller amplitude than the previous one. This approximation holds until $\sigma \sim T_c^4\delta r$, i.e., until:

$$\frac{\delta r}{r} \simeq \frac{\left(\frac{\Delta}{T_c}\right)^3 T_c^{-1}}{\epsilon r_H} \simeq \left(\frac{\Delta}{T_c}\right)^2\left(\frac{T_c}{M_{pl}}\right)^{2/3} \tag{II.14}$$

which is $O(r_*/r_i)$.

In general, ignoring dissipation, the timescale for a particular mode to bounce is $\tau_{bounce}/\tau_{QSE} \sim (\delta r/r)^{1/2}$. Therefore a bubble which is initially fairly spherical ($\delta r << r$) should be able to achieve the sphericity required by (II.14). Thus, in the approximation where we treat the background particles as a bulk fluid, bubbles can become very spherical. While this approach is extremely naive, we are encouraged by the fact that only about one in $10^8(\frac{T_c}{GeV})$ bubbles need be this spherical in order to account for $\rho_{critical}$ today.

C Cannibalism and Conclusion

It is perhaps not clear that equation (II.11) gives the final result for the mass of PBHs. One might wonder whether it is possible for many PBHs to collide and combine to

form larger black holes [21]. In this case the eventual mass distribution of PBHs might bear no resemblance to the mass distribution at formation. We will show that cannibalism will be a negligible effect for the case of PBH dark matter. There are other scenarios in which cannibalism can be an important effect and we will discuss those also.

Statistical fluctuations in the number density of PBHs can lead to density perturbations which eventually grow to nonlinearity. While PBH "galaxies" so formed can be composed of very many individual PBHs, it is very difficult for these PBHs to find each other and cannibalize. In particular, the PBHs are slow to dissipate their energies and hence only a few collisions due to intersections of orbits can occur for a long time.

Due to the stochastic nature of PBH formation, we expect fluctuations $\sim \sqrt{N}$ in regions containing N PBHs. However, a fluctuation in PBH number does not correspond to a density perturbation. The energy density ρ is the same in regions of different PBH number density. This is because failed PBHs (bubbles which had self-collisions before reaching their Schwarzschild radius) contribute to the thermal density by radiating their energy away. The total energy density $\rho = \rho_{PBH} + \rho_{thermal}$ is constant. It is only when two regions with differing $\rho_{thermal}$ come into causal contact that thermal energy equilibrates to give an actual density perturbation in PBHs $\delta\rho/\rho$. Therefore density perturbations are always inside the horizon.

It is well-known [22] that density perturbations inside the horizon do not grow while the universe is radiation dominated. Once the universe is matter dominated, however, $\delta\rho/\rho$ grows like $t^{2/3}$. When $\frac{\delta\rho}{\rho} \sim 1$ the perturbation goes nonlinear; a bound structure is formed which does not take part in the expansion of the universe. In our case, such structures will take the form of PBH "galaxies" in which many PBHs move in virialized orbits. In order to cannibalize, it is necessary for these PBH's to dissipate their kinetic energies. For a non-relativistic PBH, collisions with thermal particles cause the loss of momentum,

$$\frac{d\vec{P}}{dt} \simeq R_*^2 T_D^4 \vec{\beta} \simeq \left(\frac{T_D}{M_{pl}}\right)^4 m_{PBH}\vec{P}, \tag{II.15}$$

where T_D is the temperature of the universe at PBH "galaxy" formation. This gives the following timescale for dissipation:

$$t_D \simeq \left(\frac{M_{pl}}{T_D}\right)^4 m_{PBH}^{-1}. \tag{II.16}$$

Requiring that $t_D < \tau_{PBH} \sim m_{PBH}^3/M_{pl}^4$ yields

$$m_{PBH} > M_{pl}\left(\frac{M_{pl}}{T_D}\right). \tag{II.17}$$

Note that for a perturbation made up of many PBHs ($N >> 1$) to go nonlinear, it is clear that $T_D << T_{MD}$, where T_{MD} is the temperature at which the universe becomes matter–dominated. For $f_{PBH} \sim f_{crit} \sim 10^{-8}(\frac{GeV}{T_c})$, matter domination occurs at $T \sim eV$, so $T_D << eV$. Unless the critical temperature of the phase transition, T_c, is extremely low, it is clear that (II.17) cannot be satisfied. Although PBH structure can form, the individual PBHs typically evaporate before any significant cannibalization can occur. Or, in the case of $T_c \lesssim$ GeV, the timescale for dissipation is much longer than the age of the universe.

On the other hand, one can consider scenarios in which $f_{PBH} \sim 1$. In this case the universe is matter dominated almost immediately by PBHs, and a potentially large amount of cannibalization can occur. However, $f_{PBH} \sim 1$ is clearly unsatisfactory for dark matter. Let N_* be the size of a fluctuation that can go nonlinear and cannibalize before τ_{PBH}. If $N_* m_{PBH} \gtrsim 10^{39}$ GeV, then the lifetime of such a large PBH is longer than the age of the universe. Only a small fraction of the total number of PBHs can end up in such fluctuations, or else ρ_{PBH} (today) $>> \rho_{critical}$. Therefore, for a given volume containing N_* PBHs, a fluctuation capable of cannibalization must be very rare, and correspond to a fluctuation in PBH number many times $\sqrt{N_*}m$. This means that there will be exponentially <u>more</u> cannibalized PBHs with mass <u>less</u> than $N_* m$. Since experimental measurements of γ–ray backgrounds limit Ω in PBH of mass $\sim 10^{39}$ GeV to be less than 10^{-9}, it is not possible for cannibalism to lead to $\Omega = 1$ in PBHs with $m_{PBH} > 10^{39}$ GeV.

Interesting consequences may result from PBH production even if they evaporate while the universe is still hot. Consider a phase transition at T_c in which f is the probability that a false vacuum bubble collapses to a PBH of mass M_{pl}^2/T_c. For $f > (T_c/M_{pl})^{1/2}$, the PBHs dominate the energy density of the universe at temperature fT_c and subsequently evaporate, reheating the universe to a temperature $T_R \simeq T_c(T_c/M_{pl})^{1/2}$. A particle X, with mass $m_X < T_c$, will be produced in the evaporation with a relative abundance

$$f_X = \frac{n_X}{T_R^3} = \frac{1}{g}\left(\frac{T_c}{M_p}\right)^{1/2}, \tag{II.18}$$

where g is the number of degrees of freedom lighter than T_c [23]. This abundance can be large enough to have important consequences. For example, X particles might decay out of thermal equilibrium to generate the cosmological baryon asymmetry at low temperatures. It is easy to arrange for $T_R << m_X$, avoiding wash–out of the asymmetry. This can be used for a baryogenesis scheme where the X particles are TeV top squarks [24].

If the X particles are stable, the above abundance may be larger than the Lee-Weinberg freezeout value, and may give $\Omega_X = 1$. For example, for $T_c = 10^5$ GeV and $f > 10^{-7}$, the PBH evaporation could lead to $\Omega_X = 1$, with X being 50 GeV neutrinos. The reheat temperature of 10 MeV would be sufficient to give a fresh start to nucleosynthesis.

In this chapter we have considered the collapse of false vacuum bubbles in a model of a first order phase transition which undergoes an era of quasi–static equilibrium. We find that, during the initial era of non–relativistic collapse, a few bubbles may sphericalize to a sufficiently high degree that the work done on them during the later relativistic collapse can result in their becoming black holes. The required collapse factor and the resulting PBH mass are given in equations (II.12) and (II.11). For T_c above a TeV, the PBHs are likely to be produced, but are unlikely to survive until today. Their evaporation leads to a production of relic particles. For T_c beneath a TeV, the PBH can be heavy enough to survive until today and, with even a very small probability of a bubble becoming a PBH, can give $\Omega_{PBH} = 1$. It is possible that this might arise in QCD, although we have not studied that case here. For QCD, if holes survive until today to be the dark matter, it is likely that they would have a mass close to the observational bound of 10^{41} GeV. To detect such holes from the particles they evaporate would require their mass to be very close to the limit. Holes in the range of $10^{49} - 10^{56}$ GeV, which would occur for lower T_c, could be detected by lensing of background stars [25].

Chapter III

Percolation of the vacuum and black holes in Extended Inflation

A Introduction

The inflationary universe [6] was proposed as a solution to several cosmological puzzles including the flatness, horizon and monopole problems. Its main feature is a period of rapid expansion, in which the energy density of the universe remains roughly constant, followed by thermalization.

In the original scenario, "old inflation", a particle physics model in which the universe became trapped in a meta-stable false vacuum was utilized. The energy density of the false vacuum resulted in the Robertson-Walker scale factor growing exponentially with time. The phase transition was to have completed due to the nucleation (via quantum tunneling) of bubbles of true vacuum, which then percolate and thermalize. However, it was shown [27, 19] that a tunneling rate sufficiently low to provide enough inflation (R(t) must increase by a factor of at least 10^{27} to solve the aforementioned cosmological puzzles) also implied that percolation never occurred.

Subsequent work on inflationary models [28] has produced scenarios ("new inflation") in which second order phase transitions, involving slow-rolling of a scalar field in a flat potential, produce enough inflation while avoiding problems of percolation. These theories require extreme fine-tuning of parameters to ensure a suitably flat potential. Another alternative, "chaotic inflation", involves scalar field dynamics just below the Planck temperature. These models also involve unnatural choices of parameters, and may make unrealistic assumptions concerning the neglect of quantum gravity effects.

An interesting alternative approach has recently been suggested [29] which,

instead of focusing on exotic particle physics models, looks to minimal modifications of Einstein gravity. In a Brans-Dicke [30] or dilaton-modified [31] theory of gravity, the authors find that a first order transition of the type utilized in old inflation can be successfully completed.

This scenario, which the authors term "Extended Inflation" (EI), has several attractive features. First, the particle physics of the phase transition is natural, with no fine-tuning of parameters. Secondly, the modifications to gravity can be motivated from string theory or Kaluza-Klein theory [31].

It is important to study any inflationary model which offers the possibility of obtaining inflation without fine-tuning of parameters. Several authors [32, 33] have already studied the Brans-Dicke (BD) theory, and have found it impossible to reconcile experimental constraints on the BD parameter ($\omega > 500$) with the constraint that thermalization of large bubbles does not lead to large anisotropies in the microwave background ($1.5 < \omega < 25$). These considerations require the introduction of a potential for the BD scalar which fixes its vacuum value at $\simeq M^2_{Planck}$, at least for low energies. Alternatively, several mechanisms (dynamical or otherwise) have been developed which circumvent this difficulty [36].

In this chapter we will examine in detail the distribution of pockets of false vacuum remaining near the end of the extended inflationary era. Pockets larger than a certain critical size ($R_c \simeq M_{Planck}/T_c^2$ where T_c^4 is the false vacuum energy density) are already within their Schwarzschild radius and are likely to form black holes. We will show that for any Extended Inflationary phase transition, a nontrivial fraction of the volume of the universe after inflation is in false vacuum pockets of size R_c or larger. Similar conclusions concerning black hole production in the context of "old inflation" and Einstein gravity were arrived at by Hawking, Moss and Stewart [19], and also by K. Sato and collaborators [37].

For inflationary phase transitions which occur with critical temperatures below 10^8 GeV the universe becomes dominated by black holes, and has a reheat temperature (when black holes Hawking evaporate) of less than $\simeq$ MeV (insufficient for nucleosynthesis). Larger values of T_c also lead to black hole domination, but imply a sufficiently high reheat temperature to avoid disastrous consequences (although there may then be problems with conventional baryogenesis). Finally, we will speculate on the possibility that the interiors of false vacuum regions evolve into baby universes [34].

B Review of Extended Inflation

There are already several variants of the EI scenario [36] which reconcile seemingly contradictory solar system limits on Brans-Dicke theory ($\omega > 500$) with those from cosmology ($\omega < 25$). A common feature of all of these variants is that bubble percolation is achieved through the power law, rather than exponential, growth of the Robertson-Walker scale factor during inflation. Since the physics of percolation is essentially the same as in the original EI model [29], all of our results will be generic to the models of reference [36].

The Brans–Dicke plus matter action is given by

$$S_{BD} = \int d^4x \sqrt{-g} \left[-\Phi R + \omega g^{\mu\nu} \frac{\partial_\mu \Phi \partial_\nu \Phi}{\Phi} + \mathcal{L}_{matter} \right]. \tag{III.1}$$

Here Φ is a scalar field which plays the role of a time-dependent gravitational coupling. In the limit of infinite ω, Φ decouples and the theory reduces to Einstein gravity. The only assumption we will make concerning the matter fields is that a supercooled, first-order phase transition occurs with critical temperature T_c.

For a homogeneous and isotropic universe described by a Robertson-Walker metric, the equations of motion for this theory reduce to:

$$\left(\frac{\dot{R}}{R}\right) = \frac{8\pi\rho}{3\Phi} - \frac{k}{R^2} + \frac{\omega}{6}\left(\frac{\dot{\Phi}}{\Phi}\right)^2 - \frac{\dot{R}}{R}\frac{\dot{\Phi}}{\Phi} \tag{III.2}$$

and

$$\frac{\ddot{\Phi}}{\Phi} + 3\frac{\dot{R}}{R}\dot{\Phi} = \frac{8\pi(\rho - 3p)}{2\omega + 3}. \tag{III.3}$$

In the case of interest, that of a vacuum dominated universe, the curvature term is negligible, and we set $k = 0$. In this case we have the following solutions for R and Φ:

$$R(t) = R(0)(1 + Bt)^{\omega+1/2} \tag{III.4}$$

$$\Phi(t) = \Phi(0)(1 + Bt)^2, \tag{III.5}$$

where

$$B = \sqrt{\frac{32\pi\rho_{vacuum}}{(6\omega + 5)(2\omega + 3)\Phi(0)}} \tag{III.6}$$

plays the role of the effective Hubble parameter as a function of Φ.

In most models the nucleation rate per unit volume of true vacuum bubbles is time-independent, and given by $\Gamma = \lambda^4$. In such an approximation one neglects the initial contribution from thermal nucleations, and assumes that quantum tunneling (at effectively zero temperature) is the dominant effect. One exception to this rule is the model of Holman, Kolb and Wang [36] in which, due to the nonstandard couplings of the inflaton to the BD scalar, Γ becomes time dependent. This time dependence is of the form $\Gamma = \Gamma_0(1 + Bt)^m$ where $m \simeq -2$. For simplicity, we will present our results for the case $m = 0$. The generalization to nonzero m is straightforward, and leads to similar conclusions.

C Black hole formation

In this section we will argue that large false vacuum bubbles which remain at the end of an inflationary era are likely to become black holes. The issue is complicated by the possibly unthermalized energy density present in such an era. In a first order phase transition, the latent heat present in the false vacuum is converted primarily into kinetic energy of the domain wall-like structures which separates the two vacua. How this energy is thermalized is not fully understood, nor is the outcome of highly relativistic collisions between domain walls.

We are interested in the fate of regions of false vacuum in whose causal past no bubble nucleations have occurred. At some cosmological time t_e, which we define by the requirement that only a fraction of the universe remains in false vacuum, the remainder of the universe is clearly not homogeneous, but is in the process of becoming so as energy in bubble walls is thermalized. In what follows, we will make the assumption that the universe is still approximately described by a Robertson-Walker metric during this era.

A naive criterion for black hole formation follows from Birkhoff's theorem and the location of the Schwarzschild horizon. From Birkhoff's theorem we know that the vacuum solution to the Einstein equations outside a spherical mass is given by the Schwarzschild metric. This exterior solution will include a horizon if the mass is confined to a region of size $R < R_s = 2GM$. The above considerations lead to a criterion for black hole formation which applies whenever mass or energy is confined to a region of size R_s or less.

We would like to apply this criterion to the case of a pocket of false vacuum in the background of a Robertson-Walker (RW) metric:

$$ds^2 = dt^2 - R(t)^2(dr^2 - r^2 d\Omega^2). \tag{III.7}$$

(We take the curvature parameter k to be zero since we have just undergone a period of inflation.) For simplicity, we will take the pocket to be roughly spherical, although this assumption will not be critical to our conclusions.

Notice that the RW metric is flat on distance scales $< \dot{R}/R \simeq H^{-1}$, which is just the horizon scale. If a pocket is within the horizon, then Birkhoff's theorem is approximately valid, and our criterion holds.

At the end of inflation, the reheat temperature is at most $\simeq T_c$ and the horizon size is $\geq H^{-1} \equiv M_{Planck}/T_c^2$. Consider a pocket of false vacuum of size $r_i = NH^{-1}$ and energy density $\rho \simeq T_c^4$. Since the pocket shrinks relativistically, while the horizon grows at the speed of light, the pocket will enter the horizon after a time $\delta t = \frac{(N-1)}{2} H^{-1}$. At this point the background thermal energy density has fallen by a factor of $(2/(N+1))^2$ (for radiation dominated expansion).

Now, requiring that $r_f < 2GM = \frac{8\pi}{3} H^{-1}$ implies that any pocket with $N > 1$ will form a black hole. The mass of such a black hole is (within an order of magnitude, depending on the initial shape of the false vacuum pocket and the kinetic energy of its collapsing surface)

$$M_{BH} \approx N^3 \left(M_{Planck}/T_c\right)^2 M_{Planck}. \tag{III.8}$$

So far we have not included any effects from the scalar modifications to Einstein gravity. In fact, we have taken the value of the Brans-Dicke scalar to be $\phi = M_{Planck}^2$ everywhere in true vacuum after inflation. It appears possible that spatial variation of ϕ near the edge of the false vacuum pocket might tend to stabilize it against collapse. (An increase in ϕ corresponds to a decrease in the strength of gravitational interactions.) In fact, if the interior of the pocket is still inflating, ϕ may have a much larger value in the false vacuum than in the rest of the universe.

However, it is well known [35] that solutions in which the variation of a BD type scalar or dilaton prevents the appearance of a Schwarzschild horizon are unstable to perturbations. In studies of gravitational collapse in BD theories, it is found that any scalar charge is radiated away, leaving a black hole which is identical to that found in Einstein gravity.

In the following section, we will calculate the size distribution of false vacuum pockets at the end of an extended inflationary era. Applying the above criterion, we will assume that any pocket with $N > 1$ forms a black hole.

D Percolation in EI models

The extended inflationary era ends when a majority of the volume of the universe has been converted to true vacuum. Using the expressions given above for $R(t)$ and Γ, we can calculate $P(t)$ which is defined to be the fraction of volume still in false vacuum as a function of time:

$$P(t) = exp\left(-\frac{4\pi\Gamma}{3}\int_{t_0}^{t} dt' R^3(t') r^3(t,t')\right), \tag{III.9}$$

where

$$r(t,t') = \int_{t'}^{t} \frac{dt''}{R(t'')} \tag{III.10}$$

and t_0 is the time at which the universe first becomes vacuum dominated. We will assume that percolation occurs for $P(t) \simeq 1/2$, and that inflation ends when $P(t)$ becomes somewhat smaller. We will denote the time at which this occurs as t_e.

Next we will calculate $P(r,t)$ which is defined as the volume in false vacuum regions of coordinate size $> r$ as a function of time. We should note that such a region has physical radius strictly in excess of $R(t)r$. $P(r,t)$ is given by

$$P(r,t) = exp\left(-\frac{4\pi\Gamma}{3}\int_{t_0}^{t} dt' R^3(t')(r + r(t,t'))^3\right). \tag{III.11}$$

The above expression is merely the probability of no nucleations in the causal past of a spherical region of coordinate radius r in the time interval (t_0, t). The ratio $P(r,t_e)/P(t_e)$ then gives the fraction of false vacuum pockets of physical radius in excess of $R(t_e)r$ at the end of inflation. If the above ratio is not extremely small for $r > r_{BH}$ (i.e., for pockets that are large enough to form black holes), then the universe will inevitably become black hole dominated after suitable (radiation dominated) expansion.

Using $R(t)$ from equation (III.4), $r(t,t')$ becomes

$$r(t,t') = \frac{1}{R_0 B(\omega - 1/2)}\left[\frac{1}{x_{t'}^{\omega-1/2}} - \frac{1}{x_t^{\omega-1/2}}\right], \tag{III.12}$$

here $x_t \equiv (1 + Bt)$ and $x_{t'} \equiv (1 + Bt')$. Since our expressions for $P(t)$ and $P(r,t)$ involve integrations over t', or equivalently $x_{t'}$, it is convenient to rewrite the above expression as

$$r(t,t') = \frac{1}{R_0 B(\omega - 1/2)} \frac{1}{x_{t'}^{\omega-1/2}}\left[1 - \left(\frac{x_{t'}}{x_t}\right)^{\omega-1/2}\right]. \tag{III.13}$$

$P(r,t)$ can now be expressed as

$$
\begin{aligned}
P(r,t) =& exp(-\frac{4\pi\Gamma}{3}\int_1^{x_t}\frac{dx_{t'}}{B}R_0^3 x_{t'}^{3(\omega+1/2)} \\
& \sum_{k=1}^{3}\binom{3}{k} r^{3-k}\left(\frac{1}{R_0 B x_{t'}^{\omega-1/2}}\right)^k \left[1-\left(\frac{x_{t'}}{x_t}\right)^{\omega-1/2}\right]^k) \qquad \text{(III.14)} \\
=& exp(-\frac{4\pi\Gamma}{3}\sum_{k=1}^{3}\binom{3}{k} R_0^{3-k}B^{-(k+1)}r^{3-k}\frac{x_t^{\alpha+1}}{\alpha+1}\left[1+N(k,\omega)\right]).
\end{aligned}
$$

Here, $\alpha \equiv 3(\omega+1/2)-k(\omega-1/2)$. In the second equation we have used the approximation that $x_t >> 1$, which is necessarily true if $R(t_e) > R_0\, 10^{27}$. $N(k,\omega)$ is given by

$$
N(k,\omega) = \sum_{l=1}^{k}\binom{k}{l}(-1)^l\frac{\alpha+1}{\alpha+1+l(\omega-1/2)}. \qquad \text{(III.15)}
$$

(Note that $N(k,\omega)$ is a negative function that is strictly greater than -1 for $\omega > 1$.)

The $k=3$ term is merely $P(t)$. $P(r,t)/P(t)$ is given by the remaining terms in the expression. For large r the $k=0$ term dominates, as can be easily checked for $\omega > 1$.

Since we are interested in the period just after inflation ends, we set $P(t) \equiv exp(-n)$ (where $n \simeq 1$), which implies $n = \pi\Gamma t_e^4(1+N(3,\omega))/(3(\omega-1/2)^3)$.

We can now rewrite $P(r,t)/P(t)$ for large r (i.e., neglecting k = 1,2 terms) as:

$$
P(r,t)/P(t) = exp\left(-\frac{4n}{3\omega+1/2}\left[\frac{1}{1+N(3,\omega)}\right]\left(\frac{rR(t_e)}{L_{BH}}\right)^3\right), \qquad \text{(III.16)}
$$

where $L_{BH} = H(t_e)^{-1} \simeq x_t/(\omega B)$ is the critical size necessary to form a black hole. (To be more precise, $H(t_e) = Bx_t^{-1}(\omega + O(\frac{1}{\omega}))$.) For values of ω acceptable for inflation ($1.5 < \omega < 25$), the factor in brackets is less than ~ 10, so regions of false vacuum of size $> L_{BH}$ will comprise a large fraction of the remaining unpercolated volume. The universe just after inflation will therefore contain a large number of black holes of mass $M_{BH} \simeq (M_{Planck}/T_c)^2 M_{Planck}$.

Finally, as previously noted, it is possible in some models for $\Gamma(t) = \Gamma_0 x_t^m$ to be time dependent. In that case the results above are modified by taking $\alpha \rightarrow \alpha + m$ in equations (III.14) and (III.15). For the model of Holman, Kolb and Wang, m is $\simeq -2$, and does not significantly alter the conclusions of the preceding paragraph.

E A black hole dominated universe

We have argued above that an extended inflationary era leads to the production of black holes. Since nonrelativistic matter redshifts more slowly than radiation, even a small number density of black holes can eventually become the dominant form of energy in the universe. What are the consequences of such an era?

Black holes do not last forever, but rather evaporate due to Hawking radiation. The Hawking temperature is given by $T_H = M^2_{Planck}/(8\pi M_{BH})$ and the lifetime of the black hole is $\tau_{BH} = O(M^3 M^{-4}_{Planck})$. Since the fractional energy density in black holes redshifts like T^3, while radiation energy density redshifts like T^4, black hole domination will occur before evaporation as long as $P(r_{BH}, t_e) > (T_c/M_{Planck})^2$, where $r_{BH} \equiv L_{BH}/R(t_e)$. From (III.16), we see that this requirement is always satisfied for T_c less than GUT scales, or so, and for larger T_c if $\omega >> 1$.

If the universe is black hole dominated at evaporation, the reheat temperature is $T_R \simeq T_c(T_c/M_{Planck})$. By requiring that T_R exceed an MeV, so as to guarantee the results of standard nucleosynthesis, we have the bound $T_c > 10^8$ GeV.

Other limits can be derived by requiring that GUT-scale baryogenesis provide the matter-antimatter asymmetry that we observe today. If baryogenesis is due to the decay of GUT-mass bosons, we require that a sufficient number of such bosons be produced by the black hole in its final stages of evaporation. Any bound of this type is of course model dependent, as ϵ, the baryon asymmetry produced per heavy boson decay, is a function of CP violating phases and values of specific amplitudes in each model. However, it is straightforward to calculate the baryon asymmetry produced as a function of ϵ and g_*, the number of degrees of freedom lighter than M_{GUT}. For $T_{BH} < M_{GUT}$ this yields

$$
\frac{n_B}{n_\gamma} \simeq \frac{\epsilon}{g_*}\left[\frac{T_c}{M_{GUT}}\frac{T_c}{M_{Planck}}\right]. \qquad \text{(III.17)}
$$

The above expression yields a lower bound on ϵ as a function of T_c and M_{GUT}, since we expect $n_B/n_\gamma \sim 10^{-10}$ and $g_* \sim 10^2$. For example, $T_c \simeq M_{GUT} \simeq 10^{14}$ GeV yields $\epsilon > 10^{-3}$.

F Baby universes

Blau, Guendelman and Guth [34] have studied the question of the evolution of the interior of a false vacuum bubble. They find that for a spherical bubble larger than some critical size, the interior inflates in a very non-Euclidean way, forming its own causally separated universe. This critical size is almost identical to our criterion for black hole formation. If this type of behavior also occurs in Brans-Dicke gravity, then Extended Inflation would lead to the production of a large number of such baby universes.

Since the work of reference [34] is in the context of Einstein gravity, it cannot be directly applied to false vacuum bubbles in EI. The evolution of such a bubble in BD theories is complicated by the dynamics of the BD scalar, which would necessarily take on some spatial dependence near the surface of the bubble, as well as in its interior. In fact, we can display this spatial dependence in a simple way by momentarily neglecting gravity and considering the BD scalar alone.

Consider a spherically symmetric, uniform mass distribution of radius r_0 (a spherical false vacuum bubble would serve equally well). Using the equation of motion for the BD scalar (with $T_{\mu\nu}$ the matter stress tensor),

$$(\partial^2{}_t - \nabla^2)\Phi = T^\alpha{}_\alpha. \tag{III.18}$$

We can solve for the static configuration of the scalar field. To do so, we need to specify its value at infinity as $\Phi(\infty) = G^{-1}_{Newton} = M^2_{Planck}$ and to employ the Gauss law. Then, for $r > r_0$ we have

$$\Phi(r > r_0) = M^2_{Planck} + \frac{\int d^3x\, T^\alpha_\alpha}{4\pi r}. \tag{III.19}$$

(Hereafter $\int d^3x\, T^\alpha_\alpha \equiv TV = M$, where V is the volume of our sphere and M is its mass.) Similarly, for $r < r_0$, we integrate equation (III.18) and require continuity of Φ and Φ' at $r = r_0$. This yields

$$\Phi(r < r_0) = M^2_{Planck} + \frac{Tr_0^2}{2} - \frac{Tr^2}{6}. \tag{III.20}$$

For $r << r_0$, we have

$$\frac{\Phi}{M^2_{Planck}} \simeq 1 + \frac{3G_{Newton}M}{8\pi r_0}. \tag{III.21}$$

From equation (III.21) we see that at the center of a relatively massive sphere ($G_{Newton}M/r_0 \sim 1$), the BD scalar deviates significantly from its vacuum value. An observer at the center of this sphere would therefore find gravity relatively weak compared to an exterior observer. In fact, for such a massive sphere we clearly cannot neglect gravitational effects. This forces us to use the curved-space D'Alembertian in equation (III.18), and implies that the scalar field and Einstein equations must be solved simultaneously. This demonstrates some of the complications involved in generalizing the calculations of reference [34] to Brans-Dicke gravity. We will not comment further on this issue except to say that much work needs to be done.

G Conclusion

In this chapter we have studied the percolation of the vacuum in theories of extended inflation. While there are a number of very different EI models in existence, the common feature among them is power-law growth of the scale factor which makes percolation possible. We have argued that this leads to the production of black holes which inevitably dominate the energy density of the universe. The most conservative constraint (from nucleosynthesis) on such a universe restricts the critical temperature of the inflationary transition to be $> 10^8$ GeV. Other constraints can be derived by assuming standard baryogenesis.

An interesting, but unresolved, issue is the fate of the interiors of large false vacuum bubbles.

Chapter IV

Can a particle have a bag?

A Introduction

Recently, it has been argued [38] that non-topological solitons consisting of a single particle and a condensate of a background scalar field to which the particle couples exist in weak coupling. In other words, in a certain class of theories the lowest excitation of the system containing a single particle is one in which a bag of scalar field condensate, or "dimple", forms around the particle.

It is known that classical bag configurations can form when many particles are present. Such non-topological soliton solutions have been studied in detail [39, 40]. In this chapter we investigate whether such bags can form about a single particle. Intuitively, it appears that bag formation will occur if the energy in the scalar field gained by locally relaxing its vacuum expectation value is less than the energy liberated by the resulting decrease in particle mass. The existence of such bags around heavy quarks and gauge bosons would lead to interesting effects in both the masses and decay signature.

In section B we present semi-classical arguments for bag formation around a single particle. The validity of this semi-classical approximation is discussed in section C, and again in the language of coherent states in section D. For bags supported by one particle, we find that energetically favorable deformations are too small to be built up from the fundamental quanta. Hence, we argue that there are no dimples in the scalar field near a single, perturbatively coupled particle.

B A wrinkle in ϕ

To present the arguments for bag formation around a single particle we discuss an explicit example. Consider the Hamiltonian for a scalar field with a potential $V(\phi) = \frac{1}{4}\lambda(\phi^2 - v^2)^2$, coupled to a fermion with a mass $m_0 + g\langle\phi\rangle$, where $\langle\phi\rangle$ is the vacuum expectation value of the scalar ϕ. For time independent field configurations,

$$H = \int d^3r \left\{\frac{1}{2}(\nabla\phi)^2 + V(\phi) + \bar{\psi}(i\vec{\nabla}\cdot\vec{\gamma} + m_0 + g\phi)\psi\right\}. \tag{IV.1}$$

When there are no fermions present, the ground state of the system is just the standard vacuum $\langle\phi\rangle = v$. But, for the lowest energy solution with fermion number one, classical reasoning seems to imply that a condensate in the scalar field, where $\langle\phi\rangle$ differs from v, forms around the fermion. Ostensibly, if the Yukawa coupling to the fermion is sufficiently large, the scalar field energy gained by relaxing $\langle\phi\rangle$ towards zero near the fermion may be more than compensated by the loss in rest energy of the fermion. For a small dimple in the scalar field around a single fermion, the energy of the system is a combination of the surface energy resulting from the change in $\langle\phi\rangle = \phi_d$ in the bubble walls, the scalar field potential energy, the loss in rest mass of the fermion, and the kinetic energy of the confined fermion. For a bubble of radius R and thickness δR, and for a non-relativistic fermion,

$$E \simeq 4\pi R^2 \left(\frac{\delta v}{\delta R}\right)^2 \delta R + \Delta V(\phi_d) R^3 + m_f(\langle\phi\rangle) + \frac{p^2}{2m_0}, \tag{IV.2}$$

where ΔV is the change in potential energy density inside the bubble, δR is the thickness of the bubble wall, and m_f is the mass of the fermion in the presence of the shifted vacuum expectation value. The energy of the system will favor a bag with thick walls. In this limit, the dimple walls have a thickness comparable to the radius of the dimple $\delta R \sim R$, and the energy difference between a fermion in a dimple and one without a dimple is

$$E = \frac{2A}{3}\pi(\delta v)^2 R + 2B(\delta v)^2 m_\phi^2 R^3 - Cg\delta v + p^2/2m, \tag{IV.3}$$

where δv is the change in $\langle\phi\rangle$ at the center of the bubble, $m_\phi = \sqrt{\lambda}v$ is the mass of the scalar, and the parameters A, B, C are numerical factors of order 1. We call this energy the dimple binding energy, E_{DB}. The confinement momentum of the fermion, resulting from a spread in the fermion wave function over the dimple, will be of order $1/R$. Denoting $p = D/R$, the ground state of the system can be found by varying R and δv. When the scalar field mass is small, so that the volume term can be neglected, and provided $g \gtrsim (v/m_0)^{\frac{1}{3}}$, the binding energy of the dimple is

$$E_{DB} \simeq -\alpha^2 m_0 \left(\frac{A^3C^4}{D^2}\right), \tag{IV.4}$$

where $\alpha = g^2/4\pi$. The condition $g \gtrsim (v/m_0)^{\frac{1}{3}}$ must be imposed to ensure that δv be less than v. This is equivalent to the statement that we have a "dimple" in the scalar field rather that a bubble ($\langle\phi\rangle = 0$ at the center). Since we have optimized δv, solutions with $\delta v = v$ must give smaller energies. Our neglect of the volume term is valid provided $m_\phi \lesssim \alpha m_0 \left(\frac{C^2}{BD^2}\right)$. The dimple radius is

$$R \simeq \frac{1}{\alpha m_0}\left(\frac{D^2}{AC^2}\right). \tag{IV.5}$$

We can get a handle on the size of these factors by performing a Rayleigh-Ritz variational calculation. In this approximation we treat the bubble classically while treating the ψ-particle quantum mechanically. Below we calculate the energy of a single particle in a bag in this approximation. Writing

$$E = E_\phi + E_\psi, \tag{IV.6}$$

for $\phi_{cl} = v'$ at the center of the dimple, and denoting $\delta v = v - v'$, the scalar field contribution to the energy of the dimple is

$$E_\phi = \int d^3x \left\{\frac{1}{2}\left(\vec{\nabla}\phi_{cl}\right)^2 + V(\phi_{cl})\right\}, \tag{IV.7}$$

while the energy contribution from the ψ particle is

$$E_\psi = \int d^3r\, \psi^* \left(-\frac{\nabla^2}{2m_0} + m_0 + g\phi_{cl}(r)\right)\psi. \tag{IV.8}$$

Using various sets of trial wave functions, we minimize the binding energy with respect to R_D, R_ψ and v'. This results in $R_D \sim R_\psi$, which maximizes the decrease in the fermion mass while minimizing its kinetic energy.

The optimal ansatz was

$$\begin{aligned}\phi_{cl}(r) &= v\left(1 - \delta e^{-r/R_D}\right)\\ \psi(r) &= \sqrt{\frac{1}{\pi R_\psi^3}}\, e^{-r/R_\psi}.\end{aligned} \tag{IV.9}$$

For these trial wave functions we find

$$\begin{aligned}E_{DB} &\simeq -\frac{1}{28}\alpha^2 m_0\\ R_D &\simeq 9\frac{1}{\alpha m_0}.\end{aligned} \tag{IV.10}$$

For future reference we note that the energy of these "would be" non-topological solitons vanishes as the couplings are turned off.

C The Classical Criteria

Although the classical reasoning used above seems to suggest that the ϕ-field dimple is the minimum energy solution in the presence of an external source (eg/ the fermion), this does not mean that we should place the dimple on an equal footing with the actual ground state, $\langle\phi\rangle = v$. The arguments used to find this classical, minimum energy solution to the field equations treated the scalar field ϕ as a continuous, c-number field. Quantum mechanics tell us that this is not in general correct. As a result of quantization, changes in the value of a field are discrete. These changes may be viewed as continuous when differences in the field considered represent the addition or subtraction of many quanta. When a deformation in the value of an field operator is small compared to differences induced by the emission or absorption of many quanta, we shouldn't trust classical arguments. Another way of saying this is that the expectation value of an operator can be replaced by its classical value only when the fluctuations in that expectation value are small. This is only the case when the field contains many quanta.

Armed with the results of section B, we can check our treatment for self-consistency by comparing the energy in the scalar-field dimple to that of an individual scalar quantum. We find, unfortunately, that the approximation of treating the dimple classically is a bad one because its energy is *less* than that of a single quantum. Consider a Fourier decomposition of the scalar-field dimple. Since the size of the dimple is $\sim \frac{1}{\alpha m_0}$ (where, for simplicity, we drop numerical factors), we expect the fourier modes to have momentum $\sim \alpha m_0$. This implies that the typical energy of a quantum in the dimple is

$$E_{quantum} \simeq (m_\phi^2 + \alpha^2 m_0^2)^{1/2}. \tag{IV.11}$$

Comparing this to E_{DB} we find,

$$\frac{E_{DB}}{E_{quantum}} \sim \alpha. \tag{IV.12}$$

Hence, for *perturbative* coupling our solution is composed of much *less* than one quantum.

The invalidity of the classical treatment of a dimple in a scalar field surrounding a particle can be seen directly. In what follows, we denote the total energy in the scalar field by E_ϕ. The criterion that the deformation in ϕ_{cl} be made up of at least one quantum is

$$E_\phi \geq E_{quantum} \simeq \sqrt{m_\phi^2 + R^{-2}} > 1/R. \tag{IV.13}$$

In order for a bound state to exist the total energy in the scalar field, E_ϕ, must be less than the change in rest mass of the fermion:

$$E_\phi \leq g\delta v \tag{IV.14}$$

Putting this together with (IV.13) we have $1 < g(\delta v)R$. In particular the energy in the "surface" terms must be less than the change in rest mass of the fermion,

$$4\pi(\delta v)^2 R < g\delta v, \tag{IV.15}$$

so that, $\delta v R < g/4\pi$. Putting these together gives $\alpha = g^2/4\pi > 1$. Requiring that the dimple be made up of N quanta implies that

$$\alpha = \frac{g^2}{4\pi} > N. \tag{IV.16}$$

Hence there are no bubbles for perturbative couplings.

It is instructive to compare the behavior of these "would be" non-topological solitons to the behavior of classically valid topological solitons [41, 42]. For definiteness, we recall the properties of the "kink" (domain wall) solitons of ϕ^4 theory in $1+1$ dimensions. Rescaling the scalar field ϕ by the small coupling g, $\phi' = g\phi$, the ϕ^4 theory Lagrangian is

$$\mathcal{L}/\hbar = \frac{1}{g^2\hbar}\int dx \left\{ \frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{\mu^2}{2}(1-\phi'^2)^2 \right\}. \tag{IV.17}$$

In addition to the degenerate vacua of $\langle\phi'\rangle = \pm 1$, the Lagrangian (3.7) has a stable classical solution

$$\phi'_{cl} = \tanh \mu x. \tag{IV.18}$$

Quantizing around this solution, to order g, the energy is given by

$$E = \frac{4\mu}{3g} - \mu g\left(\frac{3}{\pi} - \frac{1}{2\sqrt{3}}\right) + \mathcal{O}(g^3\mu). \tag{IV.19}$$

Hence, we see that, in the region of validity, the "classical" contribution to the energy of this topological soliton $E \sim \mu/g$ is many times larger than the energy of a quantum fluctuation $E_q \sim g\mu$. The classical limit $\hbar \to 0$ is equivalent to $g \to 0$. So, for weak coupling, the energy of the classical solution is much larger than the energy of fluctuations around the solution (IV.18). For large values of g the energy of the classical solution is not the dominant term in (IV.19) and there is no more reason to believe the classical analysis than there is to trust perturbation theory [41]. The same remarks would apply to the soliton structures of sine-Gordon theory, the 'tHooft-Polyakov monopole, or other topological solitons in field theory.

D Classical Values, Fluctuations and Coherent States

For small shifts $\delta\phi$ in the field ϕ about the minimum of the potential $V(\phi) = \lambda(\phi^2 - v^2)^2$, we can perturbatively define our theory so that to lowest order it is merely the theory of a heavy fermion coupled to a free, massive scalar field. The Lagrangian for such a theory is simply

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \overline{\psi}(i\partial\!\!\!/ - m_o)\psi - g\phi\overline{\psi}\psi, \tag{IV.20}$$

where m is given by the second derivative of $V(\phi)$ at its minimum. Since the scalar field is free except for its coupling to the fermion, we can now study its behavior using the formalism of coherent states.

Coherent states are constructed by the application of creation operators (in the interaction picture) on the vacuum state $|0\rangle$ so as to yield a prescribed expectation value for the field operator $\hat{\phi}$. We define coherent states for the positive frequency part of the quantum field ϕ. The field operator $\hat{\phi}$ can be expanded in on-shell Fourier modes in the usual way:

$$\hat{\phi} = \int d\tilde{k}\{a_k e^{i\vec{k}\cdot\vec{x}} + a_k^\dagger e^{-i\vec{k}\cdot\vec{x}}\} \tag{IV.21}$$

where $d\tilde{k} = d^3k/(2\pi)^3 2\omega_k$. Since we work in the interaction picture, the operators a_k and $a_k^\dagger$ are time dependent (ie/ $a_k(t) = a_k(0)e^{i\omega_k t}$). The creation and annihilation operators also satisfy the usual commutation relation

$$[a_k, a_{k'}^\dagger] = (2\pi)^3 2\omega_k \delta^3(k-k'). \tag{IV.22}$$

Consider the coherent state $|\eta\rangle$ given by

$$|\eta\rangle = C_N \exp\{\int d\tilde{k}\eta(k)a_k^\dagger\}|0\rangle, \tag{IV.23}$$

where $\int d\tilde{k}|\eta(k)|^2 < \infty$, and $C_N = \exp\{-\frac{1}{2}\int d\tilde{k}|\eta(k)|^2\}$ is chosen so that $|\eta\rangle$ has a norm of one. It is easy to verify that

$$\langle\eta|\hat{\phi}|\eta\rangle = \int d\tilde{k}\{\eta(k)e^{i\vec{k}\cdot\vec{x}} + \eta^*(k)e^{-i\vec{k}\cdot\vec{x}}\}, \tag{IV.24}$$

so that by proper choice of $\eta(k)$ one can ensure that $\langle\eta|\hat{\phi}|\eta\rangle$ has the desired form.

The average number of particles in the coherent state, $\overline{N} = \langle\hat{N}\rangle$, is

$$\langle\eta|\hat{N}|\eta\rangle = \langle\eta|\int d\tilde{k}\, a_k^\dagger a_k|\eta\rangle = \int d\tilde{k}\, |\eta(k)|^2, \tag{IV.25}$$

so $C_N = \exp\{-\frac{1}{2}\overline{N}\}$. The coherent state has energy

$$\langle\eta|\hat{H}|\eta\rangle = \langle\eta|\int d\tilde{k}\,\omega_k a_k^\dagger a_k|\eta\rangle = \int d\tilde{k}\,\omega_k|\eta(k)|^2. \tag{IV.26}$$

Fluctuations in $\langle\eta|\hat{H}|\eta\rangle$ can be defined by

$$\langle\eta|\hat{H}^2|\eta\rangle = \langle\eta|\hat{H}|\eta\rangle^2 + \int d\tilde{k}\,\omega_k^2|\eta(k)|^2. \tag{IV.27}$$

The classical limit is attained by choosing $\eta(k)$ so that the average number of particles $\overline{N}$ is large. As the number of particles is increased,

$$\frac{\langle\eta|\hat{H}^2|\eta\rangle - \langle\eta|\hat{H}|\eta\rangle^2}{\langle\eta|\hat{H}|\eta\rangle^2} \propto \frac{1}{\overline{N}}, \tag{IV.28}$$

where $\overline{N}$ is the total number of quanta as defined in (IV.25). Note that when $\overline{N}$ is small, the energy of the state $|\eta\rangle$ is not well defined, and undergoes large fluctuations compared to its mean value. This is to be expected as $\hat{H}$ and $\hat{\phi}$ do not commute. The semi-classical approximation used in section B requires that both $\langle\hat{\phi}\rangle$ (which determines the fermion mass) and $\langle\hat{H}\rangle$ (which gives the scalar field energy) be known. This is only possible in the classical limit where $\overline{N} \to \infty$. We can therefore conclude that when $\overline{N}$ is small, there is no reason to expect that the calculation of section B has any validity whatsoever.

Moreover, when $\overline{N}$ is small the non-classical nature of the coherent state is manifest since

$$|\eta\rangle = e^{-\frac{1}{2}\overline{N}}\{|0\rangle + \int d\tilde{k}\,\eta(k)|k\rangle + \ldots\}, \tag{IV.29}$$

so that the coherent state is mostly vacuum with small amplitudes to be in an n particle state.

It is important to reiterate that there is no uncertainty whatsoever in this system as to the identity of the ground state. The ground state is clearly $|0\rangle$, upon which all excitations of ϕ and ψ are constructed. By "turning on" a source term gradually, one sees that the resulting ϕ field is a superposition of scalar excitations above the vacuum. It is only when the scalar excitations consist of many quanta that a classical description of the ϕ field is valid.

Quantum fluctuations of the type considered above can also be understood as a finite volume effect. A measurement of the value of a field must always be taken in some smeared volume. The value of the field can only be determined to within fluctuations which are $\sim 1/L$, where L is roughly the size of the smeared volume. (This can easily be seen by considering the quantity $\langle\phi(x)\phi(x')\rangle$, which diverges for $x \to x'$ even in a free theory.) It is straightforward to show that for the dimple system,

with $L \sim R_D$, the fluctuations $1/R_D$ are always larger than the classical shift in the vev, δv. Thus, the quantum fluctuations overwhelm the purported classical shift, and it is meaningless to ascribe a physical significance to a scalar dimple in a volume of size R_D^3.

Now we construct the exact solution to the scalar field equations in the presence of a normalizable, localized source which we can take to represent the fermion wave function in the dimple system. We can then rigorously show that, for perturbative coupling between the fermion and scalar, $\langle\hat{N}\rangle$ for this solution is less than one.

The time-independent equation of motion for the scalar field of (IV.20) in the presence of a source $J(x) \equiv \psi^\dagger(x)\psi(x)$ is given by

$$(\nabla^2 - m^2)\phi(x) = gJ(x). \tag{IV.30}$$

The coherent state corresponding to the solution of (4.10) is given by (4.4), with

$$\eta(k)/(2\omega_k) = -gJ(k)/(k^2 + m^2). \tag{IV.31}$$

where $J(k)$ satisfies $J(x) = \int \frac{d^3k}{(2\pi)^3}\{J(k)e^{i\vec{k}\cdot\vec{x}} + h.c.\}$. The expectation value of the number operator is then

$$\overline{N} = 4g^2\int d\tilde{k}\frac{|J(k)|^2}{(k^2+m^2)}. \tag{IV.32}$$

Inserting for ψ our best ansatz (IV.9), we can now directly calculate $\langle\hat{N}\rangle$. For $R_\psi m_\phi < 1/2$, the total number of quanta in the dimple is

$$\overline{N} = \frac{\alpha}{2\pi}\left[\ln(16/\xi^2) - \frac{23}{6}\right]\{1 + \mathcal{O}(\xi^2)\}, \tag{IV.33}$$

where $\xi = R_\psi m_\phi$. We note that most of these quanta have wavelengths of $\sim 1/m_\phi$.

The apparent divergence for $\xi \to 0$ does not concern us, as it is just the standard infrared divergence due to long wavelength modes. This is the familiar result for the massless limit of a Yukawa field (i.e., a Coulomb field). Long wavelength modes are irrelevant to the question of whether the dimple should be treated classically. The relevant modes are those of wavelength $\lesssim R_\psi$. Therefore, for extremely small values of m_ϕ, we should impose an infrared cutoff $1/L$ in (IV.32) to obtain the number of physically relevant quanta. This corresponds to placing the system in a box of size L. For $L \sim R_\psi$, it is easy to see that the number of quanta (IV.33) is always less than α.

We note that by suitable choice of the source term $J(x)$, it is possible to obtain a valid classical state $\phi(x)$. Since $\overline{N}$ is proportional to $g^2|J(k)|^2$, it can be increased either by increasing g, or by increasing the overall magnitude of $J(x)$. The former corresponds to strong coupling, while the latter corresponds to increasing the number of fermions contained in the probability distribution $J(x) \equiv \psi^{\dagger}(x)\psi(x)$. The solutions obtained in that case correspond to many-particle nontopological solitons, as discussed in [39].

Although we have argued against the existence of a classical condensate surrounding the fermion, this does not imply that there is no vestige of collective behavior for non-zero g. In fact, the collective effects which do appear are well understood [43]. For non-zero coupling g, the lowest energy eigenstate of the Hamiltonian with fermion number one is just the unperturbed free fermion state "dressed" by a cloud of scalar quanta. This "dressing" can be understood simply as wave function and mass renormalization of the fermion, yielding a correction to the fermion mass which is divergent. A prescription for subtracting this divergence (e.g., by the addition of counterterms to the Lagrangian) is necessary to ensure a finite physical mass for the fermion. This *is* the mass of the physical (renormalized) fermion, which is in fact the lowest energy state of fermion number one.

E Conclusion

To summarize, we have considered here putative non-topological solitons consisting of a single particle and a deformation in the scalar field whose vacuum expectation value gives a mass to the particle. Although a purely classical treatment of the scalar field suggests that the solution in which a dimple forms in the scalar field near the particle is energetically favored, this is overshadowed by the fact that for the bubble solution found, the classical treatment is not valid. The classical computation of a quantity is only reliable when the fluctuations about the solution are small compared to the classical value. A necessary condition for the fluctuations to be small is that the deformations in a field must be made of many quanta. Unlike topological solitons in weak coupling, the dimple solution discussed in this chapter and in [38] is not well described by classical physics. For weak coupling, the collective effects of the interaction are understood in terms of renormalization.

We should also mention some phenomenological work concerning a possible "bag" around the top quark in the standard model. The existence of such a bag might result in an interesting signature in top quark decays. The work of S. Dimopoulos, B.W. Lynn, S. Selipsky and N. Tetradis [44] entitled "The vacuum abhors top bags", studies the possibility of a Higgs dimple around the top quark in the standard model. The authors of this paper apply a semi-classical approximation to study the properties of a Higgs-top dimple, and find that such an object is energetically favorable only for fairly large Higgs-top couplings ($g_{top} > 2$), which they claim are ruled out

phenomenologically. We remark here that insofar as any calculations are reliable in a theory with such large couplings, the arguments presented in our chapter are relevant to the Higgs dimple that they consider. Therefore, it can be concluded that such an object does not exist without resorting to the phenomenological arguments presented in their work.

Chapter V

Instantons and vacuum tunneling via particle collisions

A Introduction

There has been a great deal of interest recently in the question of (B+L) violation mediated by instantons in the standard model [46, 47, 48, 50, 51]. Investigations along these lines have focused on the calculation of exclusive and inclusive (B+L) violating cross sections in the one-instanton sector. Khlebnikov, Rubakov and Tinyakov (KRT)[48] have recently given a functional integral expression for the inclusive cross section, which reproduces earlier results from calculations using one-instanton Green's functions and the LSZ procedure [46, 50, 51].

In this chapter we will address the closely related issue of producing a vacuum bubble in a theory with a metastable false vacuum by colliding energetic particles. The cross section for this process can also be expressed in terms of a KRT-like functional integral. We show that the cross section for vacuum tunneling increases with center of mass energy, although we are unable to extract its exact energy dependence. There have been several previous works on so-called "induced" vacuum decay in the presence of heavy particles [52].

The simple model we will study is that of a scalar field in 4 dimensions with two nondegenerate vacua (the analysis can readily be extended to theories in an arbitrary number of dimensions). Tunneling at low energies in this theory is mediated by fixed-size, $O(4)$ symmetric instantons (the "bounce" [45]) and results in the production of a bubble of true vacuum which expands relativistically after nucleation. These models have more than an academic relevance, as there is a large region of standard model parameter space in which our universe lives in just such a metastable false vacuum

[49] (i.e., when $m_{top} > m_{Higgs}, m_{W,Z}$). Therefore, a calculation of the rate of vacuum tunneling at high energies is important for determining limits on the Higgs-top sector of the standard model. (A somewhat heuristic analysis of vacuum tunneling at high energies is presented by J. Ellis et al. in the second reference of [49].) The calculations performed in this chapter will be directly applicable to the case mentioned above, except that in the standard model the instanton action will not be calculable in the "thin-wall" approximation, and should be evaluated numerically.

As we will discuss below, the physical interpretation of the instantons (bounces) is in terms of their relation to vacuum decay. However, we will also show that the existence of the instanton solutions implies some potentially interesting behavior in the inclusive cross section for scalar-scalar scattering in the false vacuum. We find that this cross section grows exponentially with center of mass energy, reminiscent of (B+L) violation in the standard model.

The organization of this chapter is as follows. In section B we exhibit the instanton solutions of our theory, and also discuss properties of its sphaleron. In section C we review the KRT formalism and apply it in a calculation of inclusive scattering between Fock states of the metastable vacuum. We find that the nonperturbative contribution to this cross section grows exponentially with center of mass energy, in agreement with (B+L) violating cross sections in the standard model. In section D we modify the KRT formalism to calculate the cross section for vacuum bubble production at low energies, which apparently is also enhanced at high energies. Our results can also be used to reproduce the rate for spontaneous decay of the false vacuum, previously deduced by other methods. Section E summarizes our results.

B Instantons and sphalerons

The system we consider in this chapter is the theory of a real scalar field in four dimensions, whose potential exhibits two nondegenerate minima. Although simple, this system exhibits many of the features present in the Gauge-Higgs sector of the standard model. That is, it possesses both instanton and sphaleron solutions. On the other hand, we will see that there are also interesting physical phenomena, such as the instability of the higher energy vacuum, which have no analog in the standard model, unless one considers the Higgs sector alone in the limit of a heavy top quark [49]. In this section we study the instanton and sphaleron solutions of the theory, and derive expressions which will be necessary for subsequent analysis.

The theory can be written in terms of the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{8}\left(\phi^2 - \frac{\mu^2}{\lambda}\right)^2 + V_{sb}(\phi), \tag{V.1}$$

where $V_{sb}(\phi)$ breaks the symmetry between the vacua at $\phi = +a, -a$. (Here $a^2 = \mu^2/\lambda$.) We will take $V'(\phi)$ to be

$$V_{sb}(\phi) = \frac{\epsilon}{2a}(\phi - a), \tag{V.2}$$

where ϵ has dimensions of (mass)^4.

Henceforth we will refer to $\sigma_+ = +a$ as the false vacuum, and $\sigma_- = -a$ as the true vacuum. If the lifetime of the false vacuum is long enough (i.e., the rate for spontaneous decay is sufficiently small), we can build states perturbatively as harmonic excitations which know nothing of the other lower minimum. We can then define an S-matrix and discuss scattering between these states of the false vacuum. Since there are higher order (i.e., $\lambda\phi^4$) interactions among these states, there exist tree graphs which contribute to such scattering. We will show that instantons also contribute to this scattering, and may even give the dominant contribution at high energies. *

Spontaneous vacuum decay via instantons in this theory was first considered in [45]. The instanton (or "bounce") solutions are O(d) symmetric functions of $\rho = (t^2 + x^2)^{1/2}$. (It has been proven that non-O(d) symmetric solutions all have higher Euclidean action than the O(d) symmetric solution [45].) The bounce obeys the following Euclidean equations of motion and satisfies the following boundary conditions:

$$\begin{gathered}\frac{d^2\Phi}{d\rho^2} + \frac{3}{\rho}\frac{d\Phi}{d\rho} = V'(\Phi) \\ \Phi(\rho \to \infty) = \sigma_+.\end{gathered} \tag{V.3}$$

Note that the equation of motion for Φ is equivalent to the equation of motion for a particle moving in a one-dimensional potential, with a ρ dependent drag. The one-dimensional potential is equal to the negative of the scalar potential. Using this physical analogy, one can demonstrate that a solution of (V.3) will always exist. We should note that if $\Phi(x)$ is a solution to (V.3), then so is $\Phi(\xi, x) \equiv \Phi(x - \xi)$. That is, there are many instantons, each centered at a different point in Euclidean four-space.

*We mean this only in the sense that if one believes that (B+L) violating instanton effects in the standard model become strong at high energies, one would also conclude that similar instanton mediated processes here will eventually be dominant. At present, however, there is no rigorous evidence that this is the case in either model.

The action of the bounce solution can be calculated analytically in the thin-wall approximation [45]. In that approximation the energy of the true vacuum is only slightly less than that of the false vacuum. In that case one can show that the bounce looks like a four-dimensional spherical bubble of radius R (see Figure 2). For, $\rho >> R$, $\Phi(\rho) = \sigma_+$, while for $\rho << R$, $\Phi(\rho) = \sigma_-$. The transition between σ_+ and σ_- occurs at $\rho \simeq R$, and the shape of $\Phi(\rho \simeq R)$ is approximately that of a one-dimensional kink of thickness μ^{-1}.

Given this information, one can perform a variational calculation in R to minimize the action of the bounce. For the theory specified in (V.1), this yields $R = \mu^3/\lambda\epsilon$ and $S_B = (\pi^2\mu^{12})/(6\epsilon^3\lambda^4)$. For self consistency of the approximation, we must have $\mu R >> 1$, or $\mu^4/\lambda >> \epsilon$.

In subsequent calculations we will need to know the asymptotic ($\rho \to \infty$) behavior of the bounce. This is easily obtained by linearizing equation (V.3) about the minima Φ_+. One obtains

$$\left(\frac{d^2}{d\rho^2} + \frac{3}{\rho}\frac{d}{d\rho} - \mu^2\right)(\Phi - \sigma_+) = 0. \tag{V.4}$$

This has the solution

$$\Phi(\rho) - \sigma_+ = AK_1(\mu\rho)/\mu\rho, \tag{V.5}$$

where K_1 is a modified Bessel function, and A is determined by matching (2.5) onto the exact bounce. Although a precise determination of A requires a numerical calculation, in the thin-wall approximation $A \sim a\mu R/K_1(\mu R)$. In any case, the ρ dependence of (2.5) will be sufficient to determine the energy dependence of processes we wish to consider.

Again for future use we will need to compute the projection of the bounce onto plane wave states for $|T| \to \infty$. For this purpose we need the three dimensional Fourier transform of $\Phi(\rho)$ at time T. This can be expressed [48] in terms of the coefficient of the on-shell pole of the four dimensional Fourier transform of $\Phi(\rho)$:

$$\begin{aligned}\tilde{\Phi}(\xi, T_i, k) &= \frac{e^{i\omega_k T_i}}{(2\pi)^3 2\omega_k} R(k,\xi) \\ \tilde{\Phi}(\xi, T_f, k) &= \frac{e^{-i\omega_k T_f}}{(2\pi)^3 2\omega_k} R(k,\xi),\end{aligned} \tag{V.6}$$

where $R(k,\xi)$ is defined by

$$\lim_{k^2 \to -\mu^2} (k^2 + \mu^2) \int dx_0 d\vec{x}\, e^{ik_\mu x^\mu} \Phi(x - \xi) \equiv R(k,\xi). \tag{V.7}$$

The quantity $R(k,\xi)$ also appears in the LSZ reduction formulas used in [46]. The necessary Fourier transforms of quantities like (2.5) are given in those references, and we shall merely state the result:

$$R(k,\xi) = 4\pi^2(A/\mu^2)exp(ik_\mu\xi^\mu). \tag{V.8}$$

We note that the magnitude of R is independent of k. This implies, as we will see in section C, that the overlap of plane wave states with the asymptotic limit of the instanton is *not* suppressed when the momentum k of the plane waves is large. This contradicts the arguments of [47], where it is claimed that there is an exponential suppression associated with the overlap of the instanton with high energy states.

The sphaleron † in this theory is nothing more than a particular critical vacuum bubble. A bubble is critical if it can decrease its energy by expanding. The sphaleron sits at an unstable point in configuration space and can decrease its energy by either expanding or shrinking. If perturbed, it will either shrink to zero or expand, depending on the sign of the perturbation. We can illustrate some of its properties by studying how the energy of a vacuum bubble scales with its radius. For simplicity, we will again work in the thin-wall approximation.

Consider a spherical vacuum bubble $\Phi(r)$ of radius R in the theory specified by (V.1), where ϵ is small. At large distances from the center of the bubble, $\Phi(r >> R) = \sigma_+$, while $\Phi(r << R) = \sigma_-$. At $r \simeq R$, the field makes a transition from σ_+ to σ_- over a distance of order μ^{-1}. This transition region again has the shape of a kink. The surface energy of the transition region can be easily calculated as $\Sigma \simeq \mu^3/\lambda$.

The total energy of the bubble can be written as the sum of surface and volume terms:

$$E_B(R) \sim -\epsilon R^3 + \Sigma R^2. \tag{V.9}$$

Notice that $E_B(R)$ goes to zero as $R \to 0$, and $-\infty$ as $R \to \infty$. However, in an intermediate region of R there is an energy barrier with a height equal to $E_{sphaleron}$. Consider the behavior of $\frac{dE_B}{dR} \sim -3\epsilon R^2 + 2\Sigma R$. This function is initially positive, but goes through zero at $R_* \sim \Sigma/\epsilon$. Bubbles of radius larger than R_* will expand, while those smaller will contract to zero. The sphaleron for this class of configurations is a thin-walled bubble of radius R_*. Its energy is given by

$$E_{sphaleron} \sim (\Sigma^3/\epsilon^2) \sim (\mu^8/\lambda^3\epsilon^2). \tag{V.10}$$

One can also verify that the Minkowski rotation of the $T = 0$ slice of the bounce is actually a critical bubble, with $R > R_*$.

† The author is grateful to G. Anderson for many of the observations concerning sphalerons.

We note that the scaling of $E_{sphaleron}$ with the various parameters of the theory is very different from the scaling of S_B. Many authors [48, 50, 51] have stated that the sphaleron energy is somehow the relevant scale at which tunneling rates become large, but this claim seems unfounded in general. The suppression factor that must be overcome by energy dependence is actually e^{-S_B}, and there is no simple relation of that factor to the sphaleron energy in this theory.

Finally, we note that the presence of the symmetry breaking interaction $V_{sb}(\phi)$ is crucial for the existence of the bounce solution. In the absence of this term the action of the instanton goes to infinity, as seen in the expression for S_B. In fact, one can see that in the limit that $\epsilon \to 0$, there are no O(d) symmetric solutions satisfying the boundary conditions given in (V.3) except the trivial solution, $\Phi = \sigma_+$. It is also clear that there are no sphaleron solutions in this limit.

C KRT redux

In this section we will briefly review the coherent state path integral used by KRT to give the inclusive cross section in the one instanton sector. We will then apply the formalism using the bounce solution found in section B to give the instanton contribution to scattering between particle states of the false vacuum.

We begin with the transition amplitude between arbitrary states $\langle i|$ and $|f\rangle$:

$$\langle i|U|f\rangle = \int D\phi_i D\phi_f \langle i|\phi_i\rangle\langle\phi_i|U|\phi_f\rangle\langle\phi_f|f\rangle, \tag{V.11}$$

where $|\phi_{i,f}\rangle$ are eigenstates of the field operator $\hat{\phi}(x,T_{i,f})$ with eigenvalues $\phi(x,T_{i,f})$ and $|i\rangle,|f\rangle$ are coherent states [53]. The transition amplitude is given by

$$\langle\phi_i|U|\phi_f\rangle = \int_{\phi(T_i)=\phi_i}^{\phi(T_f)=\phi_f} D\phi \;\; e^{iS[\phi]}. \tag{V.12}$$

Equations (V.11) and (V.12) are completely general. Since we are interested in the process $2 \to all$, where *all* means asymptotic particle states of the false vacuum, we must square (V.11) and sum over all $|f\rangle$'s which are superpositions of plane wave states,

$$\sigma_{inc} \sim \sum_f |\langle i|U|f\rangle|^2 = \sum_f |\int D\phi_i D\phi_f \langle i|\phi_i\rangle\langle\phi_i|U|\phi_f\rangle\langle\phi_f|f\rangle|^2. \tag{V.13}$$

We now rotate to Euclidean space and restrict the paths which contribute to (V.12) to be the instantons, taking the limit $T_i, T_f \to \pm\infty$. The path integral is then reduced to an integral over instanton coordinates, ξ:

$$\sigma_{inc} \sim \sum_f \int D\xi D\xi' \langle i|\Phi_i(\xi)\rangle\langle\Phi_i(\xi')|i\rangle\langle\Phi_f(\xi)|f\rangle\langle f|\Phi_f(\xi')\rangle exp(-2S_B), \tag{V.14}$$

where S_B is the bounce action, and $\Phi_i = \Phi_{bounce}(T_i \to -\infty)$, $\Phi_f = \Phi_{bounce}(T_f \to +\infty)$ are the asymptotic limits of the instanton.

The overlaps can be calculated using standard results for coherent states. In general, for an overlap between a coherent state $\langle\eta|$ and $|\phi\rangle$, evaluated at time T,

$$\begin{aligned}\langle\eta|\phi\rangle = exp(&- 1/2\int dk\ \eta(k,T)\eta(-k,T) - 1/2\int dk\ \omega_k\tilde{\phi}(k,T)\tilde{\phi}(-k,T) \\ &+ \int dk\sqrt{\omega_k}\eta(k,T)^*\tilde{\phi}(k,T).\end{aligned} \tag{V.15}$$

Here $\tilde{\phi}$ and η are functions in momentum space. (In particular, $\tilde{\phi}(k,T)$ is the three dimensional Fourier transform of $\phi(x,T)$.)

For the case of interest, we wish to evaluate $\langle i|\Phi_i\rangle$, where $\langle i|$ is a 2 particle plane wave state of four momenta $(E/2, \pm k)$, and $|\Phi_i\rangle$ is the asymptotic tail of the instanton. In that case [48], we have

$$\langle i|\Phi_i(\xi)\rangle = R(k,0)R(-k,0)e^{-iE\xi_0}. \tag{V.16}$$

The sum over $|f\rangle$'s in (V.14) yields an overlap which can be expressed as

$$\langle\Phi_f(\xi)|\Phi_f(\xi')\rangle = exp(\int \tilde{dk}e^{i\omega_k(\xi_0-\xi_0')-i\vec{k}\cdot(\vec{\xi}-\vec{\xi}')}R(k,0)R^*(k,0)). \tag{V.17}$$

We note that the Euclidean time dependence in $\tilde{\Phi}_{i,f}(k,\xi)$ is such that the first two terms in the exponential of (V.15) (the normalization terms) go to zero as $T_{i,f} \to \pm\infty$, leaving the third term in which these dependences cancel except for a relative factor of $\xi - \xi'$. At the moment we have written the $\xi - \xi'$ terms as Minkowski (real) separations, which explains the factors of i.

Using (V.16) and (V.17), (V.14) can be written as

$$\begin{aligned}\sigma_{inc} \sim &VT exp(-2S_B)\int D(\xi-\xi')|R(k,0)|^2|R(-k,0)|^2 \\ &\cdot exp\left(-iE(\xi_0-\xi_0') + \int \tilde{dk}e^{i\omega_k(\xi_0-\xi_0')-i\vec{k}\cdot(\vec{\xi}-\vec{\xi}')}R(k,0)R^*(k,0)\right),\end{aligned} \tag{V.18}$$

where we have rewritten the integration over ξ, ξ' as a total four volume VT times an integral over $\xi - \xi'$. The volume term is necessary for the definition of the cross section as $\sigma = (1/VTj)|Amplitude|^2$, where j is the flux of incoming particles.

We can now evaluate the integral (V.18) over $\xi - \xi'$ in the saddlepoint approximation and extract the dependence of the total cross section on energy. We neglect the effect of the non-exponential prefactors, and simply extremize the exponent. Inserting the expression for $R(k,\xi)$ from (V.8), and defining $x \equiv (\xi - \xi')$, we have for the exponential

$$e^W \equiv exp(-iEx_0 + C\int \tilde{dk}e^{i\omega_k x_0 - i\vec{k}\cdot\vec{x}}), \tag{V.19}$$

where $C = |R(k,0)|^2$ is a constant given by $16\pi^4(A^2/\mu^4)$.

The saddlepoint occurs at $\frac{\partial W}{\partial x}|_{x^*} = 0$. This yields $\vec{x}^* = 0$, and for x_0^* the equation

$$E = C\int dk e^{i\omega_k x_0^*} = C/(\pi^2 i(x_0^*)^3). \tag{V.20}$$

Thus, $ix_0^* = -(C/E\pi^2)^{1/3}$, and we have for the total cross section

$$\sigma_{inc} \sim exp(-2S_B + (C/8\pi^2)^{1/3}E^{2/3}). \tag{V.21}$$

Note that at the saddlepoint x_0^* the separation $\xi_0 - \xi_0'$ is actually Euclidean. (The last equality in (V.20) is only valid in the limit $x_0 << \mu^{-1}$, but for $E >> \mu$ the approximation is self-consistent. Alternatively, the integral can be evaluated explicitly in terms of a Bessel function.)

What is the physical interpretation of what we have found? Consider a physicist who lives in the metastable vacuum, completely oblivious to the existence of the lower vacuum. (This may indeed be the case with our universe, if m_t is sufficiently large!) That physicist can conduct scalar-scalar scattering experiments at higher and higher energies. Our results show that the existence of the lower minimum affects the scattering cross section of scalars on scalars. If one extrapolates the exponential rise in cross section to sufficiently high energies [46, 50], one could even conclude that this is the dominant contribution. [1] This is somehow disturbing, unless other processes more directly related to the instability of the false vacuum also become strong. One such process is vacuum tunneling, which we will consider in section D.

D Semi-bounces

In this section we will generalize the analysis in the previous section to include scattering into more complicated final states. Previously, we restricted our final states $|f\rangle$ to be simple superpositions of plane wave states. They are therefore suitable asymptotic states of the theory, whose time evolution is given by the free Hamiltonian. However, we know that an object such as a critical vacuum bubble is *not* an asymptotic state. Rather, its evolution is complicated and depends on the nonlinear interactions of the full scalar potential.

[1] In calculating the inclusive cross section σ_{inc}, we should legitimately have summed over classically allowed contributions to ϕ-ϕ scattering (i.e., from tree graphs) before squaring the amplitude in (V.13). However, if the dominant contribution at high energies is from instantons, we can neglect interference due to perturbative processes.

We will now drop the requirement on $|f\rangle$ that it be a simple coherent state. Instead, we will take the set of final states to be a complete set of states for the full theory, which must therefore include complicated objects such as vacuum bubbles, and other many-particle semiclassical configurations.

Given the above considerations, it is clear that the full bounce solution will not give a contribution to vacuum bubble production. That is, it will overlap asymptotically only with free particle states. To remedy this fact, we will use the bounce solution in the region $[T_i = -\infty, 0]$. This describes a Euclidean path that connects plane wave states to a snapshot (or O(3) slice, see Figure 2) of the instanton at $T = 0$. The Minkowski rotation of this snapshot is a true vacuum bubble in a sea of false vacuum, which is energetically favored to expand. This is precisely the type of configuration which we would expect to contribute to nucleation of a vacuum bubble due to particle collisions.

The advantage of these "semi-bounces" is that they are still solutions to the Euclidean equations of motion, and hence provide a semiclassical expansion for the path integral with suitable boundary conditions. Returning to (V.11)-(V.13), we will now generalize the calculation to make use of these configurations.

We wish to evaluate the general expression (V.13), but now keeping T_f fixed while $T_i \to -\infty$. The paths about which we will expand are the semi-bounces, each labeled by a Euclidean coordinate ξ_μ. The boundary conditions for a semi-bounce are $\Phi_i(\xi, \vec{x}) = \Phi_{bounce}(T \to -\infty, \vec{x} - \vec{\xi}) \to \sigma_+$ and $\Phi_f(\xi, \vec{x}) = \Phi_{bounce}(T = 0, \vec{x} - \vec{\xi})$, where the latter is simply an O(3) slice through the center of the instanton.

The contribution of semi-bounces to the path integral (V.13) is given by

$$\begin{aligned}\sigma_{inc} \sim \sum_f \int D\xi D\xi' \; &\langle i|\Phi_i(\xi)\rangle\langle\Phi_i(\xi')|i\rangle\langle\Phi_f(\xi)|f\rangle\langle f|\Phi_f(\xi')\rangle \\ &\cdot \langle\Phi_i(\xi)|U|\Phi_f(\xi)\rangle\langle\Phi_f(\xi')|U|\Phi_i(\xi')\rangle,\end{aligned} \tag{V.22}$$

where the last three factors can be expressed as Euclidean path integrals. In particular, if we do the sum over $|f\rangle$'s, we have

$$\sum_f \langle\Phi_f(\xi)|f\rangle\langle f|\Phi_f(\xi')\rangle = \langle\Phi_f(\xi)|\Phi_f(\xi')\rangle = \int_{\phi(T=\xi_0,\vec{x})=\Phi_f(\vec{x}-\vec{\xi})}^{\phi(T=\xi_0',\vec{x})=\Phi_f(\vec{x}-\vec{\xi}')} D\phi \; e^{-S_E[\phi]}. \tag{V.23}$$

The last two amplitudes in (V.22) are evaluated semiclassically about the semi-bounces. Up to a determinant each yields a factor of $exp(-S_{SB})$, where $S_{SB} = S_B/2$. Thus (V.22) can be written as

$$\sigma_{inc} \sim \int D\xi D\xi' \langle i|\Phi_i(\xi)\rangle\langle\Phi_i(\xi')|i\rangle exp(-2S_{SB})\langle\Phi_f(\xi)|\Phi_f(\xi')\rangle. \tag{V.24}$$

Using the overlap result for initial states (V.16), and rewriting

$$\langle\Phi_f(\xi)|\Phi_f(\xi')\rangle \equiv exp(-f(\xi - \xi')), \tag{V.25}$$

we have

$$\begin{aligned}\sigma_{inc} &\sim \int D(\xi - \xi') exp(-2S_{SB}) exp(E(\xi - \xi') - f(\xi - \xi')) \\ &\equiv exp(-2S_{SB}) \int Dx \;\; e^{-W(x)}.\end{aligned} \tag{V.26}$$

While we are at a loss to explicitly evaluate $exp(-f(\xi - \xi'))$, we expect that it is a rapidly decreasing function of $(\xi - \xi')$. This being the case, we expect the largest contribution in the integral over $(\vec{\xi} - \vec{\xi}')$ to come from $\vec{\xi} = \vec{\xi}'$. On the other hand, for Euclidean time separations $x_0 = \xi_0 - \xi_0'$, the first term in $W(x)$ is increasing. If the form of $f(x)$ is such that a saddlepoint evaluation of the integral is possible (i.e., $\frac{\partial W}{\partial x_0}|_{x_0^*} = E - \frac{\partial f}{\partial x_0}|_{x_0^*} = 0$ for some x_0^*), then we will recover the exponential growth with energy of the cross section. In any case, increasing E will always yield a larger cross section, although if the fall-off in $f(x)$ is precipitous enough, the increase may be inconsequential.

How is the quantity we have calculated related to the cross section for bubble production? We know that $\sigma_{bubble} \leq \sigma_{inc}$, and if the $|f\rangle$'s that dominate the sum in (V.23) correspond to critical bubbles, then $\sigma_{bubble} \simeq \sigma_{inc}$. This will certainly be the case if the integral (V.26) is dominated by small x_0, which will be true for $E << E_{sphaleron}$. In other words, for sufficiently low energies the cross section for bubble production is approximately equal to the total cross section due to semi-bounces.

It is possible to give an intuitive argument for why the cross section for bubble production must grow with energy. Consider the amplitude connecting an initial two particle state $\langle i|$ to a final state $|f\rangle$, which we leave arbitrary except to specify that it is an eigenstate of energy and momentum. Then in the one-instanton approximation

$$\langle i|U|f\rangle = \int D\xi \; \langle i|\Phi_i(\xi)\rangle\langle\Phi_f(\xi)|f\rangle exp(-S_{SB}). \tag{V.27}$$

This can be rewritten as

$$\langle i|U|f\rangle = \int D\xi e^{(i\xi\cdot(P_i - P_f))}\langle i|\Phi_i(0)\rangle\langle\Phi_f(0)|f\rangle exp(-S_{SB}). \tag{V.28}$$

Performing the ξ integral merely gives a delta function which preserves energy and momentum. The total cross section can then be written as

$$\sigma_{inc} = \sum_f exp(-2S_{SB})|\delta^4(P_i - P_f)\langle i|\Phi_i(0)\rangle\langle\Phi_f(0)|f\rangle|^2. \tag{V.29}$$

This expression for σ_{inc} is equivalent to previous ones (e.g., (V.24),(V.26)), except that here the sum over final states is explicitly shown. In this form, it is easy to see that the total cross section has the form of an almost constant matrix element, summed over a phase space which grows with energy. At zero energy, we expect the dominant final state to be a vacuum bubble of zero energy (the $T = 0$ slice of the bounce). However, at higher energies, we can easily imagine final states corresponding to vacuum bubble plus scalars or slightly excited vacuum bubbles which overlap equally well with $\langle\Phi_f(0)|$. Thus we see that the growth in energy of σ_{inc}, as expressed in (V.26) as the integral over $e^{-W(x)}$, is mainly due to phase space.

Finally, we note that our techniques can reproduce earlier results for the rate of decay of the false vacuum. If we had taken our initial state $\langle i|$ as the vacuum state (i.e., such that $\langle i|\phi_i\rangle = 1$), the result for $\sigma_{inc} = \sigma_{bubble}$ in (V.22) can be interpreted as a decay rate per unit volume, per unit time, which agrees to leading order with the previous result in [45].

E Conclusion

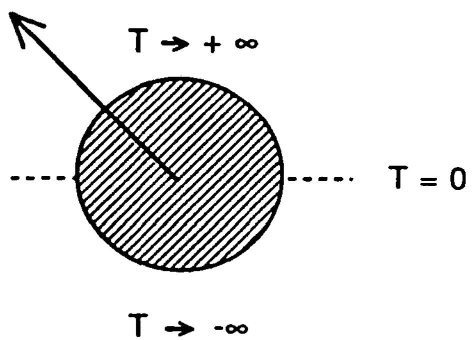
Our purpose in this chapter has been to investigate whether or not indications of strong behavior at high energies were present in theories with a metastable false vacuum. One would have naively expected this to be so from the very presence of instanton solutions with the correct asymptotic behavior (that is, on-shell poles with constant or slowly varying residue). This can be seen most naively by simply calculating the one instanton contribution to a multiparticle Green's function, and then applying the LSZ procedure to arrive at an S-matrix element. Since the operators so arrived at are local, the extrapolation to high energies of scattering due to those operators will eventually become large and even violate unitarity.

As expected, we find exactly this type of behavior in the scalar-scalar scattering cross section, and also suspect that the cross section for vacuum tunneling (production of a vacuum bubble) grows exponentially with energy. Unfortunately, no strong conclusions can be drawn from these calculations because, as shown by KRT [48], corrections to the one instanton cross section grow large before the suppression factor e^{-S_B} is overcome. (We must quibble slightly, as mentioned in section B, with the general claim that corrections are proportional to powers of $(E/E_{sphaleron})$, since for the theory considered here there is no simple relation between the sphaleron energy and the bounce action.)

The phenomenological and cosmological implications of the tunneling rate becoming large due to high energy collisions are myriad. Models of inflation involving

supercooling would have to be reexamined. Stringent limits on top and Higgs masses could be derived from the fact that cosmic ray collisions have yet to destabilize our universe. Perhaps we would even be forced to reconsider our picture of the dominant dynamics of the hot early universe. But, before any further speculation, much more work needs to be done to understand the true behavior of these semiclassical processes at high energy.

Figure 2: A schematic representation of the bounce, $\Phi(\rho)$, where $\rho = (t^2 + \vec{x}^2)^{1/2}$. In the shaded region, $\Phi \simeq \sigma_-$, while $\Phi \simeq \sigma_+$ at large ρ. The bounce is sliced into two semi-bounces along the $T = 0$ line.



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