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Metric Dependence of Partition Function at High Temperature

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Abstract

This paper is about thermal field theory in a background gravitational field (in space-time which is asymptotically Minkowskian). We use the analytically continued imaginary-time formalism in one-loop order, and restrict ourselves to high temperatures. An all-orders, but implicit, expression is given for the partition function.

1. Introduction

There has been quite a lot of work on hot quantum field theory in the presence of a gravitational field [1,2]. Much of this work has been concerned with the region

$$T \gg \omega, |\mathbf{k}|, m \tag{1}$$

where T is the temperature, and ω and $\mathbf{k}$ are a typical frequency and a typical wave-number of the gravitiational field, and m is the mass of the thermal particles. (Since thermal masses are normally $O(eT)$, where e is a coupling constant, there is such a region.) We assume that we may set $m = 0$. We believe that this is a justified approximation except near exceptional values of the graviton momenta, like $k^2 = 0$. In the region (1), the partition function, Z is proportional to T^4, and is a non-local functional of the metric [2,3]. We also assume that the chemical potential is negligible. In previous work [2,3], Z has been expanded in powers of the deviation of the metric $g_{\mu\nu}$ from the Minkowki metric $\eta_{\mu\nu}$. For instance, writing

$$g^{\mu\nu} = \sqrt{(-g)}(\eta^{\mu\nu} + \kappa\phi^{\mu\nu}), \tag{2}$$

the terms of order κ^2 are well known [2], and the terms of order κ^3 have been studied [3]. We are concerned to find an all-orders expression for the partition functional $Z(g)$, albeit in an implicit integral representation.

We work at zero chemical potential and neglect all masses. In order to have an absolutely clear meaning to thermal equilibrium, we assume that the space is asymptotically Minkowskian,

$$g^{\mu\nu} \sim \eta^{\mu\nu}, \tag{3}$$

and we choose the time-axis in the asymptotic Minkowski space to be the one defining the thermal equilibrium. We work to one-loop order in the thermal fields (which may be of spin 0, $\frac{1}{2}$ or 1). We use the analytically continued imaginary-time formalism (in which the condition $T \gg \omega$ makes sense), which is known to be closely connected to the real-time formalism [4].

The method we use is an extension of that in [5], in which the background field was a gauge field instead of gravity. First, using an idea of Barton's [6], we relate the partition function to a momentum integral of the forward-scattering amplitude for the thermal particle in the gravitational field. In the high-temperature limit we require the high-energy limit for the scattering amplitude. Then the relation is

$$Z(g) \sim a \int d\Omega J_L(g; \mathbf{Q})/\mathbf{Q}^2, \tag{4}$$

where J_L is the forward-scattering amplitude (spin-summed) for a high-energy particle of momentum $\mathbf{Q}$, and the angular integration is over the direction of $\mathbf{Q}$ (the right-hand side of (4) is independent of $|\mathbf{Q}|$). In d space-time dimensions,

$$a = (2\pi)^{1-d} T^d \Gamma(d)\zeta(d) \tag{5}$$

(with an extra factor $(1 - 2^{1-d})$ for fermions).

The next step (Section 2) is to show that J_L has the following five properties:-

(a) It is invariant under coordinate transformations which preserve (3).
(b) Under Lorentz transformations in the asymptotic Minkowski space, it is a covariant function of the null-vector Q^μ, where $Q^0 = |\mathbf{Q}|$.
(c) It has dimension $(\text{mass})^{2-d}$.
(d) It is a homogeneous function of Q of degree 2.
(e) The only nonlocalities in the perturbation expansion of J_L are line-integrals in the direction of Q; so that in momentum space the only denominators are of the form $(k.Q)^{-b}$ (b an integer), where k is a linear combination ef external graviton momenta.

Then, in Section 2, we argue that these properties are sufficient to determine J_L uniquely in terms of the $O(\kappa)$ contribution (which is very simple and well-known).

Finally, in Section 3, we display a form, (31), for J_L which satisfies the above five properties and which has the correct $O(\kappa)$ term; and which therefore, by the above arguments, must be the correct unique expression for J_L.

In Appendix B, we remark on the Weyl invariance of J_L, and on the trace identities which express this invariance.

2. The forward-scattering amplitude and its properties

The derivation of the general relationship between Z and the forward-scattering amplitude J [6] is exactly the same as in [5]. When we consider the high-temperature limit, there are new complications. In order to get the leading term (5), we must in general keep several high $|\mathbf{Q}|$ terms in J. We call the necessary asymptotic terms J_L. If we expand in powers of κ (using (2)), in the term $O(\kappa^n)$ we must keep terms which are homogeneous in

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Q of degree p where $p = 2, 3, ...2n$. The term with $p = 2$, which we call 'leading' is the one which actually gives (5). We will show that the other terms, which we call 'super-leading' cancel in the complete set of graphs.

Let us consider the first two values of n. For $n = 1$ (the one-graviton term) we have simply

$$J_L^{(1)} = w\kappa Q_\mu Q_\nu \int d^d x \phi^{\mu\nu}, \tag{7}$$

where ϕ is defined in (2) and w is a numerical weight factor depending on the spin and other quantum numbers of the thermal particles (see [5]).

For $n = 2$, a simple calculation gives

$$2J_L^{(2)} = w\kappa^2 (2\pi)^{-d} \int d^d x d^d y d^d k e^{-ik.(x-y)} C_{\mu\nu\alpha\beta}(k) \phi^{\mu\nu}(x) \phi^{\alpha\beta}(y), \tag{8}$$

where

$$2C_{\mu\nu\alpha\beta} = \frac{k^2 Q_\mu Q_\nu Q_\alpha Q_\beta}{(Q.k)^2} - \frac{Q_\mu Q_\nu Q_\alpha k_\beta + Q_\mu Q_\nu k_\alpha Q_\beta + Q_\mu k_\nu Q_\alpha Q_\beta + k_\mu Q_\nu Q_\alpha Q_\beta}{Q.k}. \tag{9}$$

To obtain this, one needs to expand the Feynman denominator:

$$\frac{1}{k^2 + 2Q.k} = \frac{1}{2Q.k} - \frac{k^2}{(2Q.k)^2} + ... \tag{10}$$

In this case there is a 'super-leading' term

$$\frac{Q_\mu Q_\nu Q_\alpha Q_\beta}{Q.k} \tag{11}$$

which cancels between the two graphs in Fig.1.

Note that (9) is in some ways more complicated than a previous form (see eq (2.33) of [3]), but this did not satisfy properties (a) and (b) above. Eq (9) is what is obtained using the relation to the forward-scattering amplitude. Of course the two forms give the same Z after the angular integration is done in (4).

As a first, simple example, let us show that (9) is uniquely determined by (7), given the five properties (a) to (e) of Section 1. Using properties (b) to (e) it is fairly easy to see that C in (8) must have the tensor structure

$$AQ_\mu Q_\nu Q_\alpha Q_\beta + B(Q_\mu Q_\nu Q_\alpha k_\beta + ...)$$
$$+C(Q_\mu Q_\nu \eta_{\alpha\beta} + Q_\alpha Q_\beta \eta_{\mu\nu}) + D(Q_\mu Q_\alpha \eta_{\nu\beta} + ...), \tag{12}$$

where ... denotes the obvious terms to symmetrize and A, B, C and D are dimensionless invariant functions of k^2 and $Q.k$. (In fact, C and D are constants.) As an example of the reasoning which leads to (12), the tensor structure $Q_\mu Q_\nu k_\alpha k_\beta$ is impossible because no coefficient satisfying (d) and (e) can make it dimensionless.

Now we use property (a), invariance under coordinate transformations. Then there is a Ward identity connecting (8) to (7). This (see (3.2) and (3.3) of [3]) is

$$2k^\mu C_{\mu\lambda\alpha\beta} - k_\lambda C_{\nu\mu\alpha\beta}\eta^{\mu\nu} = -Q_\lambda(k_\alpha Q_\beta + k_\beta Q_\alpha).$$

With the use of the trace identity in (B4), this may be rewritten

$$2k^\mu C_{\mu\lambda\alpha\beta} = -Q_\lambda(k_\alpha Q_\beta + k_\beta Q_\alpha) - k_\lambda Q_\alpha Q_\beta. \tag{13}$$

Inserting (12), the lefthand side of (13) yields

$$2(Bk.Q + D)Q_\lambda(Q_\alpha k_\beta + Q_\beta k_\alpha)$$
$$+2Ck.QQ_\lambda\eta_{\alpha\beta} + 2Dk.Q(Q_\alpha\eta_{\beta\lambda} + Q_\beta\eta_{\alpha\lambda}) + 2(Ak.Q + Bk^2)Q_\lambda Q_\alpha Q_\beta. \tag{14}$$

There are 5 independent tensors in (14), so the Ward identity suffices to determine the 4 coefficients in (12). In fact, since the Minkowski metric tensor does not appear on the right of the identity, $C = D = 0$. So the first term in (14) fixes B, and then the last gives A. The result is (9).

A particular consequence of this argument is that (12) can contain no 'super-leading' term. This is because the Ward identity connects terms of the same degree in Q, and (7) is of degree 2. In general, the same arguments which lead to the uniqueness of J_L show that it contains no 'super-leading ' terms.

We now sketch how the same type of argument can be used to show how the κ^3 term in J_L is determined by the Ward identity in terms of (9). We define

$$6J_L^{(3)} = w\kappa^3(2\pi)^{-2d} \int \prod_0^3 d^d x_i d^d k_i \delta^d(k_1 + k_2 + k_3) C_{\mu\nu\alpha\beta\rho\sigma}(k_i)$$
$$\times e^{-i(k_1x_1 + k_2x_2 + k_3x_3)} \phi^{\mu\nu}(x_1)\phi^{\alpha\beta}(x_2)\phi^{\rho\sigma}(x_3). \tag{15}$$

The most general form allowed for the 3-graviton function by properties (b) to (e) is

$$C_{\mu\nu\alpha\beta\rho\sigma} = AQ_\mu Q_\nu Q_\alpha Q_\beta Q_\rho Q_\sigma + B^1_\mu Q_\nu Q_\alpha Q_\beta Q_\rho Q_\sigma + (\mu \leftrightarrow \nu) + B^2_\alpha Q_\mu Q_\nu Q_\beta Q_\rho Q_\sigma + ...$$
$$+C^1_{\mu\nu} Q_\alpha Q_\beta Q_\rho Q_\sigma + ... + D^{12}_{\mu\alpha} Q_\nu Q_\beta Q_\rho Q_\sigma + ...$$
$$E(\eta_{\mu\nu}\eta_{\alpha\beta}Q_\rho Q_\sigma + ...) + E'(\eta_{\mu\nu}\eta_{\alpha\rho}Q_\beta Q_\sigma + ...) + E''(\eta_{\mu\alpha}\eta_{\nu\beta}Q_\rho Q_\sigma + ...) + E'''(\eta_{\mu\alpha}\eta_{\nu\rho}Q_\beta Q_\sigma + ...)$$
$$+F^{12}_\mu \eta_{\alpha\beta} Q_\nu Q_\rho Q_\sigma + ... + G^{12}_\mu \eta_{\nu\alpha} Q_\beta Q_r Q_\sigma + ... + H^1_\mu \eta_{\alpha\sigma} Q_\nu Q_\beta Q_\rho + ...$$
$$+I^1 \eta_{\mu\nu} Q_\alpha Q_\beta Q_\rho Q_\sigma + ... + K^{12} \eta_{\mu\alpha} Q_\nu Q_\beta Q_\rho Q_\sigma + ... \tag{16}$$

Here E, E', E'', E''' are constants, A, I^i and K^{ij} are scalar functions of Q and the external momenta, $B^i_\mu, F^{ij}_\mu, G^{ij}_\mu, H^i_\mu$ are vector functions of Q and the external momenta not proportional to Q_μ, and $C^i_{\mu\nu}, D^{ij}_{\mu\nu}$ are tensor functions containing neither Q_μ nor

and hence by symmetry all the B^i. Finally we are left with the maximum number of Q's. The term containing $Q_\alpha Q_\beta Q_\rho Q_\sigma Q_\lambda$ arises only from the A and B^1 terms on the right of (17) (and not from the right side), and implies that

$$A = \frac{k_1^2 k_2.k_3}{4(k_1.Q)^2 k_2.Q k_3.Q} + (\text{cyclic permutations}). \tag{22}$$

Thus $J_L^{(3)}$ is given by (15), (16), (19), (20), (21) and (22). Of course. it could alternatively be found by starting from its definition in terms of a forward scattering amplitude, and making expansions like (10) (but now to one more term) to find the high-energy limit. The point of deriving it from the Ward identities is to demonstrate that these are sufficient to fix it uniquely.

We believe that this type of argument would work for the higher terms, though it would be very long to write out in detail. The basic point is that any tensor allowed by properties (b) to (e) must contain at least two Q vectors, and so will contribute at least one term to the Ward identity in which a Q in the tensor is contracted with k, the remainder of the tensor being intact. Thus all the coeficients will be determined. If the terms are considered in order of increasing numbers of Q vectors, then each term will fix one further type of coefficient.

It is interesting that none of the terms (9), (19), (20), (21), (22) depends explicitly on the Minkowski metric η or on the dimension d of space-time. We have not found a simple explanation of this fact.

3. The Partition Function

We have argued that there is a unique functional $J_L(\mathbf{Q}, g)$ with the properties (a) to (e) of Section 1, and with the correct one-graviton component (7). We now display such a functional. In order to do so, we must start with some definitions. Given a space-time point x, we define a null-geodesic, $y(\theta, x)$ through it (where θ is a parameter along the geodesic) to have the properties

$$y^\mu(0, x) = x^\mu, \qquad \frac{dy^\mu}{d\theta} \to Q^\mu \quad \text{as} \quad \theta \to -\infty. \tag{23}$$

(Note this is possible since we are assuming (3).) We further define y to satisfy the geodesic equation in the form

$$\frac{d^2 y^\mu}{d\theta^2} + \Gamma^\mu_{\alpha\beta}(y)\frac{dy^\alpha}{d\theta}\frac{dy^\beta}{d\theta} = 0, \tag{24}$$

which fixes the parameter θ up to a constant scale factor, which in turn is fixed in terms of Q by (23).

The choice $-\infty$ (rather than $+\infty$) in (23) is arbitrary. Strictly speaking, one should start by confining oneself to metrics such that

$$\frac{dy^\mu}{d\theta} \to Q^\mu \quad \text{as} \quad \theta \to +\infty$$

Q_ν. The .... denote the addition of as many terms as are necessary to symmetrize under $\mu \leftrightarrow \nu, \alpha \leftrightarrow \beta, \rho \leftrightarrow \sigma$, and under the permutations of

$$(\mu\nu, 1) \quad (\alpha\beta, 2) \quad (\rho\sigma, 3)$$

The important point about (16) is that there are at least two vectors Q in each term, otherwise condtions (b) to (e) cannot be satisfied.

Now consider the Ward identity obtained by relating (15) to (8) (again simplified by using (B5)):

$$2k^\mu C_{\mu\lambda\alpha\beta\rho\sigma}(k, k', k'') = [-k_\alpha C_{\lambda\beta\rho\sigma}(k'') - (\alpha \leftrightarrow \beta) + k'_\lambda C_{\alpha\beta\rho\sigma}(k'')] + [k', \alpha, \beta \leftrightarrow k'', \rho, \sigma]. \tag{17}$$

We claim that there are enough independent tensors in this identity to fix (16) uniquely. For example, the contribution to the lefthand side of (17) from the C^i terms in (16) is

$$2k^\mu C^1_{\mu\lambda} Q_\alpha Q_\beta Q_\rho Q_\sigma + 2k.Q C^2_{\alpha\beta} Q_\lambda Q_\rho Q_\sigma + 2k.Q C^3_{\rho\sigma} Q_\alpha Q_\beta Q_\lambda. \tag{18}$$

First we show that all the terms in (16) containing the Minkowski metric η are zero. This is because there are no such terms on the right of the identity (there are none in (9)). In using the identity, it is convenient to consider terms with least number of Q vectors first, and then go to terms with increasing numbers of Q's. Thus the term in the identity proportional to $\eta_{\alpha\beta}\eta_{\rho\sigma}Q_\lambda$ gives $E = 0$, $\eta_{\alpha\beta}\eta_{\rho\lambda}Q_\sigma$ gives $E' = 0$, $\eta_{\alpha\sigma}\eta_{\beta\rho}Q_\lambda$ gives $E'' = 0$. $\eta_{\alpha\lambda}\eta_{\beta\rho}Q_\sigma$ gives $E''' = 0$. Then the term containing $\eta_{\alpha\beta}Q_\rho Q_\sigma$ gives $F^{12}_\lambda = 0$ and hence by symmetry all $F^{ij}_\mu = 0$, $\eta_{\rho\alpha}Q_\beta Q_\lambda$ gives $G^{32}_\sigma = 0$ and hence all $G^{ij}_\mu = 0$. Next, the term containing $\eta_{\sigma\alpha}Q_\beta Q_\rho$ gives $H^1_\lambda = 0$ and hence all $H^i_\mu = 0$. Last, the term containg $\eta_{\alpha\beta}Q_\rho Q_\sigma Q_\lambda$ gives $I^2 = 0$ and hence all $I^i = 0$, and the term with $\eta_{\alpha\rho}Q_\beta Q_\sigma Q_\lambda$ gives $K^{23} = 0$ and hence all $K^{ij} = 0$.

All that remain now are the A, B, C, D terms in (16). Again we begin with the terms with the fewest Q's, ie the C and D terms. There is only one term on the left of the Ward identity containing $Q_\alpha Q_\beta Q_\lambda$ and no other Q vector, the one displayed in (16). Comparing with the right of (17) and using (9) ,this gives

$$C^3_{\rho\sigma} = \frac{k_{1\rho}k_{2\sigma} + k_{2\rho}k_{1\sigma}}{4Q.k_1 Q.k_2}, \tag{19}$$

and hence by symmetry all the C^i. (The other terms in (18), containing more Q vectors, will now be known when we come to consider the terms with more Q's in the identity.) The terms containing $Q_\alpha Q_\rho Q_\lambda$ and no other Q determine

$$D^{23}_{\beta\sigma} = \frac{k_{1\beta}k_{3\sigma}}{4k_1.Q k_3.Q} + \frac{k_{1\sigma}k_{2\beta}}{4k_1.Q k_2.Q}, \tag{20}$$

and hence all the D^{ij}. Next we go to terms containing one more Q. Terms containing $Q_\beta Q_\rho Q_\sigma Q_\lambda$ arise from B^2 and D^{12} in the left of (17), and imply

$$B^2_\alpha = -\frac{k_1.k_3 k_{2\alpha}}{4k_1.Q k_2.Q k_3.Q} - \frac{k_1^2 k_{3\alpha}}{4(k_1.Q)^2 k_3.Q} - \frac{k_3^2 k_{1\alpha}}{4(k_3.Q)^2 k_1.Q}, \tag{21}$$

also. Such would be the case if the fourier transform of the metric had support in a certain region R of k-space (see eq (10) of [7]). Then, in general, there would be several analytic continuations out of R. The choice in (23) corresponds to one such analytic continuation.

Having defined the geodesic y, we define $Y(x)$ by

$$y^\mu(x,\theta) - Q^\mu\theta \to Y^\mu \quad \text{as} \quad \theta \to -\infty. \tag{25}$$

The relation between x, y and Y is displayed schematically in Fig. 2.

The above equations may be solved iteratively, starting with the flat-space form

$$y^\mu = Q^\mu\theta + x^\mu, \qquad Y^\mu = x^\mu. \tag{26}$$

The first iteration then gives

$$Y^\mu = x^\mu + \int_{-\infty}^{0} d\theta \int_{-\infty}^{\theta} d\theta' \Gamma^\mu_{\alpha\beta}(Q\theta' + x)Q^\alpha Q^\beta + \ldots. \tag{27}$$

We may also invert the relation to get $x(Y)$, which we can likewise find iteratively. The first iteration gives

$$x^\mu = Y^\mu - \int_{-\infty}^{0} d\theta \int_{-\infty}^{\theta} d\theta' \Gamma^\mu_{\alpha\beta}(Y - Q\theta')Q^\alpha Q^\beta + \ldots \tag{28}$$

Note that, under an infintesimal co-ordinate transformation

$$x^\mu \to x^\mu + \varepsilon^\mu(x), \tag{29}$$

which has the property

$$\varepsilon^\mu \to 0 \tag{30}$$

at infinity (so that (3) is preserved), Y is invariant. Also, Y transforms as a 4-vector in the asymptotic Minkowski space, so that its index may be raised and lowered with the Minkowski metric η.

With these definitions, the form of J_L which we propose is

$$J_L = 2w \int d^dY \frac{\partial x^\mu}{\partial Y^\alpha} Q_\mu Q^\alpha. \tag{31}$$

To see that this is invariant under co-ordinate transformations, note that under (29)

$$\delta J_L = 2w \int d^dY \frac{\partial \varepsilon^\mu[x(Y)]}{\partial Y^\alpha} Q_\mu Q^\alpha = 0, \tag{32}$$

using (30).

The other properties, (b) to (e) of Section 1, are almost obvious from (31) and (24). The denominators $(k.Q)^{-b}$ come from the line integrals in the Q direction, like those in (28). These are the only non-localities.

Finally, we must check the one-graviton term in (31). This can be got from the first iteration (28):

$$\begin{aligned} w^{-1}J_L &\approx -2\int d^dY \int_{-\infty}^{0} d\theta \Gamma^\mu_{\alpha\beta}(Y - Q\theta)Q^\alpha Q^\beta Q^\mu \\ &\approx -\kappa \int d^{s+1}Y \int_{-\infty}^{0} d\theta \frac{\partial \phi^{\alpha\beta}(Y - Q\theta)}{\partial x_\mu} Q_\alpha Q_\beta Q_\mu \\ &\approx \kappa \int d^dY \int_{-\infty}^{0} d\theta \frac{\partial}{\partial\theta}[\phi^{\alpha\beta}(Y - Q\theta)]Q_\alpha Q_\beta \\ &\approx \kappa \int d^dY \phi^{\alpha\beta}(Y)Q_\alpha Q_\beta, \end{aligned} \tag{34}$$

in agreement with (7). Note that (31), unlike (32), is *not* zero, since $\frac{\partial x^\mu}{\partial Y^\alpha}$ does not tend to zero asymptotically in the Q direction.

We will now derive an alternative form for J_L, equivalent to (31), which is more useful for some purposes.

First we define

$$G_{\alpha\beta}(Y) \equiv g_{\mu\nu}[x(Y)]\frac{\partial x^\mu}{\partial Y^\alpha}\frac{\partial x^\nu}{\partial Y^\beta}. \tag{35}$$

At first sight, one might regard the transformation $x \to Y$ as a co-ordinate transformation, and G as the metric in the new system. However, G does not satisfy condition (3), so this interpretation should be used with caution. In Appendix A, we prove a property of G:

$$G_{\alpha\beta}Q^\beta = Q_\alpha. \tag{36}$$

It follows from (36) that

$$G_{\alpha\beta}Q^\alpha Q^\beta = 0. \tag{37}$$

The alternative form of (31) is

$$J_L = w \int d^dY \left(2\frac{\partial x^\mu}{\partial Y^\alpha}Q^\alpha Q_\mu - G_{\alpha\beta}Q^\alpha Q^\beta\right). \tag{38}$$

To see the usefulness of (38), we define

$$x^\mu \equiv Y^\mu + \Delta^\mu(Y), \quad g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}. \tag{39}$$

Then (38) gives

$$J_L = -w \int d^dY \left(h_{\alpha\beta} + \frac{\partial\Delta^\mu}{\partial Y^\alpha}\frac{\partial\Delta^\nu}{\partial Y^\beta}\eta_{\mu\nu} + 2h_{\mu\alpha}\frac{\partial\Delta^\mu}{\partial Y^\beta} + h_{\mu\nu}\frac{\partial\Delta^\mu}{\partial Y^\alpha}\frac{\partial\Delta^\nu}{\partial Y^\beta}\right)Q^\alpha Q^\beta. \tag{40}$$

There are two advantages of this form over (31). First, there is a practical advantage. In order to calculate J_L to $O(\kappa^2)$, we would require Δ to $O(\kappa^2)$ in (31), but only to $O(\kappa)$ in (40) (though $h_{\mu\nu}$ is required to $O(\kappa^2)$ in (40)).

The second advantage is more important. The term which has cancelled in (38) to give (40) is

$$\int d^dY \frac{\partial \Delta^\mu}{\partial Y^\alpha} Q_\mu Q^\alpha. \tag{41}$$

Terms like this could lead to ambiguities. As it stands, (41) looks as if it requires the values of Δ at infinity. To $O(\kappa)$ it has a well-defined value, as shown in (34). But to higher orders one needs to arrange cancellations between different terms in order to get a sensible result. This problem is avoided in (40).

The expression (31) for J_L has a simple geometrical interpretation, which we now describe. First, introduce light-cone coordinates in the asymptotic Minkowski space:

$$Y_\lambda = Y_\lambda^T + uQ_\lambda + vQ'_\lambda,$$

or

$$Y = (\mathbf{Y}^\mathbf{T}, v, u),$$

where Q' is a vector satisfying $Q'^2 = 0, Q.Q' = 1$ and where $Q.Y^T = Q'.Y^T = 0$. Now consider the curve given by

$$x(Y) = x(\mathbf{Y}^\mathbf{T}, v, u)$$

as u varies for fixed $\mathbf{Y}^\mathbf{T}, v$. From the relation between Y and x given in (23) and (25), it follows that this curve is the null geodesic satisfying

$$x_\lambda(\mathbf{Y}^\mathbf{T}, v, u) \sim Y_\lambda \text{ as } u \to -\infty,$$

that is it is asymptotic to the null line which has direction Q and which contains the point$(\mathbf{Y}^\mathbf{T}, v, 0)$ in the asymptotic Minkowski space. With this notation, (31) gives

$$J_L = 2w \int d^{d-2}\mathbf{Y}^\mathbf{T} dv \lim_{u\to\infty} [Q.x(\mathbf{Y}^\mathbf{T}, v, u) - v]. \tag{41}$$

Thus we have an integral, over all null geodesics which asymptotically have direction Q, of the difference between the v-coordinates at each end. The finiteness of the limit in (41) (i.e. the fact that the geodesics have direction Q at *both* ends) is a consequence of the assumption mentioned before (25).

In conclusion, then, the partition function at high temperature is given by (4), with any of the forms (31), (40) or (41) inserted for J_L.

Appendix A

Here we prove the relation (36). To this end, we define

$$f_\nu(\theta, Y) \equiv g_{\alpha\beta}(y)\dot{y}^\alpha \frac{\partial y^\beta}{\partial Y^\nu}, \tag{A1}$$

where y is regarded as a function of Y and θ, and dot denotes differentiation with respect to θ (for fixed Y). Then (36) is the same as

$$f_\nu(0, Y) = Q_\nu, \tag{A2}$$

and since $f_\nu(-\infty, Y) = Q_\nu$, we need to show that f_ν is independent of θ.

Differentiating (A1) with respect to θ gives

$$\dot{f}_\nu = \frac{\partial g_{\alpha\beta}}{\partial y^\rho}\frac{\partial y^\beta}{\partial Y^\nu}\dot{y}^\rho\dot{y}^\alpha + g_{\alpha\beta}\frac{\partial y^\beta}{\partial Y^\nu}\ddot{y}^\alpha + g_{\alpha\beta}\dot{y}^\alpha\frac{\partial \dot{y}^\beta}{\partial Y^\nu}. \tag{A3}$$

We now use the geodesic equation (18) to eliminate $\ddot{y}$ and (remembering that the geodesic y is null)

$$\frac{\partial}{\partial Y^\nu}[g_{\alpha\beta}\dot{y}^\alpha\dot{y}^\beta] = 0$$

to eliminate

$$\frac{\partial \dot{y}^\beta}{\partial Y^\nu}$$

and thus show that (A3) is indeed zero.

It is worth remarking that (36) looks a at first sight like a gauge-fixing condition, defining a 'light-cone' gauge. However, as remarked after (35), the transformation $x \to Y$ is not a legal transformation preserving the condition (3), so this interpretation is not right.

Appendix B

Here we discuss the invariance under the Weyl transformation

$$g^{\mu\nu} \to e^{2\sigma(x)}g^{\mu\nu} \tag{B1}$$

(where $\sigma \to 0$ at infinity) or

$$\sqrt{(-g)}g^{\mu\nu} \to e^{(2-d)\sigma(x)}g^{\mu\nu} \tag{B2}$$

From the point of view of the high-temperature limit or of the high-energy limit of the forward scattering amplitude, we expect invariance under (B2), since the thermal particles are effectively massless. Using (2) and (B2), this leads to the identity

$$(\eta^{\mu\nu} + \kappa\phi^{\mu\nu})\frac{\delta J_L}{\delta\phi^{\mu\nu}} = 0. \tag{B3}$$

In terms of (7) and the definitions (8) and (15), (B3) gives

$$C_{\mu\nu\alpha\beta}(k)\eta^{\alpha\beta} = -Q_\mu Q_\nu, \tag{B4}$$

$$C_{\mu\nu\alpha\beta\rho\sigma}(k, k', k'')\eta^{\rho\sigma} = -C_{\mu\nu\alpha\beta}(k) - C_{\mu\nu\alpha\beta}(k'). \tag{B5}$$

These trace identities may easily be verified from (9), (16), (19), (20), (21) and (22).

We now verify that (31) does have this Weyl invariance. The null geodesic y is invariant under (B1), but it is reparametrized and so, for an infintesimal Weyl transformation,

$$\delta Y^\alpha = Q^\alpha \epsilon, \tag{B6}$$

where ϵ is infintesimal. Then (using the formula for the variation of the inverse of a matrix)

$$\delta\left(\frac{\partial \Delta^\mu}{\partial Y^\alpha}\right) Q^\alpha Q_\mu = -\frac{\partial \Delta^\mu}{\partial Y^\beta}\frac{\partial \delta Y^\beta}{\partial x^\nu}\frac{\partial \Delta^\nu}{\partial Y^\alpha} Q^\alpha Q_\mu$$

$$= -Q_\mu \frac{\partial \Delta^\mu}{\partial Y^\beta} Q^\beta \frac{\partial \epsilon}{\partial x^\nu}\frac{\partial \Delta^\nu}{\partial Y^\alpha} Q^\alpha = -Q_\mu \frac{\partial \Delta^\mu}{\partial Y^\beta} Q^\beta \frac{\partial \epsilon}{\partial Y^\alpha} Q^\alpha. \tag{B7}$$

On the other hand, the Jacobian of the transformation from Y to $Y + \delta Y$ is, by (B6),

$$1 + \frac{\partial \epsilon}{\partial Y^\alpha} Q^\alpha. \tag{B8}$$

Thus, using (B7) and (B8), we see from (31) that $\delta J_L = 0$, as required.

Finally, we remark upon the fact that the gravitational interaction of a (massless) scalar field is Weyl invariant only of the term $\frac{1}{6}R\phi^2$ is included in the Lagrangian. However, we have verified that, at least to order κ^3, this term does not contribute to the leading high-temperature limit (4), (5).

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Figure captions

Fig. 1. The two forward-scattering graphs contributing to (9). The solid line is the thermal particle and the wavy lines represent the external gravitational field.

Fig. 2. A schematic representation of the quantities defined in Section 3. Y lies on the asymptote (in the direction of Q) to the null geodesic y through x. The geodesic is parameterized by θ.

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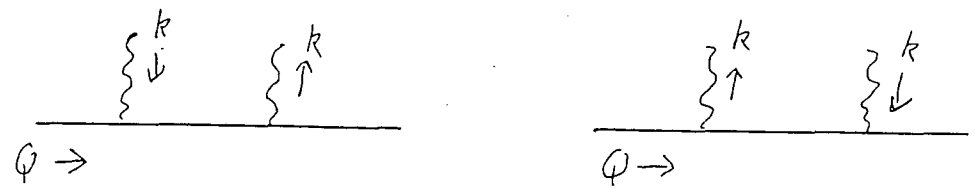

Fig. 1

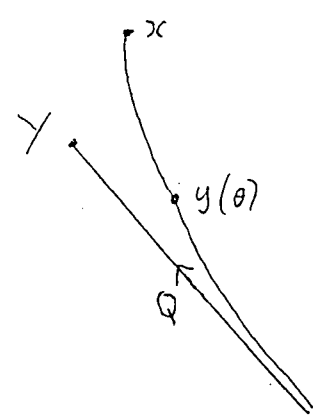

Fig. 2