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Electroweak Phase Transitions

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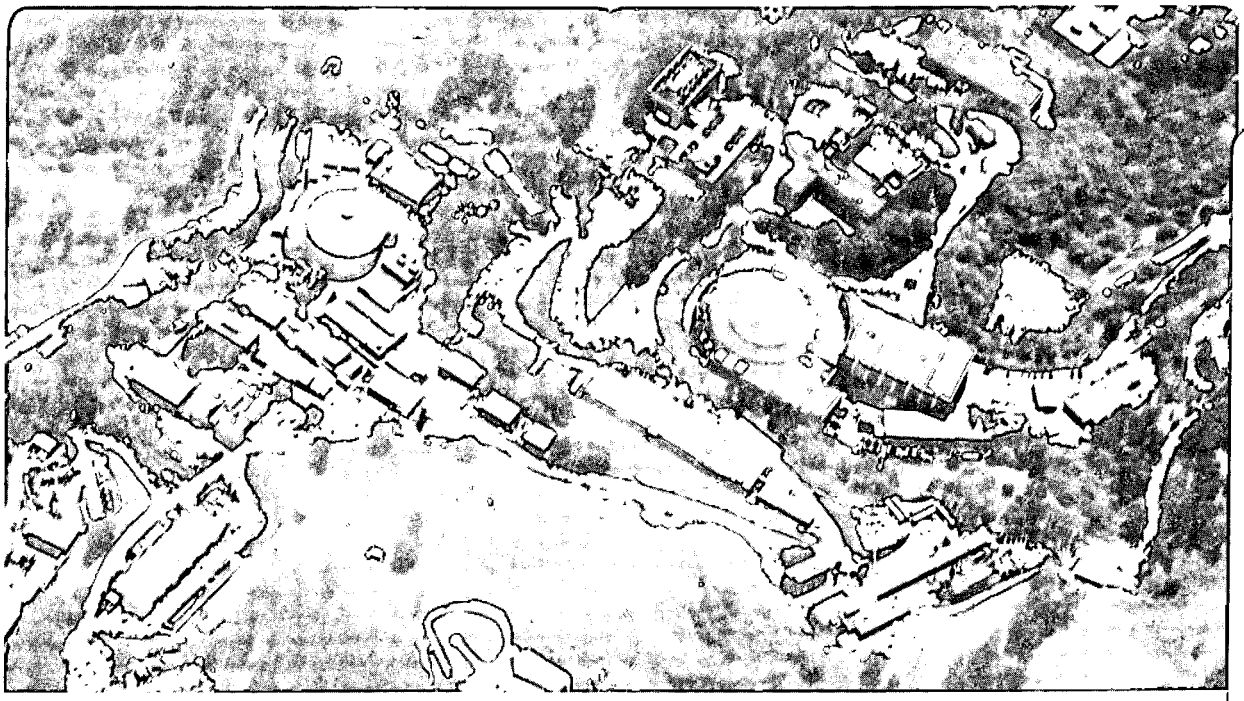

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Electroweak Phase Transitions *

Ph.D. Thesis

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Abstract

An analytic treatment of the one Higgs doublet, electroweak phase transition is given. The phase transition is first order, occurs by the nucleation of thin walled bubbles and completes at a temperature where the order parameter, $\langle\phi\rangle_T$ is significantly smaller than it is when the origin becomes absolutely unstable. The rate of anomalous baryon number violation is an exponentially sensitive function of $\langle\phi\rangle_T$. In very minimal extensions of the standard model it is quite easy to increase $\langle\phi\rangle_T$ so that anomalous baryon number violation is suppressed after completion of the phase transition. Hence baryogenesis at the electroweak phase transition is tenable in minimal extensions of the standard model. In some cases additional phase transitions are possible. For a light Higgs boson, when the top quark mass is sufficiently large, the state where the Higgs field has a vacuum expectation value $\langle\phi\rangle = 246$ GeV is not the true minimum of the Higgs potential. When this is the case, and when the top quark mass exceeds some critical value, thermal fluctuations in the early universe would have rendered the state $\langle\phi\rangle = 246$ GeV unstable. The requirement that the state $\langle\phi\rangle = 246$ GeV is sufficiently long lived constrains the masses of the Higgs boson and the top quark. Finally, we consider whether local phase transitions can be induced by heavy particles which act as seeds for deformations in the scalar field. Semi-classical reasoning suggests that, when a particle receives a contribution to its mass from the vacuum expectation value of a scalar, under certain conditions, the ground state of particle number one contains a 'dimple' or shallow scalar field condensate around the particle. We argue that this is not the case. A careful analysis, taking into account quantum mechanics, shows that the semi-classical approximation is a poor one. We find that there are no energetically favored one-particle dimple solutions for perturbative couplings.

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Chapter I

The Electroweak Phase Transition

1 Introduction

In the minimal standard model, electroweak symmetry breaking is induced by the ground state of a single doublet scalar field. We can write the potential for the real scalar component of the doublet which acquires a vacuum expectation value as

$$U(\phi) = \frac{1}{4}\lambda\left(\phi^2 - \sigma^2\right)^2. \tag{1.1}$$

In a cold and empty (or relatively empty) universe, the Higgs field ϕ can minimize its energy and hence the free energy of the system by choosing a vacuum expectation value $\langle\phi\rangle = \sigma$. However, the early universe was neither cold nor empty, and the presence of an ambient, thermal distribution of particles changes this picture. Although the vacuum energy of the system is still reduced by shifting the classical value of the Higgs field away from $\langle\phi\rangle = 0$, we now pay the price of adding free energy to the particles in the surrounding plasma as they acquire a mass (see Figure 2). When the temperature is high enough, the free energy required to give mass to a thermal distribution of particles exceeds the vacuum energy liberated by displacing the Higgs field vacuum expectation

value from the origin. This occurs for temperatures above a critical temperature we call T_1, where the configuration of the Higgs field that minimizes the free energy of the system is $\langle\phi\rangle = 0$. Thus, at temperatures large compared to the scale of electroweak physics, the minimum of the effective Higgs potential is at the origin. This is the origin of symmetry restoration at high temperature [1-3]. As the universe cools, when the temperature drops to the critical temperature T_1, a new minimum appears, separated from the origin by a hump. When the free energy barrier separating the two extrema is small enough, bubbles of true vacuum are nucleated and grow. The thermal decay rate from the unstable state carries a suppression [4, 5],

$$\Gamma/\mathcal{V} \propto e^{-F_c(T)/T}, \tag{1.2}$$

where $F_c(T)$ is the surplus free energy needed to create a bubble of true vacuum large enough to grow indefinitely. Bubbles of true vacuum smaller than this critical size collapse under surface tension. For a bubble larger than this critical size, as the radius of the bubble increases the free energy liberated by the expanding volume of true vacuum exceeds the free energy required to increase the bubble's surface area. Such a bubble will grow and convert space to true vacuum. A static bubble which is exactly the critical size is in unstable equilibrium, it is a saddle point solution of the free energy functional.

At temperature T_2, where the second derivative of the potential at the origin vanishes, fluctuations can classically roll towards the global minimum without surmounting an energy barrier. If the phase transition has not yet completed by the time the temperature drops to T_2, the transition no longer occurs through bubble nucleation. We call the transition first-order if it proceeds by bubble nucleation. So a necessary condition for a first order phase transition is that bubbles occupy most of space before the temperature drops to T_2. We will satisfy a slightly more stringent condition. At temperatures very close to T_2 the loop expansion parameter becomes large[2]. So in addition, a reliable analytical determination of the phase transition requires that it complete while the effective loop expansion parameter is small.

In the next section we write the finite temperature effective potential for the one Higgs doublet standard model in a form which is useful for the analytic understanding of the nature of the phase transition. We take the Higgs boson to be lighter than about 150 GeV, and use the high temperature approximation, which we will show is highly accurate for all aspects of the phase transition. In section three we discuss the nature of bubbles which could be nucleated to trigger the phase transition. The scaling arguments of section three suggest that an examination of whether the phase transition occurs via nucleation of thin walled bubbles is warranted. In section four, the free energy of these thin walled bubbles is calculated, and in section five it it shown that the thin walled bubble free energy is small enough that the phase transition does complete by the rapid nucleation of exclusively thin walled bubbles. Our analysis is purely analytic, and we obtain formulae for such quantities as the temperature of the universe at the completion of the phase transition and the number of bubbles nucleated per horizon volume.

Section six provides an important application of our results to the question of depletion of the baryon asymmetry after the phase transition. The standard model contains an anomaly which is baryon number violating [6]. At high temperatures the rate of anomalous baryon number violation can be quite large [5,7-10]. This has stimulated a great deal of interest in the possibility of creating the baryon asymmetry at the electroweak phase transition, (EWPT), [11-17] A successful scenario must explain why the baryon excess created at the electroweak phase transition is not washed out after the phase transition completes. We show that extremely simple additions to the standard model avoid washout for any Higgs mass up to 150 GeV.

2 Evolution of the Potential

The tree level potential for the physical Higgs scalar is

$$U(\phi) = \frac{\lambda_0}{4}\left(\phi^2 - \sigma^2\right)^2, \tag{2.1}$$

where λ_0 is related to the Higgs boson mass by $m_H^2 = 2\lambda_0\sigma^2$. To reliably analyze the

dynamics of this field, we need to include the interactions of the Higgs field with virtual particles and with the heat bath.

The one loop, zero temperature potential, $V(\phi)$ can be written as the sum of the classical potential and a one loop correction $V(\phi) = U(\phi) + \overline{V}_1(\phi)$. If we adopt the renormalization prescriptions *i)* $V''(\sigma) = m_H^2$, and *ii)* $V'(\sigma) = 0$, for each degree of freedom to which the Higgs boson is coupled, the zero temperature one loop correction to the effective potential is (see Appendix A)

$$\begin{aligned}\overline{V}_1(\phi) = \pm\frac{1}{64\pi^2}\Big\{ & m^4(\phi)\ln\left(m^2(\phi)/m^2(\sigma)\right) \\ & -\frac{3}{2}m^4(\phi) + 2m^2(\phi)m^2(\sigma) - \frac{1}{2}m^4(\sigma)\Big\},\end{aligned} \tag{2.2}$$

where the $\pm$ is for bosons (fermions) and $m(\phi)$ is the mass of the particle in the presence of a background field ϕ. Equation (2.2) is valid for particles which have a mass of the form $m^2 = a + b\phi^2$ in the mass eigenstate basis. In addition to these quantum corrections, we must also include the interaction between the Higgs field and the hot electroweak plasma. Taking the Higgs boson sufficiently light that we can ignore the contribution of scalar loops, it is shown in Appendix A that, using the high temperature expansion, the effective potential for the standard model can be reliably written

$$V(\phi, T) = D(T^2 - T_2^2)\phi^2 - ET\phi^3 + \frac{1}{4}\lambda_T\phi^4, \tag{2.3}$$

where $D = \frac{1}{24}\left\{6(m_W/\sigma)^2 + 3(m_Z/\sigma)^2 + 6(m_t/\sigma)^2\right\}$, and the coefficient of the term linear in temperature is $E = \frac{1}{12\pi}\left[6(m_W/\sigma)^3 + 3(m_Z/\sigma)^3\right] \simeq 10^{-2}$. The temperature dependent ϕ^4 coupling is:

$$\lambda_T = \lambda - \sum_B g_B\frac{m_B^4}{16\sigma^4\pi^2}\ln(m_B^2/c_BT^2) + \sum_F g_F\frac{m_F^4}{16\sigma^4\pi^2}\ln(m_F^2/c_FT^2), \tag{2.4}$$

where the $B(F)$ denotes bosons (fermions), $g_{B(F)}$ is the number of degrees of freedom, c_F and c_B are constants which can be found in Appendix A, and the masses in (2.4) are evaluated at $\langle\phi\rangle = \sigma$. The physical Higgs mass is related to λ by

$$m_H^2 = (2\lambda + 12B)\sigma^2, \tag{2.5}$$

where $B = \frac{1}{64\pi^2\sigma^4}\left(6m_W^4 + 3m_Z^4 - 12m_t^4\right)$. We define T_2 as the temperature where $V''(\phi = 0) = 0$. From Appendix A,

$$T_2^2 = \frac{m_H^2 - 8B\sigma^2}{4D} \equiv \chi^2(m_t, m_H)m_H^2. \tag{2.6}$$

Because this result was obtained at the origin, it is valid to all orders in m/T. The quantity χ is plotted against m_H for various values of m_t in Figure 1.

At temperatures well above T_2, the only minimum of the potential is $\langle\phi\rangle = 0$. As the early universe cools down from this high temperature, a second local minimum of the potential first appears (as an inflection point) when the temperature reaches

$$T_*^2 = T_2^2 \frac{1}{1 - 9E^2/8\lambda_T D} \tag{2.7}$$

at a value of the field $\phi_* = 3ET_*/2\lambda_T$. At lower temperatures, this point splits into a barrier ϕ_- and a local minimum ϕ_+ which subsequently evolve as

$$\phi_\pm = \frac{3ET}{2\lambda_T} \pm \frac{1}{2\lambda_T}\sqrt{9E^2T^2 - 8\lambda_T D\left(T^2 - T_2^2\right)}. \tag{2.8}$$

The evolution of $\phi_\pm$ is shown in Figure 2. We define the temperature T_1 to be the temperature at which the second minimum becomes degenerate with the origin, $V(\phi_+(\mathrm{T}_1)) = 0$. Hence, if we divide equation (2.3) by ϕ^2, T_1 occurs where the resulting quadratic equations has two real equal roots. This gives the relation

$$T_1^2 = \frac{1}{1 - \frac{E^2}{\lambda_T D}} T_2^2. \tag{2.9}$$

For Higgs Boson masses above the current experimental limit, the difference in temperature between T_1 and T_2 is small compared to the temperature. Writing $T_1 = T_2 + \tau$, we find $\tau << T_2$ provided $m_H \gtrsim 10$ GeV, where

$$\tau = \left(\frac{E^2}{2\lambda_T D}\right) T_2. \tag{2.10}$$

From equations (2.8) and (2.9) we see that

$$\begin{aligned} \phi_\pm(T_1) &= \frac{3ET_1}{2\lambda_T} \pm \frac{ET_1}{2\lambda_T} \\ \phi_\pm(T_2) &= \frac{3ET_2}{2\lambda_T} \pm \frac{3ET_2}{2\lambda_T}. \end{aligned} \tag{2.11}$$

It will prove convenient to write the potential in terms of the scaled field ϕ', where $\phi = \left(\frac{ET}{\lambda_T}\right)\phi'$, for which

$$V(\phi) = \lambda_T \left(\frac{ET}{\lambda_T}\right)^4 \left(\frac{T_1^2(T^2 - T_2^2)}{T^2(T_1^2 - T_2^2)}\phi'^2 - \phi'^3 + \frac{1}{4}\phi'^4\right). \tag{2.12}$$

If $T_1 - T_2 << T_1 + T_2$, then for any T such that $|T_1 - T| << T_1$, the potential reduces to the simple form

$$V(\phi) = \lambda_T \left(\frac{ET}{\lambda_T}\right)^4 \left((1-\epsilon)\phi'^2 - \phi'^3 + \frac{1}{4}\phi'^4\right), \tag{2.13}$$

where $\epsilon = (T_1 - T)/(T_1 - T_2)$. In terms of the scaled field

$$\phi'_{\pm} = \frac{1}{2}\left(3 \pm \sqrt{1+8\epsilon}\right). \tag{2.14}$$

The critical temperatures T_*, T_1, and T_2 correspond to the following values of ϵ:

$$\epsilon_* = -3/8, \ \epsilon_1 = 0, \ \epsilon_2 = 1. \tag{2.15}$$

The largest values of m/T which are of importance in this chapter correspond to $T \simeq T_1$ and $\phi \simeq \phi_+(T_1)$. Is the high temperature expansion a good approximation in this case? If the high temperature approximation is valid for the top quark, it will be valid for all other particles as well. In the temperature region of interest, for the top quark we have

$$\frac{m_t(T)}{T} \simeq \lambda_t \frac{\phi_+(T_1)}{T_1} \simeq \lambda_t \frac{2E}{\lambda_T}. \tag{2.16}$$

In Appendix A we show that the high temperature approximation is valid to better than 5 percent provided this quantity is less than 1.6 . An inspection of Figures three and four demonstrates that our use of the high temperature approximation is well justified.

3 A Heuristic Discussion of the Saddle Point

After the universe cools down to a temperature below T_1, the previously global minimum $\langle\phi\rangle = 0$ becomes metastable. The subsequent conversion of the universe to the true vacuum state $\langle\phi\rangle = \phi_+(T)$ takes place by the nucleation of true vacuum bubbles.

Accordingly, we need to determine the free energy barrier such bubbles must surmount in order to grow. Consider a true vacuum bubble in a sea of false vacuum $\langle\phi\rangle = 0$. Let $\langle\phi\rangle = \phi'$ at the center of the bubble (see Figure 5).

By convention we choose the state $\langle\phi\rangle = 0$ to have free energy zero, $V(0) = 0$. Then the surplus free energy of a true vacuum bubble is

$$F = \int d^3x \left\{ \frac{1}{2} \left(\vec{\nabla}\phi \right)^2 + V(\phi, T) \right\}. \tag{3.1}$$

The free energy of a true vacuum bubble has two contributions: a surface free energy F_S, coming from the derivative terms in (3.1), and a volume term F_V, which arises from the difference in free energy density inside and outside the bubble. These two contributions scale like

$$\begin{aligned} F_S &\sim 2\pi R^2 \left(\frac{\delta\phi}{\delta R} \right)^2 \delta R, \\ F_V &\sim -\frac{4\pi}{3} R^3 \overline{V}, \end{aligned} \tag{3.2}$$

where R is the radius of the bubble, δR is the thickness of the bubble wall, $\delta\phi = \phi'$,* and $\overline{V}$ is minus the average value of the potential inside the bubble. When $V(\phi_-)$ is large compared to $-V(\phi')$, it is important to minimize the contribution to F_V coming from regions near $\phi \simeq \phi_-$. More precisely, when the height and the width of the barrier near ϕ_- are not small compared to the depth and width of the well at ϕ_+, for the optimal solution, ϕ will change quickly between 0 and ϕ', and δR will be small. This is the situation for temperatures just below T_1. Hence, the first bubbles which could be formed are thin wall bubbles.

As the temperature subsequently drops towards T_2, the barrier in ϕ space tends to zero, and the difference in free energy density between the states $\langle\phi\rangle = 0$ and $\langle\phi\rangle = \phi_+$ increases. When the size of the hump in the potential at ϕ_- becomes small compared to the depth of the well at ϕ_+, it is favorable to make δR as large as possible so as to minimize the 'surface' term F_S. Hence, $\delta R \sim R$, and we should work in a thick wall

*For thin walled bubbles ϕ' will lie at the absolute minimum, while for thick walled bubbles we must allow for the possibility that ϕ' is somewhat less than ϕ_+.

approximation. So, whether the EWPT proceeds by the nucleation of thick walled or thin walled bubbles depends on how large the rate of bubble nucleation becomes before thick walled bubbles are energetically preferred. Our purpose, throughout this section, is to gain a qualitative understanding of the dependence of bubble free energies on the shape of the effective potential. Accordingly, we shall not be too concerned with the precise value of the numerical prefactors which occur in estimates of bubble parameters. Simple qualitative and semi-quantitative estimates of these two cases will tell us what kind of bubble to examine with closer scrutiny. Let's first consider the simpler case of thick walled bubbles.

• **Thick Walled Bubbles**

For thick walled bubbles $\delta R \sim R$, and the surface energy of the bubble grows like R. In contrast, the negative volume term increases in magnitude like R^3. Thus, a thermal fluctuation producing a bubble of true vacuum, which starts from a radius of zero and expands in radius to envelope the system, must have a free energy greater than or equal to some critical value. The critical radius of the bubble, R_c, is the bubble radius where the total free energy of the bubble reaches a maximum. Differentiating (3.2), $R_c \sim \delta\phi/\sqrt{2\overline{V}}$, and the thick wall bubble free energy is

$$\begin{aligned} F_c &\simeq 2\pi R_c(\delta\phi)^2 - \frac{4\pi}{3}R_c^3\overline{V} \\ &\sim \frac{(\delta\phi)^3}{\sqrt{\overline{V}}}. \end{aligned} \tag{3.3}$$

Note that for thick walled bubbles, the magnitude of F_c depends on the relative sizes of the shift in ϕ and the potential difference between the center and the outside of the bubble and not the height of the barrier in ϕ -space. [†] As a quick estimate for the potential (2.3), defining $r = 1 - \epsilon$, as the temperature approaches T_2 the critical free energy scales like

$$F_c \sim \frac{ET}{\lambda_T^{3/2}} r^{3/2}. \tag{3.4}$$

[†]While it is true that a large barrier in $V(\phi)$ vs. ϕ space makes it hard to fluctuate a bubble of true vacuum, and a smaller barrier makes it easier, we should remember that the real barrier is in configuration space (see Figure 6), and we should be wary of intuition based on the one dimensional mechanics.

• **Thin Walled Bubbles**

For thin walled bubbles, in addition to the contribution of the derivative term to the bubble wall free energy, inside the bubble walls there is also a positive contribution to the free energy from the barrier in ϕ-space, $V_b \sim V_-$. As a function of the bubble radius and thickness, the thin wall bubble free energy is

$$F(R) \sim 2\pi R^2 \delta R \left\{ \left(\frac{\delta\phi}{\delta R} \right)^2 + 2V_b \right\} - \frac{4\pi}{3} R^3 \overline{V}. \tag{3.5}$$

The saddle point corresponds to $\delta R \simeq \delta\phi/\sqrt{2V_b}$ and $R_c \sim \sqrt{\frac{V_b}{\overline{V}}} \frac{\delta\phi}{\sqrt{\overline{V}}}$ (see Figure 6). So the critical free energy is

$$F_c \sim \frac{(\delta\phi)^3}{\sqrt{\overline{V}}} \left(\frac{V_b}{\overline{V}} \right)^{3/2}. \tag{3.6}$$

So the thin wall bubble radius is found by scaling the thick wall radius by $\sim \sqrt{V_b/\overline{V}}$ and the thin wall free energy increases by a factor of $\sim \left(\frac{V_b}{\overline{V}}\right)^{3/2}$ relative to the thick wall case. For the potential of (2.3), $\overline{V} \simeq -V(\phi_+)$, $V_b \simeq V(\phi_-)$, and $\delta\phi = \phi_+$. Since the first bubbles to form as the universe cools to a temperature below T_1 will be thin walled bubbles, substituting $\phi_\pm$ into equation (3.6), recalling $\epsilon = (T_1 - T)/(T_1 - T_2)$, and expanding for small ϵ, the first bubbles to form must surmount a free energy barrier which scales like

$$F_c(\epsilon) \sim \frac{ET_1}{\epsilon^2 (2\lambda_T)^{3/2}} \tag{3.7}$$

As we will discuss in section 5, a free energy barrier $\mathcal{O}(100T)$ is small enough to allow bubbles to nucleate. Accordingly, the estimate (3.7) tells us we should make a careful study of bubble free energies in the thin wall approximation the instant after the temperature drops below T_1.

4 Saddle Point Free Energies in the Thin Wall Approximation

We can adapt Coleman's thin wall approximation[18] to derive an analytic formula for the critical free energy valid in the limit that the temperature approaches T_1 from

below.[‡] The thin wall analysis has been applied to the general case of thermal vacuum stability by Linde [19]. However we find his conclusions regarding the applicability of the thin wall approximation to realistic gauge theories overly pessimistic. For the potential of equation (2.3), if we shift our field $\phi \to \phi' = \phi + (ET/\lambda_T)$, we can cast the potential in a form where the validity of Coleman's thin wall approximations will be transparent. Recall $\delta = E^2/\lambda_T D$, $\epsilon = (T_1 - T)/(T_1 - T_2)$, and define $v = ET/\lambda_T$. In terms of the shifted field the potential becomes

$$V(\phi) = \left\{ \frac{\lambda_T}{4} \left[\phi^2 - v^2(1+2\epsilon)\right]^2 - \epsilon \left(2\lambda_T v^3\right)(\phi + v) \right\} \{1 + \mathcal{O}(\delta\epsilon)\} \tag{4.1}$$

plus terms independent of ϕ. For a universe filled with false vacuum $\langle\phi\rangle = -v$, the true vacuum bubble of minimum free energy, which is just large enough to grow, is a static $O(3)$ invariant solution to the equations of motion. Hence it satisfies

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = V'(\phi, T), \tag{4.2}$$

where the $'$ denotes differentiation with respect to ϕ. Integrating the appropriate solution to (4.2) gives the critical free energy

$$F_c = 4\pi \int r^2 dr \left\{ \frac{1}{2}\left(\frac{d\phi}{dr}\right)^2 + V(\phi, T) \right\}. \tag{4.3}$$

In the limit of small ϵ, the bubble wall thickness is negligible when compared to the bubble radius. So for small ϵ, inside the bubble walls we can neglect the term in (4.2) linear in spatial derivatives. This, together with the boundary conditions that the derivative of ϕ and the free energy density vanish outside of the bubble, implies

$$\frac{d\phi}{dr} = \sqrt{2V} \tag{4.4}$$

inside the bubble walls. We have defined the zero of $V(\phi)$ in (4.1) so that $V(-v) = 0$. From (4.3) and (4.4) we can write the resulting bubble free energy as

$$F \simeq 4\pi R^2 \int_{\phi(R-\delta R)}^{\phi(R+\delta R)} \sqrt{2V(\phi)}\, d\phi + 4\pi \int_0^R dr\, r^2 V(\phi_+, T). \tag{4.5}$$

[‡]The result we obtain here will be exact in the limit ϵ goes to zero, for the potential of (2.3).

As a function of the bubble radius, the free energy is given by[§]

$$F(R) = \frac{8\pi}{3}\sqrt{2\lambda_T}v^3R^2 - \epsilon\frac{16\pi}{3}\lambda_T v^4 R^3. \tag{4.6}$$

Varying with respect to R, we find that $R_c = 2/\left(\sqrt{2\lambda_T}(3\epsilon v)\right)$. Hence, in the limit of small ϵ, the critical free energy to temperature ratio is

$$\begin{aligned} F_c/T_1 &= \left(\frac{64\pi}{81}\right)\frac{E}{\epsilon^2(2\lambda_T)^{3/2}} \\ &\simeq (2.85)\left(\frac{50\,GeV}{m_H}\right)^3\epsilon^{-2}. \end{aligned} \tag{4.7}$$

As we will discuss in section 5, the phase transition completes when the free energy to temperature ratio is on the order of 100. This is achieved for $\epsilon \simeq (1/6)(50\,GeV/m_H)^{3/2}$. For completeness, we note that to lowest order in ϵ, inside the bubble walls ϕ behaves like a domain wall and the well known solution is

$$\phi(r) = -v\tanh\left(\sqrt{\frac{\lambda_T}{2}}v(r-R)\right). \tag{4.8}$$

5 Bubble Production, Evolution, and Number

Having determined the free energy of a bubble large enough to grow indefinity, we examine the rate for producing such bubbles. As the early universe cooled from temperature T_1 to temperature T_2, a point was reached where classical thermal fluctuations were large enough to nucleate bubbles of true vacuum. These thermal fluctuations produced bubbles of true vacuum at a rate per unit volume

$$\Gamma/\mathcal{V} = \Lambda^4(T)e^{-F_c(T)/T}, \tag{5.1}$$

where $F_c(T)$ is the free energy of a fluctuation large enough to pass over the energy barrier separating the two vacua, and Λ is a characteristic scale in the theory. For definiteness we take $\Lambda^4 = \omega T^4$. As we shall see, the temperatures we are interested in are on the order of the particle masses. Moreover, because the nucleation rate is dominated by the

[§]The reader is cautioned that formulas (4.12) and (4.21) of reference [18] are off by a factor of two.

exponential, the exact value of the prefactor is not very important, so the effect of $\omega \neq 1$ will be negligible.

• **The Onset of Nucleation**

Let's begin by determining when the onset of bubble nucleation occurs. In the radiation dominated era, the time-temperature relationship is $t = \xi M_{Pl}/T^2$, where t is the age of the universe, $M_{Pl} = 1.22 \times 10^{19}$ GeV is the Planck mass, and $\xi \simeq 1/34$ near the electroweak phase transition. ¶ Because the horizon size scales like $d_H = 2t$, the size of a causal volume at a temperature T is

$$V_H(T) \sim 8\xi^3 M_{Pl}^3/T^6. \tag{5.2}$$

Without being overly precise about the numerical prefactor, we take (5.2) as our definition of a causal volume. For probabilities small compared to one, the probability that a bubble was nucleated inside a causal volume during a temperature interval dT is given by

$$dP \sim 16\,\omega\xi^4 \frac{M_{Pl}^4}{T^4} e^{-F_c(T)/T} \frac{dT}{T}. \tag{5.3}$$

In this section, it will be convenient to make use of the fact that the critical bubble free energy to temperature ratio, (4.7), has a Taylor expansion about a temperature T_0 given by

$$F_c(T)/T = F_c(T_0)/T_0 \left(1 + 2x + 3x^2 + ...\right), \tag{5.4}$$

where $x = \frac{T-T_0}{T_1-T_0}$. Between the temperatures T_1 and T_2 the free energy is a quickly changing function of temperature but the change in temperature itself between T_1 and T_2 is insignificant. Define T_N as the temperature reached when the first bubble is nucleated in a typical horizon. Using the expansion (5.4), we integrate equation (5.3) and note that the integrand is sharply peaked about $x = 0$. The first bubble nucleation in a typical horizon when $P \sim 1$ giving

$$F_c(T_N)/T_N \sim 134. \tag{5.5}$$

¶ $\xi = \sqrt{\left(\frac{90}{32\pi^3}\right) \frac{1}{\sum_B g_B + (7/8)\sum_F g_F}}$

This ratio varies by about 2 percent as the Higgs Boson mass increases from 50 GeV to 100 GeV.

• **The Phase Transition**

In order to claim that the phase transition really proceeds by bubble nucleation, it is necessary but not sufficient to require that T_2 is small enough relative to T_N to ensure that nucleated bubbles grow enough to convert the universe to true vacuum before the temperature drops to T_2, where the free energy barrier disappears. A reliable determination of the phase transition requires that it completes before the loop expansion parameter in the $SU(2) \times U(1)$ becomes large[2]. Moreover, because the anomalous baryon number violating processes which persist after the EWPT completes are very sensitive to the value of $\phi_+(T_\Omega)$ [‖], a careful determination of the temperature at which the phase transition completes is required.

Define $dN(T)$ as the number of bubbles per unit volume nucleated between temperatures T and $T + dT$. From (5.1),

$$dN(T) = 2\omega\xi M_{Pl} T e^{-F_c(T)/T}\, dT. \tag{5.6}$$

Consider a true vacuum bubble expanding with a constant terminal velocity β with respect to the plasma. There are two contributions to the bubble wall expansion. One comes from the propagation of the bubble wall through the fluid and the other is from the expansion of the universe. Hence,

$$dR = \beta dt - a(T)\frac{dT}{T}, \tag{5.7}$$

where $a(T)$ is the Robertson-Walker scale factor. In the radiation dominated era, this leads to the differential equation

$$\frac{dR}{dT} = -2\beta\xi\frac{M_{Pl}}{T^3} - \frac{a(T)}{T}. \tag{5.8}$$

[‖] ϕ_+ increases by a factor of 3/2 as the temperatures drops from T_1 to T_2.

During the electroweak phase transition a bubble nucleated at a temperature T', and expanding with a velocity β, has a radius

$$R_B(T,T') = R_N(T')\frac{T'}{T} + 2\beta\xi\frac{M_{Pl}}{T}\left(\frac{1}{T} - \frac{1}{T'}\right), \tag{5.9}$$

where $R_N(T')$ is the radius of a true vacuum bubble when it is produced at a temperature T'.

We can imagine two qualitatively different types of first order phase transitions. In a bubble expansion dominated first order phase transition, a typical observer in the fluid sees the surrounding volume converted to true vacuum as a nearby bubble of true vacuum expands and a bubble wall passes over the volume. A second type of first order phase transition occurs if the nucleation rate turns on very suddenly and is so rapid that a typical volume element in the fluid is converted to true vacuum by a nucleation event. We call this second type of first order phase transition nucleation dominated. The nucleation rate increases on a time scale set by the expansion rate of the universe. At a time when the bubble nucleation rate is large enough to produce bubbles copiously, but small enough that the conversion of space to true vacuum by the nucleation event itself is still negligible, previously nucleated bubbles will grow appreciably in size before there is a substantial increase in the nucleation rate. This is a result of the smallness of the temperature relative to the Planck mass. As a result, the vast majority of space is converted to true vacuum by the expansion of bubbles as opposed to being on the site of a nucleation event. So, the volume of a bubble produced at temperature T' can be safely written

$$V(T,T') = \frac{32\pi}{3}\beta^3\xi^3\frac{M_{Pl}^3}{T^3}\left(\frac{1}{T} - \frac{1}{T}\right)^3. \tag{5.10}$$

Integrating over the duration of bubble production, the fraction of a typical horizon that

is true vacuum at time t is *

$$f \simeq \int dt' V(t',t)\Gamma/\mathcal{V}$$
$$\simeq \frac{64\pi}{3}\omega\beta^3\xi^4(M_{Pl}/T)^4 \int_T^{T_1} \frac{T dT'}{T'^2}\left(1-\frac{T}{T'}\right)^3 e^{-F_c(T')/T'}. \tag{5.11}$$

• **End of the Phase Transition**

Define T_Ω as the temperature reached when the phase transition has completed. In the region of interest, the temperature change is insignificant compared to the change in $F_c(T)$. Defining $x = (T - T_\Omega)/T_\Omega$, using the Taylor expansion (5.4) about T_Ω, we see that the integrand of (5.11) is sharply peaked at temperatures very near T_Ω. The phase transition completes when the fraction of true vacuum approaches unity, giving

$$F_c(T_\Omega)/T_\Omega \simeq 96 - 7\ln\left(\frac{2\lambda_T\sigma^2}{(50\,GeV)^2}\right) - 4\ln\left(\frac{100\,GeV}{T_1}\right) - \ln\beta^3. \tag{5.12}$$

The expanding bubbles are not extremely non-relativistic, giving $F_c(T_\Omega)/T_\Omega \simeq 100$. Hence $\epsilon_\Omega \simeq (1/7)(50\,GeV/m_H)^{3/2}$, and the phase transition completes the instant after the universe cools to temperature T_1. The precise value of ϵ_Ω is shown as a function of the Higgs boson mass for several top quark masses in Figure 7.

• **The Number of Bubbles**

Define T_n as the temperature reached by the time n bubbles have been nucleated inside the comoving volume coincident with the horizon volume at the end of the phase transition.

$$n \simeq 16\omega\xi^4\left(\frac{M_{Pl}}{T}\right)^4 \int_{T_n}^{T_1} e^{-\beta F_c(T)/T}\frac{dT}{T}. \tag{5.13}$$

Equating T_n with T_Ω, the number of bubbles produced per horizon by the end of the phase transition satisfies the simple relation:

$$n \simeq \frac{1}{16\pi}\beta^{-3}\left(\frac{dF_c}{dT}\right)^3_\Omega$$
$$= \frac{1}{2\pi}\beta^{-3}\left(\frac{T_\Omega}{T_1 - T_\Omega}\right)^3\left(\frac{F_c(T_\Omega)}{T_\Omega}\right)^3. \tag{5.14}$$

*This expression for the fraction of space containing true vacuum over counts the volume where bubbles overlap. The exact formula [20] is $1 - f = e^{-\int \Gamma/\mathcal{V}V(t',t)dt'}$. Hence, our definition of the end of the phase transition corresponds to the era where only $1/e$ of space is still false vacuum.

It is easy to understand why the number of bubbles per horizon increases as the derivative of the saddle point free energy increases. At the end of the phase transition, the faster the nucleation rate is changing, the shorter the duration of bubble nucleation. A larger number of bubbles needs to be produced for the phase transition to complete in a shorter time interval.

6 Avoiding Washout of the Baryon Asymmetry

Can baryogenesis occur at the one Higgs doublet EWPT? The picture of the phase transition which we have given, valid for Higgs masses from the experimental limit of 46 GeV up to about 150 GeV, certainly shows that the universe is far from thermal equilibrium after the temperature drops below T_N. At high temperature, anomalous baryon number violation can be quite rapid [5, 7, 8, 9, 10]. Hence the possibility of successful baryogenesis rests on two issues:

- sufficient CP violation
- avoiding washout of the $B + L$ asymmetry, after T_Ω.

The numerical size of CP violation in the standard model is insufficient for baryogenesis. This is true even if the strong CP parameter $\overline{\theta}$ were of order unity at high temperatures, because the physical effects of $\overline{\theta}$ are repressed by light quark Yukawa couplings. Nevertheless, it is simple to add new physics to the standard model which yields sufficient CP violation without changing the behavior of the EWPT. One possibility is that this new physics yields operators of the form $\frac{1}{M^2}\phi^*\phi F\widetilde{F}$ in the low energy theory, where F is the electroweak field strength [11]. Hence it may be possible to create a significant baryon asymmetry at the one Higgs EWPT.

We are left with the problem of how to avoid the $B + L$ asymmetry being depleted just after T_Ω as soon as thermal equilibrium is re-established. Several authors [12, 16] have argued that anomalous baryon number violation will washout any baryon asymmetry for Higgs masses larger than some critical value, m_{H_c}. Although there are

uncertainties in the calculation of m_{H_c}, it is in the vicinity of 50 GeV, perilously close to the experimental lower bound. Hence, the one Higgs EWPT of the minimal standard model, extended only to include CP violating operators of the form $\frac{1}{M^2}\phi^*\phi F\widetilde{F}$, does not yield an acceptable baryon asymmetry, with the possible exception that the Higgs boson mass is very close to 46 GeV. (There is also the possibility that m_H is much larger than 100 GeV, in which case we do not know the nature of the phase transition.).

In this section we show that the problem of baryon washout in one Higgs doublet models can be solved in a way similar to the solution of the problem of sufficient CP violation. Particles can be added to the standard model such that our analysis of the one Higgs EWPT persists, but baryon washout is avoided for Higgs masses all the way up to 150 GeV. We find that these additional particles affect the EWPT indirectly, by changing the numerical values of the parameters of the effective potential (B, D, E, λ_T). However, our formulae for quantities of interest $(T_{1,2}, \phi_\pm(T_{1,2}), \epsilon_\Omega$, etc.) are still correct when written in terms of these parameters B, D, E, λ_T.

As we go beyond the minimal standard model, why not go to the two Higgs doublet model? We do not do this because this greatly complicates the EWPT. In general, one cannot just define a single linear combination of the Higgs as the one which gets a vev, because this combination is T dependent. No complete analysis of the phase transition exists. For example, we do not know the quantities $\phi_{1+}(T_\Omega)$ and $\phi_{2+}(T_\Omega)$ which are relevant for baryogenesis. The advantage of the one Higgs doublet EWPT is that we know essentially everything about the phase transition, so that we can use the requirement of avoiding baryon washout as a guide to what new physics should exist.

The analysis of this section will be valid for a whole class of models. This class of models has a single Higgs doublet and has the EWPT proceed by the nucleation of thin wall bubbles. More precisely, the class is defined by three criteria:

- The EWPT is induced by a single Higgs doublet, and the coefficient of the tree level ϕ^4 term is sufficiently small that the Higgs contribution to $V(\phi, T)$ can be neglected.
- All particle masses are such that at temperatures near T_1 the high T approxi-

mation, (A2.6), for $V(\phi, T)$ is valid, with the exception of particles so massive that their thermal contributions to the effective potential are Boltzmann suppressed to the point that they can be ignored.

• The EWPT proceeds by the thermal fluctuation of thin wall bubbles. This means that †

$$\epsilon_\Omega \simeq \frac{1}{12} \frac{E^{1/2}}{\lambda_T^{3/4}} << 1. \tag{6.1}$$

These one Higgs models all have the EWPT proceed as we have described, except now the parameters B, D, E, λ_T can differ significantly from their standard model values. We now calculate how the baryon washout rate depends on these parameters. We will not attempt to calculate the numerical value for the washout rate, however.

At $T < T_\Omega$ the rate for anomalous baryon number violation is proportional to $\exp(-F_{sp}(T)/T)$, where $F_{sp}(T)$ is the sphaleron free energy at temperature T. In the region of interest to us $T \simeq T_\Omega$ (which is just below T_1), and we do not know what $F_{sp}(T)$ is. This is because the usual approximation of keeping only the $T^2\phi^2$ terms in the high T expansion of $\Delta V_1(\phi, T)$ [21] is not good at temperatures near T_Ω. In particular it is clear that the term $-ET\phi^3$ cannot be neglected. Although the $\phi^2 T^2$ terms are the largest ϕ dependent terms in the m/T expansion, they combine with the zero temperature ϕ^2 terms to cancel when the temperature is T_2. One possibility is to try to find some lower temperature where the ϕ^3 term can be dropped, but where the high T expansion is still good. However, since the baryon washout rate decreases as T is lowered, this will only yield a lower bound on the amount of depletion. Alternatively one can do a numerical analysis for the sphaleron energy at temperatures just below T_1 [12]. We will assume that the sphaleron free energy is linear in $\phi(T)/T$, a form motivated by the inclusion of

†In order to use this formula for ϵ_Ω it is actually necessary to also require that $T_1 - T_2 << T_1 + T_2$. This is because our thin wall analysis for ϵ_Ω was based on the effective potential of (2.13) which assumes $T_1 - T_2 << T_1 + T_2$. However if this constraint on $T_1 - T_2$ is not satisfied, equation 6.1 still holds provided ϵ is replaced by $F\epsilon$, where $F = \frac{T_2}{T_1}\frac{2T_2}{T_1+T_2} \leq 1$. In the majority of models where F is not close to 1, we find that F is not small either.

the $T^2\phi^2$ terms. Hence we take

$$\frac{dn_B}{n_B} = \frac{\Gamma}{HT}dT \tag{6.2a}$$

where

$$\frac{\Gamma}{HT} = C_1 \exp\left\{-\left(C_2\frac{\phi_+(T)}{T}\right)\right\} \tag{6.2b}$$

and $H = H(T)$ is the Hubble parameter. C_2 is a large dimensionless constant which we assume has a more mild dependence on (B, D, E, λ_T) than does $\phi_+(T)$ in the region of $T = T_\Omega$. We take C_1 to be a constant, although our results are unchanged if it has a large power dependence on T.

The most likely T region for baryon washout is immediately below T_Ω. If the washout is to be limited, the washout rate must have frozen out well before $T_2 : (\frac{\Gamma}{H})_{T_2} << 1$. Taking $\phi_+(T) = \frac{ET}{\lambda_T}\phi'_+(T)$ where ϕ'_+ is given in equation (2.14), and integrating (6.2a) gives a logarithmic depletion of

$$\ln\left(\frac{n_B(T_\Omega)}{n_B(0)}\right) \simeq \frac{\lambda_T}{2C_2E}\left(\frac{T_1 - T_2}{T_1}\right)\left(\frac{\Gamma}{H}\right)_{T_\Omega}, \tag{6.3}$$

where

$$\left(\frac{\Gamma}{H}\right)_{T_\Omega} \simeq C_1 T_1 e^{-C_2\frac{E}{2\lambda_T}\left(3+\sqrt{1+8\epsilon_\Omega}\right)}. \tag{6.4}$$

When $\epsilon_\Omega << 1$, it can be dropped from equation 6.4. Whether depletion is significant is largely a question of whether the baryon washout rate freezes out before or after T_Ω, i.e. of whether $(\Gamma/H)_{T_\Omega}$ is greater or less than unity. This entire class of one Higgs doublet models has a baryon washout rate which is exponentially sensitive to the ratio E/λ_T, but is relatively insensitive to B and D, and therefore to T_1 and T_2. There is significant sensitivity to m_H because λ_T is related to m_H by

$$\lambda_T = \frac{m_H^2}{2\sigma^2} - (\Delta\lambda + 6B) \tag{6.5}$$

where we have defined the logarithmic terms of equation 2.4 to be $\Delta\lambda : \lambda_T = \lambda - \Delta\lambda$.

Others have found a critical Higgs mass of about 50 GeV [12, 16] in the minimal standard model. This corresponds to

$$\left(\frac{E}{\lambda_T}\right)_c \simeq \frac{1}{2} \tag{6.6}$$

for low top quark masses.

We can now see how simple it is to avoid baryon washout at the one Higgs EWPT. Particles should be added to the standard model so that $E/\lambda_T > \frac{1}{2}$ for any desired value of m_H in the region of 45-150 GeV. This can be accomplished in two ways:

- add bosons with small $SU(2) \times U(1)$ preserving masses to increase E
- add bosons so that $\Delta\lambda + 6B$ is increased. For a given Higgs mass this decreases λ_T as can be seen from (6.5).

It is interesting that both possibilities involve additional bosons. Although fermions never contribute to E, they do contribute to $\Delta\lambda + 6B$. However, they tend to increase baryon depletion. This can be seen from the fact that a heavy top quark mass in the standard model decreases the critical Higgs mass:

$$m_{H_c}^2(m_t) = m_{H_c}^2(0) - \frac{3\sigma^2}{4\pi^2}\left(\frac{m_t}{\sigma}\right)^4 \left(4\ln(\frac{m_t}{\sqrt{c_F}T_1}) + 3\right). \tag{6.7}$$

We now give a specific simple extension of the standard model which avoids baryon washout even for Higgs boson masses up to 150 GeV. We add a spin 0 multiplet S which is a singlet, so that the Lagrangian of the standard model is augmented by

$$\mathcal{L}_S = \partial^\mu S^* \partial_\mu S - M^2 S^* S - \lambda_S (S^* S)^2 - 2\zeta^2 S^* S H^* H, \tag{6.8}$$

where H is the Higgs doublet field: $|H^0| = \frac{\phi}{\sqrt{2}}$. We take M^2 and ζ^2 to be positive so that $< S >= 0$ at all temperatures, and the EWPT is that of the single Higgs.

A simple possibility is to take M^2 sufficiently small that where $\phi \simeq \phi_+(T_1)$ the scalar mass $m_S^2 = M^2 + \zeta^2\phi_+^2 \simeq \zeta^2\phi_+^2$. In this case the S field contributes to E. If it gives the dominant contribution to E, then

$$\frac{E}{\lambda_T} = \frac{g_s}{\pi}\zeta^3 \left(\frac{100\,GeV}{m_H}\right)^2 \tag{6.9}$$

where g_s is the degeneracy of the S multiplet, and we have chosen parameters such that $\Delta\lambda + 6B$ can be dropped in equation (6.5). The size of E/λ_T is restricted by requiring

that the high T expansion be valid at T_1:

$$\frac{E}{\lambda_T} \leq \frac{1}{\zeta}. \tag{6.10}$$

Hence we must take $\zeta \leq 1$. Note that the criterion (6.1) for the thin wall approximation is easily satisfied for E/λ_T much bigger than we need. There are a wide range of parameters that avoid baryon washout for all Higgs masses up to 150 GeV.

As an example, consider $\zeta = 1, g_S = 1$, and $m_H = 100$ GeV. In this case $E/\lambda_T \simeq 2/\pi$, and all constraints and approximations are satisfied. Taking $m_t = 125$ GeV gives $T_2 \simeq 130$ GeV and $T_1 \simeq 148$ GeV. How small must the $SU(2)$ invariant S mass be in order that $m_S(T_1) \simeq \zeta\phi_+(T_1)$ is a valid approximation? We find that $M \leq \zeta\frac{E}{\lambda_T}T_1 \simeq 94$ GeV, which does not involve any more fine tuning than for the Higgs doublet.

The simple extension of the standard model described by equation (6.8) can also avoid baryon washout, even if the S particle has a mass in the TeV range. At first sight this is surprising since S does not contribute to E. This is because $M >> \zeta\phi_+(T_1)$, and because the S particles have an exponentially suppressed number density at T_1. The contribution of the S field to the effective potential, $V_S(\phi)$, can then be obtained by expanding A1.6 in a power series in $\frac{\zeta\phi}{M}$:

$$V_s(\phi) = \frac{g_s\zeta^4}{64\pi^2}\left(-\phi^2(\phi^2-\sigma^2) + \frac{1}{3}\frac{\phi^6}{\sigma^2}\right)\frac{\zeta^2\sigma^2}{M^2} + \mathcal{O}\left(\frac{\zeta^4\sigma^4}{M^4}\right). \tag{6.11}$$

The decoupling behavior as $M \to \infty$ is manifest. Since $V_s'(\sigma) = V_s''(\sigma) = 0$, this contribution does not alter the minimum of the potential or the relation for the Higgs mass: $m_H^2 = 2(\lambda_{T_{SM}} + 6B_{SM})\sigma^2$. For our purposes the most important consequence of V_s is to correct the coefficient of ϕ^4:

$$\lambda_T = \lambda_{T_{SM}} - \frac{g_s\zeta^4}{16\pi^2}\frac{\zeta^2\sigma^2}{M^2}. \tag{6.12}$$

This is a crucial correction since it changes the m_H/λ_T relation, increasing the critical Higgs mass:

$$m_{H_c}^2(\zeta) = m_{H_c}^2(\zeta=0) + \frac{g_s\zeta^4}{8\pi^2}\sigma^2\frac{\zeta^2\sigma^2}{M^2}. \tag{6.13}$$

Hence if $(g_s\zeta^4)(\frac{\zeta^2\sigma^2}{M^2}) = 3$, the critical Higgs mass squared is increased by $(50\ \text{GeV})^2$. For example taking $g_s\zeta^4 = 15$ and $\frac{\zeta^2\sigma^2}{M^2} = \frac{1}{5}$ would give a bare S mass of $M = 0.56\,\zeta$ TeV. Baryon depletion can be avoided even if the additions to the standard model have masses in the TeV region.

We conclude that successful baryogenesis at the one Higgs doublet EWPT can occur provided the Higgs doublet is given two new interactions: one to violate CP and the other to enhance E/λ_T to avoid baryon workout.

7 Conclusion

We have presented a completely analytic treatment of the electroweak phase transition (EWPT), valid for all Higgs boson masses from the experimental limit of 46 GeV up to about 150 GeV. The electroweak phase transition is first order and proceeds by the nucleation of thin walled bubbles. We give the precise value of the temperature at which the phase transition completes as a function of the top quark and Higgs boson masses. In addition to characterizing the electroweak phase transition, we determine the value of the Higgs field vev after the phase transition completes, the number of bubbles nucleated per horizon, and many other quantities. Our formulae also apply to many extensions of the standard model which have the EWPT occur through just one field acquiring a vev. Additional particles can significantly alter quantities such as the temperature at which the phase transition completes, through their virtual effects on parameters in the Higgs potential. An important result of our analysis is determination of $\langle\phi\rangle_T$ at the end of the phase transition. The temperature dependent vev, $\langle\phi\rangle_T$, increases by a factor of 3/2 as the temperature drops from T_1 to T_2. Accordingly, determination of when the phase transition completes is essential because the rate of anomalous baryon number violation is an exponentially sensitive function of $\langle\phi\rangle_T$. If the rate of anomalous baryon number violation is large after thermal equilibrium is reestablished, any $B + L$ asymmetry generated during the EWPT will be at best ephemeral.

Much attention has recently been paid to the exciting possibility that the cosmological baryon excess may be produced at the EWPT. The possibility that this occurs in a model with a single Higgs doublet has largely been ignored. This is because it has been shown that even if sufficient baryon asymmetry could be generated, immediately after the EWPT it would be destroyed by anomalous baryon number violation, at least for Higgs boson masses above about 50 GeV. As a demonstration of the utility of the analysis of the EWPT presented in this chapter, we have shown that this baryon washout is very easily avoided in simple extensions of the standard model. Perhaps the simplest is the addition of a gauge singlet scalar boson that receives a contribution to its mass from the Higgs boson vev.

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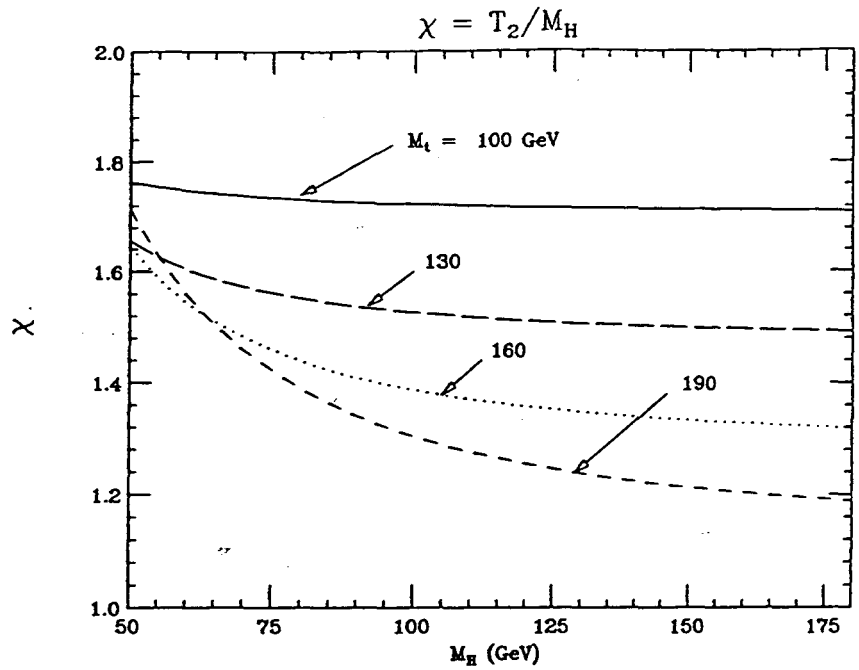

Figure 1: The ratio $\chi = T_2/m_H$ plotted as against the Higgs Boson mass for three different top quark masses.

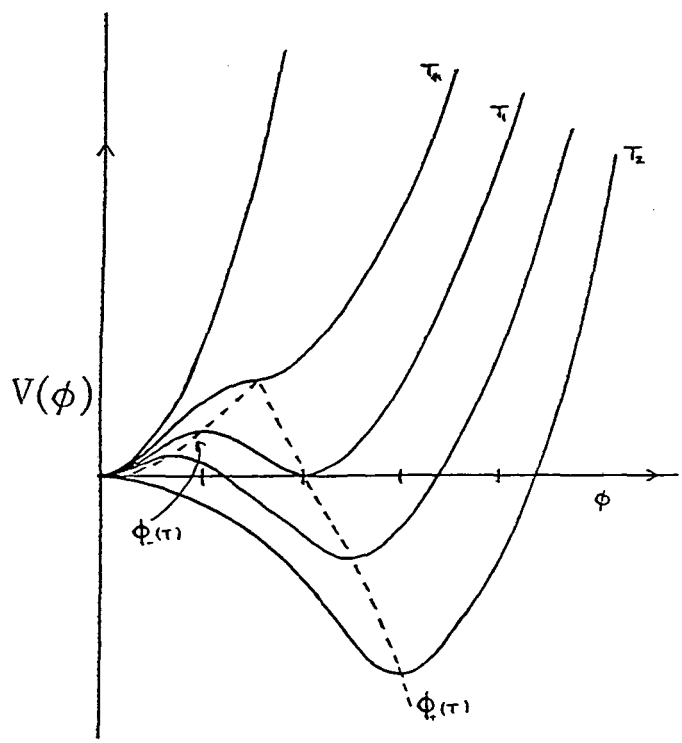

Figure 2: A schematic picture of the effective potential for temperatures T_*, T_1, $T_1 > T > T_2$ and T_2.

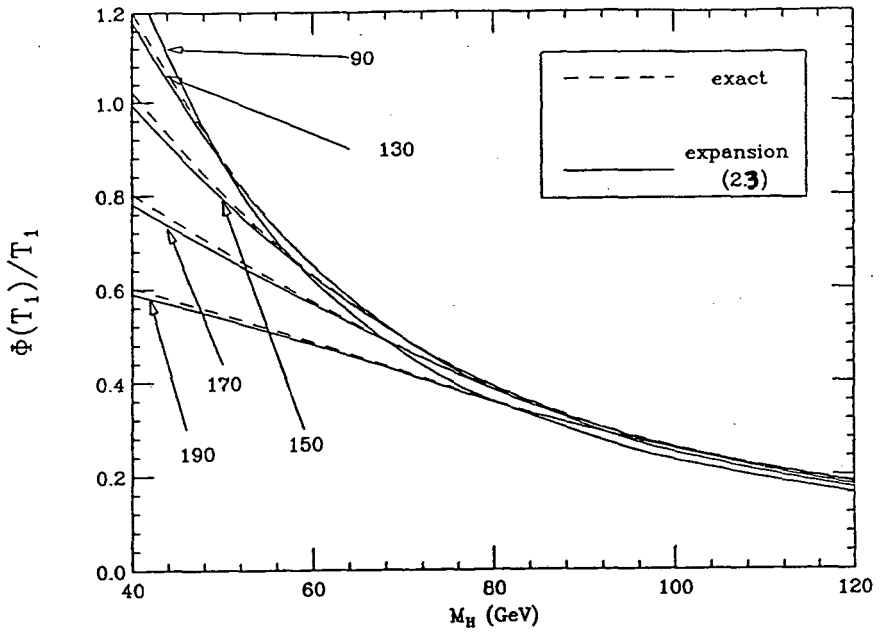

Figure 3: The temperature dependent vacuum expectation value of the Higgs field verses the Higgs Boson mass for five different top quark masses. The dashed line is the exact result, while the solid curve represents the approximaton of equation 2.3.

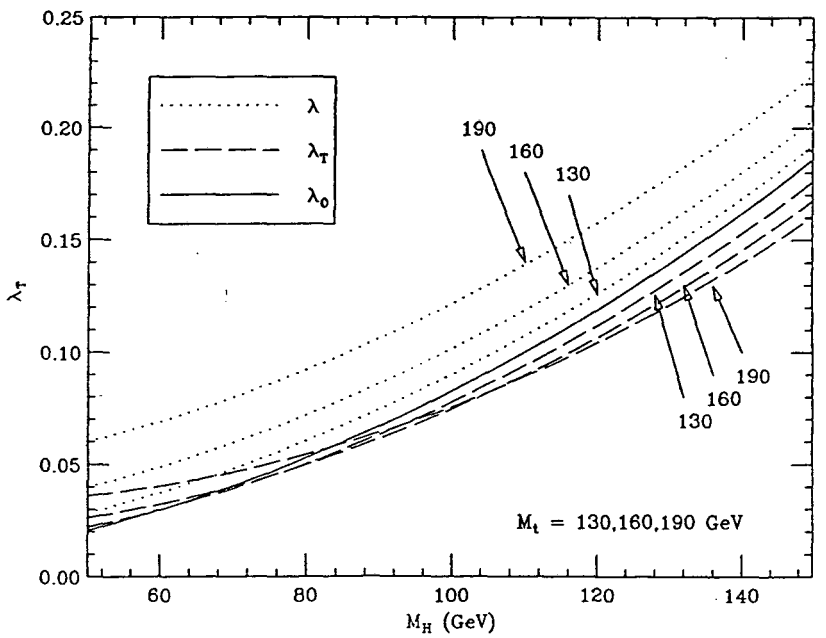

Figure 4: The tree level, one loop, and temperature dependent quartic scalar couplings verses the Higgs boson mass for three different top quark masses.

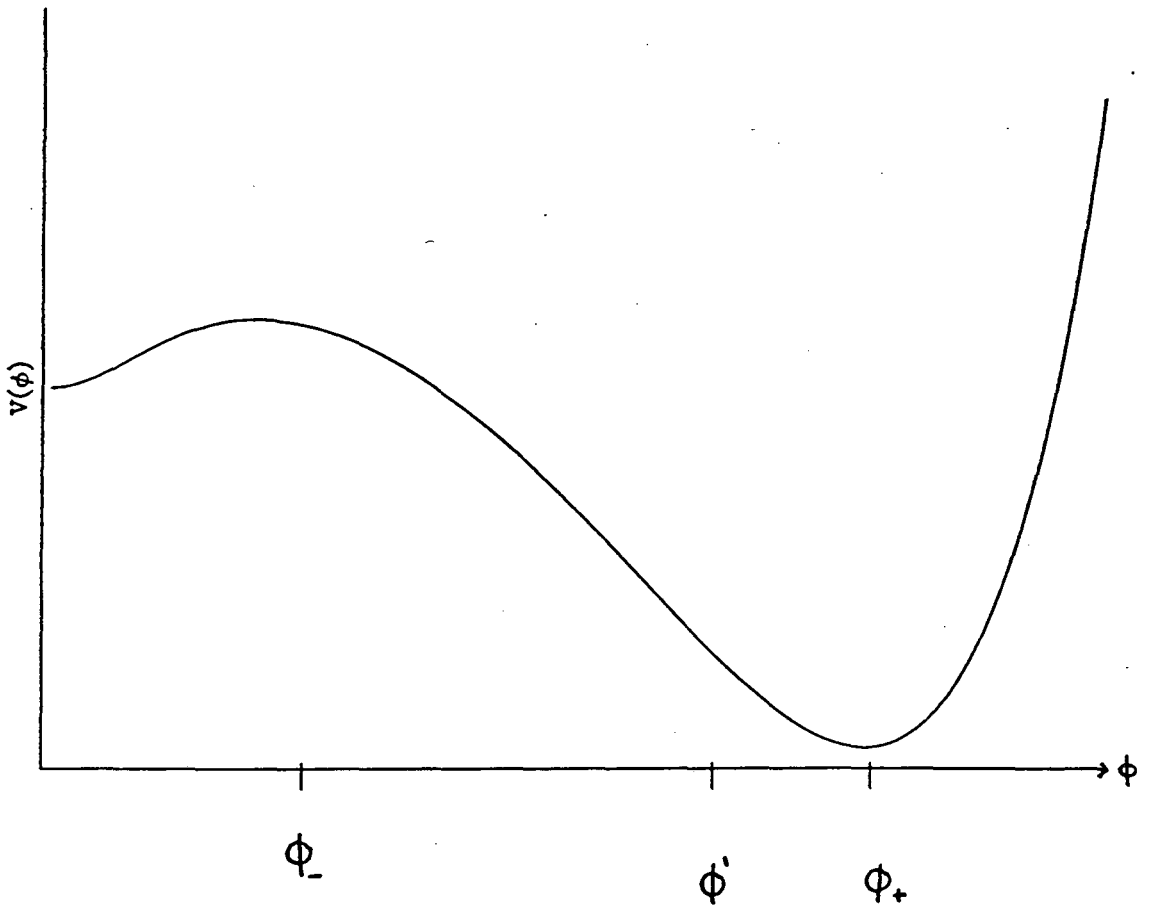

Figure 5: The effective potential

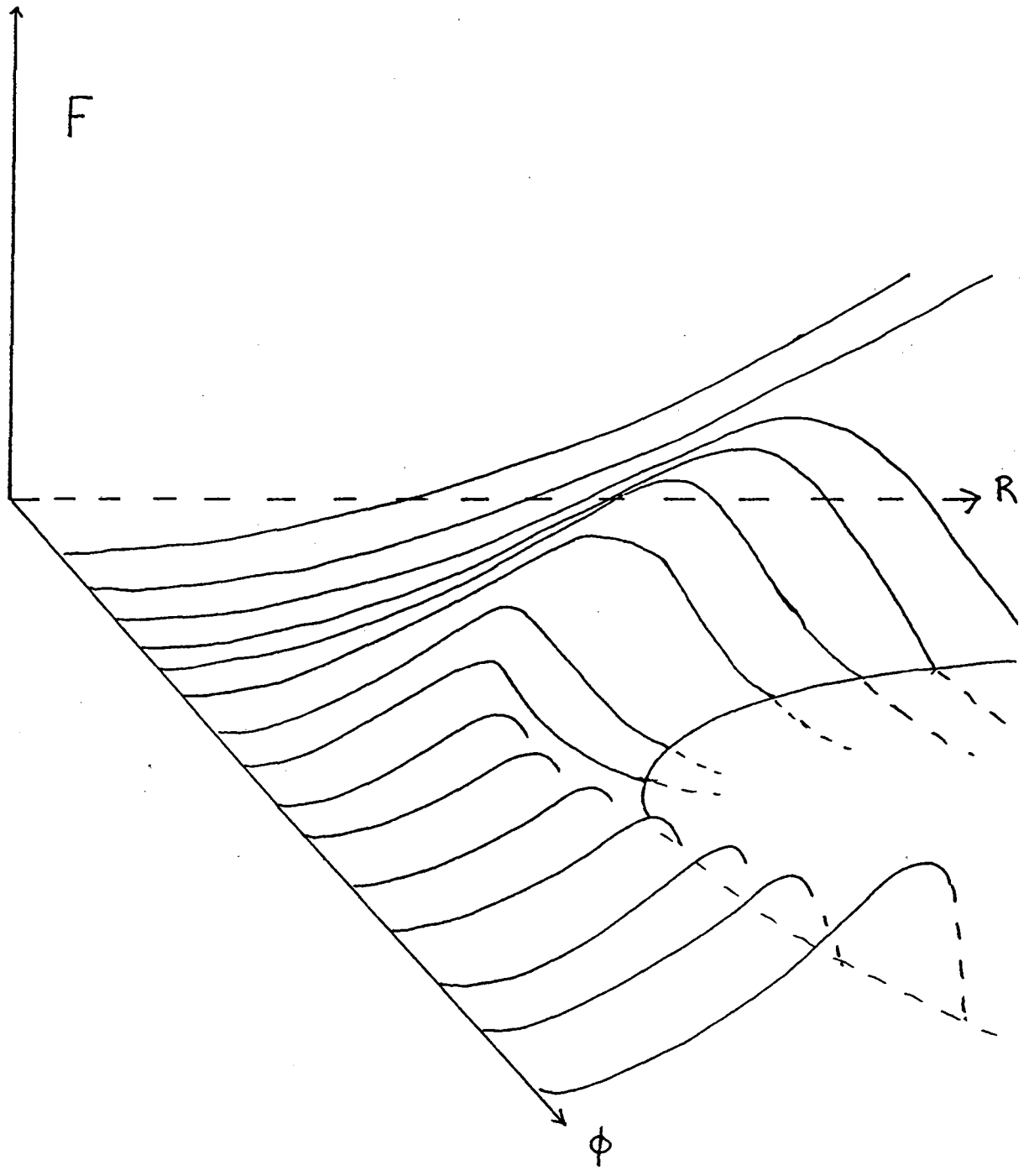

Figure 6: The Saddle in Configuration Space

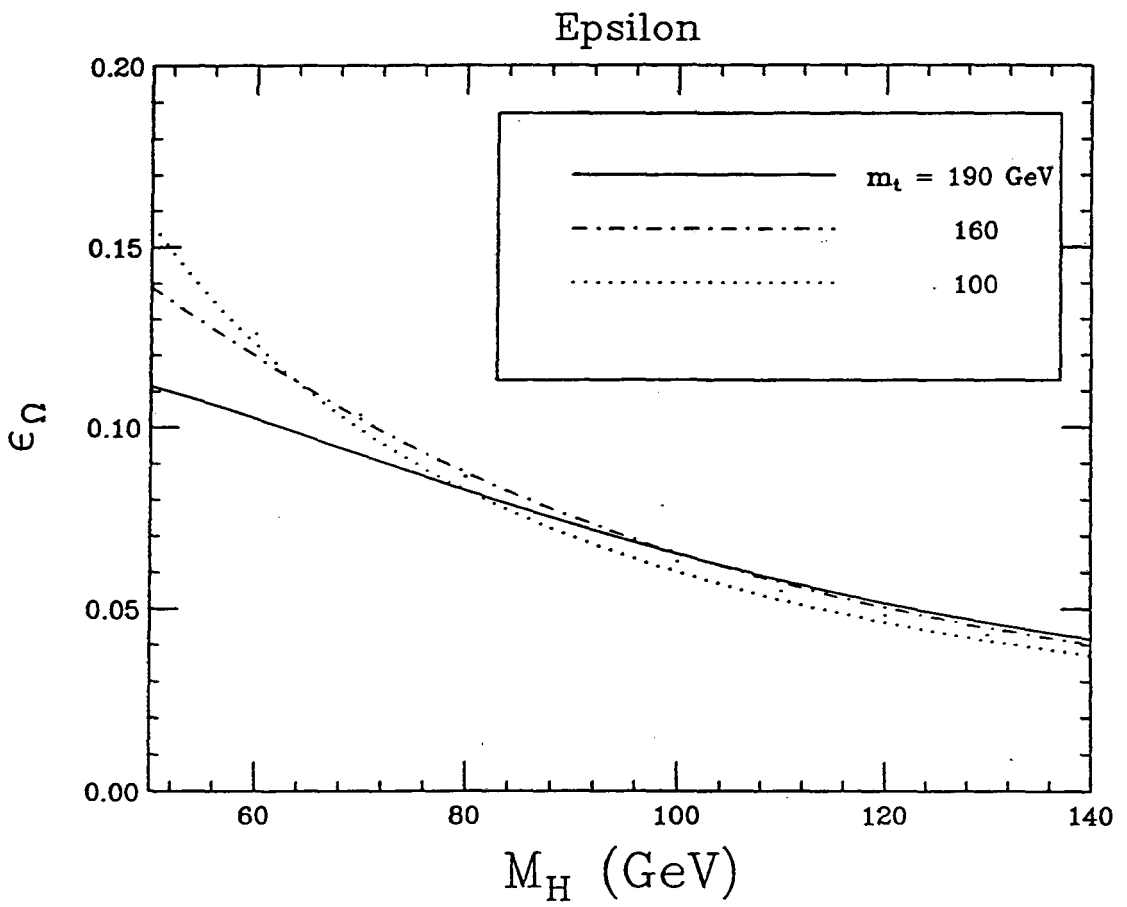

Figure 7: Epsilon at the end of the phase transition verses the Higgs boson mass for three different top quark masses.

Chapter II

Thermal Vacuum Stability and Constraints on m_H and m_t

1 Introduction

Despite a plethora of theoretical work and the vigorous experimental testing which has demonstrated the success of the standard model, we still do not know the masses of the Higgs boson or top quark. The best we can do is to exclude certain values. In addition to mass limits from accelerators, limits on the masses of the top quark and the Higgs boson can be obtained from cosmological consequences of the structure of the Higgs potential.

In the standard lore of spontaneous symmetry breaking, we live in the well of a potential shaped like a Mexican hat. The value of the Fermi constant, G_F, implies that we are in the local minimum where the Higgs field, ϕ, has a vacuum expectation value $\langle\phi\rangle = \sigma$, where $\sigma = 246$ GeV. However, the Higgs potential we often appeal to when discussing spontaneous symmetry breaking,

$$V(\phi) = \lambda\left(\phi^2 - \sigma^2\right)^2, \tag{1.1}$$

is only the tree level potential. Quantum corrections can significantly change this picture. For a fixed mass of the Higgs boson, as the mass of the top quark, m_t, increases, a new

minimum of the potential develops at $\phi > \sigma$ when m_t reaches some critical value [1] (see Figure 8).

This situation suggests that cosmological considerations may constrain the masses of the Higgs boson and top quark. For certain values of m_t and m_H, the rate of quantum tunneling is large enough to render the state $\langle\phi\rangle = \sigma$ unstable . This occurs if the lifetime of the state $\langle\phi\rangle = \sigma$ is less than the age of the universe. Flores and Sher [2] and subsequent work by Arnold [3] used this consideration to set limits on the masses of the Higgs boson and the top quark (see Figure 11). However, *quantum* tunneling in not the only vehicle by which the state $\langle\phi\rangle = \sigma$ can decay. In the early universe, when the temperature was high, thermal fluctuations may have been large enough for the system to pass over the barrier *classically*. Indeed, for many values of m_t and m_H, constraints from classical, thermal bubble nucleation are more stringent than those from quantum tunneling. In section 2, we discuss the exact form of the temperature-dependent, one-loop effective potential which is essential to calculate the bubble nucleation rate accurately. Section 3 discusses how large the primordial rate of classical, thermal bubble nucleation needs to be in order to constrain m_t and m_H. Section 4 computes critical bubble energies for two ansätze. Then we present mass limits obtained using the analysis of the previous sections.

2 Instability of the Effective Potential

The one-loop, zero temperature, effective potential for the physical Higgs scalar is discussed in the appendices. For Higgs bosons light enough to neglect scalar loops, the potential can be written [1],

$$V(\phi) = -\frac{1}{2}(\lambda + 2B)\sigma^2\phi^2 + \frac{1}{4}\lambda\phi^4 + B\phi^4 \ln(\phi^2/\sigma^2), \tag{2.1}$$

where ϕ, the physical Higgs scalar, has mass $m_H^2 = 4\sigma^2(\frac{1}{2}\lambda + 3B)$, $\langle\phi\rangle = \sigma$, and

$$B = \frac{1}{64\pi^2\sigma^4}\left(6m_W^4 + 3m_Z^4 - 12m_t^4\right). \tag{2.2}$$

For $m_t^4 \gtrsim (80\ \mathrm{GeV})^4 + \frac{1}{12} m_H^4$, the parameter B becomes negative and the Higgs potential appears to be unbounded from below as $\phi \to \infty$. Note however, that since we have neglected higher order terms in the potential of equation (2.1), this expression is not valid for large ϕ. For example, two-loop effects $\propto \phi^4 (\ln(\phi))^2$ can restore boundedness at larger values of ϕ. A more accurate determination of the mass range where $\langle \phi \rangle = \sigma$ is not the absolute minimum of the Higgs potential requires summing these higher order terms. In a recent article [5], Lindner *et al.* numerically solve the renormalization group equations for the Higgs potential at two-loop order. They find the very interesting result that, for values of m_t above the dotted line in Figure 11, we live in a false vacuum. However, as was pointed out in reference 2, there is no guarantee that we are living in the true vacuum. Sher and Zaglauer [4] argue that cosmic rays would catalyze false vacuum decay in most of the unstable region of $m_t - m_H$ space, but subsequent work by Arnold [3] disagrees with their analysis. We can only require that the lifetime of our state is greater than the age of the universe.

In general, the decay rate Γ of an unstable state is related to the imaginary part of the free energy [6]. For quantum tunneling $\Gamma = 2\,\mathrm{Im}\,F$, while for classical decay $\Gamma \propto \frac{1}{T\pi}\,\mathrm{Im}\,F$. In the path integral formulation, the quantum mechanical decay rate per unit volume of an unstable state is given by the well known formula [8]

$$\Gamma/V = A\,e^{-S}, \tag{2.3}$$

where S is the Euclidean space action for a true vacuum bubble of critical size. The coefficient A is calculated from fluctuations around the bubble [8]. The rate of quantum tunneling has been calculated [2, 3]. Flores and Sher found that zero temperature quantum tunneling will not destabilize the state $\langle \phi \rangle = \sigma$ within the age of the universe provided

$$m_H \gtrsim 3.6\,(m_t - 180\,GeV)\,, \tag{2.4}$$

while later work by Arnold [3] found more restrictive bounds (see figure 11). We shall concern ourselves with the *classical* thermal decay rate. Standard Quantum field theory

is set up to describe events that take place in the background of empty space. At the high temperatures present in the early universe, the zero temperature potential must be modified to include the interaction between the Higgs field and the hot electroweak plasma. As discussed in the appendices, the thermal contribution to the potential is the free energy of the ambient particles:

$$\begin{aligned}\Delta V_1(\phi,T) = &\sum_F \frac{-g_F T^4}{2\pi^2}\int_0^\infty dx\, x^2 \ln\left(1+e^{-\sqrt{x^2+m_F^2\beta^2}}\right)\\ &+\sum_B \frac{g_B T^4}{2\pi^2}\int_0^\infty dx\, x^2 \ln\left(1-e^{-\sqrt{x^2+m_B^2\beta^2}}\right),\end{aligned} \tag{2.5}$$

where $M_i = M_i(\phi)$ is the mass of particle i in the presence of a background field ϕ, g_i is the number of degrees of freedom of particle species i, $\beta = 1/T$, and $(F)B$ denotes a sum over (fermions) bosons respectively.

The high temperature expansion of $\Delta V_1(\phi,T)$ used in chapter 1 is not a good approximation of equation (2.5) when $M(\phi)/T$ is large. Indeed, for many cases of interest, $M(\phi)/T$ can be larger than $\mathcal{O}(1)$. In particular, the rate of thermal bubble nucleation in the early universe can be large when $M(\phi)/T$ is not small. As we shall see, the rate of thermal bubble nucleation is largest near $T_2/2$, where T_2 is the temperature at which the state $\langle\phi\rangle = 0$ becomes unstable.

The integral equation for $\Delta V_1(\phi,T)$, equation (2.5), can be written in terms of a sum. As shown in Appendix C:

$$\begin{aligned}\Delta V_1(\phi,T) = &-\sum_F N_F\left[\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2}\, a_F^2\, K_2(a_F n)\right]\\ &-\sum_B N_B\left[\sum_{n=1}^\infty \frac{1}{n^2}\, a_B^2\, K_2(a_B n)\right],\end{aligned} \tag{2.6}$$

where $N_F = g_F T^4/2\pi^2$ and $N_B = g_B T^4/2\pi^2$. Since the Bessel function K_2 falls off exponentially for large values of its argument, the expression above is well suited to numerical computation. In Figure 8 we display the full, temperature dependent, one-loop effective potential for $m_t = 160$ GeV and $m_H = 20$ GeV. We include the loop-effects of the W, Z, and top quark ($g_t = 12, g_W = 6, g_Z = 3$).

3 Of Sizes, Scales, and Temperature

The Boltzmann suppressed rate of thermal diffusion out of a metastable state has been discussed by many authors[6, 7]. The thermal escape rate for a simple system is calculated in Appendix B. For a quantum field theory, thermal fluctuations will produce bubbles of true vacuum at a rate

$$\Gamma/V = \Lambda^4(T)e^{-F_c(T)/T}, \tag{3.1}$$

where $F_c(T)$ is the free energy of a fluctuation large enough to pass over the free energy barrier separating the two vacua, and Λ is a characteristic scale in the theory. For definiteness take $\Lambda = T$. As we shall see, for a certain range of the masses m_t and m_H, the free energy of the barrier separating $\phi = \sigma$ and $\phi = \sigma'$ is small enough to allow thermal fluctuations from σ to σ' (see Figure 9). We must require that no false vacuum decay of this type occurred in our past. In other words, the probability that a phase transition occurred in a comoving volume coincident with the present horizon volume must be less than one.

To calculate the size that this comoving volume occupied at a temperature T, we must scale back our present causal volume. The ratio of the present photon temperature to the photon temperature at recombination tells us how much the cosmic scale factor has expanded since the end of the radiation dominated era:

$$T_{\gamma R}/T_{\gamma 0} = R(t_0)/R(t_R) \simeq 1300. \tag{3.2}$$

We can scale back to earlier times by noting that in the radiation dominated era, $R(T) \propto T^{-1}$. Thus, the volume that a comoving volume coincident with the horizon volume today occupied at times before recombination can be expressed as

$$\begin{aligned} \tilde{V}(T) &\simeq t_0^3 \left[R^3(t)/R^3(t_0)\right] \\ &\simeq \left(\frac{t_0}{1300}\right)^3 T_R^3/T^3, \end{aligned} \tag{3.3}$$

where $T_R \simeq 0.3$ eV and t_0 is the age of the universe. Moreover, noting that in the radiation dominated era, the time-temperature relationship is $t \sim M_{Pl}/T^2$, the probability that a bubble was nucleated during a temperature interval dT is given by

$$dP \sim \tilde{V}(T) M_{Pl} e^{-F_c(T)/T} T dT, \tag{3.4}$$

where $M_{Pl} = 1.2 \times 10^{19}$ GeV is the Plank mass. Setting $P \leq 1$, examining the bubble nucleation rate near $T \simeq 50$ GeV, and choosing a conservative temperature interval $dT \simeq \mathcal{O}(\frac{T}{10})$, no bubble nucleation implies that

$$e^{-\frac{F_c(T)}{T}} \lesssim e^{-250}. \tag{3.5}$$

Hence, bubble nucleation would not have occurred in our past horizon volume if *

$$\frac{F_c(T)}{T} \gtrsim 250. \tag{3.6}$$

4 Critical Energies for Bubble Nucleation

Consider a true vacuum bubble in a sea of false vacuum $\langle\phi\rangle_T = \sigma_T$, where σ_T is the temperature dependent local minimum of the Higgs potential destined to become σ. Let $\phi = \sigma'$ at the center of the bubble. (See Figure 9). By convention we choose this metastable state to have zero free energy. The free energy of the system is

$$F = \int d^3x \left\{ \frac{1}{2} \left(\vec{\nabla}\phi \right)^2 + V(\phi, T) - V(\sigma_T) \right\} \tag{4.1}$$

In our case, inspection of the potential reveals that $V(\sigma_b) - V(\sigma_T)$ is not large compared to $V(\sigma_T) - V(\sigma')$. In this case, as discussed in chapter I, it is favorable to make δR as

*Let R be the cosmic scale factor and let T be the temperature of the universe. The fact than RT is a constant is a consequence of entropy conservation. If entropy dumping occurs after the electroweak phase transition, the product RT will increase. If the product RT were to increases by a factor f, the right side of equation (3.5) would be reduced by a factor f^3. So, enormous entropy dumping can weaken the constraints derived from equation (3.6). For example, if $F_c(T)/T$ is reduced from 250 to 240, the bound, derived in section 4, on m_H when m_t is 200 GeV is reduced from about 46 GeV to 45 GeV. If the Higgs Boson or the top quark were to be discovered in the region excluded by this analysis, a sensible cosmology would require enormous entropy dumping at a time after the $SU(2) \times U(1)$ phase transition.

large as possible so as to minimize the 'surface' term E_S. Hence $\delta R \sim R$, and we should work in a thick wall approximation. Recall that the energy barrier that fluctuations need to surmount is a saddle point. So the energy of the barrier is the maximum free energy attained by the solution which crosses the smallest possible barrier. Hence, the energy barrier crossed by a trial function which interpolates smoothly between the phases $\langle\phi\rangle = \sigma_T$ and $\langle\phi\rangle = \sigma'$ is greater than or equal to the free energy of the saddle point. For this reason, we can obtain upper bounds on the critical bubble energy by using the trial functions of (4.2):

$$\begin{aligned} \phi_1(R) &= \sigma_T + (\sigma' - \sigma_T)e^{-r/R}, \\ \phi_2(R) &= \sigma_T + (\sigma' - \sigma_T)e^{-r^2/R^2}. \end{aligned} \tag{4.2}$$

The bubble free energy corresponding to these solutions is,

$$F_i = 4\pi \int dr\, r^2 \left\{ \frac{1}{2}\left(\frac{\partial \phi_i}{\partial r}\right)^2 + (V(\phi_i(r)) - V(\sigma_T)) \right\}. \tag{4.3}$$

For each of the two ansätze (4.2), the resulting bubble energy, calculated from (4.3), was maximized with respect to R and minimized with respect to σ', subject to the constraint that the energy of the bubble be less than a bubble of false vacuum as the radius of the bubble approaches infinity. An example of the dependence of $F_c(T)$ on temperature is shown in Figure 10 for each of the two ansätze. Figure 11 displays the limits on the m_H and m_t resulting from the requirement that the bubble energies calculated from equation (4.3) satisfy the inequality of equation (3.6).

5 Conclusion

In this chapter we have investigated constraints on the Higgs boson and top quark masses arising from the requirement that the ground state of the Higgs potential near $\langle\phi\rangle = 246$ GeV is stable at the high temperatures present in the early universe. For certain values of Higgs boson and top quark masses, thermal fluctuations would have destabilized the local minimum of the Higgs potential near $\langle\phi\rangle = \sigma$ when the temperature was roughly

half the temperature where the $SU(2) \times U(1)$ phase transition occurred. The fact that today, $\langle\phi\rangle = \sigma$, excludes the mass range above the solid curve in Figure 11. In particular, if the Higgs boson has a mass of 25 GeV, the top quark must be lighter than 170 GeV.

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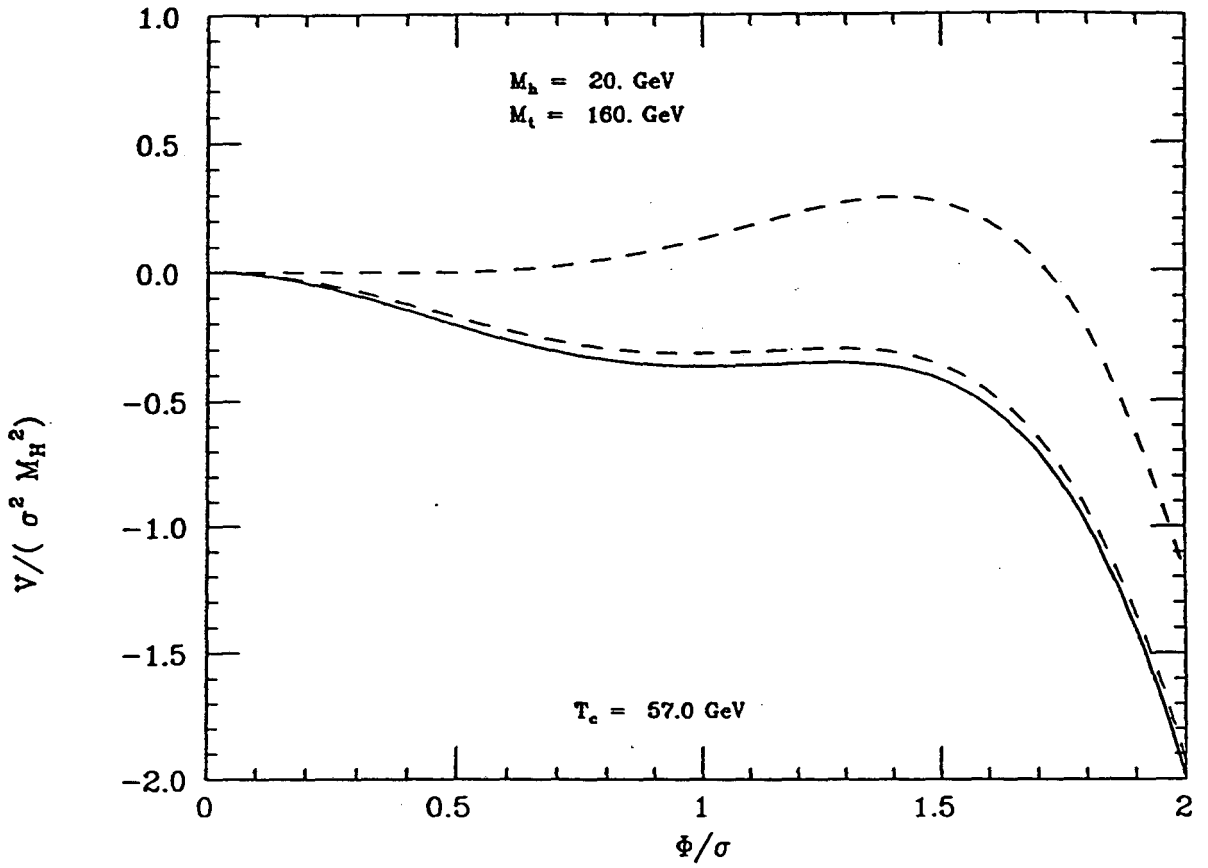

Figure 8: The effective potential for the Higgs field with $m_t = 160.$ GeV and $m_H = 20.$ GeV. For each curve, the zero of the potential has been chosen to be at $\phi = 0$. The solid curve is the zero temperature contribution of equation (2.1). The dashed curves represent the temperature dependent potential at temperatures $T = \frac{1}{2}T_2$ and $T = T_2$, where T_2 is the temperature at which the state $\langle\phi\rangle = 0$ becomes unstable. The dashed curves include the one-loop effects of the W and Z gauge bosons and the top quark.

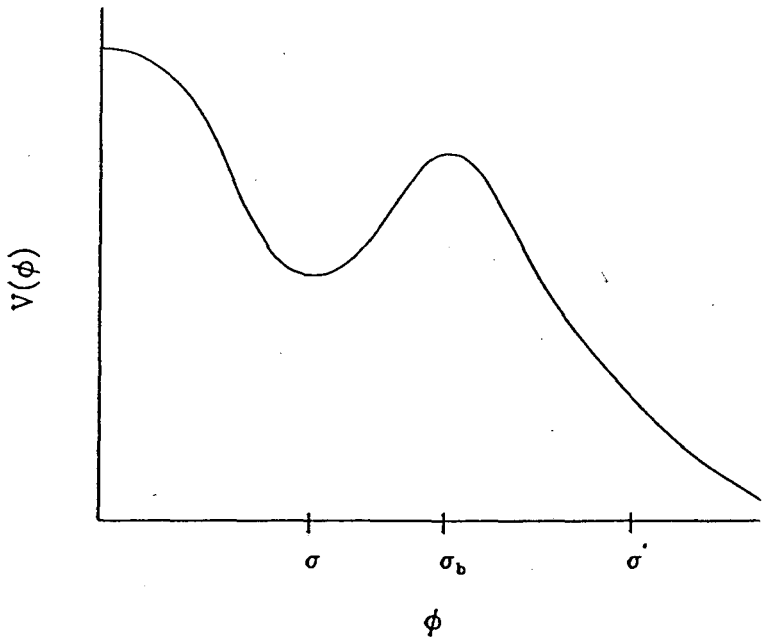

Figure 9: The effective potential for ϕ

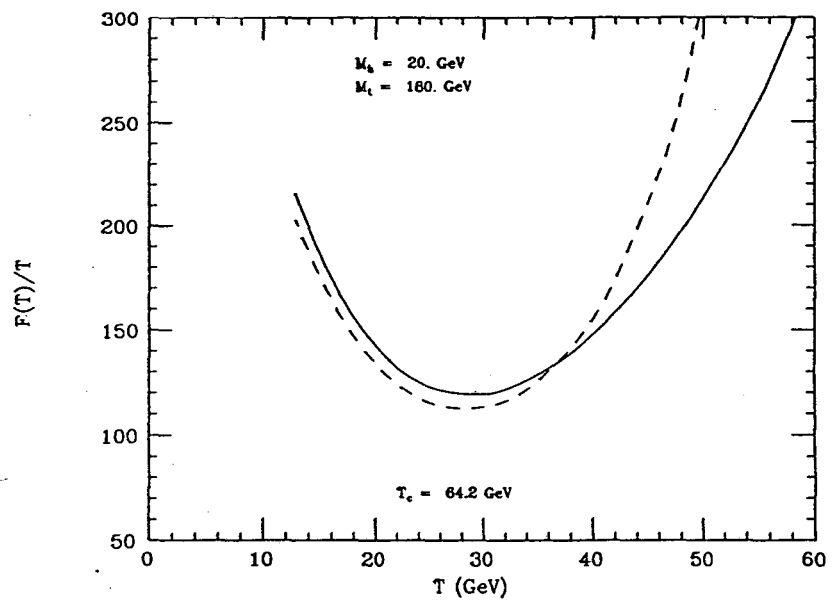

Figure 10: The critical bubble energy calculate from (4.3) for each of the two ansätze of (4.2) is potted as a function of temperature. The dashed (solid) curve is the result for ansatz 1 (2).

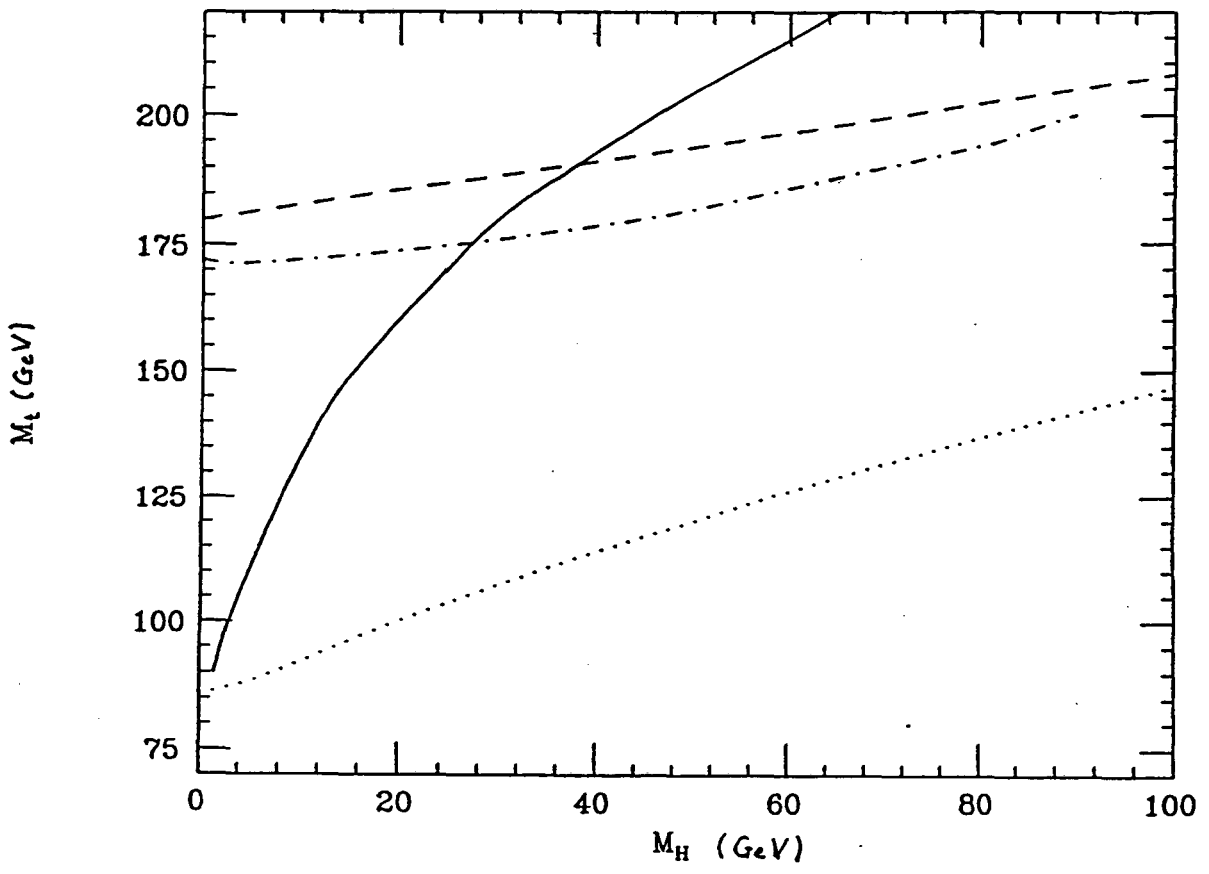

Figure 11: Excluded regions of $m_H - m_t$ space. The analysis of this chapter excludes the region above the solid line in figure 11. The region above the dashed line is ruled out by the quantum tunneling constraint of Flores and Sher [2]. Points above the alternately dotted and dashed curve are excluded by the quantum tunneling constraint of Arnold[3], this curve assumes the renormalization group improved Higgs potential can be trusted up to a scale of 10^4 GeV. In the region below the dotted line, $\langle\phi\rangle = \sigma$ is the true minimum of the Higgs potential.

Chapter III

Can a Particle Have a Bag?

1 Introduction

In addition to the phase transitions we have discussed, it is interesting to consider whether a small scale deformation in the scalar field can be induced by the presence of a heavy particle. Specifically, can a heavy particle act as seed for a local nucleation event which leaves the particle enclosed in a bag of scalar field condensate? Recently, it has been argued [1] that when a particle receives a contribution to its mass from the vacuum expectation value, v, of a scalar ϕ, under certain conditions the lowest excitation of the system containing a single particle is one in which a bag of scalar field condensate, or 'dimple', forms around the particle.

It is known that classical bag configurations can form when many particles are present. Such non-topological soliton solutions have been studied in detail [2, 3]. In this chapter we investigate whether such bags can form about a single particle. Intuitively, it appears that bag formation will occur if the energy in the scalar field gained by locally relaxing its vacuum expectation value is less than the energy liberated by the resulting decrease in particle mass. The existence of such bags around heavy quarks and gauge bosons would lead to interesting effects in both the masses and decay signatures.

In section 2 we present semi-classical arguments for bag formation around a single

particle. The validity of this semi-classical approximation is discussed in section 3, and again in the language of coherent states in section 4. For bags supported by one particle, we find that energetically favorable deformations are too small to be built up from the fundamental quanta. Hence, we argue that there are no dimples in the scalar field near a single, perturbatively coupled particle.

2 A Wrinkle in ϕ

To present the arguments for bag formation around a single particle we discuss an explicit example. Consider the Hamiltonian for a scalar field with a potential $V(\phi) = \frac{1}{4}\lambda(\phi^2 - v^2)^2$, coupled to a fermion with a mass $m_0 + g\langle\phi\rangle$, where $\langle\phi\rangle$ is the vacuum expectation value of the scalar ϕ. For time independent field configurations,

$$H = \int d^3r \left\{ \frac{1}{2}(\nabla\phi)^2 + V(\phi) + \overline{\psi}(i\vec{\nabla}\cdot\vec{\gamma} + m_0 + g\phi)\psi \right\}. \tag{2.1}$$

When there are no fermions present, the ground state of the system is just the standard vacuum $\langle\phi\rangle = v$. But, for the lowest energy solution with fermion number one, classical reasoning seems to imply that a condensate in the scalar field, where $\langle\phi\rangle$ differs from v, forms around the fermion. Ostensibly, if the Yukawa coupling to the fermion is sufficiently large, the scalar field energy gained by relaxing $\langle\phi\rangle$ towards zero near the fermion may be more than compensated for by the loss in rest energy of the fermion. For a small dimple in the scalar field around a single fermion, the energy of the system is a combination of the surface energy resulting from the change in $\langle\phi\rangle = \phi_{cl}$ in the bubble walls, the scalar field potential energy, the loss in rest mass of the fermion, and the kinetic energy of the confined fermion. For a bubble of radius R and thickness δR, and for a non-relativistic fermion,

$$E \simeq 4\pi R^2 \left(\frac{\delta v}{\delta R}\right)^2 \delta R + \Delta V(\phi_{cl})\, R^3 + m_f(\langle\phi\rangle) + \frac{p^2}{2m_0}, \tag{2.2}$$

where ΔV is the change in potential energy density inside the bubble, δR is the thickness of the bubble wall, and m_f is the mass of the fermion in the presence of the shifted vacuum

expectation value. The energy of the system will favor a bag with thick walls. In this limit, the dimple walls have a thickness comparable to the radius of the dimple $\delta R \sim R$, and the energy difference between a fermion in a dimple and one without a dimple is

$$E = \frac{2A}{3}\pi(\delta v)^2 R + 2B(\delta v)^2 m_\phi^2 R^3 - Cg\delta v + p^2/2m, \tag{2.3}$$

where δv is the change in $\langle\phi\rangle$ at the center of the bubble, $m_\phi = \sqrt{\lambda}v$ is the mass of the scalar, and the parameters A, B, C are numerical factors of order unity. We call this energy the dimple binding energy, E_{DB}. The confinement momentum of the fermion, resulting from a spread in the fermion wave function over the dimple, will be of order $1/R$. Denoting $p = D/R$, the ground state of the system can be found by varying R and δv. When the scalar field mass is small, so that the volume term can be neglected, and provided $g \gtrsim (v/m_0)^{\frac{1}{3}}$, the binding energy of the dimple is

$$E_{DB} \simeq -\alpha^2 m_0 \left(\frac{A^3C^4}{D^2}\right), \tag{2.4}$$

where $\alpha = g^2/4\pi$. The condition $g \gtrsim (v/m_0)^{\frac{1}{3}}$ must be imposed to ensure that δv be less than v. This is equivalent to the statement that we have a 'dimple' in the scalar field rather that a bubble ($\langle\phi\rangle = 0$ at the center). Since we have optimized δv, solutions with $\delta v = v$ must give smaller energies. Our neglect of the volume term is valid provided $m_\phi \lesssim \alpha m_0 \left(\frac{C^2}{BD^2}\right)$. The dimple radius is

$$R \simeq \frac{1}{\alpha m_0}\left(\frac{D^2}{AC^2}\right). \tag{2.5}$$

We can get a handle on the size of these factors by performing a Rayleigh-Ritz variational calculation. In this approximation we treat the bubble classically while treating the ψ-particle quantum mechanically. Below we calculate the energy of a single particle in a bag in this approximation. Writing

$$E = E_\phi + E_\psi, \tag{2.6}$$

for $\phi_{cl} = v'$ at the center of the dimple, and denoting $\delta v = v - v'$, the scalar field

contribution to the energy of the dimple is

$$E_\phi = \int d^3x \left\{ \frac{1}{2} \left(\vec{\nabla} \phi_{cl} \right)^2 + V(\phi_{cl}) \right\}, \tag{2.7}$$

while the energy contribution from the ψ particle is

$$E_\psi = \int d^3r \, \psi^* \left(-\frac{\nabla^2}{2m_0} + m_0 + g\phi_{cl}(r) \right) \psi. \tag{2.8}$$

Using various sets of trial wave functions, we minimize the binding energy with respect to R_D, R_ψ, and v'. This results in $R_D \sim R_\psi$, which maximizes the decrease in the fermion mass while minimizing its kinetic energy.

The optimal ansatz was

$$\begin{aligned} \phi_{cl}(r) &= v \left(1 - \delta e^{-r/R_D} \right), \\ \psi(r) &= \sqrt{\frac{1}{\pi R_\psi^3}} \, e^{-r/R_\psi}. \end{aligned} \tag{2.9}$$

For these trial wave functions we find

$$\begin{aligned} E_{DB} &\simeq -\frac{1}{28} \alpha^2 m_0, \\ R_D &\simeq 9 \frac{1}{\alpha m_0}. \end{aligned} \tag{2.10}$$

For future reference we note that the energy of these 'would be' non-topological solitons vanishes as the couplings are turned off.

3 The Classical Criteria

Although the classical reasoning used above seems to suggest that the ϕ-field dimple is the minimum energy solution in the presence of an external source (e.g., the fermion), this does not mean that we should place the dimple on an equal footing with the actual ground state, $\langle \phi \rangle = v$. The arguments used to find this classical, minimum energy solution to the field equations treated the scalar field ϕ as a continuous, c-number field. Quantum mechanics tell us that this is not in general correct. As a result of quantization, changes in the value of a field are discrete. These changes may be viewed as continuous when

differences in the field considered represent the addition or subtraction of many quanta. When a deformation in the value of an field operator is small compared to differences induced by the emission or absorption of many quanta, we shouldn't trust classical arguments. Another way of saying this is that the expectation value of an operator can be replaced by its classical value only when the fluctuations in that expectation value are small. This is only the case when the field contains many quanta.

Armed with the results of section 2, we can check our treatment for self-consistency by comparing the energy in the scalar-field dimple to that of an individual scalar quantum. We find, unfortunately, that the approximation of treating the dimple classically is a bad one because its energy is *less* than that of a single quantum. Consider a Fourier decomposition of the scalar-field dimple. Since the size of the dimple is $\sim \frac{1}{\alpha m_0}$ (where, for simplicity, we drop numerical factors), we expect the Fourier modes to have momentum $\sim \alpha m_0$. This implies that the typical energy of a quantum in the dimple is

$$E_{quantum} \simeq (m_\phi^2 + \alpha^2 m_0^2)^{1/2}. \tag{3.1}$$

Comparing this to E_{DB} we find,

$$\frac{E_{DB}}{E_{quantum}} \sim \alpha. \tag{3.2}$$

Hence, for *perturbative* coupling our solution is composed of much *less* than one quantum.

The invalidity of the classical treatment of a dimple in a scalar field surrounding a particle can be seen directly. In what follows, we denote the total energy in the scalar field by E_ϕ. The criteria that the deformation in ϕ_{cl} be made up of at least one quantum is

$$E_\phi \geq E_{quantum} \simeq \sqrt{m_\phi^2 + R^{-2}} > 1/R. \tag{3.3}$$

In order for a bound state to exist, the total energy in the scalar field, E_ϕ, must be less than the change in rest mass of the fermion:

$$E_\phi \leq g\delta v. \tag{3.4}$$

Putting this together with (3.3) we have $1 < g(\delta v)R$. In particular, the energy in the 'surface' terms must be less than the change in rest mass of the fermion,

$$4\pi(\delta v)^2 R < g\delta v, \tag{3.5}$$

so that $\delta v R < g/4\pi$. Putting these together gives $\alpha = \frac{g^2}{4\pi} > 1$. Requiring that the dimple be made up of N quanta implies that

$$\alpha = \frac{g^2}{4\pi} > N. \tag{3.6}$$

Hence there are no bubbles for perturbative couplings.

It is instructive to compare the behavior of these 'would be' non-topological solitons to the behavior of classically valid topological solitons [4, 5]. For definiteness, we recall the properties of the 'kink' (domain wall) solitons of ϕ^4 theory in $1+1$ dimensions. Rescaling the scalar field ϕ by the small coupling g, $\phi' = g\phi$, the ϕ^4 theory Lagrangian is

$$L/\hbar = \frac{1}{g^2\hbar}\int dx \left\{\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{\mu^2}{2}(1-\phi'^2)^2\right\}. \tag{3.7}$$

In addition to the degenerate vacua of $\langle\phi'\rangle = \pm 1$, the Lagrangian (3.7) has a stable classical solution

$$\phi'_{cl} = \tanh \mu x. \tag{3.8}$$

Quantizing around this solution, to order g, the energy is given by

$$E = \frac{4\mu}{3g} - \mu g(\frac{3}{\pi} - \frac{1}{2\sqrt{3}}) + \mathcal{O}(g^3\mu). \tag{3.9}$$

Hence, we see that in the region of validity, the 'classical' contribution to the energy of this topological soliton $E \sim \mu/g$ is many times larger than the energy of a quantum fluctuation $E_q \sim g\mu$. The classical limit $\hbar \to 0$ is equivalent to $g \to 0$. So, for weak coupling, the energy of the classical solution is much larger than the energy of fluctuations around the solution (3.8). For large values of g the energy of the classical solution is not the dominant term in (3.9), and there is no more reason to believe the classical analysis than there is to trust perturbation theory [4]. The same remarks would apply

to the soliton structures of sine-Gordon theory, the 't Hooft-Polyakov monopole, or other topological solitons in field theory.

4 Classical Values, Fluctuations, and Coherent States

For small shifts $\delta\phi$ in the field ϕ about the minimum of the potential $V(\phi) = \lambda(\phi^2 - v^2)^2$, we can perturbatively define our theory so that to lowest order it is merely the theory of a heavy fermion coupled to a free, massive scalar field. The Lagrangian for such a theory is simply

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \overline{\psi}(i\partial\!\!\!/ - m_o)\psi - g\phi\overline{\psi}\psi, \tag{4.1}$$

where m is given by the second derivative of $V(\phi)$ at its minimum. Since the scalar field is free except for its coupling to the fermion, we can now study its behavior using the formalism of coherent states.

Coherent states are constructed by the application of creation operators (in the interaction picture) on the vacuum state $|0\rangle$ so as to yield a prescribed expectation value for the field operator $\hat{\phi}$. We define coherent states for the positive frequency part of the quantum field ϕ. The field operator $\hat{\phi}$ can be expanded in on-shell Fourier modes in the usual way:

$$\hat{\phi} = \int d\tilde{k}\{a_k e^{i\vec{k}\cdot\vec{x}} + a_k^\dagger e^{-i\vec{k}\cdot\vec{x}}\}, \tag{4.2}$$

where $d\tilde{k} = d^3k/(2\pi)^3 2\omega_k$. Since we work in the interaction picture, the operators a_k and $a_k^\dagger$ are time dependent (i.e., $a_k(t) = a_k(0)e^{i\omega_k t}$). The creation and annihilation operators also satisfy the usual commutation relation

$$[a_k, a_{k'}^\dagger] = (2\pi)^3 2\omega_k \delta^3(k - k'). \tag{4.3}$$

Consider the coherent state $|\eta\rangle$ given by

$$|\eta\rangle = C_N \exp\{\int d\tilde{k}\eta(k)a_k^\dagger\}|0\rangle, \tag{4.4}$$

where $\int d\tilde{k}|\eta(k)|^2 < \infty$, and $C_N = \exp\{-\frac{1}{2}\int d\tilde{k}|\eta(k)|^2\}$ is chosen so that $|\eta\rangle$ has a norm of one. It is easy to verify that

$$\langle\eta|\hat{\phi}|\eta\rangle = \int d\tilde{k}\{\eta(k)e^{i\vec{k}\cdot\vec{x}} + \eta^*(k)e^{-i\vec{k}\cdot\vec{x}}\}, \tag{4.5}$$

so that by proper choice of $\eta(k)$ one can ensure that $\langle\eta|\hat{\phi}|\eta\rangle$ has the desired form.

The average number of particles in the coherent state, $\overline{N} = \langle\hat{N}\rangle$, is

$$\langle\eta|\hat{N}|\eta\rangle = \langle\eta|\int d\tilde{k}\, a_k^\dagger a_k|\eta\rangle = \int d\tilde{k}\,|\eta(k)|^2, \tag{4.6}$$

so $C_N = \exp\{-\frac{1}{2}\overline{N}\}$. The coherent state has energy

$$\langle\eta|\hat{H}|\eta\rangle = \langle\eta|\int d\tilde{k}\,\omega_k a_k^\dagger a_k|\eta\rangle = \int d\tilde{k}\,\omega_k|\eta(k)|^2. \tag{4.7}$$

Fluctuations in $\langle\eta|\hat{H}|\eta\rangle$ can be defined by

$$\langle\eta|\hat{H}^2|\eta\rangle = \langle\eta|\hat{H}|\eta\rangle^2 + \int d\tilde{k}\,\omega_k^2|\eta(k)|^2. \tag{4.8}$$

The classical limit is attained by choosing $\eta(k)$ so that the average number of particles $\overline{N}$ is large. As the number of particles is increased,

$$\frac{\langle\eta|\hat{H}^2|\eta\rangle - \langle\eta|\hat{H}|\eta\rangle^2}{\langle\eta|\hat{H}|\eta\rangle^2} \propto \frac{1}{\overline{N}}, \tag{4.9}$$

where $\overline{N}$ is the total number of quanta as defined in (4.6). Note that when $\overline{N}$ is small, the energy of the state $|\eta\rangle$ is not well defined, and undergoes large fluctuations compared to its mean value. This is to be expected as $\hat{H}$ and $\hat{\phi}$ do not commute. The semi-classical approximation used in section 2 requires that both $\langle\hat{\phi}\rangle$ (which determines the fermion mass) and $\langle\hat{H}\rangle$ (which gives the scalar field energy) be known. This is only possible in the classical limit where $\overline{N} \to \infty$. We can therefore conclude that when $\overline{N}$ is small, there is no reason to expect that the calculation of section 2 has any validity whatsoever.

Moreover, when $\overline{N}$ is small the non-classical nature of the coherent state is manifest since

$$|\eta\rangle = e^{-\frac{1}{2}\overline{N}}\{|0\rangle + \int d\tilde{k}\,\eta(k)|k\rangle + ...\}, \tag{4.10}$$

so that the coherent state is mostly vacuum with small amplitudes to be in an n particle state.

It is important to reiterate that there is no uncertainty whatsoever in this system as to the identity of the ground state. The ground state is clearly $|0\rangle$, upon which all excitations of ϕ and ψ are constructed. By 'turning on' a source term gradually, one sees that the resulting ϕ field is a superposition of scalar excitations above the vacuum. It is only when the scalar excitations consist of many quanta that a classical description of the ϕ field is valid.

Quantum fluctuations of the type considered above can also be understood as a finite volume effect. A measurement of the value of a field must always be taken in some smeared volume. The value of the field can only be determined to within fluctuations which are $\sim 1/L$, where L is roughly the size of the smeared volume. (This can easily be seen by considering the quantity $\langle\phi(x)\phi(x')\rangle$, which diverges for $x \to x'$ even in a free theory.) It is straightforward to show that for the dimple system, with $L \sim R_D$, the fluctuations $1/R_D$ are always larger than the classical shift in the vev, δv. Thus, the quantum fluctuations overwhelm the purported classical shift, and it is meaningless to ascribe a physical significance to a scalar dimple in a volume of size R_D^3.

Now we construct the exact solution to the scalar field equations in the presence of a normalizable, localized source which we can take to represent the fermion wave function in the dimple system. We can then rigorously show that, for perturbative coupling between the fermion and scalar, $\langle\hat{N}\rangle$ for this solution is less than one.

The time-independent equation of motion for the scalar field of (4.1) in the presence of a source $J(x) \equiv \psi^\dagger(x)\psi(x)$ is given by

$$(\nabla^2 - m^2)\phi(x) = gJ(x). \tag{4.11}$$

The coherent state corresponding to the solution of (4.10) is given by (4.4), with

$$\eta(k)/(2\omega_k) = -gJ(k)/(k^2 + m^2), \tag{4.12}$$

where $J(k)$ satisfies $J(x) = \int \frac{d^3k}{(2\pi)^3}\{J(k)e^{i\vec{k}\cdot\vec{x}} + h.c.\}$. The expectation value of the number operator is then

$$\overline{N} = 4g^2 \int d\tilde{k} \frac{|J(k)|^2}{(k^2+m^2)}. \tag{4.13}$$

Inserting for ψ our best ansatz (2.9), we can now directly calculate $\langle \hat{N} \rangle$. For $R_\psi m_\phi < 1/2$, the total number of quanta in the dimple is

$$\overline{N} = \frac{\alpha}{2\pi}\left[\ln(16/\xi^2) - \frac{23}{6}\right]\{1 + \mathcal{O}(\xi^2)\}, \tag{4.14}$$

where $\xi = R_\psi m_\phi$. We note that most of these quanta have wavelengths of $\sim 1/m_\phi$.

The apparent divergence for $\xi \to 0$ does not concern us, as it is just the standard infrared divergence due to long wavelength modes. This is the familiar result for the massless limit of a Yukawa field (i.e., a Coulomb field). Long wavelength modes are irrelevant to the question of whether the dimple should be treated classically. The relevant modes are those of wavelength $\lesssim R_\psi$. Therefore, for extremely small values of m_ϕ, we should impose an infrared cutoff $1/L$ in (4.13) to obtain the number of physically relevant quanta. This corresponds to placing the system in a box of size L. For $L \sim R_\psi$, it is easy to see that the number of quanta (4.14) is always less that α.

We note that by suitable choice of the source term $J(x)$, it is possible to obtain a valid classical state $\phi(x)$. Since $\overline{N}$ is proportional to $g^2|J(k)|^2$, it can be increased either by increasing g, or by increasing the overall magnitude of $J(x)$. The former corresponds to strong coupling, while the latter corresponds to increasing the number of fermions contained in the probability distribution $J(x) \equiv \psi^\dagger(x)\psi(x)$. The solutions obtained in that case correspond to many-particle nontopological solitons, as discussed in [2].

Although we have argued against the existence of a classical condensate surrounding the fermion, this does not imply that there is no vestige of collective behavior for non-zero g. In fact, the collective effects which do appear are well understood [6]. For non-zero coupling g, the lowest energy eigenstate of the Hamiltonian with fermion number one is just the unperturbed free fermion state 'dressed' by a cloud of scalar quanta. This 'dressing' can be understood simply as wave function and mass renormalization of

the fermion, yielding a correction to the fermion mass which is divergent. A prescription for subtracting this divergence (i.e., by addition of counterterms to the Lagrangian) is necessary to ensure a finite physical mass for the fermion. This *is* the mass of the physical (renormalized) fermion, which is in fact the lowest energy state of fermion number one.

5 Conclusion

To summarize, although a purely classical treatment of the scalar field suggests that the solution in which a dimple forms in the scalar field near a particle getting a mass is energetically favored, this is overshadowed by the fact that for the bubble solution found, the classical treatment is not valid. The classical computation of a quantity is only reliable when the fluctuations about the solution are small compared to the classical value. A necessary condition for the fluctuations to be small is that the deformations in a field must be made of many quanta. Unlike topological solitons in weak coupling, the dimple solution discussed in this chapter and in [1] is not well described by classical physics. For weak coupling, the collective effects of the interaction are understood in terms of renormalization.

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Appendix A

The Effective Potential

1 The Zero Temperature, One-Loop Effective Potential

Consider an ensemble of particles i, which receive a contribution to their mass from the vacuum expectation value of a scalar field ϕ. In the mass eigenstate basis, the unrenormalized one-loop contribution to the effective potential is

$$\Delta V_1 = \pm\left\{-\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left(-k^2+m^2(\phi)-i\epsilon\right)\right\}, \tag{A1.1}$$

where the $\pm$ is for bosons (fermions) respectively. Going to Euclidean space, introducing a cutoff Λ, and integrating we find

$$\Delta V_1 = \pm\frac{1}{32\pi^2}\left[\frac{1}{2}m^4(\phi)\ln\left(\frac{m^2(\phi)}{\Lambda^2}\right)-\frac{1}{4}m^4(\phi)+m^2(\phi)\Lambda^2\right]. \tag{A1.2}$$

We write the renormalized potential V_1 as the sum of the tree level potential $U(\phi)$, the one loop correction ΔV_1, and a counter term potential V_{ct}:

$$V_1 = U(\phi)+\Delta V_1(\phi)+V_{ct}(\phi). \tag{A1.3}$$

We denote the renormalized one loop correction by $\overline{V}_1 = \Delta V_1(\phi)+V_{ct}(\phi)$. At $\phi=\sigma$, we choose to impose renormalization conditions that preserves the tree level values of m_H

and σ:

$$\begin{aligned} i) \quad & \frac{d}{d\phi}\overline{V}_1 = 0, \\ ii) \quad & \frac{d^2}{d\phi^2}\overline{V}_1 = 0. \end{aligned} \tag{A1.4}$$

For a collection of particles with masses of the form $m^2(\phi) = \mu^2 + g\phi^2$, imposing the renormalization conditions above, we find *

$$V_1(\phi) = \sum_i \pm\frac{g_i}{64\pi^2}\left[m_i^4(\phi)\ln\left(\frac{m_i^2(\phi)}{m_i^2(\sigma)}\right) - \frac{3}{2}m_i^4(\phi) + 2m_i{}^2(\phi)m_i^2(\sigma)\right] \tag{A1.5}$$

plus terms independent of ϕ. We will have occasion to use two special cases of this formula. For a particle with mass $m(\phi) = \lambda\phi$, we recover the well known result

$$\overline{V}_1(\phi) = \pm\frac{\lambda^4}{64\pi^2}\left\{\phi^4\ln(\phi^2/\sigma^2) - \frac{3}{2}\phi^4 + 2\phi^2\sigma^2\right\}. \tag{A1.6}$$

Finally, for a particle with a large $SU(2)$ conserving mass $m^2(\phi) = M^2 + g\phi^2$, where $M^2 >> g\sigma^2$, the one loop contribution decouples as

$$V(\phi) = \pm\frac{g^3}{64\pi^2}\left(\sigma^4\phi^2 - \sigma^2\phi^4 + \frac{1}{3}\phi^6\right)M^{-2} + \mathcal{O}(M^{-4}). \tag{A1.7}$$

2 The Effective Potential of the Standard Model

For the tree level potential

$$U(\phi) = \frac{\lambda_0}{4}\left(\phi^2 - \sigma^2\right)^2, \tag{A2.1}$$

λ_0 is related to the Higgs mass by $m_H^2 = 2\lambda_0\sigma^2$. If we adopt the renormalization prescription $i)$ $V''(\sigma) = m_H^2$, and $ii)$ $V'(\sigma) = 0$, the relation between m_H and λ_0 will

*Strictly speaking, the tree level Higgs boson mass preserved by (A1.4) is the Higgs boson mass at zero Euclidean momentum. The formulas we obtain here will have corrections coming from running the mass from zero euclidean momentum up to m_H. This fact has been noted by the authors of reference 19 in chapter 1. The divergence in (A1.5) for Goldstone bosons, is an artifact of this running. This is because the Goldstone boson contribution gives an infinite running of the Higgs mass between $p^2 = m_H^2$ and zero euclidean momentum. Alternatively one could impose renormalization conditions at a different value of ϕ. In that case, a singularity in V'' would still exist for the reasons mentioned above, and parameters of the effective potential would be related to measurable quantities by both a running in field space and momentum space. In this appendix we work with Higgs Boson self couplings small enough that we can neglect scalar loops which contain Goldstone Bosons.

be preserved. In this case, for each degree of freedom, the one loop corrections to the effective potential is given by (A1.5) †

$$\begin{aligned}\overline{V}_1(\phi) = \pm\frac{1}{64\pi^2}\{&m^4(\phi)\ln\left(m^2(\phi)/m^2(\sigma)\right)\\ &-\frac{3}{2}m^4(\phi)+2m^2(\phi)m^2(\sigma)-\frac{1}{2}m^4(\sigma)\},\end{aligned} \tag{A2.2}$$

where the $\pm$ is for bosons (fermions) and $m(\phi)$ is the mass of the particle in the presence of a background field. Neglecting the Higgs doublet contribution to $\overline{V}_1$, the one-loop, zero temperature, effective potential for the physical Higgs scalar is given by the well known expression (see, for example, reference 18 of chapter 1)

$$V(\phi) = -\frac{1}{2}(\lambda+2B)\sigma^2\phi^2+\frac{1}{4}\lambda\phi^4+B\phi^4\ln(\phi^2/\sigma^2) \tag{A2.3}$$

plus terms independent of ϕ. Here ϕ, the physical Higgs scalar, has mass $m_H^2 = 2\lambda_0\sigma^2 = (2\lambda+12B)\sigma^2$, $\langle\phi\rangle = \sigma$, and $B = \frac{1}{64\pi^2\sigma^4}\left(6m_W^4+3m_Z^4-12m_t^4\right)$. When the system is in contact with a hot thermal reservoir, such as in the early universe, the effective potential for the Higgs boson must be modified to include the interactions between the Higgs field and the hot ambient plasma. The thermal one-loop corrections to the effective potential for the Higgs boson is just the free energy of the Bose-Einstein and Fermi-Dirac distributions of particles getting a mass from ϕ,

$$\begin{aligned}\Delta V_1(\phi,T) = &-\sum_F\frac{g_F T^4}{2\pi^2}\int_0^\infty dx\, x^2\ln\left(1+e^{-\sqrt{x^2+\beta^2 m_F^2}}\right)\\ &+\sum_B\frac{g_B T^4}{2\pi^2}\int_0^\infty dx\, x^2\ln\left(1-e^{-\sqrt{x^2+\beta^2 m_B^2}}\right),\end{aligned} \tag{A2.4}$$

where $m_{B(F)}$ is the mass of a boson (fermion) in the presence of a background field ϕ, $g_{B(F)}$ is the number of degrees of freedom, $\beta = 1/T$, and $(F)B$ denotes a sum over (fermions) bosons respectively. Expanding the argument of the logarithm and integrating, the integral equation for $\Delta V_1(\phi,T)$ can be written in terms of a sum. As shown in

† We have added a constant so that the one loop contribution to the cosmological constant vanishes at $\langle\phi\rangle = \sigma$.

appendix C,

$$\begin{aligned}\Delta V_1(\phi,T) = &\sum_F \frac{g_F T^4}{2\pi^2}\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(\beta m_F)^2 K_2(\beta m_F n)\right] \\ &-\sum_B \frac{g_B T^4}{2\pi^2}\left[\sum_{n=1}^{\infty}\frac{1}{n^2}(\beta m_B)^2 K_2(\beta m_B n)\right].\end{aligned} \tag{A2.5}$$

Since the modified Bessel function K_2 falls off exponentially for large values of its argument, the expression (A2.5) is well suited to numerical computation when m/T is large. In the high temperature limit, when $m(\phi)/T$ is small, equation (A2.4) can be expanded in powers of $m(\phi)/T$. Excluding the terms independent of ϕ, the result from reference 17 of Chapter I is

$$\begin{aligned}\Delta V_1(\phi,T) = &\sum_B g_B\left[\frac{1}{24}m_B^2T^2 - \frac{1}{12\pi}m_B^3T - \frac{m_B^4}{64\pi^2}\ln(m_B^2/c_BT^2)\right] \\ &+\sum_F g_F\left[\frac{1}{48}m_F^2T^2 + \frac{m_B^4}{64\pi^2}\ln(m_F^2/c_FT^2)\right] \\ &+\mathcal{O}\{\,m^6/T^2\,\} + \mathcal{O}\{\,m^6/T^2\ln(m/T)\,\},\end{aligned} \tag{A2.6}$$

where $\ln(c_B) = \frac{3}{2} + 2\ln 4\pi - 2\gamma \simeq 5.41$ and $\ln(c_F) = \frac{3}{2} + 2\ln\pi - 2\gamma \simeq 2.64$. ‡

Expanding to order T^0 in m/T and neglecting all one loop Higgs self interactions, we add (A2.3) and (A2.6) to obtain the one-loop, temperature dependent potential

$$\begin{aligned}V(\phi,T) &= \left[DT^2 - \left(\frac{1}{2}\lambda + B\sigma^2\right)\sigma^2\right]\phi^2 - E\phi^3T + \frac{1}{4}\lambda_T\phi^4 \\ &= D(T^2 - T_2^2)\phi^2 - ET\phi^3 + \frac{1}{4}\lambda_T\phi^4,\end{aligned} \tag{A2.7}$$

where the constants D, E, and λ_T are given in Chapter I. The absolute instability of the origin occurs when the temperature reaches T_2:

$$T_2^2 = \frac{(2\lambda + 4B)\sigma^2}{4D} = \frac{m_H^2 - 8B\sigma^2}{4D} \equiv \chi^2(m_t, m_H)m_H^2. \tag{A2.8}$$

As well as providing good qualitative behavior, the standard m/T expansion (A2.6) gives reliable quantitative results up to surprisingly large values of m/T. For what values of

‡This comes from Dolan and Jackiw, reference 17 of Chapter I, where the 2.64 was misprinted as 2.84; we thank David Brahm for bringing this to our attention.

m/T is the high temperature approximation accurate? From Figures A1 and A2 we see that the high temperature approximation agrees with the exact potential to better than 5 percent for $m/T < 1.6\,(2.2)$ for fermions (bosons). It is interesting to note that unless great accuracy is required, for any value of m/T, either the high temperature or a low temperature approximation can be used to obtain a simple formula for the free energy. For large values of x, the modified Bessel function K_2 has the asymptotic behavior

$$K_2(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left(1 + \frac{15}{8x} + ...\right). \tag{A2.9}$$

So, for large values of m/T, the temperature dependent contribution to the free energy is given by

$$\Delta V_1(\phi) = \left(\frac{m(\phi)}{2\pi T}\right)^{3/2} e^{-m(\phi)/T} \left(1 + \frac{15T}{8m(\phi)} + ...\right). \tag{A2.10}$$

For any value of m/T, the better of (A2.6) or (A2.10) will give a value for $\Delta V_1(\phi)$ which is good to better than ten percent.

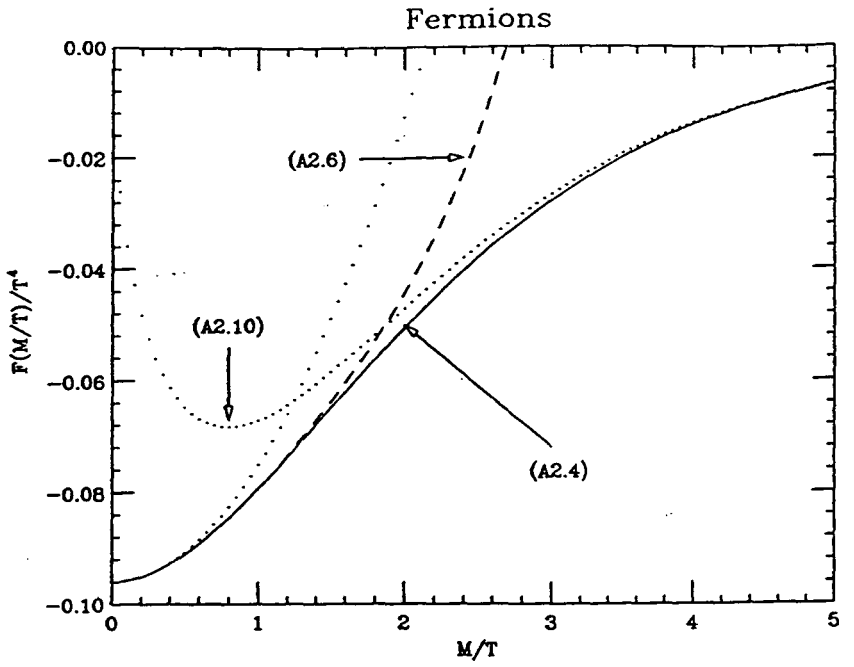

Figure A1: Exact and approximate relations for the free energy of a fermion.

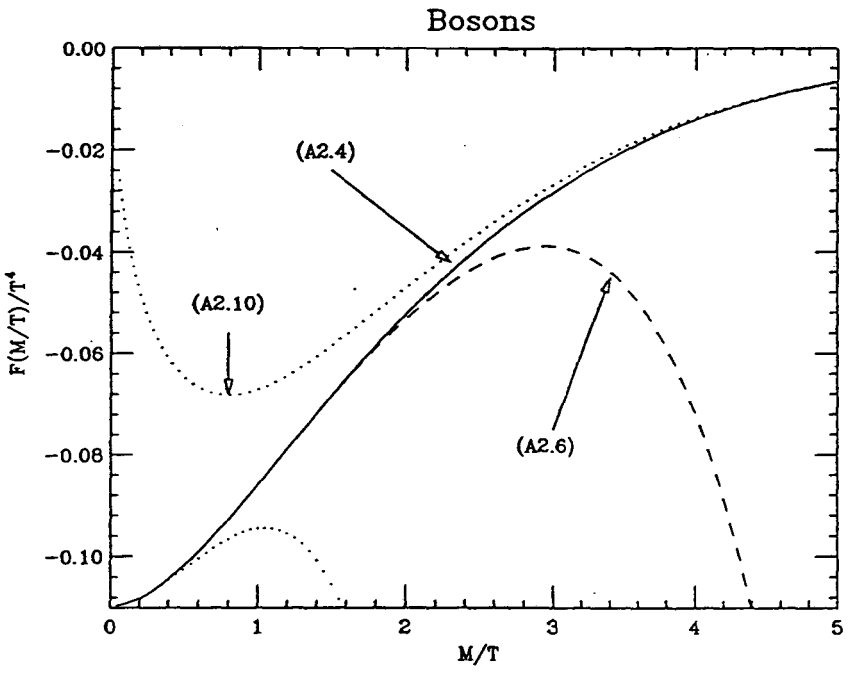

Figure A2: Exact and approximate relations for the free energy of a boson.

Appendix B

Fluctuations About the Saddle Point

1 Thermal Rates in a Simple Model

In order to develope some insight, we first consider the classical one dimensional barrier *hopping* problem, as shown in Figure B1. Following Affleck, reference 1 of Chapter I, the probability of being thermally excited out of the left well and over the barrier is a Boltzmann average of the flux:

$$\begin{aligned}\Gamma &= \langle \delta(x) p\theta(p) \rangle_\beta = \frac{\int dp\, dx\, e^{-\beta[\frac{1}{2}p^2+V(x)]}\delta(x)p\theta(p)}{\int dp\, dx\, e^{-\beta[\frac{1}{2}p^2+V(x)]}} \\ &\simeq \frac{\omega_o}{2\pi} e^{-\beta V_o}. \end{aligned} \tag{B1.1}$$

In order to extend this computation to a system with more than one degree of freedom, we relate this rate to the imaginary part of the partition function. First we note that, for an ensemble of particles localized in the left well the partition function is

$$\begin{aligned} Z &= \int dp\, dx\, e^{-\beta[\frac{1}{2}p^2+V(x)]} \\ &\simeq \int dp\, dx\, e^{-\beta[\frac{1}{2}p^2+\frac{1}{2}\omega_o^2]} \\ &\simeq \frac{2\pi}{\beta\omega_o}. \end{aligned} \tag{B1.2}$$

Next we induce an imaginary part by distorting the contour so that

$$\mathcal{Z} = \sqrt{2\pi/\beta}\left[\int_{-\infty}^{0} e^{-\beta V(z)}\, dz + \int_{0}^{-i\infty} e^{-\beta V(z)}\, dz\right]. \tag{B1.3}$$

Expanding $\mathcal{Z}$ about the barrier, we find

$$\mathcal{I}m\mathcal{Z} \simeq -\frac{\pi}{\beta\omega_-} e^{-\beta V_o}. \tag{B1.4}$$

Thus,

$$\Gamma \simeq -\frac{\omega_-}{\pi}\frac{\mathcal{I}m\,\mathcal{Z}}{Z}. \tag{B1.5}$$

We can add other degrees of freedom so the barrier becomes a saddle point instead of a maximum. In that case, the ratio of the partition function evaluated around the saddle point to the partition function around the left well, as well as and the Boltzmann average of the flux across the saddle, picks up a factor

$$\frac{\int dp_y dy e^{-\beta[\frac{1}{2}p_y^2+\frac{1}{2}\omega_y^2 y^2]}}{\int dp_y dy e^{-\beta[\frac{1}{2}p_y^2+\frac{1}{2}\omega_{oy}^2 y^2]}} = \frac{\omega_{oy}}{\omega_y}. \tag{B1.6}$$

So the relation (B1.5) is preserved. Equation (B1.5) will hold as we continue to add more degrees of freedom to the potential. For a system with an infinite number of degrees of freedom, we need to evaluate the partition function

$$Z = N\int [dx] e^{-\int_0^\beta [\frac{1}{2}(\frac{dx(\tau)}{d\tau})^2 + V(\overline{x})]d\tau}. \tag{B1.7}$$

where the integration is over all functions $x(\tau)$ satisfying $x(0) = x(\beta)$. We expand the Euclidean action about an extremum $\delta S = 0$. The extremal path satisfies $\ddot{\overline{x}} - V(\overline{x})' = 0$. Note that the $\overline{x}$'s solving this equation are not unique. We can write an arbitrary path in terms of a particular extremum, $\overline{x}$, and a variation away from the extremum, δx:

$$x(\tau) = \overline{x}(\tau) + \delta x(\tau). \tag{B1.8}$$

On the interval $(0, \beta)$ we can expand any periodic function $\delta x(\tau)$ in terms of orthonormal functions,

$$\delta x(\tau) = x_o + \sum_{1}^{\infty}[c_m x_m + c_n x_n]_\tau \tag{B1.9}$$

where $x_n = \cos(\frac{2\pi n\tau}{\beta})$, $x_m = \sin(\frac{2\pi m\tau}{\beta})$, and x_o is a constant. The measure $[dx]$ can be written

$$\begin{aligned}[dx] &= N(2\pi)^{1/2} dc_o \prod_{n=1}^{\infty} (2\pi)^{\frac{1}{2}} dc_n \prod_{m=1}^{\infty} (2\pi)^{\frac{1}{2}} dc_m \\ &\equiv N \prod_{i=-\infty}^{\infty} (2\pi)^{\frac{1}{2}} dc_i.\end{aligned} \tag{B1.10}$$

For static solutions, $\overline{x}$, our orthonormal functions are eigenfunctions of the equation:

$$-\ddot{x}_i + V''(\overline{x}) x_i = \lambda_i x_i. \tag{B1.11}$$

Expanding $V(x)$ to second order about a static solution $\overline{x}$, the Euclidean Action becomes

$$S = \beta V(\overline{x}) + \frac{1}{2} c_i^2 \lambda_i. \tag{B1.12}$$

Thus,

$$\begin{aligned}Z =& N \int \prod_{i=-\infty}^{\infty} (2\pi)^{\frac{1}{2}} dc_i e^{-\beta V(\overline{x})} e^{-\frac{1}{2} c_j^2 \lambda_j} = N e^{-\beta V(\overline{x})} \prod_{i=-\infty}^{\infty} \frac{1}{\sqrt{\lambda_i}} \\ =& N e^{-\beta V(\overline{x})} \left[det(-\partial_\tau^2 + V''(\overline{x}) \right]^{-\frac{1}{2}},\end{aligned} \tag{B1.13}$$

where

$$\lambda_n = \left[\left(\frac{2\pi n}{\beta} \right)^2 + \omega^2 \right]. \tag{B1.14}$$

For the potential of Figure B1 we expand $V(x)$ about $x(\tau) = x_0$ in Z and about $x(\tau) = x_{sp}$ in $\mathcal{Z}$. Recalling that we only integrated over half of the gaussian in our analytic continuation of $\mathcal{Z}$,

$$\frac{\mathcal{I}m\, \mathcal{Z}}{Z} \simeq \frac{1}{2} \mathcal{I}m \left[\frac{det\,(-\partial_\tau^2 + \omega_o^2)}{det\,(-\partial_\tau^2 + \omega^2)} \right]^{\frac{1}{2}} e^{-\beta V_o}. \tag{B1.15}$$

For expansion about the unstable point $V''(x_{sp}) = \omega^2 = -\omega_-^2$ from (B1.9-B1.11), we see there are always an odd number of negative eigenvalues. Evaluation of the determinant is performed in section B2, and the thermal diffusion rate is

$$\begin{aligned}\Gamma \simeq \frac{-\omega_-}{\pi} \frac{\mathcal{I}m\, \mathcal{Z}}{Z} &\simeq \frac{\omega_-}{2\pi} \mathcal{I}m \left[\frac{det\,(-\partial_\tau^2 + \omega_o^2)}{det\,(-\partial_\tau^2 + \omega^2)} \right]^{\frac{1}{2}} e^{-\beta V_o} \\ &\simeq \frac{\omega_-}{2\pi} \left[\frac{sinh(\frac{1}{2}\omega_o \beta)}{sin(\frac{1}{2}\omega_- \beta)} \right] e^{-\beta V_o}.\end{aligned} \tag{B1.16}$$

2 Evaluation of the Determinant

The determinant (B1.15) can be evaluated by the method of ζ functions. The determinant of an operator A can be written

$$det(A) = e^{-\zeta_A'(0)} = \prod_n a_n, \tag{B2.1}$$

where

$$\zeta_A(s) = \sum_n \lambda_n^{-s}. \tag{B2.2}$$

Noting that $cot(\pi z)$ has simple poles at $z = 0, \pm 1, \pm 2, ...$ with residue π^{-1}, we can make a Sommerfeld-Watson transformation

$$\zeta(s) = \frac{a^{-s}}{2i} \oint cot(\pi z) \left(z^2 + p^2\right)^{-s} dz, \tag{B2.3}$$

where $p = \omega\beta/2\pi$ and $a = (2\pi/\beta)^2$. Using the fact that the integrals above and below the contour are the same, and splitting $cot(\pi z)$ into two parts, we have

$$\begin{aligned} \zeta(s) &= 2a^{-s} \int_{C1} dz \frac{e^{i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \left(z^2 + p^2\right)^{-s} dz \ + a^{-s} \int_{C2} \left(z^2 + p^2\right)^{-s} dz \\ &= I_1(s) + I_2(s). \end{aligned} \tag{B2.4}$$

For the first integral we take the derivative with respect to s and note that the phase of the integrand changes by 2π across the branch cut (see Figures B2 and B3). So, we obtain

$$e^{-I_1'(0)} = 4sinh^2(\omega\beta/2)e^{-\omega\beta}. \tag{B2.5}$$

The second integral can be regularized by writing it in terms of a B function

$$I_2(s) = a^{-s}p^{1-2s}B(\frac{1}{2}, s - \frac{1}{2}), \tag{B2.6}$$

So that

$$e^{-I_2'(0)} = e^{\omega\beta}. \tag{B2.7}$$

The determinant is then

$$N\, det\left(-\partial_\tau^2 + \omega^2\right) = 4sinh^2(\omega\beta/2). \tag{B2.8}$$

Applying a different method than used here, Affleck and the authors of reference 11 in Chapter I find the same result.

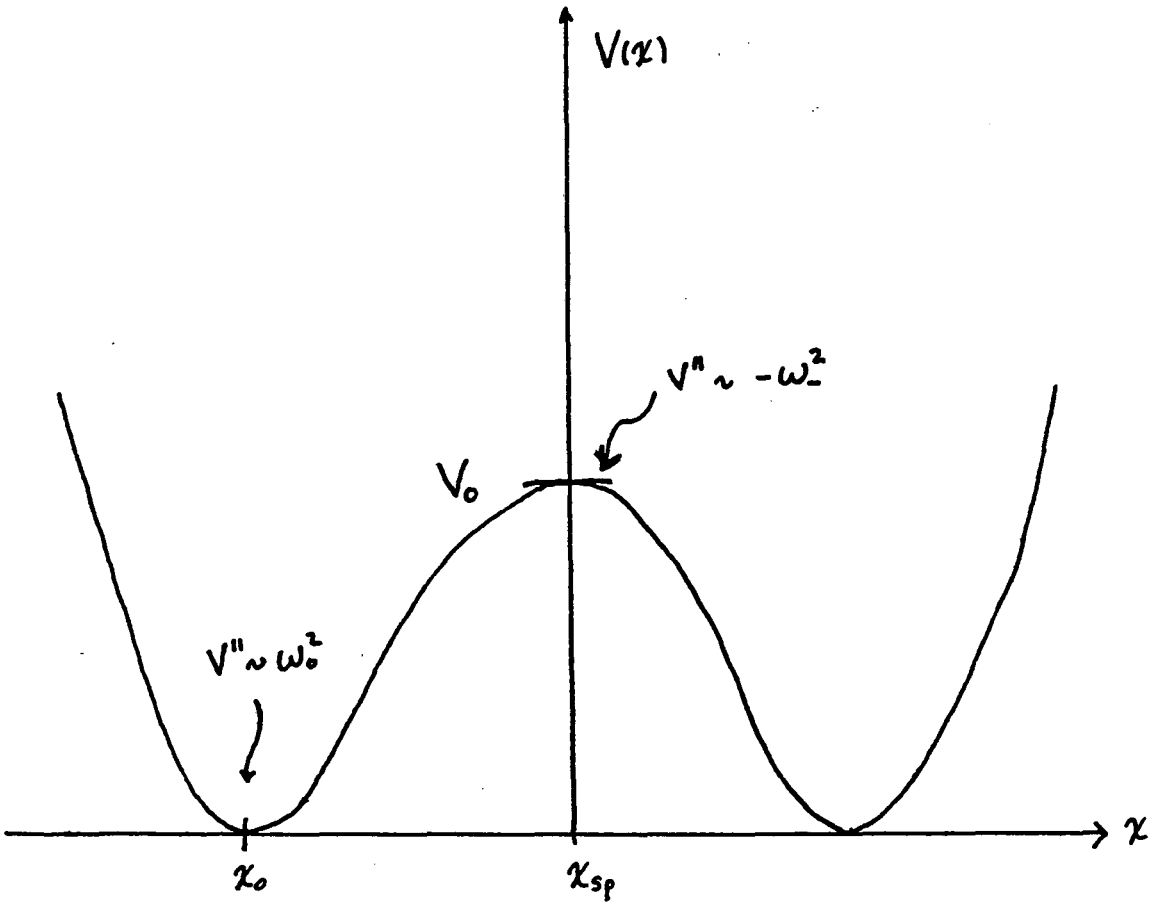

Figure B1: Thermal Diffusion Across a Double Well.

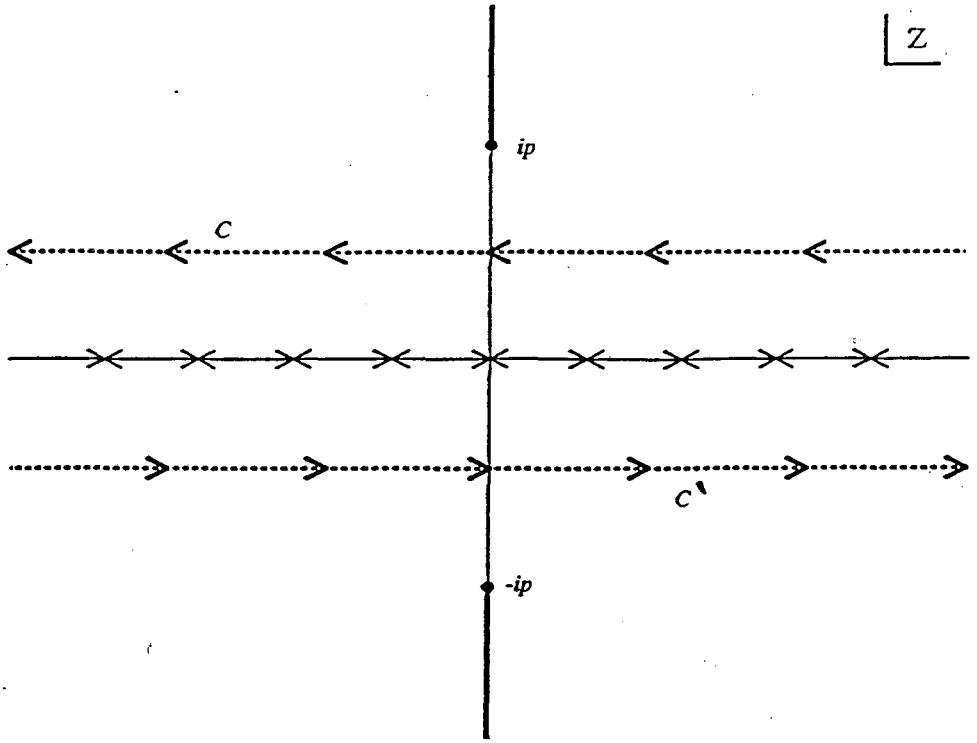

Figure B2: Contour for $\zeta(s)$

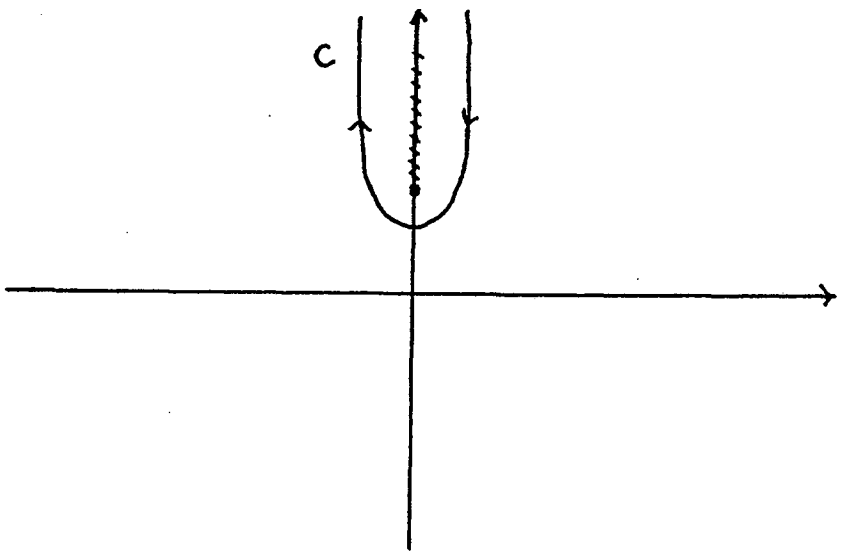

Figure B3: Distorted Contour for $\zeta(s)$

Appendix C

Evaluation of the Thermal Contribution

In this appendix we evaluate the one-loop, thermal contribution to the effective potential. Standard quantum field theory is set up to describe events that take place in a background of empty space. However, for significantly high temperatures thermal effects become important. To properly incorporate the interactions of the system with its surroundings, the background vacuum state should be replaced with a thermal bath. To one loop, the temperature dependent contribution to the effective potential is the free energy of a thermal distribution of ambient particles. Writing the full, one-loop effective potential as

$$V_1(\phi, T) = V_1(\phi) + \Delta V_1(\phi, T), \tag{C.1}$$

the thermal contributions to the effective potential, $\Delta V_1(\phi, T)$, are given by the well known formula

$$\Delta V_1(\phi, T) = \sum_i N_i \int_0^\infty dx\, x^2 \ln\left(1 \pm e^{-\sqrt{x^2 + M_i^2 \beta^2}}\right), \tag{C.2}$$

where the $+(-)$ is for fermions (bosons) respectively. $N_i = \mp \frac{g_i T^4}{2\pi^2}$, g_i is the number of degrees of freedom of particle i (i.e., $g_t = 12$, $g_w = 6$, $g_z = 3$), M_i is the mass of

particle i in the presence of a background field ϕ , and the $(+)-$ is for (fermions) bosons respectively.

Writing $a_i = M_i(\phi)/T$ and expanding the logarithm in powers of the exponential, we have

$$\Delta V_1(\phi, T) = \sum_i N_i \left\{ \sum_{n=1}^{\infty} \frac{(\pm 1)^{n+1}}{n} I_n(a_i) \right\}, \tag{C.3}$$

where the $\pm$ is for bosons (fermions) respectively. The integral I_n can be written

$$\begin{aligned} I_n(a) &= \int_0^\infty dx\, x^2 e^{-n\sqrt{x^2+a^2}} \\ &= -\frac{d}{da}\left\{ \frac{1}{an} \int_0^\infty dx\, \frac{x^4}{\sqrt{x^2+a^2}} e^{-n\sqrt{x^2+a^2}} \right\} \\ &\quad -\frac{d}{da}\left\{ \frac{a}{n} \int dx \frac{x^2}{\sqrt{x^2+a^2}} e^{-n\sqrt{x^2+a^2}} \right\}. \end{aligned} \tag{C.4}$$

This can be written in terms of the Bessel function K_2 :

$$\begin{aligned} I_n(a) = &-\frac{3}{an^3}\frac{d}{da}\left[a^2 K_2(an)\right] \\ &-\frac{1}{n^2}\frac{d}{da}\left[a^2 K_1(an)\right]. \end{aligned} \tag{C.5}$$

Hence,

$$I_n(a) = \frac{a^2}{n} K_2(an). \tag{C.6}$$

Finally, the one-loop, temperature dependent contribution to the effective potential can be written

$$\begin{aligned} \Delta V_1(\phi, T) = &-\sum_F N_F \left[\sum_{n=1}^{\infty} (-1)^{n+1} \frac{a_F^2 K_2(a_F n)}{n^2} \right] \\ &-\sum_B N_B \left[\sum_{n=1}^{\infty} \frac{a_B^2 K_2(a_B n)}{n^2} \right], \end{aligned} \tag{C.7}$$

where the $F(B)$ indicate sums over fermions (bosons).

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