LBL-31840
Preprint

Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

Physics Division

Mathematics Department

To be submitted for publication

On the Complete Pivoting Conjecture for a Hadamard Matrix of Order 12

A. Edelman and W. Mascarenhas

February 1992

U. C. Lawrence Berkeley Laboratory
Library, Berkeley

FOR REFERENCE

Not to be taken from this room

LBL-31840
Copy 1
Bldg. 50 Library.

Prepared for the U.S. Department of Energy under Contract Number DE-AC03-76SF00098

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. Neither the United States Government nor any agency thereof, nor The Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or The Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or The Regents of the University of California and shall not be used for advertising or product endorsement purposes.

Lawrence Berkeley Laboratory is an equal opportunity employer.

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

LBL-31840

ON THE COMPLETE PIVOTING CONJECTURE FOR A HADAMARD MATRIX OF ORDER 12

Alan Edelman[1]
Department of Mathematics
and Lawrence Berkeley Laboratory
University of California
Berkeley, CA 94720

Walter Mascarenhas[2]
Institute for Mathematics and Its Applications
University of Minnesota
Minneapolis, MN 55455

February 1992

[1]Supported in part by the Applied Mathematical Sciences Subprogram of the Office of Energy Research, U.S. Department of Energy under Contract DE-AC03-76SF00098.

[2]On leave from the State University of Campinas, Brazil

On The Complete Pivoting Conjecture for a Hadamard Matrix of order 12

*Alan Edelman**
Department of Mathematics
and Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

Walter Mascarenhas†
Institute for Mathematics and Its Applications
University of Minnesota
Minneapolis, MN 55455

February, 1992

Abstract

This paper settles a conjecture by Day and Peterson that if Gaussian elimination with complete pivoting is performed on a 12 by 12 Hadamard matrix, then (1,2,2,4,3,10/3,18/5,4,3,6,6,12) must be the (absolute) pivots. By way of contrast, at least 30 patterns for the absolute values of the pivots have been observed for 16 by 16 Hadamard matrices. This problem is non-trivial due to the fact that row and column permutations do not preserve pivots. A naive computer search would require $(12!)^2$ trials.

1 Introduction

Wilkinson and Cryer's conjecture [2, 16] states that if A is a real n by n matrix such that $|a_{ij}| \leq 1$, then the maximum pivot encountered during

*Supported by the Applied Mathematical Sciences subprogram of the Office of Energy Research, U.S. Department of Energy under Contract DE-AC03-76SF00098.

†On leave from the State University of Campinas, Brazil.

the process of Gaussian elimination with complete pivoting, the so called "growth factor", is bounded by n. Given a choice of pivoting strategy, the growth factor of a matrix is defined as

$$g(A) = \frac{\max_{i,j,k} |A_{ij}^{(k)}|}{\max_{i,j} |A_{ij}|},$$

where $A^{(k)}$ is the matrix obtained from A after k steps of Gaussian elimination. In this paper, we consider only "complete-pivoting", meaning that at each step the element of largest magnitude (the "pivot") is moved by row and column exchanges to the upper left corner of the current sublock.

Recently Gould [6] published a 13 by 13 matrix that exhibited in the presence of roundoff error growth slightly larger than 13.0205. He also discovered a 16 by 16 matrix which numerically has growth larger than 18. Edelman [4] found that Gould's matrices could be perturbed so as to serve as true counterexamples in exact arithmetic, though in this case finding the perturbation was not easy. Gould's discoveries were obtained computationally using sophisticated numerical optimization techniques. The key mathematical problem of finding the largest growth possible remains unsolved.

The conjecture that $g(A) \leq n$ had been supported by the observation that an **Hadamard matrix**, i.e. a $n \times n$ matrix H with entries ± 1 and $HH^T = nI$, always had growth factor at least n. In fact, the last pivot must be exactly n for an Hadamard matrix. It has long been thought unlikely that any Hadamard matrix could have any pivots greater than n. Indeed Gould's examples are not Hadamard.

We now know that the conjecture is false, but we believe a weaker version of the conjecture formulated by Cryer: We believe that $g(H) = n$ if H is a Hadamard matrix. Here we take a small step towards proving this fact about Hadamard matrices by settling a conjecture by Day and Peterson [3] that only one set of absolute pivots is possible for a 12 by 12 Hadamard matrix. Husain [10] showed that there is only one Hadamard matrix of order 12 up to Hadamard equivalence. (Two matrices are Hadamard equivalent if they can be obtained from each other by row and column permutations and by row and column sign changes.) However, the pivot pattern is not an invariant under Hadamard equivalence. For example, many pivot patterns can be observed by permuting the rows and columns of any 16 by 16 Hadamard matrix.

2 Preliminary Notation and Lemmas

Hadamard matrices are highly structured. We collect here the important properties that we will need for our proof. To begin, it is well know that a Hadamard matrix of size $n = 4t$ is equivalent to a **symmetric block design** with parameters $v = 4t - 1$, $k = 2t - 1$, and $\lambda = t - 1$ [7]. A symmetric block design with parameters v, k and λ is a collection of v objects and v blocks such that there are

- k objects in each block,
- k blocks that contain each object,
- exactly λ objects in common to a pair of blocks and
- exactly λ blocks that contain any pair of objects.

We can interpret a Hadamard matrix as a symmetric block design by first negating rows and columns of H so that its leading row and column only contains positive ones. We then have a design on the objects 1 through $n-1$ by saying that i is a member of block k iff $H_{i+1,k+1} = +1$. In particular, a 12 by 12 Hadamard matrix is equivalent to an arrangement of 11 symbols into 11 blocks containing 5 objects such that each object appears in exactly 5 blocks, every pair of distinct objects appears together exactly twice, and every pair of distinct blocks has exactly 2 elements in common.

We say that a matrix A is **completely pivoted**, or CP, if the rows and columns have been permuted so that Gaussian elimination with no pivoting satisfies the requirements for complete pivoting. Following [3], let $A[k]$ denote the absolute value of the determinant of the lower right k by k principal submatrix of A, and $A(k)$ denotes the absolute value of the determinant of the upper left k by k principal submatrix. The determinant of a 0 by 0 matrix is defined here to be 1.

Definition 2.1 *Let A be CP, we define the kth pivot as*

$$p_k = A(k)/A(k-1).$$

The usual definition of the kth pivot is the value of A_{kk} after $k-1$ steps of Gaussian elimination. Our definition is equivalent to $|A_{kk}|$ which allows us to avoid the "$\pm$" symbol in every formula.

Lemma 2.1 *If H is an n by n Hadamard matrix, then*

$$n^{n/2}\, H(k) = n^k\, H[n-k].$$

Proof See [3], Proposition 5.2.

Corollary 2.1 *If H is an n by n Hadamard matrix, then the kth pivot from the end is*

$$p_{n+1-k} = \frac{nH[k-1]}{H[k]}.$$

Proof This follows immediately from the lemma and Definition 2.1.

Corollary 2.2 *If a Hadamard matrix H is CP and $k < n$, then, for all $(k-1) \times (k-1)$ minors M_{k-1} of the $k \times k$ lower right submatrix of H, we have $H[k-1] \geq |\det(M_{k-1})|$.*

Proof This follows from Corollary 2.1 and the CP property of H, for otherwise we could permute rows and columns of the lower right $k \times k$ minor of H to obtain a larger value for p_{n+1-k}.

This corollary is useful for telling us that $H[k-1]$ is the magnitude of the largest $(k-1) \times (k-1)$ minor of the $k \times k$ lower right submatrix of H. Thus $H[k-1]/H[K]$ is the largest element in magnitude of the inverse of the $k \times k$ lower right submatrix of H.

Lemma 2.2 *Let d_n denote the largest possible value of a determinant of an n by n matrix consisting of entries ± 1. The first seven values of the sequence (d_i) are $1, 2, 4, 16, 48, 160, 576$ and for $n = 2, \ldots, 7$ if the determinant of an n by n matrix is d_n, then the matrix must have an $n-1$ by $n-1$ minor whose absolute determinant is d_{n-1}. This is not true when $n = 8$.*

Proof The values of $d_1, \ldots, d_7$ were computed in [17] who further showed that up to equivalence there is only one n by n matrix with determinant d_n for $n = 2, \ldots, 7$. It is easy to verify that each of these matrices have an $n-1$ by $n-1$ minor with absolute determinant d_{n-1}. When $n = 8$, $d_8 = 4096$ while all 7 by 7 minors have determinant of magnitude 512.

Lemma 2.3 *If H is a CP Hadamard matrix, then $H(4) = 16$ so that the 4×4 principal subminor of H is an Hadamard matrix of order 4.*

Proof See [3], Proposition 5.8.

3 Pivot Sequence for H_{12}

In this section we prove our main result: the pivots for a CP 12×12 Hadamard matrix are (1,2,2,4,3,10/3,18/5,4,3,6,6,12). The first four pivots were determined by Day and Peterson [3] as given in Lemma 2.3. In Lemma 3.1 that follows, we show that the fifth pivot must be 3 from which the remaining pivots will be determined to be unique using Lemma 2.2.

Lemma 3.1 *If H is a 12×12 CP Hadamard matrix then $H(5) = 48$.*

Proof The argument is simplified if we consider the design interpretation of a Hadamard matrix so (without loss of generality) we assume that the first row and column of H are all $+1$, and also that the upper left 4×4 submatrix of H is a 4×4 Hadamard matrix (by Lemma 2.3) which can be given by the block design $B_1 = (1)$, $B_2 = (2)$, $B_3 = (3)$.

It is of no consequence to us in what order rows and columns 5 through 12 appear, it is sufficient that we show that some 5×5 minor of H with determinant 48 includes rows and columns 1 through 4. In fact, since by Lemma 2.2, 48 is the maximum value of the determinant of a 5×5 matrix of ± 1's, if there were such a minor M_5 with determinant 48 then, because of complete pivoting, we must have $48 = det(M_5) \leq H(5) \leq 48$, implying $H(5) = 48$. It is also of no consequence how rows and columns 2 through 4 are ordered among themselves.

We proceed to find conditions on four blocks in the symmetric design corresponding to H. Each block has five elements, each pair of blocks has two elements in common, and we are building from the upper left 4×4 submatrix of H. From these conditions, there is no loss in generality by labeling

$$B_1 = (1\ 4\ 5\ 6\ 7), \text{ and } B_2 = (2\ 4\ 5\ 8\ 9).$$

Now $B_1 \cap B_2 \cap B_3$ is either empty or consists of one object which we can call 5 without loss of generality. (There could not be three blocks containing the same pair.) This leads to only two distinct possibilities for B_1, B_2 and B_3 either

$$(1\ 4\ 5\ 6\ 7),\ \ (2\ 4\ 5\ 8\ 9),\ \ (3\ 5\ 6\ 8\ 10),$$

or

$$(1\ 4\ 5\ 6\ 7),\ \ (2\ 4\ 5\ 8\ 9),\ \ (3\ 6\ 7\ 8\ 9).$$

Let B_4 be a block that contains 1 and 2 but not 3. (There are two blocks that contain any pair such as 1 and 2, and they both could not contain 3 for

otherwise there would be too many elements in common.) It is easy to verify that B_4 can not contain a 4 for if it did, it would not be possible to choose the last two elements to be consistent with the either of the possibilities above. Thus B_4 contains 1 and 2 but not 3 and 4.

The information from B_1 through B_4 about the objects 1 through 4 tells us that we have a five by five minor that includes rows and columns 1 through 4 of the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 \end{pmatrix}.$$

This matrix has determinant 48 and thus $H(5) = 48$. □

Corollary 3.1 *If H is a 12×12 CP Hadamard matrix then*

$$p_5 = H(5)/H(4) = 3.$$

Theorem 3.1 *No matter how the rows and columns of a CP 12 by 12 Hadamard matrix H are ordered, the pivots must be 1, 2, 2, 4, 3, 10/3, 18/5, 4, 3, 6, 6, 12.*

Proof From Lemmas 3.1 and 2.1, it follows that $H[7] = 576$. Observe that, from Lemma 2.2, this is the maximum value attained by the absolute value of the determinant of a 7×7 matrix with entries ± 1. Lemma 2.2 also tell us that the 7×7 lower right corner has a 6×6 minor with maximal determinant 160. As a consequence of Corollary 2.2, $H[6] = 160$. Similarly, we conclude $H[5] = 4$, $H[4] = 16$, $H[3] = 4$, $H[2] = 2$, and $H[1] = 1$. The last seven pivots now follow from Corollary 2.1.

4 Hadamard matrices of order 16 and open problems

It is known (see [14]) that there are five equivalence classes of Hadamard matrices of order 16. Unfortunately, the pivot pattern is not an invariant of the equivalence class, and thus the number of equivalence classes may offer little useful information. We found over 30 possible pivot patterns for

Hadamard matrices of order 16, though not all patterns appeared for each equivalence class. However, we found the number of possible values for $H(k)$ to be quite small. For example the only values that appeared in practice for $H(8)$ were 1024, 1536, 2048, 2304, 2560, 3072, and 4096. For $H(7)$ the values that appeared were 256, 384, 512, and 576. It may be necessary to understand these possible values to prove that the growth factor for a 16 by 16 Hadamard matrix, must be 16.

An interesting conjecture by Day and Peterson [3] that the fourth from last pivot must be $n/4$ remains unsolved. We performed extensive experiments beyond those reported in [3] for a large variety of Hadamard matrices including some that were only discovered in the last seven years. We too strongly believe this conjecture, though we have not attempted to prove this at this time. Hadamard matrix problems sound tantalizingly easy, yet still the existence of relatively small Hadamard matrices ($n = 428$) is not known.

References

[1] A.M.Cohen, A note on pivot size in Gaussian elimination, *Lin. Alg. Appl. 8* (1974), 361–368.

[2] C.W.Cryer, Pivot size in Gaussian elimination, *Num. Math. 12* (1968), 335–345.

[3] J.Day and B.Peterson, Growth in Gaussian elimination, *Amer. Math. Monthly* (June 1988), 489–513.

[4] A.Edelman, Note to the editor, *SIAM J. Matrix Anal. Appl., 12* (1991).

[5] A.Y.Geramita and J.Seberry, *Orthogonal Designs*, Marcel Dekker, Inc., New York, 1979.

[6] N.Gould, On growth in Gaussian elimination with pivoting, *SIAM J. Matrix Anal. Appl., 12* (1991), 354–361.

[7] M.Hall, *Combinatorial Theory, 2nd ed.*, John Wiley, New York, 1986.

[8] N.J.Higham and D.J.Higham, Large growth factors in Gaussian elimination with pivoting, *SIAM J. Matrix Anal. Appl., 10* (1989), 155–164.

[9] N.Higham and N.Trefethen, Complete pivoting conjecture is disproved, *SIAM News 24* (January 1991), 9.

[10] Q.M.Husain, On the totality of the solutions for the symmetrical incomplete block designs: $\lambda = 2, k = 5$ or 6, *Sankhya 7* (1945), 204–208.

[11] L.Tornheim, Pivot size in Gauss reduction, Tech Report, Chevron Research Co., Richmond CA., 1964.

[12] L.Tornheim, Maximum pivot size in Gaussian elimination with complete pivoting, Tech Report, Chevron Research Co., Richmond CA., 1970.

[13] L.N.Trefethen and R.S.Schreiber, Average-case stability of Gaussian elimination, *SIAM J. Matrix Anal. Appl., 11* (1990), 335–360.

[14] W.D.Wallis, A.P.Street, and J.S.Wallis, *Combinatorics: Room Squares, Sum-Free Sets, Hadamard Matrices* Lecture Notes in Math. 292, Springer Verlag, 1970.

[15] J.H.Wilkinson, Error analysis of direct methods of matrix inversion, *J. Assoc. Comp. Machinery 8* (1961), 281–330.

[16] J.H.Wilkinson, *The Algebraic Eigenvalue Problem*, Oxford Univ. Press, London, 1965.

[17] J.Williamson, Determinants whose elements are 0 and 1, *American Math Monthly 53* (1946), 427–434.

LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
TECHNICAL INFORMATION DEPARTMENT
BERKELEY, CALIFORNIA 94720