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Space Experiments to Test Mathematical Results on Capillary Surfaces

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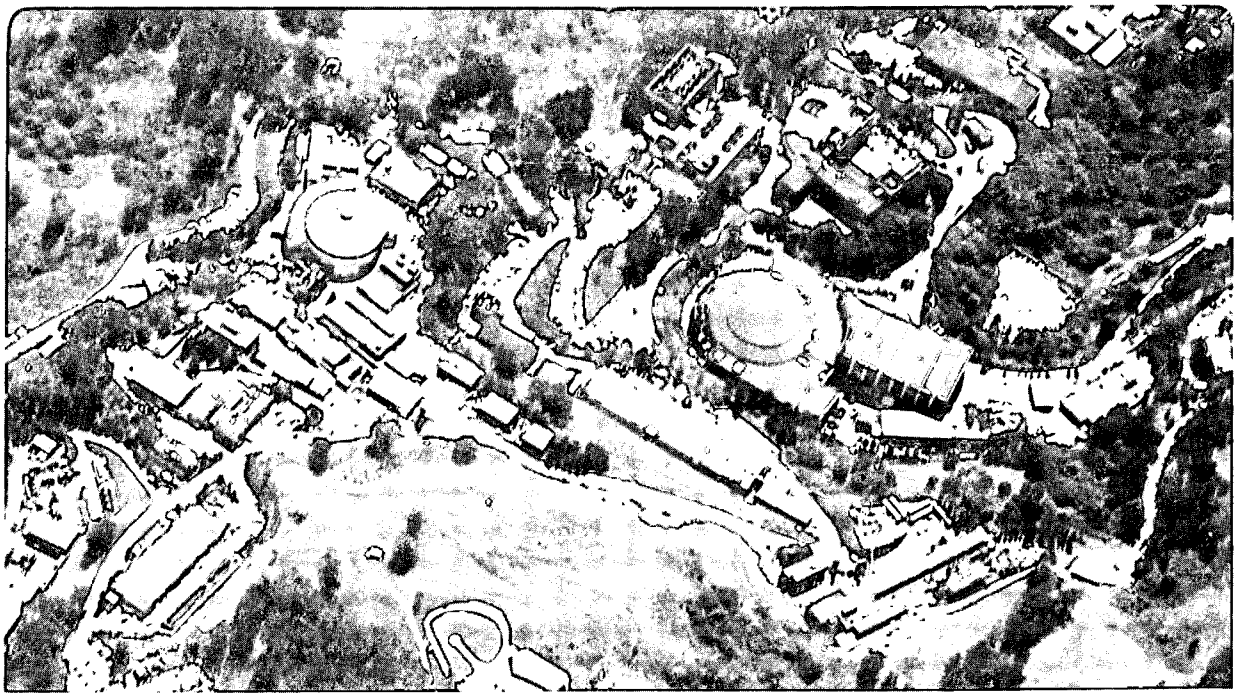
January 1992



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SPACE EXPERIMENTS TO TEST MATHEMATICAL RESULTS ON CAPILLARY SURFACES*

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SPACE EXPERIMENTS TO TEST MATHEMATICAL RESULTS ON CAPILLARY SURFACES

The behavior of capillary free surface interfaces can change qualitatively and sometimes discontinuously, depending on boundary conditions and on the presence or absence of a gravity field. Some of these effects are the basis for experiments on procedures for the precise determination of contact angle, to be conducted through the European Space Agency, and were discussed in a previous SIAM News article (June 1978). In the meantime, we have uncovered an instability phenomenon of a different kind that leads to a remarkable symmetry breaking. Preliminary drop-tower experiments made by Mark Weislogel at the NASA Lewis Zero Gravity Facility have contributed to the design of an apparatus for a detailed study of the phenomenon, which will be carried out in space within the next few months on the first NASA United States Microgravity Laboratory (USML-1) flight.

The study of equilibrium capillary surfaces was shown by Young, Laplace, and Gauss during the early 19th century to reduce (at least under idealized conditions) to a purely geometric problem: to find a surface S of which the mean curvature is a prescribed function of position in space and which meets prescribed rigid boundary walls in a prescribed constant angle γ. In modern notation, the equation for the surface becomes

$$\Delta \mathbf{x} = 2H\mathbf{N},$$

where Δ is the surface Laplacian (Laplace-Beltrami operator), $\mathbf{N}$ is the (appropriately oriented) unit normal to S, and $H > 0$ is the scalar mean curvature of S; in a gravity field,

$$2H = \kappa z + \lambda,$$

where κ is the capillarity constant (which vanishes at zero gravity), z is the surface height, and λ is a Lagrange parameter arising from the constraint of fixed liquid volume. These equations characterize the equilibrium surfaces corresponding to stationary values of the mechanical energy of the system; for stability, an additional requirement is that the configuration yield a local minimum for the energy.

Under conditions that were studied classically by the early investigators, these equations, together with the boundary condition that $\gamma =$ constant, completely determine an energy-minimizing surface (remarkably, the formal proofs of that basic fact were not obtained until quite recently). But it is not necessary to vary the conditions very much to encounter bifurcations and nonuniqueness. We consider some simple examples.

Of two axially symmetric containers, each partially filled with a given volume of liquid, such as in Figure 1, which configuration will be stabler under small disturbances? If you choose the convex container, you might well be seriously off-base. To put the matter in perspective, consider the container section shown in Figure 2. If we choose the materials so that $\gamma = \pi/4$, then the conical portion can be filled to any height h and the horizontal plane $z = h$ yields an explicit physical surface that satisfies the basic equations. This is the case for any gravity g. We first imagine the container filled from the bottom, until a height just below the altitude of the cone is reached, obtaining the flat surface indicated in the figure, with volume V_0 just below the volume of the cone. Having done that, we take an identical container and start by placing a significantly larger volume V of fluid into it, so that a meniscus is formed high up in the cylindrical part of the container. Because of the choice of contact angle, the meniscus must be curved, as indicated in the figure; its shape can be shown to be uniquely determined and independent of V, so long as the contact line lies on the cylinder. We now remove fluid until $V = V_0$, thus obtaining the two distinct configurations with the same volume and contact angle that are shown in the figure.

Such behavior is of interest in a number of contexts. If we consider a volume of fluid, covering the base of a circular cylindrical container closed at the bottom and making constant contact angle γ with the walls, then the shape of the meniscus is known to be determined uniquely and to be symmetric. The same volume of fluid would form a unique "drop" when resting on a horizontal plane that it meets with angle γ. It is not known whether the plane can be continuously deformed into the cylinder so as to preserve uniqueness throughout the deformation, but the example just considered shows that the uniqueness can easily be lost. On the other hand, Finn has shown that if the deformation is made so that the sense of boundary curvature on the contact line is always that of Figure 1a, then the configuration will yield at every step a local minimum for the mechanical

energy with respect to symmetric deformations; there is heuristic evidence to suggest that the energy is, in fact, locally minimized without regard to the symmetry of the deformation.

In the example of Figure 2, it is not immediately clear which, if either, of the two menisci for volume V_0 yields a lower energy. But *the example can be modified so as to produce an "exotic" container shape that yields an entire continuum of solution surfaces, all with the same volume, contact angle, and mechanical energy.* This was done first by Gulliver and Hildebrandt in the special case of zero gravity and contact angle $\pi/2$. We have shown that the phenomenon occurs for any gravity and contact angle and that the shape is always convex, in the sense of Figure 1b; moreover, *the particular horizontal surface in the family is always unstable, in the sense that a neighboring (asymmetric) surface can be found that yields lower energy.* Thus, depending on the specific container shape, convex configurations such as that shown in Figure 1b can turn out to be quite unstable.

If $g \neq 0$, then the exotic shape is completely determined and has been computed over the range of possible contact angles; in the earth's gravitational field, this shape has miniscule dimensions, on the scale of millimeters. Thus, if water is being carried in a pitcher in your house, there is little danger that it will spill of its own accord up out of the pitcher and onto the floor.

The minuteness of the exotic shapes in the earth's gravitational field preclude accurate observation and measurement of the phenomenon. If $g = 0$, however, the shapes can be scaled to any size without changing the effect. Even in a microgravity environment that is not strictly zero-gravity, length scales large enough for accurate physical observation and measurement are possible. *For $g = 0$, closed axially symmetric containers can be exhibited that admit entire continua of symmetric equilibrium solutions, but for which no energy-minimizing solution can be symmetric.* Several such containers modeled on this idea have been designed and built in collaboration with Weislogel for our forthcoming experiment on the USML-1 Glovebox Experiment Module. A typical container, corresponding to $\gamma = 60°$, is shown in Figure 3, along with members of the continuum of symmetric solution surfaces. These and other containers were subjected to preliminary drop-tower tests by Weislogel; in some of the tests the meniscus tended to "stick" to its initial symmetric shape, but in others it assumed a clearly asymmetric form in the five seconds of free fall, as shown

in Figure 4. The USML-1 experiment will provide a unique opportunity to observe such geometries over extended time intervals.

As a further verification of the theory, Michael Callahan used a modification of Kenneth Brakke's Surface Evolver program to seek locally stable surfaces numerically. The procedure yielded three distinct candidates with varying degrees of asymmetry. The one with the lowest energy looks remarkably similar to the configuration of Figure 4b. The others have two or three excursions from the top to the bottom of the bulge (and back), instead of just one.

Although the existence of capillary surfaces as solutions of the formal mathematical equations has been shown under varying and quite general conditions, and there can be no question of the physical existence of such entities, both theoretical and experimental investigations increasingly suggest that uniqueness is a relatively rare event, to be found only under very particular circumstances. Especially for zero-gravity situations, as encountered in space travel, it does not suffice for design requirements to determine only particular equilibrium surfaces, as many such surfaces may occur in practice. Even configurations that are locally stable may change with time into others with lower energy, due to evaporation from the original surface and condensation elsewhere (Kelvin instability). The examples discussed here show clearly that the most innocent-looking configurations (e.g., the horizontal surface in the exotic containers) can be deceptive and treacherous illusions of reality. In space travel the stabilizing effect of gravity, which leads to the free surface configurations to which we have all become accustomed in our daily lives, is no longer present; the behavior that we must learn to expect certainly will be at first strikingly (and in some ways disconcertingly) at variance with our everyday experience.

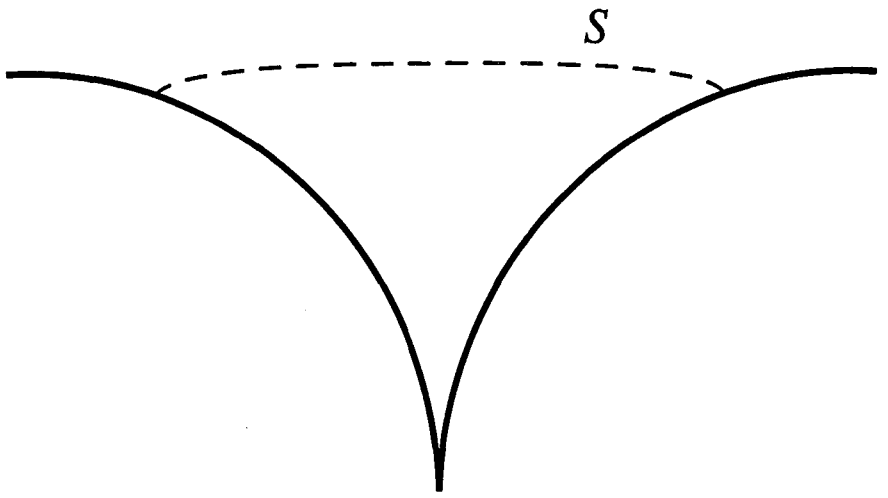


Figure 1a. Concave container section

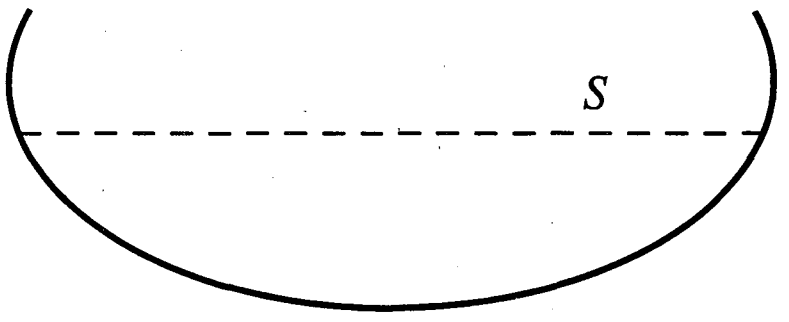


Figure 1b. Convex container section

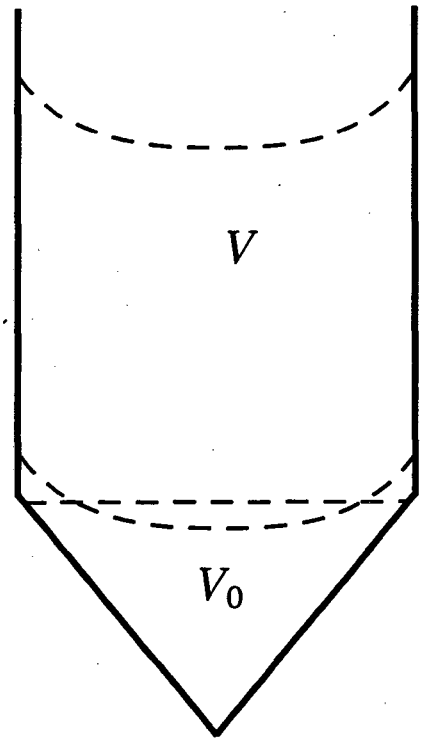


Figure 2. Nonuniqueness

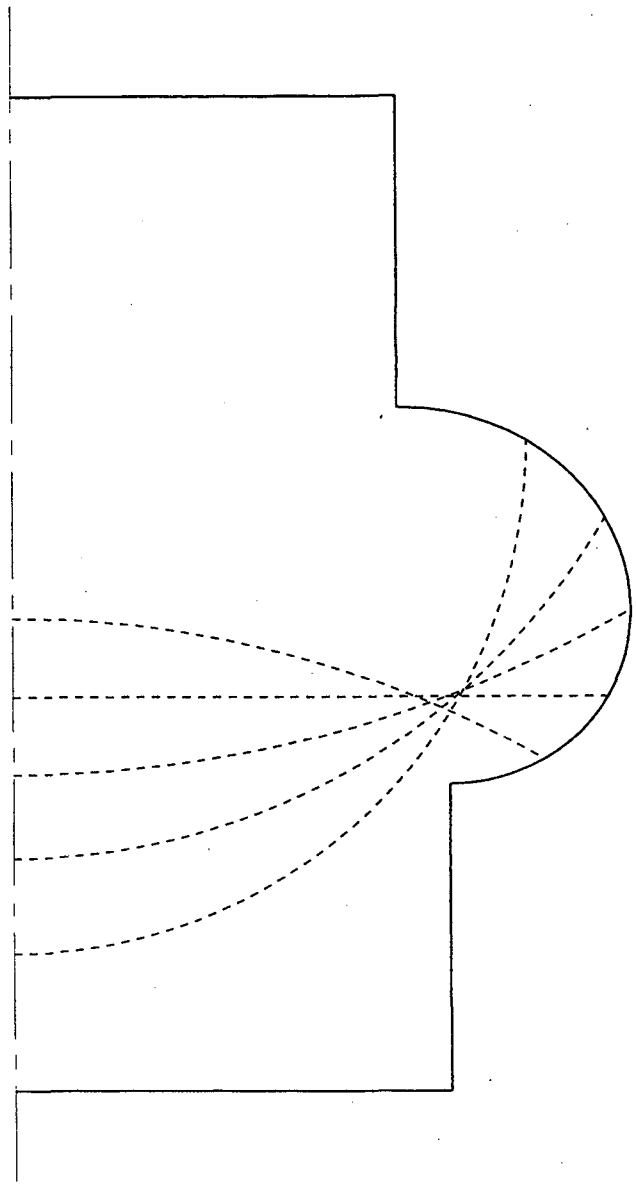
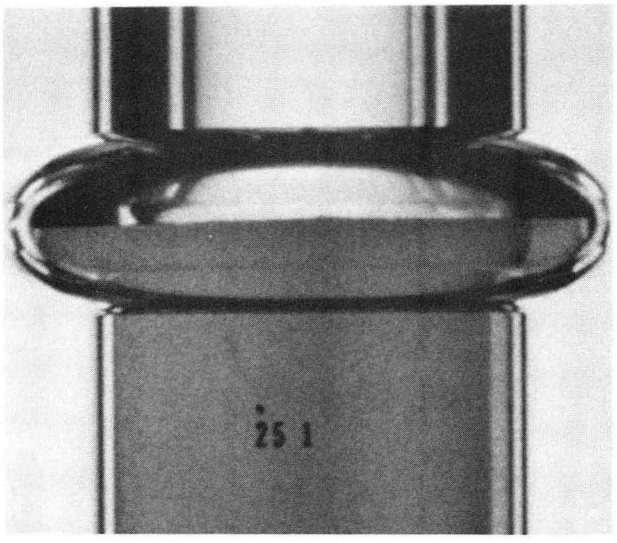


Figure 3. Radial section of a container (solid curve) with selected free surfaces (dashed curves) from the continuum having the same energy, contact angle, and enclosing the same liquid volume with the container. $\gamma = 60°$, $g = 0$.



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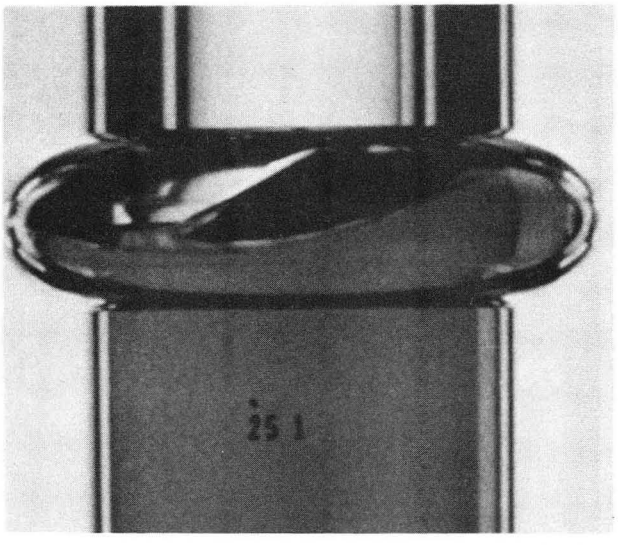

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Figures 4a-b. Drop-tower experiment: 4a. (upper) Flat interface initial configuration; 4b. (lower) Non-symmetric terminal configuration.

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