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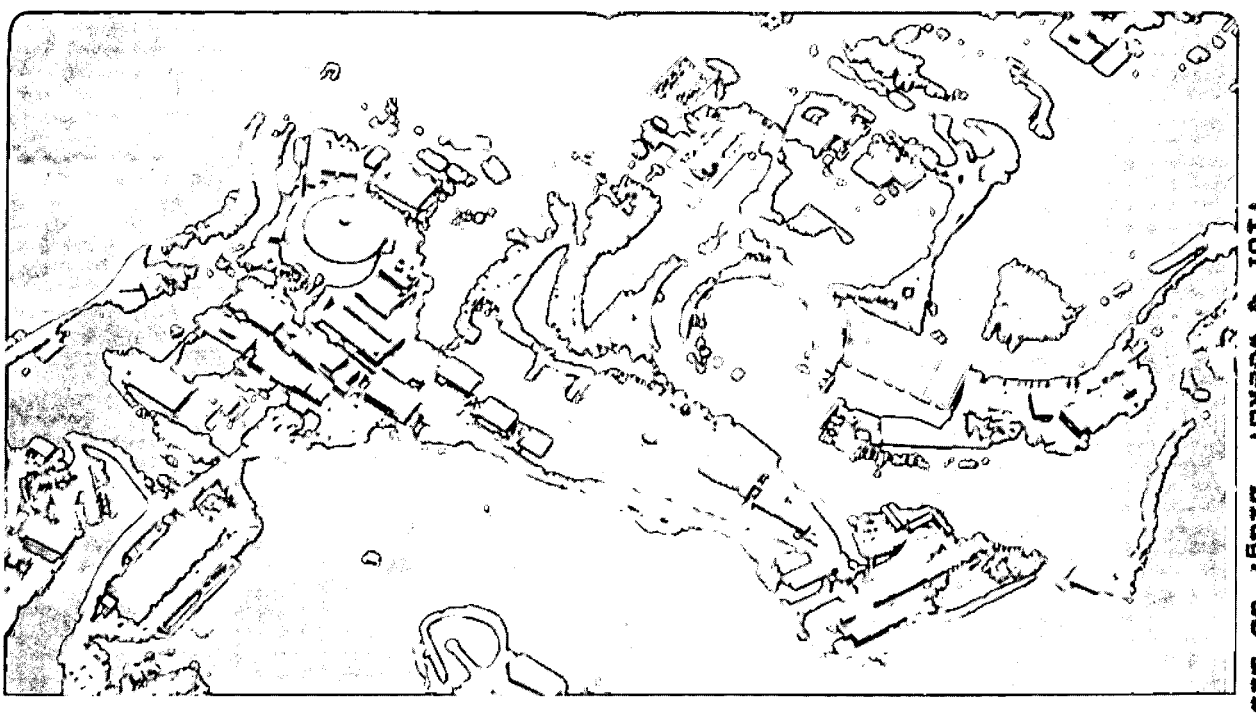

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# Infinite–Mode Squeezed Coherent States and Non–equilibrium Statistical Mechanics (Phase–Space–Picture Approach) *

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## Abstract

The phase–space–picture approach to quantum non–equilibrium statistical mechanics via the characteristic function of infinite–mode squeezed coherent states is introduced. We use quantum Brownian motion as an example to show how this approach provides an interesting geometrical interpretation of quantum non–equilibrium phenomena.

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# 1 Introduction

The standard approach of quantum statistical mechanics uses the density operator $\hat{\rho}$ to describe the (mixed) state of the physical system of interest. Since $\hat{\rho}$ is an operator in the Hilbert space, we usually need some representation to perform the practical calculations. There are many equivalent representations in the literature, e.g., the coordinate representation, P–representation, Q–representation, Fock space representation, Wigner function and characteristic function (Chi function hereafter), etc. In this paper we will use the last two representations since they provide a "phase–space picture" for the quantum–mechanical problems [1].

In quantum equilibrium statistical mechanics, a system (A) immersed in a heat bath (B) with temperature $T$ is described by the canonical ensemble. According to ensemble theory, the density operator is:

$$\hat{\rho} = \frac{\exp(-\beta\hat{H})}{Tr[\exp(-\beta\hat{H})]}, \tag{1}$$

where $\beta = \frac{1}{kT}$ is the inverse temperature and $\hat{H}$ is the Hamiltonian of (A). The structure of (B) and the interaction between (A) and (B) are irrelevant to the density operator. If $\hat{H}$ is (inhomogeneously) quadratic and with a finite number of degrees of freedom, the density operator will correspond to a finite–mode thermal Squeezed Coherent State (SqCS) [2].

In quantum non–equilibrium statistical mechanics, ensemble theory is no longer valid and we have to build a model for the heat bath (B) and consider (A)+(B) as a total system. The total Hamiltonian then contains three parts—the Hamiltonian of (A) and (B) and the interaction Hamiltonian.

It is well known that the number of degrees of freedom of a heat bath must be infinite (the thermodynamic limit), otherwise, due to the Poincaré recurrence theorem there will be no phenomena such as approach to equilibrium, damping or dissipation. The simplest model of a heat bath is an assembly of infinitely many harmonic oscillators with linear couplings to (A). In this kind of model, the total Hamiltonian is quadratic if the Hamiltonian of (A) is quadratic. Since for quadratic Hamiltonians we have a phase–space picture of quantum mechanics with the help of Wigner and/or Chi function, we can construct an infinite–mode

(pure) SqCS for the total system in an infinite–dimensional phase space using these functions. After reduction, i.e., ignoring the heat bath but keeping its "influence", we will get a finite–mode SqCS for (A). In the limit as time approaches infinity, it can be shown that (A) will approach equilibrium, and the finite–mode SqCS will become a thermal SqCS consistent with the fluctuation–dissipation theorem [3, 4, 5, 6, 7, 8].

In this paper we introduce a geometric interpretation of these non–equilibrium phenomena via the Chi–function representation of infinite–mode SqCS. In Sec. 2 notations, conventions and a lemma on matrix are introduced for the mathematics used in this paper. In Sec. 3 we study finite–mode SqCS's by Wigner and Chi functions and then extend them to infinite mode. In Sec. 4 we use the quantum Brownian motion as an example to illustrate geometrical reduction in phase space.

## 2 Mathematical Preliminaries

Throughout this paper, $\hbar$ is set equal to 1; "$\top$" denotes the transpose of a matrix and "$-\top$"denotes the inverse of the transpose of a matrix. The physical system under consideration is of $N = n + 1$ degrees of freedom, where $n$ is either finite or equal to infinity. The symbols $\vec{x} = (x^0, x^1, x^2, \ldots, x^n)$ and $\vec{k} = (k^0, k^1, k^2, \ldots, k^n)$ denote the $N$–dimensional canonical coordinate and momentum respectively, thus $\vec{z} \equiv (\vec{x}; \vec{k})$ is a vector in $2N$–dimensional phase space. $\vec{\hat{q}}$ and $\vec{\hat{p}}$ denote $N$–dimensional position and momentum operators corresponding to the canonical variables $\vec{x}$ and $\vec{k}$.

The metaplectic group **Mp(2N,R)** is an $N(2N+1)$–dimensional Lie group. It is the quantum analogue of symplectic group **Sp(2N,R)**. The elements of the Lie algebra of **Mp(2N,R)** can be organized as anti–hermitian operators in the following form:

$$\begin{aligned}
\hat{\Phi}(m) &= \frac{i}{2}\sum_{i,j=0}^{n}[\alpha_{ij}\hat{q}_i\hat{q}_j + \beta_{ij}\hat{p}_i\hat{p}_j + \gamma_{ij}(\hat{q}_i\hat{p}_j + \hat{p}_j\hat{q}_i)] \\
&= \frac{i}{2}(\vec{\hat{q}};\vec{\hat{p}})\begin{pmatrix}\alpha & \gamma \\ \gamma^\top & \beta\end{pmatrix}(\vec{\hat{q}};\vec{\hat{p}})^\top \\
&= \frac{i}{2}(\vec{\hat{q}};\vec{\hat{p}})Jm(\vec{\hat{q}};\vec{\hat{p}})^\top, \qquad (2)
\end{aligned}$$

where $\alpha_{ij} = \alpha_{ji}$, $\beta_{ij} = \beta_{ji}$ and

$$m = \begin{pmatrix} -\gamma^{\mathsf{T}} & -\beta \\ \alpha & \gamma \end{pmatrix} \in \mathrm{sp(2N,r)} \tag{3}$$

is a $2N \times 2N$ real matrix [9], while

$$J = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \mathbf{1} = N \times N \text{ unit matrix.} \tag{4}$$

The Lie algebra of $\mathrm{Mp(2N,R)}$ is isomorphic to that of $\mathrm{Sp(2N,R)}$. The action of $\exp[\hat{\Phi}(m)] \in \mathrm{Mp(2N,R)}$ on $(\vec{\hat{q}}; \vec{\hat{p}})$ is:

$$\exp[\hat{\Phi}(m)](\vec{\hat{q}}; \vec{\hat{p}})^{\mathsf{T}} \exp[-\hat{\Phi}(m)] = \exp(-m)(\vec{\hat{q}}; \vec{\hat{p}})^{\mathsf{T}}, \tag{5}$$

where $\exp(-m) \in \mathrm{Sp(2N,R)}$.

**Lemma** [10]

If $M$ is a symmetric and positive definite $2N \times 2N$ matrix, then there exist two matrices $S_1, S_2 \in \mathrm{Sp(2N,R)}$ (but not unique), such that

$$M = S_1^{\mathsf{T}} \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} S_1 = S_2^{\mathsf{T}} \begin{pmatrix} \omega^2 & 0 \\ 0 & 1 \end{pmatrix} S_2, \tag{6}$$

where $\omega = \mathrm{diag}(\omega_0, \omega_1, \omega_2, \ldots, \omega_n)$, $\omega_j > 0$ for all $j$, and

$$S_2 = \begin{pmatrix} \omega^{-\frac{1}{2}} & 0 \\ 0 & \omega^{\frac{1}{2}} \end{pmatrix} S_1. \tag{7}$$

**Remarks:**

(1) $S \in \mathrm{Sp(2N,R)}$ if and only if $S^{\mathsf{T}} J S = J$ by definition.

(2) $\omega_j$'s are not eigenvalues of $M$ in general. We will call them the "symplectic eigenvalues" of matrix $M$.

(3) The eigenvalues of $JM$ are $\pm i\omega_j$'s, hence we can calculate $\omega_j$'s from $JM$ as an ordinary eigenvalue problem.

(4) If the matrix $C_j$ corresponds to a 2–dimensional rotation on the $(x_j, k_j)$ plane, then

$$C_j^{\mathsf{T}} \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} C_j = C_j^{\mathsf{T}} C_j \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}. \tag{8}$$

Therefore $S_1$ in (6) can be replaced by $C_j S_1$ and hence is not unique.

# 3 Squeezed Coherent States in Phase Space

## 3.1 Wigner Function

The Wigner function of an N–mode density operator $\hat{\rho}$ is defined as [11, 12]:

$$W(\vec{x};\vec{k}) = \pi^{-N}\int_{-\infty}^{\infty} d\vec{y}\exp(2i\vec{k}\cdot\vec{y})\rho(\vec{x}-\vec{y},\vec{x}+\vec{y}), \tag{9}$$

where $\rho(\vec{x},\vec{x'})$ is the coordinate representation of the density operator $\hat{\rho}$.

The Wigner function is real and normalized by definition:

$$\int_{-\infty}^{\infty} d\vec{x}d\vec{k}W(\vec{x};\vec{k}) = 1. \tag{10}$$

However, it is not always positive–definite and is thus called the quasi–probability distribution function over the "phase space" $\vec{z} = (\vec{x};\vec{k})$.

If the density operator is an exponential of a quadratic form of position and momentum operators, then the Wigner function is a Gaussian distribution in $\vec{z}$:

$$W(\vec{z}) = C_N \exp[-(\vec{z}-\vec{z}_c)M(\vec{z}-\vec{z}_c)^{\mathsf{T}}]. \tag{11}$$

where $C_N = \pi^{-N}\sqrt{\det(M)}$ is the normalization constant, $\vec{z}_c$ is a constant vector in the $2N$–dimensional phase space, and $M$ is a symmetric, positive definite matrix with all its symplectic eigenvalues smaller or equal to 1. (Otherwise (11) will not correspond to a physical state.) The Gaussian Wigner function (11) corresponds to the multimode thermal SqCS in general [2]. It contains the ordinary coherent states (when $M$ is a unit matrix) and the ordinary SqCS (when all the symplectic eigenvalues of $M$ equal to 1) as special cases.

The "Wigner ellipsoid" of (11) is defined as:

$$(\vec{z}-\vec{z}_c)M(\vec{z}-\vec{z}_c)^{\mathsf{T}} = 1, \tag{12}$$

which is an ellipsoid in the $2N$–dimensional phase space with its center at $\vec{z}_c$ and its shape determined by $M$. We can take the Wigner ellipsoid as a geometric representation of the Gaussian Wigner function.

## 3.2 Characteristic Function (Chi Function)

The Chi function of a density operator $\hat{\rho}$ is defined as:

$$\chi(\vec{x};\vec{k}) = Tr[\hat{\rho}\hat{D}(-\vec{x};-\vec{k})], \tag{13}$$

It can be shown that Chi function is the symplectic Fourier transformation of the Wigner function:

$$\chi(\vec{x};\vec{k}) = \int_{-\infty}^{\infty} d\vec{x'}d\vec{k'}W(\vec{x'};\vec{k'})\exp[-i(\vec{x'}\cdot\vec{k}-\vec{k'}\cdot\vec{x})]. \tag{14}$$

The normalization condition of the Wigner function corresponds to $\chi(\vec{0};\vec{0}) = 1$ in the Chi function. Since the operator $\hat{D}(-\vec{x};-\vec{k})$ is unitary, $\chi(\vec{x};\vec{k})$ is complex in general.

The Chi function which corresponds to the Gaussian Wigner function (11) is also Gaussian:

$$\chi(\vec{z}) = \exp[-\frac{1}{4}\vec{z}JM^{-1}J^{\mathsf{T}}\vec{z}^{\mathsf{T}} + i\vec{z}J\vec{z}_c^{\mathsf{T}}]. \tag{15}$$

Analogue to the Wigner ellipsoid, we can also define the "Chi ellipsoid" for a Gaussian Chi function as:

$$(\vec{z}-\vec{z}_c)JM^{-1}J^{\mathsf{T}}(\vec{z}-\vec{z}_c)^{\mathsf{T}} = 1. \tag{16}$$

The center of Chi ellipsoid is the same as that of the Wigner ellipsoid, while the shape of this ellipsoid is determined by the matrix $JM^{-1}J^{\mathsf{T}}$.

## 3.3 Mean Vector and Covariance Matrix

For an $N$–mode (mixed) state with the density operator $\hat{\rho}$, the mean vector is defined as $(\langle\vec{\hat{q}}\rangle;\langle\vec{\hat{p}}\rangle)$ in the $2N$–dimensional phase space, where $\langle\hat{q}_i\rangle \equiv Tr(\hat{\rho}\hat{q}_i)$, etc. The covariance matrix is defined as a $2N\times 2N$ matrix of the form:

$$\begin{pmatrix} U & Q \\ Q^{\mathsf{T}} & V \end{pmatrix}, \tag{17}$$

$$U_{ij} \equiv \langle(\hat{q}_i - \langle\hat{q}_i\rangle)(\hat{q}_j - \langle\hat{q}_j\rangle)\rangle = \langle\hat{q}_i\hat{q}_j\rangle - \langle\hat{q}_i\rangle\langle\hat{q}_j\rangle, \tag{18}$$

$$V_{ij} \equiv \langle(\hat{p}_i - \langle\hat{p}_i\rangle)(\hat{p}_j - \langle\hat{p}_j\rangle)\rangle = \langle\hat{p}_i\hat{p}_j\rangle - \langle\hat{p}_i\rangle\langle\hat{p}_j\rangle, \tag{19}$$

$$
\begin{aligned}
Q_{ij} &\equiv \frac{1}{2}\langle(\hat{q}_i - \langle\hat{q}_i\rangle)(\hat{p}_j - \langle\hat{p}_j\rangle) + (\hat{p}_j - \langle\hat{p}_j\rangle)(\hat{q}_i - \langle\hat{q}_i\rangle)\rangle \\
&= \langle\frac{1}{2}(\hat{q}_i\hat{p}_j + \hat{p}_j\hat{q}_i)\rangle - \langle\hat{q}_i\rangle\langle\hat{p}_j\rangle. \qquad (20)
\end{aligned}
$$

For the Gaussian Wigner function (11) or Gaussian Chi function (15), the mean vector is $\vec{z}_c$, and the covariance matrix takes the form:

$$
\begin{pmatrix} U & Q \\ Q^{\mathsf{T}} & V \end{pmatrix} = \frac{1}{2}M^{-1}. \qquad (21)
$$

Therefore (15) can be re–written as:

$$
\chi(\vec{z}) = \exp[-\frac{1}{2}\vec{z}\begin{pmatrix} V & -Q^{\mathsf{T}} \\ -Q & U \end{pmatrix}\vec{z}^{\mathsf{T}} + i\vec{z}J\vec{z}_c^{\mathsf{T}}]. \qquad (22)
$$

## 3.4 Time Evolution of Wigner and Chi Functions

Consider an $N$–mode Hamiltonian:

$$
\hat{H} = \frac{1}{2}(\vec{\hat{q}};\vec{\hat{p}})K(\vec{\hat{q}};\vec{\hat{p}})^{\mathsf{T}}, \qquad (23)
$$

where $K$ is a $2N \times 2N$ positive definite symmetric matrix . According to the lemma introduced in Sec. 1, this kind of Hamiltonian can be transformed into the following form:

$$
\hat{H} = \frac{1}{2}(\vec{\hat{q}};\vec{\hat{p}})S^{\mathsf{T}}\begin{pmatrix} \omega^2 & 0 \\ 0 & 1 \end{pmatrix}S(\vec{\hat{q}};\vec{\hat{p}}), \qquad (24)
$$

where $\omega = \mathrm{diag}(\omega_0, \omega_1, \ldots, \omega_n)$ and the $\omega_i$'s are symplectic eigenvalues of the matrix $K$. The time–evolution operator $\exp(-it\hat{H})$ is an element in **Mp(2N, R)** and the time evolution of $(\vec{\hat{q}};\vec{\hat{p}})$ is a special case of (5):

$$
\exp(it\hat{H})(\vec{\hat{q}};\vec{\hat{p}})^{\mathsf{T}}\exp(-it\hat{H}) = R(t)(\vec{\hat{q}};\vec{\hat{p}})^{\mathsf{T}}, \qquad (25)
$$

where

$$
R(t) = \exp(tJK) = S^{-1}\begin{pmatrix} \cos(\omega t) & \omega^{-1}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{pmatrix}S \qquad (26)
$$

is an element in **Sp(2N,R)**. $\{R(t)|\, t \in \mathrm{R}\}$ forms a one–parameter subgroup of **Sp(2N,R)** describing the phase flow in the $2N$–dimensional classical phase space:

$$
(\vec{x}(t);\vec{k}(t))^{\mathsf{T}} = R(t)(\vec{x}(0);\vec{k}(0))^{\mathsf{T}}. \qquad (27)
$$

It is well known that for the quadratic Hamiltonian (23), the time evolution of Wigner function and Chi function follow exactly this phase flow:

$$W(\vec{z};t) = C_N \exp[-(\vec{z}-\vec{z}_c(t))R(t)^{-\mathsf{T}}MR(t)^{-1}(\vec{z}-\vec{z}_c(t))^{\mathsf{T}}], \tag{28}$$

$$\begin{aligned}\chi(\vec{z};t) &= \exp[-\frac{1}{2}\vec{z}R(t)^{-\mathsf{T}}\begin{pmatrix} V & -Q^{\mathsf{T}} \\ -Q & U \end{pmatrix} R(t)^{-1}\vec{z}^{\mathsf{T}} + i\vec{z}J\vec{z}_c^{\mathsf{T}}(t)] \\ &\equiv \exp[-\frac{1}{2}\vec{z}\begin{pmatrix} V(t) & -Q(t)^{\mathsf{T}} \\ -Q(t) & U(t) \end{pmatrix} \vec{z}^{\mathsf{T}} + i\vec{z}J\vec{z}_c^{\mathsf{T}}(t)],\end{aligned} \tag{29}$$

where $\vec{z}_c^{\mathsf{T}}(t) = R(t)\vec{z}_c^{\mathsf{T}}$.

## 3.5 Reduction of Multimode Squeezed Coherent States

Consider the quantum system (A)+(B) discussed in Sec. 1 whose density operator is $\hat{\rho}_{AB}$. If we reduce this system by ignoring (B), the expectation value of an operator $\hat{O}_A$ which corresponds to a measurement on (A) will become:

$$\langle \hat{O}_A \rangle = Tr[\hat{\rho}_A \hat{O}_A], \tag{30}$$

where $\hat{\rho}_A = Tr_{(B)}(\hat{\rho}_{AB})$ is a well–defined reduced density operator for (A) which contains the "influence" of (B) on (A), $Tr_{(B)}$ represents the "partial trace" which only takes trace with respect to the degrees of freedom of (B).

If the Wigner function $W(\vec{x}_A, \vec{x}_B; \vec{k}_A, \vec{k}_B)$ corresponds to the original density operator $\hat{\rho}_{AB}$, then the reduced Wigner function corresponding to $\hat{\rho}_A$ will be [12]:

$$W_A(\vec{x}_A; \vec{k}_A) = \int_{-\infty}^{\infty} d\vec{x}_B d\vec{k}_B W(\vec{x}_A, \vec{x}_B; \vec{k}_A, \vec{k}_B). \tag{31}$$

As for the Chi function, if $X(\vec{x}_A, \vec{x}_B; \vec{k}_A, \vec{k}_B)$ corresponds to $\hat{\rho}_{AB}$, the reduced Chi function corresponding to $\hat{\rho}_A$ will take the form:

$$X_A(\vec{x}_A; \vec{k}_A) = X(\vec{x}_A, \vec{0}; \vec{k}_A, \vec{0}), \tag{32}$$

which is a restriction of the original $X(\vec{x}_A; \vec{x}_B, \vec{k}_A; \vec{k}_B)$ to a subspace in the $2N$–dimensional phase space. From the mathematical point of view, it is easier to use the Chi function to perform the reduction.

Now let us use $N$–mode to one–mode reduction as an example. For a given $N$–mode Gaussian Chi function (22), we want to make a reduction by ignoring all the degrees of freedom which correspond to modes $1, 2, \ldots, n$ and leave only the 0–th mode. Without any calculation, we can write down the reduced Chi function directly :

$$\chi(x^0, k^0) = \exp[-\frac{1}{2}(x^0, k^0) \begin{pmatrix} V_{00} & -Q_{00} \\ -Q_{00} & U_{00} \end{pmatrix} (x^0, k^0)^\top + i(x^0 k_c^0 - k_0 x_c^0)], \quad (33)$$

which is a one–mode Gaussian Chi function.

The geometrical interpretation of this reduction process is cutting the original Chi ellipsoid in the $2N$–dimensional phase space by a "shifted $(x^0, k^0)$ plane"—the plane which is parallel to $(x^0, k^0)$ plane and passes through the center of the Chi ellipsoid. The section on the Chi ellipsoid gives the "Chi ellipse" on the shifted $(x^0, k^0)$ plane which represents the reduced one–mode Gaussian Chi function. A schematic graph of this geometrical reduction is shown in Fig. 1.

### 3.6 Infinite–Mode Squeezed Coherent States

The infinite–mode SqCS is a naive generalization of finite–mode SqCS. Comparing the three equivalent representations of finite–mode SqCS's: (11) , (15) and (22), we see that (22) can be directly generalized to infinite mode without any ambiguity or convergence problem. So we will take (22) in the infinite–dimensional phase space as the definition of infinite–mode SqCS, all the above formulas which involve (22) can be applied to infinite–mode case.

## 4 Quantum Brownian Motion

In this section we shall study quantum Brownian motion of a harmonic oscillator. The physical picture is a quantum harmonic oscillator immersed in a dissipative heat bath. In classical statistical mechanics, this problem can be studied via the Langevin equation:

$$M\,\ddot{X} + M\gamma\,\dot{X} + M\Omega^2 X = 0, \quad (34)$$

where $X$, $M$ and $\Omega$ are the coordinate, mass and frequency of the oscillator, and $M\gamma$ is the friction constant.

The quantum analogue of the Langevin equation can be achieved by several quantum–mechanical heat–bath models, e.g., the FKM model [3], linear coupling model [4, 5], independent–oscillator model [13], etc. Actually it can be proved that all these models are equivalent [14]. In this paper we will use the independent–oscillator model since it is the simplest and most intuitive.

## 4.1 Independent–Oscillator Heat–Bath Model

Consider the Brownian particle to be a harmonic oscillator immersed in a dissipative heat bath with inverse temperature $\beta$. Using the independent–oscillator heat–bath model, the total Hamiltonian of the system is [13]:

$$H = \frac{\hat{P}^2}{2M} + \frac{1}{2}M\Omega^2\hat{Q}^2 + \sum_{i=1}^{\infty}[\frac{\hat{p}_i^2}{2m_i} + \frac{1}{2}m_i\omega_i^2(\hat{q}_i - \hat{Q})^2], \tag{35}$$

where $\hat{Q}$ and $\hat{P}$ are the position and momentum operators of the Brownian particle; $\hat{q}_i$, $\hat{p}_i$, $m_i$ and $\omega_i$ are the position operator, momentum operator, mass and frequency of the $i$–th heat–bath oscillator, $i = 1, 2, 3, \ldots$. This Hamiltonian is a special case of (23).

It can be proved that in order to make the Brownian particle satisfy the quantum Langevin equation:

$$M\ddot{\hat{Q}} + M\gamma\dot{\hat{Q}} + M\Omega^2\hat{Q} = 0, \tag{36}$$

the spectral distribution of heat–bath oscillators must obey:

$$\sum_{i=1}^{n} m_i\omega_i^2\delta(\omega - \omega_i) = \frac{2}{\pi}M\gamma. \tag{37}$$

## 4.2 Quantum Brown Motion in Phase Space

In the following we will study time evolution of the Brownian particle by the reduced Chi function. The initial condition is chosen to be $\hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B$, where $\hat{\rho}_A$ is the initial density operator of the Brownian particle which corresponds to an

arbitrary Gaussian Wigner/Chi function, and $\hat{\rho}_B$ is the initial density operator of heat bath which is in thermal equilibrium at the inverse temperature $\beta$. Since the detailed calculations can be obtained by combining the calculations in [6] and [13], here we will only discuss the result and the geometrical interpretation.

Let the degree of freedom of the Brownian particle correspond to the 0–th mode, and those of the heat bath correspond to other modes. The infinite–mode Chi function for the initial condition state is (22) with the following parameters: $U_{00}$, $V_{00}$, $Q_{00}$ and $\vec{z}_c^0$, which correspond to the initial conditions of the Brownian particle, are arbitrary; $\vec{z}_c$ has only two non–zero components corresponding to $\vec{z}_c^0$ since the mean vectors for all heat–bath oscillators equal to zero; and other elements in the covariance matrix are:

$$U_{ij} = \frac{1}{2m_i\omega_i}\coth(\frac{1}{2}\beta\omega_i)\delta_{ij}, \tag{38}$$

$$V_{ij} = \frac{1}{2}m_i\omega_i\coth(\frac{1}{2}\beta\omega_i)\delta_{ij}, \tag{39}$$

$$U_{i0} = U_{0j} = V_{i0} = V_{0j} = Q_{i0} = Q_{0j} = Q_{ij} = 0, \tag{40}$$

for all $i,j = 1,2,3,\ldots$.

Combining (29) and (33), we get the time–dependent reduced Chi function of the Brownian particle (the index 0 for the canonical coordinates is suppressed):

$$\chi(x,k;t) = \exp[-\frac{1}{2}(x,k)\begin{pmatrix} V_{00}(t) & -Q_{00}(t) \\ -Q_{00}(t) & U_{00}(t)\end{pmatrix}(x,k)^{\mathsf{T}} + i(xk_c(t) - kx_c(t))]. \tag{41}$$

It is easy to write down the corresponding Wigner function by comparing (11) and (15):

$$W(x,k;t) = C_1\exp[-(x - x_c(t), k - k_c(t))M(t)(x - x_c(t), k - k_c(t))^{\mathsf{T}}], \tag{42}$$

where

$$M(t) = \frac{1}{2[U_{00}(t)V_{00}(t) - Q_{00}^2(t)]}\begin{pmatrix} V_{00}(t) & -Q_{00}(t) \\ -Q_{00}(t) & U_{00}(t)\end{pmatrix}. \tag{43}$$

Unlike ordinary reduction methods [5, 6, 7], we obtained this reduced Wigner function without using integration over the heat–bath degrees of freedom.

Comparing (41) with (42) and (43), we see that at any moment the Wigner ellipse and the Chi ellipse are similar and their areas inversely proportional to

each other. (Although both areas are time–dependent in general.) When time approaches infinity, the Brownian particle will approach the equilibrium state which is independent of its initial condition and consistent with the fluctuation–dissipation theorem. In Fig. 2, we plot the time evolution of Wigner ellipse and Chi ellipse of the Brownian particle in phase space.

# 5 Conclusion and Outlook

The method and result discussed in this paper are valid as long as: (1) The initial state of (A) is a finite–mode (not necessary one–mode) Gaussian Wigner/Chi function. (2) The Hamiltonian of (A) is quadratic and with finite number of degrees of freedom.

If (1) no longer holds, then we will not be able to use an ellipsoid in phase space to represent the state. However, the phase–space picture continues to be valid since time evolution of the Wigner/Chi function will still follow the phase flow in classical phase space. On the contrary if (2) is not true, e.g., as in quantum tunneling problems, then time evolution of the Wigner/Chi function will not follow the phase flow exactly and the phase–space picture will fail. In order to relieve this limitation, some authors introduced the idea of "effective potential"[15, 16] so that time evolution of the Wigner/Chi function can be still expressed in terms of the phase flow. Integration of this modified phase–space picture with the dissipation mechanism is an interesting question and worth pursuing further.

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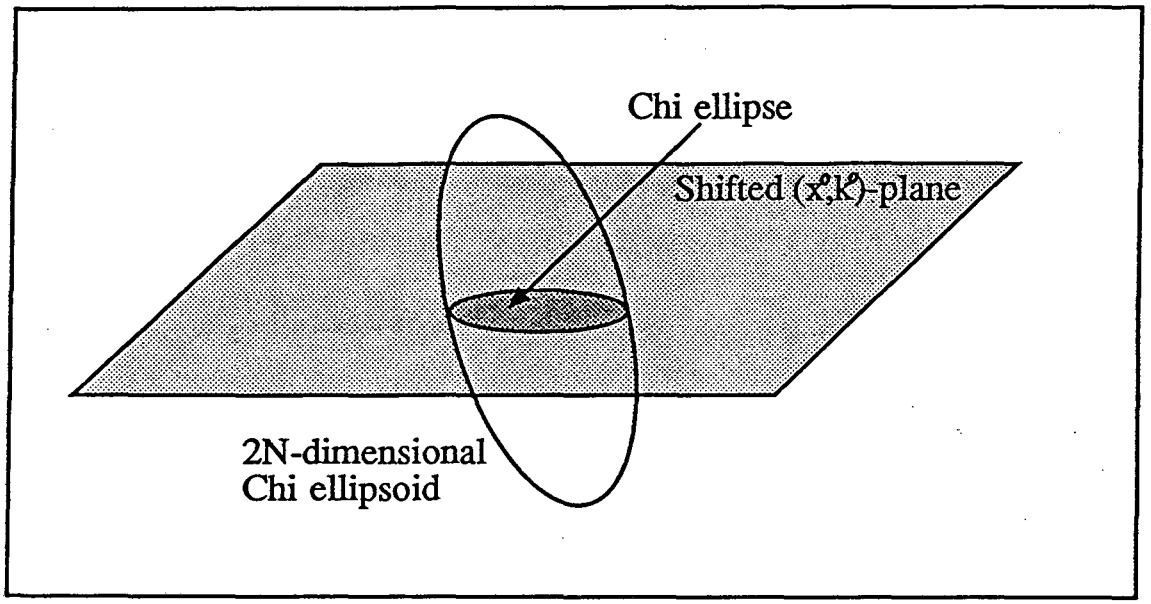


Fig. 1. Reduction as a geometrical operation in phase space.

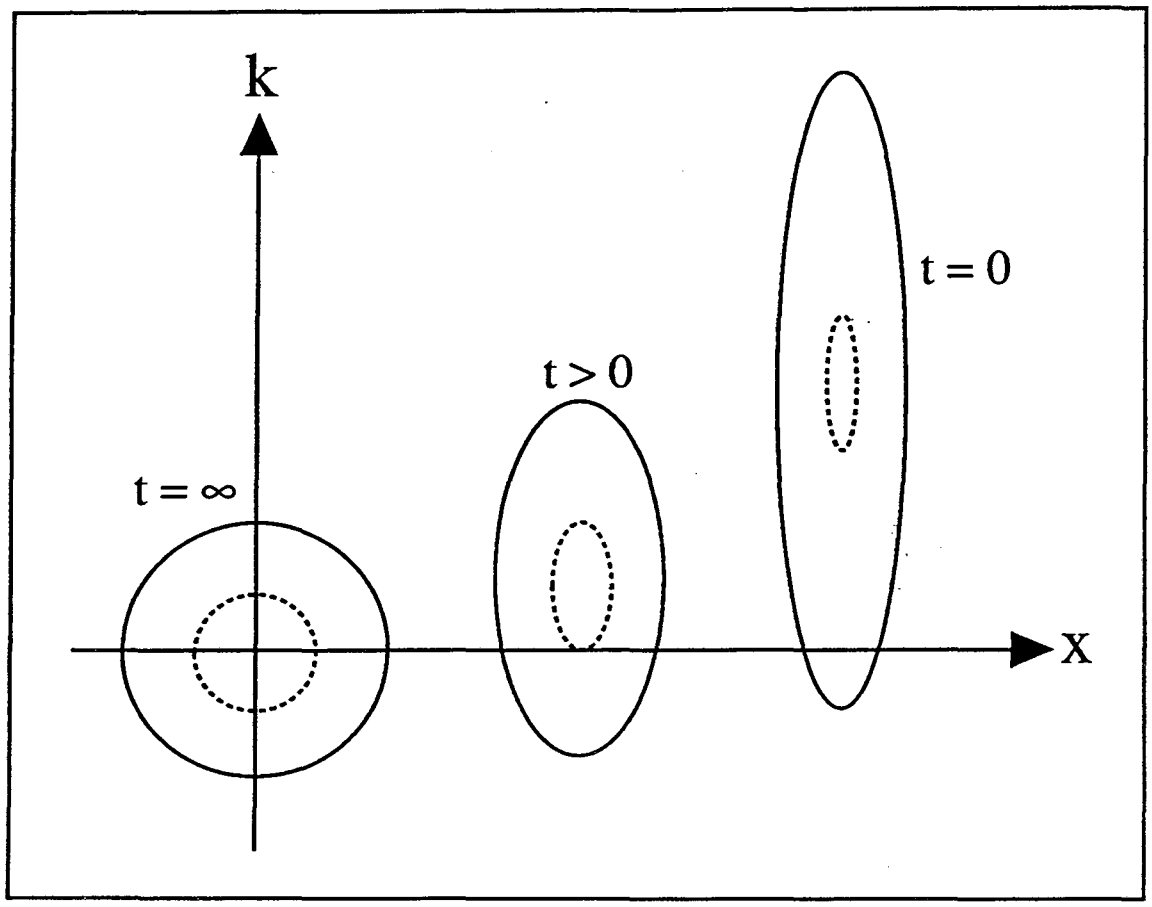


Fig. 2. Time evolution of Wigner ellipse (solid) and Chi ellipse (dotted) of the Brownian particle. Note that the area of the former is always larger than that of the latter.

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