May 1992

MPI-Ph/92-39
LMU-TPW 1992-6
UCB-PTh 92/16
LBL-32354

The Hopf Algebra of Vector Fields on Complex Quantum Groups

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Abstract

We derive the equivalence of the complex quantum enveloping algebra and the algebra of complex quantum vector fields for the Lie algebra types A_n, B_n, C_n and D_n by factorizing the vector fields uniquely into a triangular and a unitary part and identifying them with the corresponding elements of the algebra of regular functionals.

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1. Introduction

In [CDSWZ] we constructed the algebra of vector fields on complex quantum groups [CW, DSWZ, OSWZ, SWZ]. For the special case of A_1 we proved that the algebra of vector fields generates the algebra of regular functionals and coincides in some sense with $U_q sl(2, \mathbb{C})$ thus showing in particular that the algebra of vector fields admits a Hopf-structure. In this paper we want to prove that these facts generalize to A_n, B_n, C_n and D_n.

In the real case it is known that the Hopf algebra of regular functionals is generated by the vector fields which appear in the bicovariant differential calculus on quantum groups [CSWW, Jur, Wor, Zum]. This is proved in [Bur] using the fact that the matrices $L^{+i}{}_j$ and $L^{-i}{}_j$ generating the algebra of regular functionals are upper and lower triangular respectively. A similar procedure involving the Gram-Schmidt-orthogonalization scheme, also used for quantum groups in [JS], is applied here to the vector fields of the complex quantum groups [CDSWZ] leading to the result mentioned above that vector fields generate the algebra of regular functionals.

2. The Algebra of Vector Fields

Throughout this paper we are using the notations and conventions of [DSWZ].
Set $(I) := (i, \bar{i})$, $\bar{I} := (\bar{i}, \bar{\bar{i}}) = (\bar{i}, i)$, $(i, \bar{i} = 1, \ldots, N)$ where $N = n$ for A_{n-1}, $N = 2n+1$ for B_n and $N = 2n$ for C_n, D_n. Define then the $2N \times 2N-$matrix

$$(T^I{}_J) := \begin{pmatrix} t & 0 \\ 0 & \hat{t} \end{pmatrix} \tag{2.1}$$

and the $\hat{\mathcal{R}}-$matrix

$$(\hat{\mathcal{R}}_q^{IJ}{}_{KL}) := \begin{pmatrix} \beta\hat{R}_q & 0 & 0 & 0 \\ 0 & 0 & \beta\hat{R}_q & 0 \\ 0 & \beta^{-1}\hat{R}_q{}^{-1} & 0 & 0 \\ 0 & 0 & 0 & \beta^{-1}\hat{R}_q{}^{-1} \end{pmatrix} \tag{2.2}$$

with the corresponding $\hat{R}_q$-matrix [FRT] and with

$$\beta := \begin{cases} q^{-\frac{1}{n}} > 0 & \text{for } A_{n-1} \\ 1 & \text{for } B_n,\, C_n \text{ and } D_n. \end{cases} \tag{2.3}$$

For convenience we fixed the parameters α_i from [DSWZ] to the special values $\alpha_0 = \alpha_1 = \alpha_2^{-1} = \alpha_3^{-1} = \beta$.

We are considering the following quantum group

$$\mathcal{A} := \mathbb{C} < T^I{}_J > \Big/ ((2.5), (2.6), (2.7)) \ . \tag{2.4}$$

where the ideal is generated by

$$I^{IJ}_{ST} := \hat{\mathcal{R}}_q^{IJ}{}_{KL}\, T^K{}_S\, T^L{}_T - T^I{}_V\, T^J{}_W\, \hat{\mathcal{R}}_q^{VW}{}_{ST}\,, \tag{2.5}$$

$$\begin{aligned} &\det(t^i{}_j) - \mathbf{1} = \frac{(-1)^{n-1}}{[n]_q!} q^{-\binom{n}{2}} \varepsilon^{k_1 \ldots k_n}\, t^{l_1}{}_{k_1} \cdot \ldots \cdot t^{l_n}{}_{k_n}\, \varepsilon_{l_1 \ldots l_n} - \mathbf{1} \quad \text{for } A_{n-1}\,, \\ &t^i{}_s\, (C^{-1})^{sk}\, t^l{}_k\, C_{lj} - \delta^i_j \mathbf{1}, \quad (C^{-1})^{ik}\, t^l{}_k\, C_{ls}\, t^s{}_j - \delta^i_j\, \mathbf{1} \quad \text{for } B_n,\, C_n,\, D_n\,, \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\det(\hat{t}^{\bar{i}}{}_{\bar{j}}) - \mathbf{1} = \frac{(-1)^{n-1}}{[n]_q!} q^{-\binom{n}{2}} \varepsilon^{\bar{k}_1 \ldots \bar{k}_n}\, \hat{t}^{\bar{l}_1}{}_{\bar{k}_1} \cdot \ldots \cdot \hat{t}^{\bar{l}_n}{}_{\bar{k}_n}\, \varepsilon_{\bar{l}_1 \ldots \bar{l}_n} - \mathbf{1} \quad \text{for } A_{n-1}\,, \\ &\hat{t}^{\bar{i}}{}_{\bar{s}}\, (C^{-1})^{\bar{s}\bar{k}}\, \hat{t}^{\bar{l}}{}_{\bar{k}}\, C_{\bar{l}\bar{j}} - \delta^{\bar{i}}_{\bar{j}} \mathbf{1}, \quad (C^{-1})^{\bar{i}\bar{k}}\, \hat{t}^{\bar{l}}{}_{\bar{k}}\, C_{\bar{l}\bar{s}}\, \hat{t}^{\bar{s}}{}_{\bar{j}} - \delta^{\bar{i}}_{\bar{j}}\, \mathbf{1} \quad \text{for } B_n,\, C_n,\, D_n \end{aligned} \tag{2.7}$$

with $\varepsilon_{i_1 \ldots i_n} = (-1)^{n-1}\, \varepsilon^{i_1 \ldots i_n} = (-q)^{l(\sigma)}$, $l(\sigma)$ is the length (minimal number of transpositions) of the permutation $\sigma = \begin{pmatrix} 1 & \ldots & n \\ i_1 & \ldots & i_n \end{pmatrix}$, $[n]_q = (q^n - q^{-n})/(q - q^{-1})$, $[n]_q! = [1]_q \cdot \ldots \cdot [n]_q$ [CSWW] and C_{ij} is the usual metric [FRT].

With the involution

$$(t^i{}_j)^* := \kappa(\hat{t}^{\bar{j}}{}_{\bar{i}}) \tag{2.8}$$

$\mathcal{A}$ becomes a $*$-Hopf algebra with comultiplication Φ, counit e and antipode κ [DSWZ].

The dual space $\mathcal{A}^*$ of the Hopf algebra $\mathcal{A}$ is an algebra with the convolution product. One can introduce an antimultiplicative involution " $*$ " on $\mathcal{A}^*$: For $f \in \mathcal{A}^*$ one sets

$$\forall a \in \mathcal{A}: \quad f^*(a) := \overline{f(\kappa^{-1}(a^*))}\,. \tag{2.9}$$

We define functionals $L^{\pm I}{}_J \in \mathcal{A}^*$ through their action on the generators of $\mathcal{A}$:

$$\begin{aligned} L^{\pm I}{}_J(\mathbf{1}) &:= \delta^I_J\,, \\ L^{\pm I}{}_J(T^K{}_L) &:= \hat{\mathcal{R}}_q^{\pm 1\,IK}{}_{LJ} \end{aligned} \tag{2.10}$$

and their comultiplication

$$\forall a, b \in \mathcal{A}: \quad L^{\pm I}{}_J(ab) = L^{\pm I}{}_K(a)\, L^{\pm K}{}_J(b)\,. \tag{2.11}$$

The algebra $U_{\mathcal{R}}$ of regular functionals on $\mathcal{A}$ is the unital algebra generated by $\{L^{\pm I}{}_J\}$ [DSWZ]. It is shown in [DSWZ] that $U_{\mathcal{R}}$ is a $*$-Hopf algebra with comultiplication Δ, counit ϵ and antipode S.

Now we introduce the matrices

$$\begin{aligned} Y &:= L^+ S(L^-) = \begin{pmatrix} y & 0 \\ 0 & \hat{y} \end{pmatrix}, \\ Y^{-1} &:= L^- S(L^+) = \begin{pmatrix} y^{-1} & 0 \\ 0 & \hat{y}^{-1} \end{pmatrix} \end{aligned} \tag{2.12}$$

with the matrix entries $Y^I{}_J$ and $Y^{-1\,I}{}_J \in U_{\mathcal{R}}$. It follows from the commutation relations of $L^{\pm I}{}_J$ derived in [DSWZ] that

$$\hat{\mathcal{R}}_q(\mathbf{1} \otimes Y)\hat{\mathcal{R}}_q(\mathbf{1} \otimes Y) = (\mathbf{1} \otimes Y)\hat{\mathcal{R}}_q(\mathbf{1} \otimes Y)\hat{\mathcal{R}}_q\,. \tag{2.13}$$

The $Y^I{}_J$ have the comultiplication

$$\Delta(Y^I{}_J) = \Theta^{+IK}{}_{LJ} \otimes Y^L{}_K\,, \tag{2.14}$$

where

$$\Theta^{+IK}{}_{LJ} = L^{+I}{}_L\, S(L^{-K}{}_J)\,. \tag{2.15}$$

We set for any matrix M, $M^I{}_J \in U_{\mathcal{R}}$ the hermitian involution "$\dagger$" with $M^{\dagger\bar{I}}{}_{\bar{J}} := (M^J{}_I)^*$.

Using the involution properties of the $L^{\pm I}{}_J$ (see (3.13) of [DSWZ]) one obtains

$$Y^{\dagger J}{}_I = Y^{-1 J}{}_I. \tag{2.16}$$

For A_{n-1} it is possible to define an invariance relation for antisymmetric n–fold products of the $y^i{}_j$ and $\hat{y}^{\bar{i}}{}_{\bar{j}}$ (–from that one gets a modified determinant for the $y^i{}_j$ and the $\hat{y}^{\bar{i}}{}_{\bar{j}}$ resp.). By observing that the $t^i{}_k y^k{}_j$ behave like the $t^i{}_j$ under coaction of the group and have the same commutation relations we make the ansatz

$$\varepsilon_{i_1 \ldots i_n} t^{i_1}{}_{k_1} y^{k_1}{}_{j_1} \ldots t^{i_n}{}_{k_n} y^{k_n}{}_{j_n} = \varepsilon_{j_1 \ldots j_n} \det(ty) \tag{2.17}$$

Formula (3.23) from [CDSWZ] and the relation corresponding to (2.17) for the $t^i{}_j$ yield the following invariance relation

$$\begin{aligned} q^{-(n^2-n)} \varepsilon^{i_n \ldots i_1} \cdot \mathrm{e} &= (y)_1 \, (\hat{R}_{q\,2,1} \, (y)_1 \, \hat{R}_{q\,2,1}) \cdot \ldots \\ \ldots &\cdot (\hat{R}_{q\,n,n-1} \cdot \ldots \cdot \hat{R}_{q\,2,1} \, (y)_1 \, \hat{R}_{q\,2,1} \cdot \ldots \cdot \hat{R}_{q\,n,n-1})^{i_1 \ldots i_n}{}_{j_1 \ldots j_n} \varepsilon^{j_n \ldots j_1} . \end{aligned} \tag{2.18}$$

For the $\hat{y}^{\bar{i}}{}_{\bar{j}}$ one obtains analogously

$$\begin{aligned} q^{(n^2-n)} \varepsilon^{\bar{i}_n \ldots \bar{i}_1} \cdot \mathrm{e} &= (\hat{y})_1 \, (\hat{R}_q^{-1}{}_{2,1} \, (\hat{y})_1 \, \hat{R}_q^{-1}{}_{2,1}) \cdot \ldots \\ \ldots &\cdot (\hat{R}_q^{-1}{}_{n,n-1} \cdot \ldots \cdot \hat{R}_q^{-1}{}_{2,1} \, (\hat{y})_1 \, \hat{R}_q^{-1}{}_{2,1} \cdot \ldots \cdot \hat{R}_q^{-1}{}_{n,n-1})^{\bar{i}_1 \ldots \bar{i}_n}{}_{\bar{j}_1 \ldots \bar{j}_n} \varepsilon^{\bar{j}_n \ldots \bar{j}_1} . \end{aligned} \tag{2.19}$$

The matrices $\hat{R}_q^{\pm 1}{}_{l,l-1}$ are defined as in [FRT]. From (2.18) and (2.19) it is now possible to derive the modified determinant $\mathrm{Det}(y)$ and the $\mathrm{Det}(\hat{y})$ and to express the $(y^{-1})^i{}_j$ and the $(\hat{y}^{-1})^{\bar{i}}{}_{\bar{j}}$ in terms of the $Y^I{}_J$. If we normalize $\mathrm{Det}(y)$ and $\mathrm{Det}(\hat{y})$ to e we obtain

$$\begin{aligned} \det(ty) &= q^{-(n^2-1)} \det(t) \, \mathrm{Det}(y) , \\ \det(\hat{t}\hat{y}) &= q^{(n^2-1)} \det(\hat{t}) \, \mathrm{Det}(\hat{y}) . \end{aligned} \tag{2.20}$$

For B_n, C_n and D_n we get similar results.

$$\begin{aligned} C^{kt} &= q^{N-1} \, y^t{}_a y^s{}_u \, \hat{R}_q^{ka}{}_{vs} \, C^{vu} , \\ C^{\bar{k}\bar{t}} &= q^{-(N-1)} \, \hat{y}^{\bar{t}}{}_{\bar{a}} \hat{y}^{\bar{s}}{}_{\bar{u}} \, \hat{R}_q^{-1\,\bar{k}\bar{a}}{}_{\bar{v}\bar{s}} \, C^{\bar{v}\bar{u}} . \end{aligned} \tag{2.21}$$

From (2.21) one obtains Y^{-1} in terms of Y.

Because of (2.16) the algebra generated by $\{Y^I{}_J\}$ is closed under the involution " * ".

3. Generalized Schmidt Orthogonalization

With the orthogonalization scheme of Schmidt we decompose now the y-matrix ($\hat{y}$−matrix) uniquely into the product of an upper (resp. lower) triangular and a unitary matrix ($U^\dagger = U^{-1}$) with entries in the algebra generated by $y^a{}_b$ and $\hat{y}^{\bar{a}}{}_{\bar{b}}$. These factors are then identified as $L^{+i}{}_j$ (resp. $L^{+\bar{i}}{}_{\bar{j}}$) which is upper (resp. lower) triangular with diagonalizable diagonal elements and $S(L^{-i}{}_j)$ (resp. $S(L^{-\bar{i}}{}_{\bar{j}})$) which is unitary.

For simplicity we fix the remaining ambiguity of the parameters α_i (see [DSWZ]) by $\alpha_i > 0$. In all other cases the results can be derived by just the same manner, if we restrict to $\alpha_0 = \alpha_1 = (\alpha_2)^{-1}$.

That $L^{+i}{}_j$ (resp. $L^{+\bar{i}}{}_{\bar{j}}$) is upper (respectively lower) triangular has been discussed in [DSWZ]. The diagonal elements $L^{+I}{}_I$ are algebra homomorphisms acting on $\mathcal{A}$ and

$$L^{+I}{}_I(T^K{}_L) = \hat{\mathcal{R}}_q^{IK}{}_{LI} > 0. \tag{3.1}$$

For $S(L^{-i}{}_j)$ one obtains

$$S(L^{-i}{}_j)^\dagger = S(L^{-j}{}_i)^* = (S^{-1}(L^{-j}{}_i))^* = L^{-\bar{i}}{}_{\bar{j}} = L^{-i}{}_j \tag{3.2}$$

using the equations (3.13) and part 3 from *proposition 8* in [DSWZ]. Thus we have proved the unitarity of $S(L^{-i}{}_j)$. The unitarity of $S(L^{-\bar{i}}{}_{\bar{j}})$ is proved analogously.

Write the matrix y

$$y = \begin{pmatrix} \vec{y}_1 \\ \vec{y}_2 \\ \vdots \\ \vec{y}_N \end{pmatrix} \tag{3.3}$$

with row vectors y_i.

Define for N-dimensional row vectors $\vec{a}, \vec{b} \in (U_{\mathcal{R}})^N$

$$(\vec{a}, \vec{b}) = \sum_{i=1}^{N} a^i (b^i)^*. \tag{3.4}$$

The decomposition equation reads

$$\begin{pmatrix} \vec{y}_1 \\ \vec{y}_2 \\ \vdots \\ \vec{y}_N \end{pmatrix} = \begin{pmatrix} T_{11} & \dots & \dots & T_{1N} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & T_{NN} \end{pmatrix} \begin{pmatrix} \vec{g}_1 \\ \vec{g}_2 \\ \vdots \\ \vec{g}_N \end{pmatrix} \tag{3.5}$$

where the row vectors g_i fulfill an orthonormality condition

$$(g_i, g_j) = \delta_{ij} \tag{3.6}$$

and the diagonal elements T_{ii} being positive and diagonalizable.

This equation leads uniquely to the following recursion relations (Gram-Schmidt)

Starting with $k = N$ going to $k = 1$:

$$T_{kk}\vec{g}_k = \vec{y}_k - \sum_{\nu=k+1}^{N} (\vec{y}_k, \vec{g}_\nu)\vec{g}_\nu$$

and hence leading to the unique positive root of (3.7)

$$|T_{kk}|^2 = \left| \vec{y}_k - \sum_{\nu=k+1}^{N} (\vec{y}_k, \vec{g}_\nu)\vec{g}_\nu \right|^2$$

for $i > k$: $T_{ki} = (\vec{y}_k, \vec{g}_i)$.

This proves the uniqueness of the decomposition equation (3.5). One solution to (3.5) is given by

$$\begin{pmatrix} \vec{y}_1 \\ \vec{y}_2 \\ \vdots \\ \vec{y}_N \end{pmatrix} = \begin{pmatrix} T_{11} & \dots & \dots & T_{1N} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & T_{NN} \end{pmatrix} \begin{pmatrix} \vec{g}_1 \\ \vec{g}_2 \\ \vdots \\ \vec{g}_N \end{pmatrix} = L^+ \, S(L^-) \, . \tag{3.8}$$

The analogue procedure (where T is lower triangular) leads to the construction of the $L^{+\bar{i}}{}_{\bar{j}}$ and $S(L^{-\bar{i}}{}_{\bar{j}})$ in terms of the $y^i{}_j$ and $\hat{y}^{\bar{i}}{}_{\bar{j}}$.

It is possible to obtain the $L^{-I}{}_J$ from $S(L^{-K}{}_L)$ by the following relation

$$L^{-I}{}_J = \left(S(L^{-\bar{J}}{}_{\bar{I}}) \right)^* \tag{3.9}$$

because the Y–Algebra is closed under " * ". We have thus shown how the algebra generated by $\{L^{\pm i}{}_j\}$ is reconstructed from the algebra of vector fields $\{Y^I{}_J\}$.

It is therefore proved that the algebra of regular functionals generated by $\{L^{\pm I}{}_J\}$ is equivalent to the algebra of vector fields $\{Y^I{}_J\}$ for A_n, B_n, C_n and D_n. With eqs. (2.14), (2.15) and the factorization $S(Y^I{}_J) = S^2(L^{-K}{}_J)\, L^{+I}{}_K$ (and using eq. (3.11) from [DSWZ]) it is obvious that the algebra of vector fields is a $*$-Hopf algebra.

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