LBL-34097
UC-211
Preprint

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

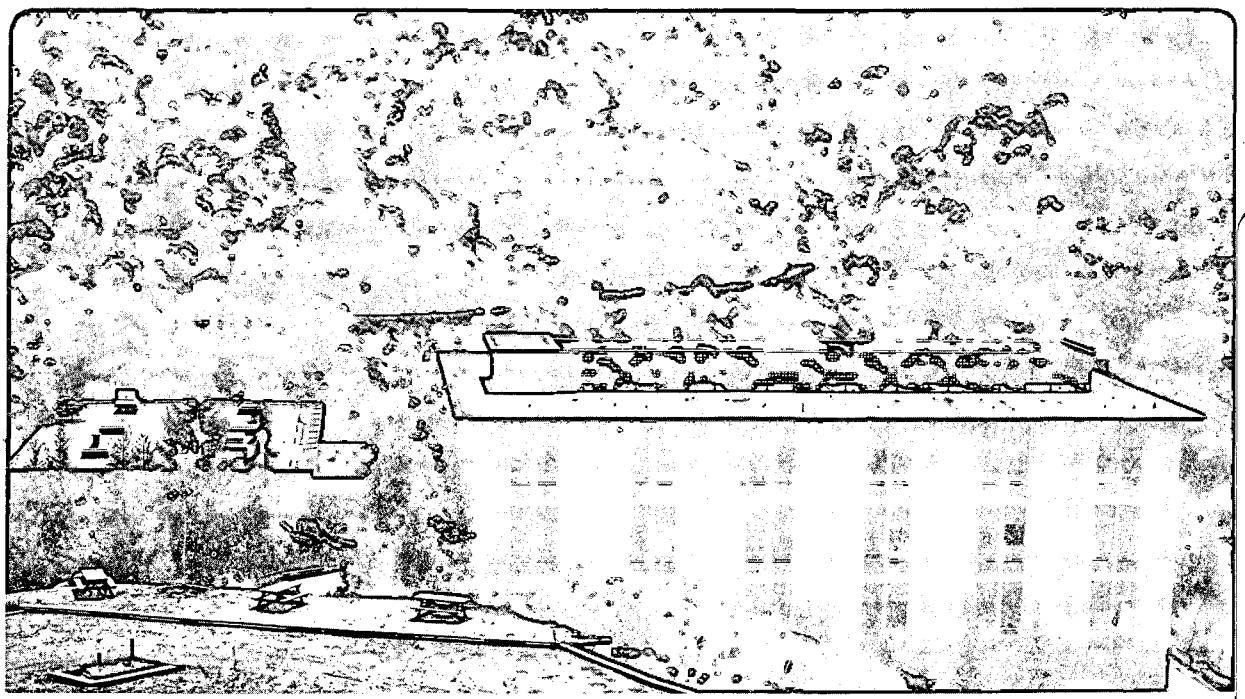
Materials Sciences Division

Submitted to the Journal of the Electrochemical Society

Magnetic Field Effects in High-Power Batteries
II. Time Constant of a Radial Circuit Terminated by a Cylindrical Cell with Inductance

V.S. Battaglia and J. Newman

May 1993



REFERENCE COPY
Does Not
Circulate
Bldg. 50 Library.
LBL-34097
Copy 1

Prepared for the U.S. Department of Energy under Contract Number DE-AC03-76SF00098

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

LBL-34097
UC-221

Magnetic Field Effects in High-Power Batteries

II. Time Constant of a Radial Circuit Terminated by a Cylindrical Cell with Inductance

Vincent S. Battaglia and John Newman

Department of Chemical Engineering
University of California

and

Materials Sciences Division
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

May 12, 1993

This work was supported by the Assistant Secretary for Conservation and Renewable Energy, Office of Transportation Technologies, Electric and Hybrid Propulsion Division of the U. S. Department of Energy under Contract No. DE-AC03-76SF00098.

Magnetic Field Effects in High-Power Batteries

II: Time Constant of a Radial Circuit Terminated by a Cylindrical Cell with Inductance

Abstract

The time constant to steady-state of an electric signal applied to the outer boundary of a radial circuit that propagates to a centrally located cylindrical conductor is determined rigorously by a transmission-line analysis. Two cases are reported: the perfect radial conductor and a radial conductor with a finite resistance. The time constant is featured as a function of the ratio of the radius of the radial conductive leads to the radius of the inner conductor. As the ratio is increased, the time constant departs from that of an electric field penetrating a cylindrical conductor and approaches the inductive time constant of the leads. An analytic approximation of the time constant of the system is also provided. This analysis should assist in the development of high-power circuitry design when the discharge time is on the order of the inductive time constant.

Introduction

The intent of this research is to determine the inductive time constant of a circuit that contains a battery. As discussed by Rosser,[1] the rate at which the current rises in one portion of a cir-

cuit is a function of the configuration and properties of the rest of the circuit. Therefore, this problem cannot be solved in parts. The rigorous method for handling the inductance effects is to solve Maxwell's equations of electromagnetism throughout the entire circuit and surrounding medium, simultaneously with the equations that govern battery performance. This would require an enormous effort. As such, we direct our investigation to the simpler case of determining rigorously the rate of discharge of a radial circuit with an inductive, coaxial, cylindrical core. Applying the same arguments as found in part I, the cylindrical core is assumed to approximate the behavior of a bipolar battery. Furthermore, the electromotive force is assumed rooted at some radial distance, r_o, away.

In part I we developed the solution to the rate at which an electric field penetrates a cylindrical conductor. As discussed, a solid conductor is a poor approximation of the inner workings of a battery. However, that analysis provided an adequate first approximation of the inductive behavior of the battery. In this section, we shall use a transmission-line analysis in conjunction with the preceding work to determine the time constant of the total circuit configuration. The solution to this full problem may suggest where the limitation to instantaneous discharge resides and may then be used for design criteria for systems intended to be discharged at high rates.

Analytic Approach

An analytic approach is possible if a simplified circuit geometry is proposed. The following configuration is considered: a cylindrical conductor of finite length d, conductivity σ_i, and radius r_i sandwiched between two circular plates of radius r_o, where $r_o > r_i$ (see figure 1.) The system is initially at open-circuit, and the potential is zero everywhere. A constant voltage source is then applied at time zero at $r = r_o$. We wish to determine the time it takes the system to reach steady-state. A transmission-line analysis shall be used to characterize the propagation of the signal along the radial conductors to the inner cylindrical conductor. The solution derived in part I section for the penetration of an electric field into a conductor shall be converted into a boundary condition at $r = r_i$. The problem is solved using Laplace transforms. Two cases are considered: the first is where the outer radial conductors are assumed to have an infinite conductivity; and the second is where they have a finite conductivity. Before proceeding to the solution, we shall first develop, through the elementary laws of statics and electrodynamics, the transmission-line equations.

Transmission-Line Analysis

A measure of the ease by which charge q migrates through a medium is characterized by its resistivity, ρ. This parameter is lowest for conductors, of moderate value for semiconductors, and highest for insulators. The total resistance of a bar of a conducting material of length l and area A is

$$R = \frac{\rho l}{A}. \tag{1}$$

Thus the resistance is defined by two parameters: the physical geometry and resistivity.

According to Gauss's law, an electric field outside of a conductor acting perpendicular to the conductor is given by

$$\mathbf{E}\cdot\mathbf{n} = \frac{q}{\epsilon A}. \tag{2}$$

The potential is defined as

$$-\nabla\Phi = \mathbf{E}. \tag{3}$$

For two parallel plates of equal and opposite charge and distance d apart we obtain

$$\Phi = \frac{qd}{\epsilon A}, \tag{4}$$

or

$$q = C\Phi, \tag{5}$$

where C is the capacitance. The capacitance, like the resistance, is a function of the physical geometry and a parameter, ϵ, that describes the medium separating the conductive material. For the parallel plates, capacitance per unit area is defined as

$$c \equiv \frac{C}{A} = \epsilon/d. \tag{6}$$

Current is defined as the amount of charge that passes a particular point per unit of time. In differential form

$$I = \frac{\partial q}{\partial t} = C\frac{\partial \Phi}{\partial t} \tag{7}$$

for a capacitive current.

Ampere's law shows that with the passing of any current there is an associated magnetic field,

$$\nabla \times \mathbf{B} = \mu \mathbf{i}. \tag{8}$$

For a current flowing in a wire, the magnetic field is proportional to the amount of current and inversely proportional to the square of the distance from the center of the wire,

$$B \propto \mu I_L / r^2. \tag{9}$$

The magnetic flux, ϕ, is the sum of the magnetic fields of current-carrying elements from different positions in the conductor. The magnetic linkage, λ, is the sum of the total magnetic flux in the system. The magnetic linkage is proportional to the current,

$$\lambda = L I_L. \tag{10}$$

L, the self-inductance, is a proportionality constant between the flux linkage and the current and again is a function of the physical geometry and μ, a parameter describing the surrounding medium. For a parallel-plate configuration, it can be shown that the self-inductance is

$$L = \mu d. \tag{11}$$

To relate the flux linkage back to a potential, we turn to Faraday's law,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{12}$$

which mathematically states that a magnetic field that is varying in time has associated with it an electric field. These definitions

ultimately lead to the expression

$$-\nabla\Phi = L\frac{\partial i_L}{\partial t}. \tag{13}$$

We can now discuss the transmission of a signal in the radial direction between two conducting plates. If the conductors are "perfect," *i.e.* have a resistivity of zero, charge travels between the plates on the surfaces and is referred to as the skin current. If the plates have a finite resistance, the current penetrates within the plates and it can be integrated over the thickness ℓ. In either case, the current density will have units of A/cm. Charge flowing in the radial direction behaves as a purely inductive current, i_L, if the plates are perfect, and will contain an ohmic term if they are not. Some of the charge goes to charging the plates and is referred to as the capacitive current, i_c, with units of A/cm^2. A shell balance of the current traveling in the radial direction along a plate with no resistance entering the shell at r and leaving at $r+\Delta r$ appears as

$$2\pi r i_L\big|_r = 2\pi r i_L\big|_{r+\Delta r} + 2\pi r\Delta r i_c. \tag{14}$$

Dividing through by $r\Delta r$ and taking the limit as Δr approaches zero gives the differential form

$$\frac{1}{r}\frac{\partial(r i_L)}{\partial r} = -i_c. \tag{15}$$

Differentiation of equation 15 with respect to time and the subsequent substitution of the previously derived current-potential relationships,

$$\frac{\partial \Phi}{\partial r} = -L\frac{\partial i_L}{\partial t} \tag{16}$$

and

$$i_c = c\frac{\partial \Phi}{\partial t}, \tag{17}$$

gives the transmission-line equation for a radial circuit of infinite conductivity,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) = Lc\frac{\partial^2 \Phi}{\partial t^2}. \tag{18}$$

If the conductor maintains a finite resistivity, the equation

$$-\frac{\partial \Phi}{\partial r} = L\frac{\partial i_L}{\partial t} + \frac{i_L}{\sigma l}, \tag{19}$$

(σ, the conductivity, is equal to $1/\rho$.) which contains an additional term for the ohmic drop, is substituted into equation 15 in the place of equation 16. This substitution gives the transmission-line equation for a radial circuit with finite conductivity,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) = Lc\frac{\partial^2 \Phi}{\partial t^2} + \frac{c}{\sigma l}\frac{\partial \Phi}{\partial t}. \tag{20}$$

Case 1:

Outer Conductors of Infinite Conductivity.

As shown above, the following transmission-line equation for the potential applies to radial conductors of infinite conductivity:

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} = Lc\frac{\partial^2 \Phi}{\partial t^2}. \tag{21}$$

Substitution of the dimensionless variables,

$$\xi = r/r_i, \quad \tau = t/r_i^2\mu_i\sigma_i, \quad \text{and} \quad \phi = \Phi/\Phi_o, \tag{22}$$

leads to the equation

$$\frac{\partial^2\phi}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\phi}{\partial\xi} = \delta_o^2\frac{\partial^2\phi}{\partial\tau^2}, \tag{23}$$

where $\delta_o = r_i\sqrt{Lc}/r_i^2\mu_i\sigma_i$.

We shall solve this equation by means of a Laplace transformation. The Laplace transform of equation 23 is

$$\frac{\partial^2\bar{\phi}}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\bar{\phi}}{\partial\xi} = \delta_o^2 s^2\bar{\phi} - s\phi(+0) - \phi'(+0). \tag{24}$$

The following is a list of the initial and boundary conditions.

Initial Conditions

$$\phi = 0 \quad \text{at} \quad \tau = 0, \tag{25}$$

$$\frac{\partial\phi}{\partial\tau} = 0 \quad \text{at} \quad \tau = 0. \tag{26}$$

The first condition mathematically states that the potential is zero everywhere at time zero. The second condition states that the change of the potential with respect to time at time zero is also zero. This implies that some type of "inertia" or inductance must be overcome before the potential will change at time zero.

Boundary Conditions

$$\phi = 1 \qquad \text{at} \quad \xi = r_o/r_i, \tag{27}$$

$$2\pi\int i_i r dr = 2\pi r_i i_o \qquad \text{at} \quad \xi = 1. \tag{28}$$

The first boundary condition states that at $r = r_o$ the potential is

instantaneously set to Φ_o. The second condition is interpreted as the integral of the current density over the area in the inner conductor is equal to the line integral of the skin-current density that enters the inner conductor at $r = r_i$.

The Laplace transformation of equation 27 is

$$\bar{\phi} = 1/s \qquad \text{at} \quad \xi = r_o/r_i. \tag{29}$$

The Laplace transformation of equation 28, carried out in the Appendix, is

$$\frac{\bar{\phi} I_i(\delta_i^2 s^2 + s)^{\frac{1}{2}}}{(\delta_i^2 s^2 + s)^{\frac{1}{2}} I_o(\delta^2 s^2 + s)^{\frac{1}{2}}} = \frac{\mu_i}{Ld}\frac{1}{s}\frac{\partial\bar{\phi}}{\partial\xi} \qquad \text{at} \quad \xi = 1. \tag{30}$$

Solution

Solution of equation 24 with the conditions of 25, 26, 29, and 37 is

$$\bar{\phi} = A I_o(\delta_o s\xi) + BK_o(\delta_o s\xi). \tag{31}$$

$$A = \frac{\frac{1}{s} - BK_o\left(\delta s\frac{r_o}{r_i}\right)}{I_o\left(\delta_o s\frac{r_o}{r_i}\right)}, \tag{32}$$

and

$$B = \frac{I_o(a)Y(s) - cI_i(a)}{s\,[K_o(b)I_o(a) - K_o(a)I_o(b)]Y(s) - c\,[K_o(b)I_1(a) - K_1(a)I_o(b)]}, \tag{33}$$

where

$$a = \delta_o s, \quad b = \delta_o s r_o / r_i, \quad c = \delta_o \mu_i / Ld, \tag{34}$$

and

$$Y(s) = \frac{I_1(\delta_i^2 s^2 + s)^{\frac{1}{2}}}{(\delta_i^2 s^2 + s)^{\frac{1}{2}} I_o(\delta_i^2 s^2 + s)^{\frac{1}{2}}}. \tag{35}$$

Inversion of this solution to position and time coordinates is formidable. The transformation may be performed by the method of residues, application of which requires the poles of the equation. The poles are also equal to the negative of the time constants.

Time Constants

Since our primary concern in this investigation is the rate at which a conductor in a circuit can be brought to full current, we shall focus our efforts on deriving the time constant of the above solution. The poles of the solution are the roots of the denominator of B. To emphasize the effect of circuit size on the time constant, we rearrange the denominator of B to

$$c[K_o(b)I_1(a) + K_1(a)I_o(b)]Y(s)\left\{\frac{K_o(b)I_o(a) - K_o(a)I_o(b)}{[K_o(b)I_1(a) - K_1(a)I_o(b)]\,c} - \frac{1}{Y(s)}\right\} \tag{36}$$

The term on the left within the brackets, from here on referred to as K, is a function of s, the properties of the radial part of the circuit, and r_o/r_i. The term on the right within the brackets is a function of s and the properties of the inner conductor. A plot of both terms within the brackets is provided in figure 2, where the ratio of r_o/r_i is a parameter. K appears as a straight line with a slope which becomes more negative as r_o/r_i is increased. $Y(s)$ is

unaltered by changes in the radius ratio. The root of equation 43 is the value of s where the two terms in the brackets are equal (the point of intersection in figure 2). As the size ratio approaches infinity, one sees in figure 2 that the point of intersection approaches zero, and, thus, the time constant approaches infinity. Figure 2 further shows that $1/Y(s)$ intersects the abscissa at the value of $s = s_o$ where $\delta_i^2 s_o^2 - s_o = \lambda_o^2$. λ_o is the first zero of J_o, the Bessel function of the first kind, of order 0. $\lambda_o^2 = 5.783186.^2$ K intersects the abscissa at $s = 0$. The time constant, $\tau' = -1/root \equiv -1/s_r$, is therefore bound between $-1/s_o$ and infinity, and is a function of the size ratio r_o/r_i.

We now wish to develop an approximate analytic expression of the time constant as a function of the size ratio. The limit of K as s approaches zero is

$$\lim_{s \to 0} K = -\frac{s\delta_o}{c} \ln \frac{r_o}{r_i}. \tag{37}$$

(c is defined in equation 34.) An approximation of $1/Y(s)$ for small s is

$$\lim_{s \to 0} \frac{1}{Y(s)} \approx 2\left(1 - \frac{s}{s_o}\right). \tag{38}$$

Combination of the above equations leads to

$$\frac{\tau'}{\tau_o} = \frac{s_o}{s_r} = 1 + \frac{|s_o|\delta_o}{2c} \ln \frac{r_o}{r_i}. \tag{39}$$

This approximation of the time constant as a function of the size ratio is plotted in figure 3 along with the roots of equation 43.

This figure shows that the time constant increases without bound as the ratio of the radius of the perfectly conducting outer leads to the radius of the inner conductor is increased.

Case 2:

Outer Conductors of Finite Conductivity.

The radial circuit geometry is again used here; however, this time the circuit maintains a finite conductivity. The transmission-line equation for a radial conductor with a finite conductivity is

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} = Lc\frac{\partial^2 \Phi}{\partial t^2} + \frac{c}{\sigma_o \ell}\frac{\partial \Phi}{\partial t}, \tag{40}$$

as derived earlier. Substituting into the above equation the previously defined dimensionless parameters from equation 22 gives

$$\frac{\partial^2 \phi}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \phi}{\partial \xi} = \delta_o^2\left(\frac{\partial^2 \phi}{\partial \tau^2} + \beta\frac{\partial \phi}{\partial \tau}\right), \tag{41}$$

where

$$\beta = \frac{r_i^2 \mu_i \sigma_i}{\sigma_o \ell L}. \tag{42}$$

The self-inductance of the inner conductor is proportional to its permeability, μ_i. The resistance of the inner conductor is inversely proportional to the radius squared and the conductivity, $1/r_o^2\sigma_i$. Likewise, the resistance of the outer conductor is inversely proportional to the conductivity and the thickness of the plates, $1/\ell\sigma_o$. Thus, β can be thought of as the ratio of the inductive time constants, L/R, of the inner conductor and outer radial leads.

The Laplace transform of equation 23 is

$$\frac{\partial^2\bar{\phi}}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\bar{\phi}}{\partial\xi} = \delta_o^2[s^2\bar{\phi} - s\phi(+0) - \phi'(+0) + \beta(s\bar{\phi} - \phi(+0))]. \quad (43)$$

The same initial and boundary conditions apply as in case 1; however, i_o obeys the equation from the transmission-line analysis for a conductor of finite conductivity, equation 19, which is rewritten here as

$$-r\frac{\partial\Phi}{\partial r} = Lr\frac{\partial i_o}{\partial\tau} + \frac{r}{\sigma l}i_o. \quad (44)$$

The dimensionless Laplace transform form of this equation is

$$-\frac{\Phi_o}{r_i}\frac{\partial\bar{\phi}}{\partial\xi} = \left(\frac{Ls}{r_i^2\mu_i\sigma_i} + \frac{1}{\sigma_o l}\right)\bar{i}_o. \quad (45)$$

Following the same arguments as developed in the appendix with the appropriate substitution of equation 45 provides the boundary condition

$$\frac{\mu_i}{(s+\beta)Ld}\frac{\partial\bar{\phi}}{\partial\xi} = \frac{I_1(\delta_i^2 s^2 + s)^{1/2}}{s(\delta_i^2 s^2 + s)^{1/2} I_o(\delta_i^2 s^2 + s)^{1/2}}\phi \quad \text{at} \quad \xi = 1. \quad (46)$$

Solution

Solution of equation 43 with the derived initial and boundary conditions in equations 25, 26, 29, and 46 gives the Laplace transform solution

$$\bar{\phi} = AI_o[\delta_o(s^2 + \beta s)^{1/2}\xi] + BK_o[\delta_o(s^2 + \beta s)^{1/2}\xi]. \quad (47)$$

$$A = \frac{\frac{1}{s} - B K_o(b)}{I_o(b)}. \tag{48}$$

$$B = \frac{\frac{a}{\delta_o cs} Y(s) I_o(a) - I_1(a)}{sY(s)[K_o(b)I_1(a)+K_1(a)I_o(b)]\left\{\frac{a}{\delta_o cs}\frac{K_o(b)I_o(a)-K_o(a)I_o(b)}{K_o(b)I_1(a)+K_1(a)I_o(b)} - \frac{1}{Y(s)}\right\}}. \tag{49}$$

$$a = \delta_o(s^2 + \beta s)^{\frac{1}{2}}, \quad b = \delta_o(s^2 + \beta s)^{\frac{1}{2}} r_o/r_i, \quad c = \delta_o \frac{\mu_i}{Ld}, \tag{50}$$

and Y(s) is the same as in case 1, equation 35.

As in the first case, inversion of this solution to time coordinates is difficult; we shall again focus on obtaining the time constant of the problem.

Time Constants

We show in figure 4, as we did in figure 2, a plot of the two terms within the large brackets of equation 58 for several values of β and a r_o/r_i ratio of 10/1. When β is set to zero, we recover the analysis in Case 1. As β is increased, the K line from the original analysis is offset to the left by β. Again, the poles of the solution are those values of s where the two curves intersect. The shift of K by variations in β precludes the time constant from approaching infinity even as the size of the circuit approaches this limit. We see that as β is increased, the time constant, which is the inverse of the absolute value of the roots, decreases. The dimensionless time constant of a circuit of finite conductivity is constrained between the two limits $-1/\beta$ and $-1/s_o$.

Again we search for an analytic expression of the time constant. Taking the limit of the left term in the large brackets of equation 58 as $s \longrightarrow 0$ gives

$$\lim_{s \longrightarrow 0} K = \frac{a^2}{\delta_o c s} \ln \frac{r_o}{r_i} = \frac{(s + \beta)\delta_o}{c} \ln \frac{r_o}{r_i}. \tag{51}$$

The approximation of $1/Y(s)$ for small s was derived in case 1, equation 38. Combination of the two limiting forms provides an analytic approximation of the time constant

$$\frac{\tau'}{\tau_o} = \frac{s_o}{s_r} = \frac{1 - \dfrac{s_o \delta_o}{2c} \ln \dfrac{r_o}{r_i}}{1 + \dfrac{\beta \delta_o}{2c} \ln \dfrac{r_o}{r_i}}. \tag{52}$$

The time constants derived from the roots of equation 49 and those satisfying the above approximation are plotted *versus* the size of the system in figure 5 for two values of β: one $> s_o$ and the other $< s_o$.

Discussion

δ_o and δ_i are both typically much less than 1 and were set equal to each other and to 1×10^{-5} in all of the figures. Only for very short times, s on the order of $1/\delta^2$, does the exact value of δ play a significant role. In the time frame of interest, which is the long time referred to in chapter 1, s is $O(1)$. The above equations can, therefore, all be simplified by setting the δ's to zero. This approximation has little effect on the present "long-time" results and explains why values of the δ's are implicitly eliminated from the analytic approximations of the time constant derived in cases 1 and

2.

As mentioned, β is the ratio of the inductive time constants of the inner conductor to the outer radial leads. In case 1, where the leads are of infinite conductivity or zero resistance, the inductive time constant of the leads is infinite, and β is zero. Setting β to zero in case 2 reduces the solutions to those of case 1.

For the system with infinitely conductive leads, the time constant is proportional to the logarithm of the ratio of the size of the leads to the size of the inner conductor. On the other hand, the system with leads of finite conductivity has a time constant that is bound between the time constant of the inner conductor and that of the leads. In general, the time constant of the system may either increase or decrease with the size ratio of the system, but it becomes independent of the size of the system as the size ratio approaches infinity.

The problem we have just addressed is analogous to the problem of determining the time constant of a circuit with two inductors in series. In that situation, the inductances are added, and the resistances are added. The time constant for the circuit is then[3]

$$\tau' = \frac{L_o + L_i}{R_o + R_i}. \tag{53}$$

Some rearrangement gives

$$\frac{\tau'}{L_i/R_i} = \frac{1 + L_o/L_i}{1 + R_o/R_i}, \tag{54}$$

which is analogous to the analytic approximation provided in equation

52. This investigation demonstrates that the present method of adding inductances to determine the overall time constant is an approximation (see figures 3 and 5) and that the approximation is less accurate for the smaller size ratios ($\ln r_o/r_i = O(1)$).

Conclusions

A rigorous investigation has been undertaken to determine the time constant of a radial circuit with and without resistance. The circuit with an infinite conductivity has a time constant that is proportional to the logarithm of the size of the system: as the size approaches infinity, so does the time constant. For a circuit with finite conductivity, the time constant is bound between two limits and is independent of size ratio for large systems. An analytic approximation to the time constant is provided for comparison with the rigorous evaluation and shows that the approximation, which is used in most texts of circuit analysis, is less accurate for smaller systems — circuits where the leads are approximately as wide as the cell of interest. This analysis is a first approximation to the time constant of a circuit containing an electrochemical power source and is an applicable design tool for rapidly discharging systems.

Appendix

On the right side of equation 28, i_o, the skin-current density, defined in equation 16, is

$$i_o = -\frac{1}{L}\int_0^t \frac{\partial\Phi}{\partial r}dt = -\frac{r_i\mu_i\sigma_i\Phi_o}{L}\int_0^\tau \frac{\partial\phi}{\partial\xi}d\tau. \tag{A-1}$$

The Laplace transformation of this equation is

$$\bar{i}_o = -\frac{r_i\mu_i\sigma_i\Phi_o}{L}\,\frac{1}{s}\frac{\partial\bar{\phi}}{\partial\xi}. \tag{A-2}$$

Ohm's law gives the relation for current density in the central conductor in terms of a component of electric field

$$i_i = \sigma_i E. \tag{A-3}$$

We have shown in part I that inside a cylindrical conductor the Laplace transform of the axial component of the electric field distribution due to a steady field of unit magnitude applied at the surface is

$$\bar{E} = -\frac{\Phi_o\bar{\phi}}{d} = \frac{I_o[(\delta_i^2 s^2 + s)^{1/2}\xi]}{s\,I_o[(\delta_i^2 s^2 + s)^{1/2}]}, \tag{A-4}$$

I_o is the modified Bessel function of the first kind, of order 0.[4] Duhamel's Duhamel's superposition formula,[4] an integration of the solution to the linear equations applicable to the system, is used to describe the current in the conductor

$$i_i = \sigma_i\int_0^\tau E(\xi,\tau-\tau')\frac{\partial E(\tau')}{\partial\tau}d\tau'. \tag{A-5}$$

The Laplace transform of this equation is equal to the product of the Laplace transform of its parts.[4] The Laplace transform of $E(\xi,\tau-\tau')$ is given in equation A-4. The Laplace transform of the derivative of E with respect to τ is

$$\overline{\frac{\partial E}{\partial \tau}} = s\overline{E}. \tag{6}$$

Taking the Laplace transform of equation 34 and substituting into it equations 33 and 35 gives

$$\frac{2\pi r_i^2 \sigma_i \Phi_o \overline{\phi}}{d} \int_0^1 \frac{\xi I_o[(\delta_i^2 s^2 + s)^{1/2} \xi]}{I_o(\delta_i^2 s^2 + s)^{1/2}} d\xi = \frac{2\pi r_i^2 \mu_i \sigma_i \Phi_o}{L} \frac{1}{s} \frac{\partial \overline{\phi}}{\partial \xi}. \tag{A-7}$$

After integrating and some rearrangement, we get the boundary condition,

$$\frac{\overline{\phi} I_i(\delta_i^2 s^2 + s)^{1/2}}{(\delta_i^2 s^2 + s)^{1/2} I_o(\delta^2 s^2 + s)^{1/2}} = \frac{\mu_i}{Ld} \frac{1}{s} \frac{\partial \overline{\phi}}{\partial \xi} \qquad \text{at} \quad \xi = 1. \tag{A-8}$$

Acknowledgments

This work was supported by the Assistant Secretary for Conservation and Renewable Energy, Office of Transportation Technologies, Electric and Hybrid Propulsion Division of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

List of Symbols

Roman

C	capacitance, C/V
c	capacitance per area, C/V-cm^2
d	distance between radial conductors, cm
E	electric field, V/cm
$\overline{E}$	Laplace transform of E, V/cm

i_i	current density in inner conductor, A/cm^2
i_o	current density in outer leads, A/cm
$\overline{i_o}$	Laplace transform of i_o
l	thickness of radial leads, cm
L	inductance, $V\text{-}s^2/C\text{-}cm^2$
r	radial coordinate, cm
r_i	radius of inner conductor, cm
r_o	radius of outer leads, cm
R	resistance, Ω
s	Laplace transform of dimensionless time, τ
t	time, s

Greek

β	ratio of the inductive time constants
δ	dimensionless parameter
μ	permeability, $V\text{-}s^2/C\text{-}cm$
ξ	dimensionless radial coordinate
σ	conductivity, S/cm
τ	dimensionless time
ϕ	dimensionless potential
$\overline{\phi}$	Laplace transform of ϕ
Φ	potential, V

subscripts

i	property of inner conductor
o	property of outer leads

References

1. W. G. V. Rosser, "What Makes an Electric Current Flow," *Am. J. Phys.*, 31, 884-5 (1963).

2. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, Inc., New York, p.409 (1964).

3. R. E. Thomas and A. J. Rosa, *Circuits and Signals: An Introduction to Linear and Interface Circuits*, John Wiley and Sons, Inc., New York (1984).

4. F. B. Hildebrand *Advanced Calculus for Applications*, 2nd Edition, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1976).

Figure Captions:

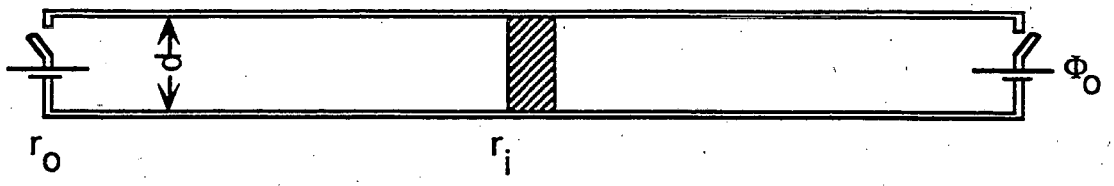
Figure 1. Radial circuit of radius r_o containing a cylindrical conductor of radius r_i.

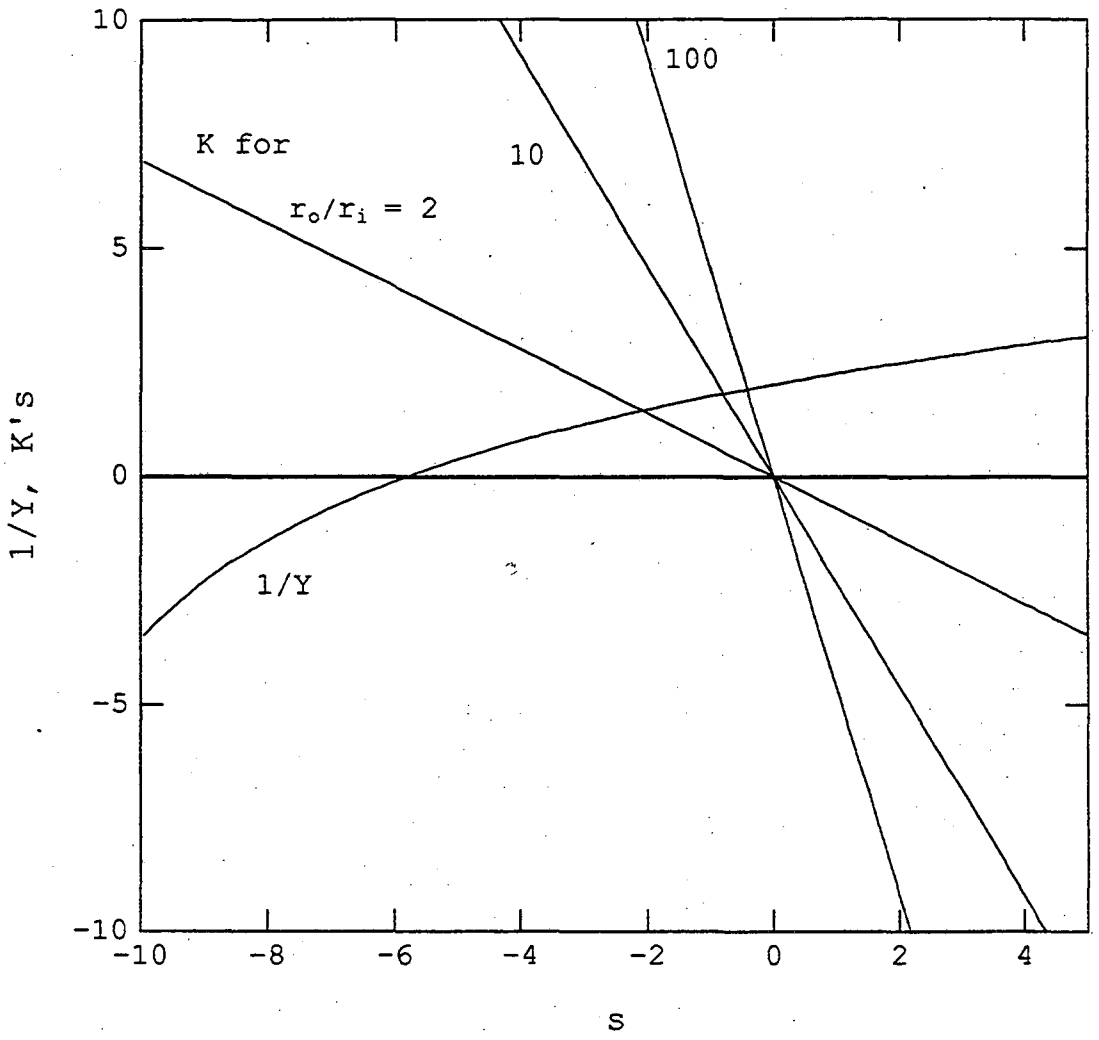
Figure 2. $1/Y$ and K *versus* s for three values of r_o/r_i. $\delta_o = \delta_i = 1\times10^{-5}$.

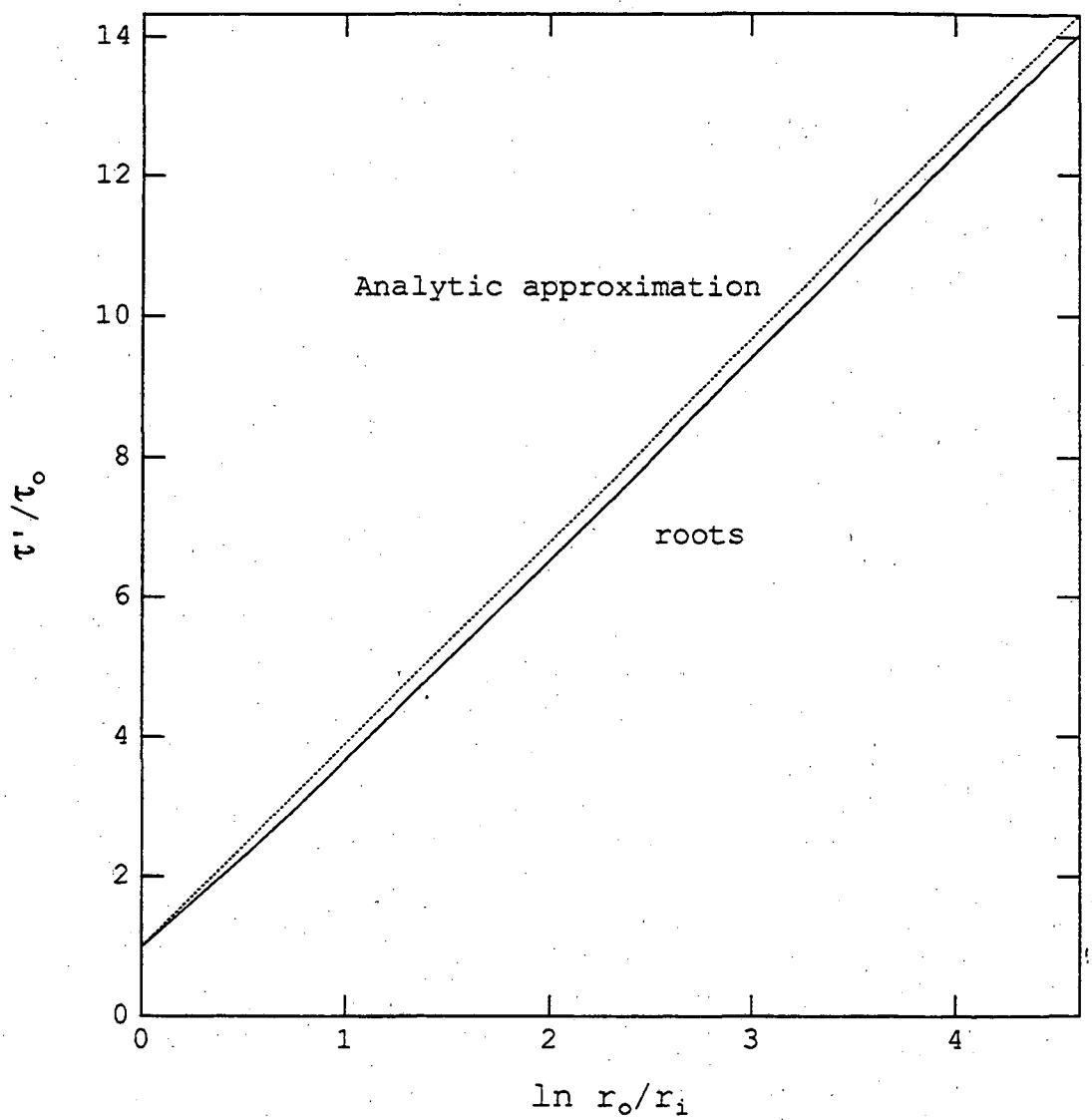
Figure 3. The time constant and an analytic approximation of the circuit of infinite conductivity *versus* the logarithm of the size ratio.

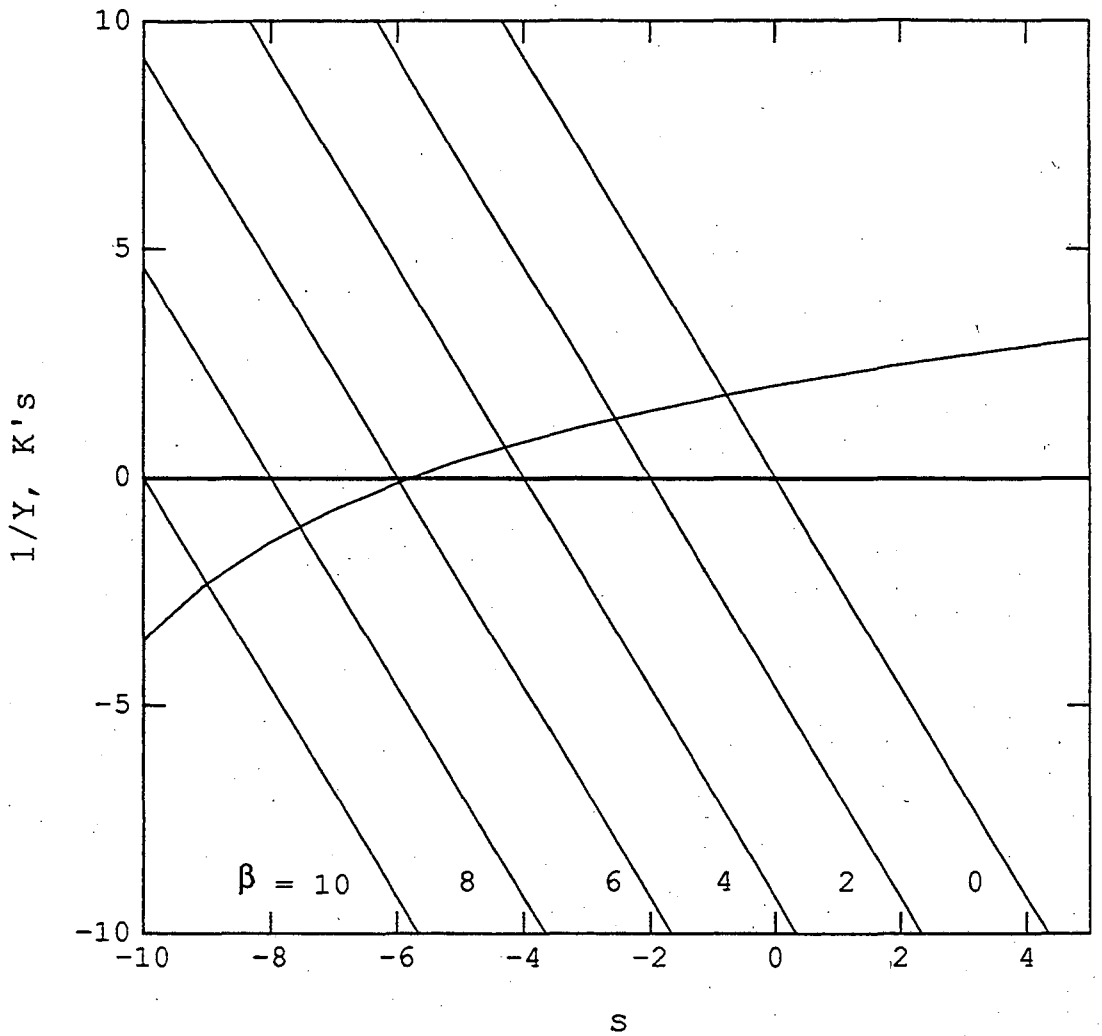
Figure 4. $1/Y$ and K *versus* s for different values of β. $r_o/r_i = 10$.

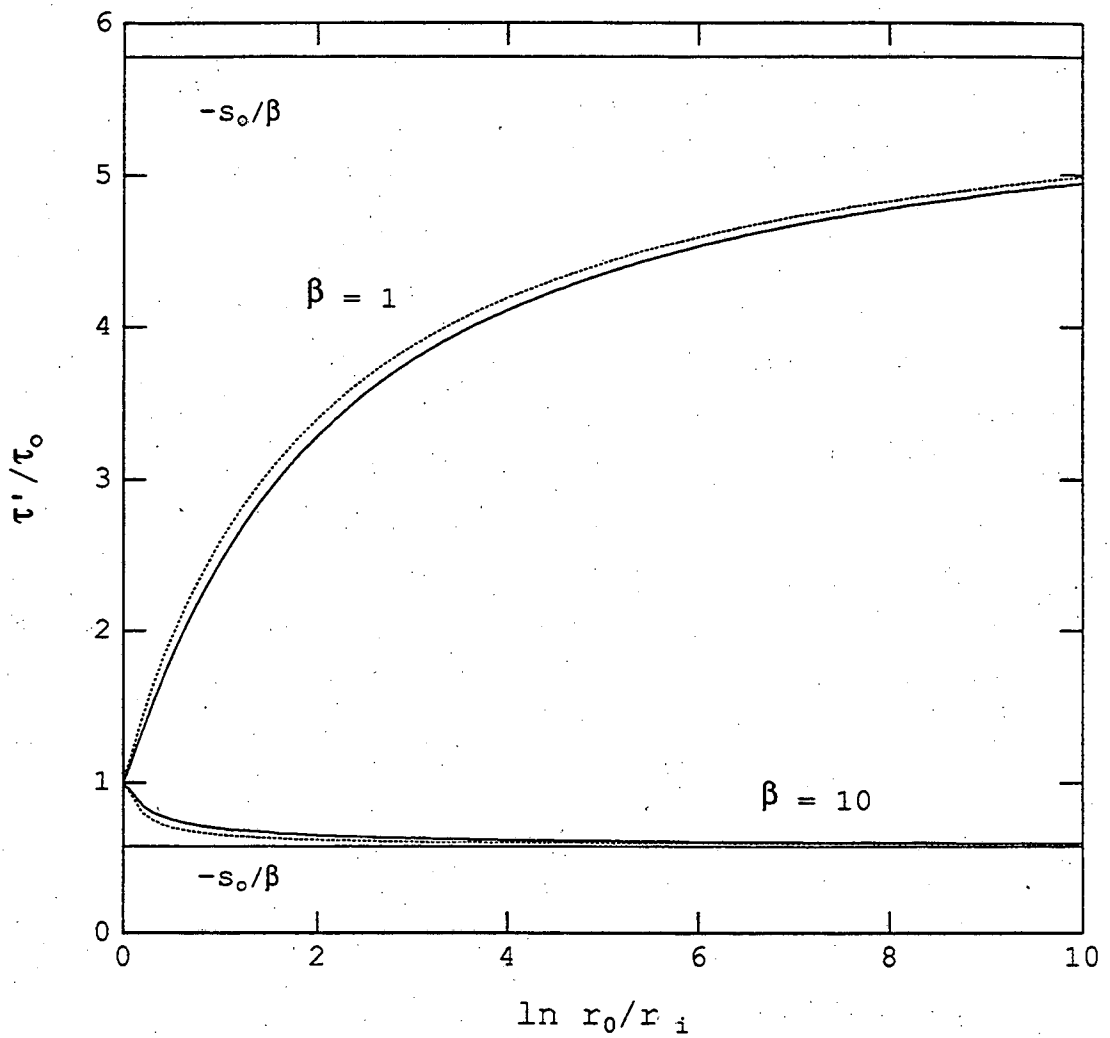
Figure 5. Time constants *versus* size for two values of β. The dashed curves are the analytic approximations.


d
Φ_o
r_o
r_i


10
5
0
-5
-10
-10
-8
-6
-4
-2
0
2
4
s
1/Y, K's
K for
r_o/r_i = 2
10
100
1/Y


14
12
10
8
6
4
2
0
0
1
2
3
4
Analytic approximation
roots
τ'/τ_o
ln r_o/r_i


10
5
0
-5
-10
1/Y, K's
β = 10
8
6
4
2
0
-10
-8
-6
-4
-2
0
2
4
s


6
5
4
3
2
1
0
$-s_o/\beta$
$\beta = 1$
$\beta = 10$
$-s_o/\beta$
τ'/τ_o
0
2
4
6
8
10
$\ln r_0/r_i$

LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
TECHNICAL INFORMATION DEPARTMENT
BERKELEY, CALIFORNIA 94720