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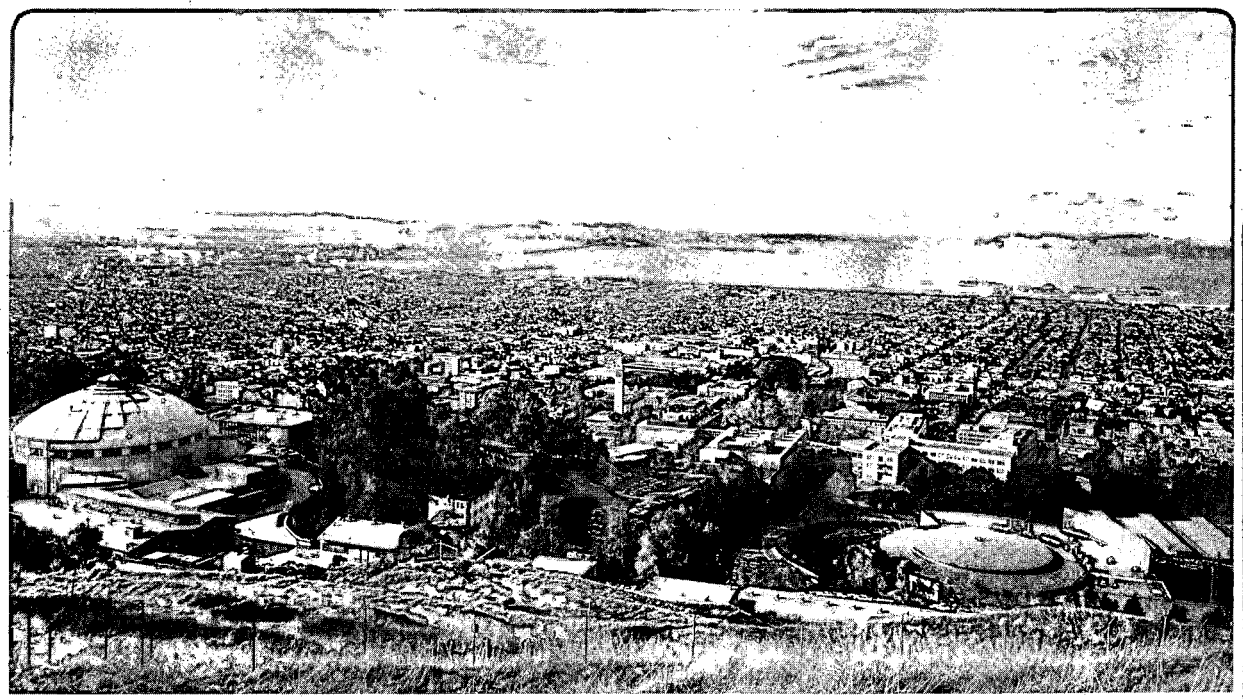

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A Numerical Dual-Porosity Model with Semi-Analytical Treatment of Fracture/Matrix Flow

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Abstract

A new dual-porosity model is developed for single-phase fluid flow in fractured/porous media. Flow is assumed to take place through the fracture network, and between the fractures and matrix blocks. The matrix blocks are treated in a lumped-parameter manner, with a single average pressure used for each matrix block. Rather than assuming that fracture/matrix flux is proportional to the difference between the fracture pressure and matrix pressure at each point, as is done in the Warren-Root model, we use a nonlinear equation which more accurately models the flux over all time regimes, including both early and late times. This flux equation is compared with analytical solutions for spherical blocks with prescribed pressure variations on their boundaries. The nonlinear flux equation is also used as a source/sink term in the numerical simulator TOUGH. The modified code allows more accurate simulations than the conventional Warren-Root method, with a large savings (about 90%) in computational time compared to methods which explicitly discretize the matrix blocks.

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Introduction

Numerical simulation of flow processes in fractured rock masses is a formidable task, due to the often complex geological and hydrological characteristics of such formations. The specific geometry and other characteristics of the fracture system are generally not known, so it is not possible to explicitly model individual fractures or individual matrix blocks. To circumvent this difficulty, so-called "double-porosity" models are often used [*Barenblatt et al.*, 1960; *Warren and Root*, 1963]. In double (or dual)-porosity models, knowledge of the actual geometric and hydrological features of the fracture network is not required, but instead only "average" properties, such as a typical fracture spacing, are needed. In a numerical simulation of a flow process in a dual-porosity system, the individual computational cells are assumed to be sufficiently large so that it is meaningful to assign suitably-averaged "effective" properties to them. Despite this simplification, numerical modeling of dual-porosity reservoirs is still a complicated and costly process. In general, fairly fine spatial discretization is needed in the matrix blocks [*cf.*, *Pruess and Narasimhan*, 1985] in order to resolve the matrix block pressure gradients. Hence modeling of a fractured reservoir requires about an order-of-magnitude more computational cells than would be needed for an unfractured porous medium simulation of a reservoir of the same overall size.

In this paper we present a new method for modeling fluid flow in fractured reservoirs that simulates reservoir behavior more efficiently and economically than methods in which the matrix blocks are discretized. The new method involves a semi-analytical treatment of fracture/matrix interflow, eliminating the need for internal discretization of matrix blocks. This allows one to perform accurate dual-porosity simulations, using a substantially smaller number of cells than would be needed in a fully-discretized simulation.

Dual-Porosity Models

When a single-phase, slightly compressible fluid flows through a macroscopically-homogeneous fractured medium, the fluid pressure in the fractures is governed by the following diffusion equation used in reservoir engineering [*Matthews and Russell,* 1967]:

$$\phi_f c_f \frac{\partial P_f(x_f,t)}{\partial t} = \frac{k_f}{\mu} \nabla^2 P_f(x_f,t) + Q(x_f,t). \tag{1}$$

In this equation, t is the time, x_f is the position vector of a point in the fracture continuum, k_f is the absolute permeability [m^2] of the fracture continuum, ϕ_f is the total fracture porosity, and c_f [Pa^{-1}] is the total compressibility of the fractures and the fluid within them. Q is a volumetric source/sink term that represents the net addition of fluid to the fracture system from the matrix blocks, per unit of total volume; its dimensions are [m^3/m^3s]. The pressure P_f represents the fluid pressure in the fractures, averaged over some sufficiently large representative elementary volume [REV; see *Chen,* 1989]. (It is not clear that such a length scale will always exist [*Long and Witherspoon*, 1985; see also *Neuman and Orr*, 1993], since heterogeneities in fracture spacing, aperture, etc. may occur at all length scales. For our present purposes, we will consider only fractured formations which are macroscopically homogeneous on some scale). The ∇^2 operator represents the Laplacian, which is the divergence of the gradient. The fracture continuum is assumed to occupy all of the physical space spanned by the variable x_f, with the actual pore volume of the fractures accounted for by the porosity factor.

A dual-porosity model can be formulated by first imagining that, at each point x_f, there is located a matrix block of some specified shape. Inside each block the fluid pressure P_m will, in general, vary from point to point. Two position variables are

needed to identify a point inside a matrix block; x_m will locate the point within the block, relative to, say, the block's center of gravity, while x_f is needed as a label to fix the location of that particular block within the fracture continuum. Fluid flow *within* each matrix block is governed by an equation of a similar form as (1), which can be written as

$$\phi_m c_m \frac{\partial P_m(x_m, t; x_f)}{\partial t} = \frac{k_m}{\mu} \nabla^2 P_m(x_m, t; x_f) . \tag{2}$$

In this equation, the parameters have meanings analogous to those in eq. (1). The derivatives implicit in the operator ∇^2 are taken with respect to the local variable x_m, while the variable x_f is merely used as a label. The fracture/matrix interflow term Q does not appear explicitly in eq. (2) since, whereas the interflow is assumed to be *distributed* throughout the fracture continuum as a source/sink term, the interflow enters the matrix blocks only at their *boundaries*. The pressures at the outer boundary of a given matrix block located at point x_f in the fracture continuum are always assumed to be equal to the fracture pressure at that point - *i.e.*, if x_m is on the boundary of the matrix block, then $P_m(x_m, t; x_f) = P_f(x_f, t)$. This is equivalent to ignoring the existence of fracture skin effects [see *Moench,* 1984], which if present would cause a finite pressure change across a small localized region of the matrix block adjacent to the fractures.

The set of equations (1) and (2) actually represents a single equation for the fracture continuum, along with a family of equations for the matrix blocks that are located at each point x_f. These equations are coupled through the term Q, which can be found by integrating the flux out of the boundary of each matrix block, using Darcy's law [see *Duguid and Lee*, 1977]:

$$Q(x_f, t) = \frac{-1}{V_m} \int_{\partial V_m} \frac{k_m}{\mu} \frac{\partial P_m}{\partial n} dA \ , \tag{3}$$

where the derivative of P_m is taken in the direction of the outward unit normal vector to the boundary ∂V_m of the block, and the integral is taken over the entire boundary. A well-posed boundary-value problem for the system of equations (1-3) would typically require initial conditions for P_m and P_f, as well as boundary conditions for the pressures at the outer boundary of the macroscopic region under investigation, *i.e.,* at the outer boundary of the x_f domain. If the initial state were one of local equilibrium, as would often be the case, we would have $P_f(x_f, t=0) = P_m(x_m, t=0; x_f)$ at each point x_f.

Dual-porosity models of the type discussed above, in which diffusion equations are solved in both the fracture and the matrix systems, are sometimes used in numerical simulations. An example is the MINC (Multiple INteracting Continua) method [*Pruess and Narasimhan,* 1985], in which the matrix blocks are discretized into nested shell-like cells. This approach is much more efficient, and under most conditions as accurate, than using a three-dimensional discretization of the matrix blocks. In order to achieve high accuracy over all time scales, however, we have found that about ten computational cells are needed in each matrix block. If only one cell is used to model each matrix block, the approach is basically a numerical implementation of the Warren-Root model, which is discussed below. As is well known, when solving problems in dual-porosity media, the Warren-Root model is inaccurate during a certain intermediate time regime [*cf., Najurieta,* 1980]. The MINC method approaches the exact response as the number of nested matrix shells increases. The accuracy of the method we have developed, which treats fracture/matrix flow with a nonlinear ordinary differential equation, will be tested by comparison with MINC-type simulations.

Warren-Root Lumped-Parameter Models

The *Warren and Root* [1963] model is a simplified form of the dual-porosity model in which no attempt is made to solve the diffusion equation within each block, but instead the blocks are treated in a "lumped parameter" fashion. The pressure in the matrix blocks is governed by an ordinary, rather than partial, differential equation. If implemented in numerical simulators in the form of a source/sink term for the fracture elements, the amount of computational time spent on solving for the matrix block pressure, and the fluid-interaction term Q, becomes negligible compared to the time spent solving the diffusion equation (1) in the fracture continuum. This model can be derived by first replacing the pressure distribution in each block, $P_m(x_m, t; x_f)$, by the average pressure within the block,

$$\bar{P}_m(x_f, t) = \frac{1}{V_m} \int_{V_m} P_m(x_m, t; x_f)\, dV \,. \tag{4}$$

A more rigorous definition of $\bar{P}_m$ would involve some sort of weighted average over the block, to account for the fact that the fluid compressibility varies with the thermodynamic state of the fluid. However, for isothermal single-phase flow, with moderate pressure variations, the fluid compressibility is nearly constant, and definition (4) suffices. Eq. (1) can still be used for the pressure within the fracture network, but eq. (2) governing the pressure distribution within the matrix blocks is no longer meaningful, since the pressure P_m is no longer defined at each point x_m within the matrix block. Instead, we integrate eq. (2) over an entire matrix block centered at point x_f, use the divergence theorem to convert the volume integral of $\nabla^2 P_m$ into a surface integral of $\partial P_m / \partial n$, and divide the resulting equation by V_m, to arrive at

$$\phi_m c_m \frac{\partial \bar{P}_m(x_f, t)}{\partial t} = \frac{1}{V_m} \int_{\partial V_m} \frac{k_m}{\mu} \frac{\partial P_m}{\partial n}\, dA \,. \tag{5}$$

Comparison of eq. (5) with eq. (3) shows that the mean pressure in the matrix block is governed by the following ordinary differential equation:

$$\phi_m c_m \frac{d\bar{P}_m(x_f,t)}{dt} = -Q(x_f,t). \tag{6}$$

Equations (1) and (6) now govern the behavior of a lumped-parameter type dual-porosity model. Note that since the local variable x_m within each matrix block has been removed by integration, Q cannot be evaluated as in eq. (3), but must somehow be related to the two pressures P_f and $\bar{P}_m$.

In order to maintain the linearity and relative simplicity of the system of differential equations, *Warren and Root* [1963] chose to model the flux term Q by assuming that it is directly proportional to the difference between P_f and $\bar{P}_m$:

$$Q(x_f,t) = \frac{-\alpha k_m}{\mu}(P_f - \bar{P}_m), \tag{7}$$

where α is a parameter that depends on block shape, and has dimensions of area^{-1} [m^{-2}]. The governing equation (6) for $\bar{P}_m$ then takes the form

$$\phi_m c_m \frac{d\bar{P}_m(x_f,t)}{dt} = \frac{\alpha k_m}{\mu}(P_f - \bar{P}_m). \tag{8}$$

Expressions (7) and (8) for the flux and the matrix pressure are often referred to as the "quasi-steady-state" approximation [*Barker*, 1985; *Chen*, 1989]. This terminology can be understood by considering the basic problem in which the fracture pressure P_f, which serves as the boundary condition for the matrix block, increases abruptly at $t = 0$

from its initial value P_i to a new value P_o. In this example, and in much of the following discussion, we assume that the matrix block is a sphere of radius a_m; extension to other block geometries is discussed in Appendix A. The exact pressure distribution within the block can then be found by solving eq. (2), subject to the conditions

$$P_m(x_m, t=0) = P_i, \tag{9}$$

$$P_m(|x_m|=a_m, t>0) = P_o. \tag{10}$$

The solution to this problem is found in many standard texts on diffusion or heat transfer, such as *Crank* [1975, p. 91]. The average pressure in the block is found by integrating the pressure distribution, as in eq. (4), to yield

$$\frac{\bar{P}_m - P_i}{P_o - P_i} = 1 - \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp(-n^2\pi^2 k_m t/\phi_m \mu c_m a_m^2). \tag{11}$$

For sufficiently large times, say $t > \phi_m \mu c_m a_m^2/\pi^2 k_m$, all terms in the series beyond the first are negligible, and the pressure varies asymptotically as

$$\frac{\bar{P}_m - P_i}{P_o - P_i} = 1 - \frac{6}{\pi^2}\exp(-\pi^2 k_m t/\phi_m \mu c_m a_m^2). \tag{12}$$

If we now differentiate eq. (12) with respect to t, and eliminate t from the resulting expression, we arrive at the following differential equation for $\bar{P}_m$:

$$\frac{d\bar{P}_m}{dt} = \frac{\pi^2 k_m}{\phi_m \mu c_m a_m^2}(P_o - \bar{P}_m). \tag{13}$$

We now make the assumption that eq. (13) will govern the mean pressure in the matrix block, regardless of whether or not P_f varies with time, to arrive at

$$\frac{d\bar{P}_m}{dt} = \frac{\pi^2 k_m}{\phi_m \mu c_m a_m^2}(P_f - \bar{P}_m). \tag{14}$$

This equation is of the same form as eq. (8), and shows that, for a spherical block, the parameter α is equal to π^2/a_m^2.

Potential difficulties with eq. (14) can be anticipated from the fact that this equation only strictly holds for large times, and even then only for step-function boundary conditions. The errors incurred by using eq. (14) will generally be quite large at "small" times, for any type of boundary condition, as will be demonstrated below. The aim of our work is to incorporate an equation analogous to eq. (14) into a dual-porosity simulator, but which will be accurate over all ranges of time scales, and for more general boundary conditions.

Fully-Transient Coupling Term

Interaction equations that are more accurate than the quasi-steady-state Warren-Root equation have appeared in the petroleum engineering literature in conjunction with the development of analytical solutions for problems such as flow to a well in a radially symmetric dual-porosity reservoir [*deSwaan,* 1976; *Najurieta,* 1980]. These interaction equations are usually found, for example, by taking the Laplace transform of the step-function response, eq. (11). Step-function responses are typically much simpler in algebraic form when written in the Laplace domain, as opposed to the time

domain [*cf., Barker,* 1985]. These methods, however, seem to be restricted to simple reservoir geometries, in which case the solution can be developed analytically. Our intention is to develop the capability of solving dual-porosity problems in macroscopically irregular geometries, which requires a numerical simulator. We therefore want to maintain the computational simplicity inherent in a lumped-parameter formulation of a dual-porosity model, but with eqs. (7) and (8) replaced by equations that more accurately account for fracture/matrix flow interactions. This approach requires the derivation of an equation for Q, which depends on P_f and $\bar{P}_m$, as well as the various physical parameters of the problem, but which does not necessarily have the same exact form as eqs. (7) and (8).

One approach to achieving this goal is that taken by *Dykhuizen* [1990,1991], who used the "integral method" [see *Goodman,* 1964; *Zimmerman and Bodvarsson,* 1989] to derive an approximate solution for diffusion into a slab-like matrix block, under the same step-function boundary conditions given by eqs. (9) and (10). This solution is given by two separate algebraic expressions, the choice of which depends on whether or not t is greater or less than some critical value t_c, which is the time at which the diffusing front reaches the center of the block. *Dykhuizen* [1990] then found the differential equation satisfied by each solution, and assumed that one or the other of these differential equations can be used for the general case of time-varying P_f. The criterion for deciding which differential equation to use is whether or not $\bar{P}_m$ is greater, in some normalized sense, than some critical value that corresponds to $\bar{P}_m(t = t_c)$ for the step-function response. This approach was found to be greatly superior to the Warren-Root model for the step-function boundary condition, and somewhat superior for a ramp-function boundary condition, in which P_f increases linearly with t. *Dykhuizen* [1990] treated only slab-like blocks, which have one dimension much smaller than the other two; extension of this approach to other geometries, such as spheres, would be considerably more complicated [see *Zimmerman and Bodvarsson,*

1989].

Another approach to more accurate treatment of fracture/matrix flow in a dual-porosity model was taken by *Pruess and Wu* [1989]. They used a version of the integral method that was developed by *Vinsome and Westerveld* [1980], in which the pressure distribution in a matrix block is approximated by a function of the form

$$P_m(z_m, t) = P_i + (P_f - P_i + rz_m + sz_m^2)e^{-z_m/\delta}, \tag{15}$$

where $\delta = \sqrt{k_m t / 4\phi_m \mu c_m}$ is a measure of the depth to which the pressure disturbance has penetrated the matrix block, and z_m is the distance from a point in the block to the outer boundary. The coefficients r and s are recalculated at each time-step so that the assumed profile (15) satisfies eq. (2) in an integrated sense over the entire matrix block, and at the outer boundary. The form chosen for the pressure profile in eq. (15) assures that no pressure change is felt *ahead* of the propagating pressure front, since the exponential term will be very small when $z_m > 4\delta$. *Pruess and Wu* [1989] verified the accuracy of this method for a step-function variation in the pressure at the boundary of a cubical block. Their approach also has the advantage of being extendable to blocks of various shapes, through the introduction of a "proximity function" that measures the amount of block volume within a certain distance from the outer boundary. Nevertheless, difficulties can be foreseen for certain types of boundary pressure variations, including cases in which the boundary pressure oscillates at a frequency high enough that the penetration distance of the pressure pulse into the block is much smaller than the block radius. At sufficiently long times, δ will become much greater than z_m at all points in the block, and $\exp(-z_m/\delta) \to 1$, so the approximate pressure profile (15) will essentially be an *undamped* quadratic polynomial. This type of function can never be localized near the outer boundary of the block, as would be required to match the true pressure profile [*cf., Carslaw and Jaeger,* 1959, p. 81].

Our approach is similar to that taken by *Dykhuizen* [1990], except that we would like to use a single differential equation for the coupling term, that would be valid for all times. An advantage of using a single differential equation is that it can be more readily incorporated into an implicit numerical reservoir simulator. The Warren-Root equation is of this form, but is only accurate in the quasi-steady-state regime. Since the Warren-Root interaction equation can be derived by differentiating the large-time approximation to the step-function pressure response, it might be thought that a more general interaction equation could be derived by differentiating the exact step-function pressure response, which is [*Crank,* 1975, p.91]

$$\frac{\bar{P}_m - P_i}{P_o - P_i} = 1 - \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp(-n^2 \pi^2 k_m t / \phi_m \mu c_m a_m^2) , \tag{16}$$

where P_i is the initial pressure in the block, and P_o is the pressure imposed at the outer boundary of the block at $t = 0$. Unfortunately, if we attempt this procedure, it is not possible to eliminate t from explicitly appearing in the resulting differential equation. A related approach is to first find an algebraically simple approximation to the step-function response, and then find the first-order differential equation that it satisfies. The resulting equation would then necessarily be accurate over all time scales for the step-function boundary condition, but would still need to be tested for other boundary conditions. This approach was initiated by *Vermeulen* [1953], who was working on the problem of absorption in ion exchange columns, which is mathematically similar to that of diffusion in a dual-porosity medium. Vermeulen found that the exact step-function pressure response (16) could be approximated, over all time scales, by

$$\frac{\bar{P}_m - P_i}{P_o - P_i} = \left[1 - \exp(-\pi^2 k_m t / \phi_m \mu c_m a_m^2) \right]^{1/2} . \tag{17}$$

Differentiating eq. (17) with respect to t, and then eliminating t from the result, leads to

$$\frac{d\bar{P}_m}{dt} = \frac{\pi^2 k_m}{2\phi_m \mu c_m a_m^2} \frac{(P_o - P_i)^2 - (\bar{P}_m - P_i)^2}{(\bar{P}_m - P_i)} . \tag{18}$$

We now generalize eq. (18) by assuming that P_o represents the fracture pressure P_f, regardless of whether or not P_f varies with time:

$$\frac{d\bar{P}_m}{dt} = \frac{\pi^2 k_m}{2\phi_m \mu c_m a_m^2} \frac{(P_f - P_i)^2 - (\bar{P}_m - P_i)^2}{(\bar{P}_m - P_i)} . \tag{19}$$

When $\bar{P}_m$ is very close to P_f, eq. (19) can be shown to reduce to eq. (14), which shows that the quasi-steady-state response of the Vermeulen equation is the same as that of the Warren-Root equation, and so will be accurate in the large-time limit. We will demonstrate below that eq. (19) is also very accurate in the small-time limit, in which case it reduces to a form very different from that of the Warren-Root equation.

For the step-function boundary conditions, eq. (19) integrates to eq. (17), which is a very close approximation to the exact step-function response, eq. (16). This is illustrated in Fig. 1, in which the exact step-function response, eq. (16), is compared to the "Vermeulen" step-function response, eq. (17). For comparison, we can also define the following "Warren-Root" step-function response as the solution of eq. (14) subject to boundary condition (10) and initial condition (9):

$$\frac{\bar{P}_m - P_i}{P_o - P_i} = 1 - \exp(-\pi^2 k_m t / \phi \mu c_m a_m^2) . \tag{20}$$

This solution is not quite the same as the large-time approximation to the exact solution, eq. (12), since that approximation does not satisfy the intial conditions. Fig. 1 shows that the Warren-Root step-function response is very inaccurate at small times, since it does not predict the correct exponent for the variation of $\bar{P}_m$ with t. This can be demonstrated by expanding eqs. (19) and (20) for small times. The Vermeulen small-time step-function response is

$$\frac{\bar{P}_m - P_i}{P_o - P_i} = \left[\frac{\pi^2 k_m t}{\phi_m \mu c_m a_m^2} \right]^{1/2} . \tag{21}$$

The Warren-Root small-time step-function response is

$$\frac{\bar{P}_m - P_i}{P_o - P_i} = \frac{\pi^2 k_m t}{\phi_m \mu c_m a_m^2} . \tag{22}$$

The exact solution (16) is not in a form that is easily interpreted for small times, since in this limit each term in the series is non-negligible. However, a different form for the exact step-function response, in terms of error functions [*Crank,* 1975, p. 91], can be used to show that

$$\frac{\bar{P}_m - P_i}{P_o - P_i} = \left[\frac{36 k_m t}{\pi \phi_m \mu c_m a_m^2} \right]^{1/2} . \tag{23}$$

The numerical coefficient in the Vermeulen solution, which is π, is smaller than the exact numerical coefficient, $6/\sqrt{\pi}$, by only 7%. More importantly, the Vermeulen equation predicts the correct $t^{1/2}$ behavior for the matrix pressure. In contrast, the

Warren-Root differential equation predicts that the rise in matrix pressure will initially be proportional to t, instead of $t^{1/2}$. This incorrect exponent leads to large errors at very small times (see Fig. 1). This error cannot be remedied by choosing a different numerical constant for α, as is sometimes done [*cf., deSwaan,* 1990], since this would merely alter the multiplicative constant in eq. (22).

It can also be shown that the Vermeulen equation offers an improvement over the Warren-Root equation for the case of a ramp-function increase in P_f; this and other boundary conditions were not considered by *Vermeulen* [1953]. Let the initial and "boundary conditions" for the matrix block be

$$\bar{P}_m(t=0) = P_i \ , \tag{24}$$

$$P_f(t>0) = P_i + Bt \ , \tag{25}$$

where B is some constant with dimensions of pressure/time. The exact solution for $\bar{P}_m$ in this case is [*Crank,* 1975, p. 93]

$$\frac{k_m(\bar{P}_m - P_i)}{\phi_m \mu c_m a_m^2 B} = \frac{k_m t}{\phi_m \mu c_m a_m^2} - \frac{1}{15} + \frac{6}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\exp(-n^2\pi^2 k_m t/\phi_m \mu c_m a_m^2) \ . \tag{26}$$

The ramp-function response predicted by the Warren-Root equation can be found by solving eq. (14), subject to conditions (24) and (25), to yield

$$\frac{k_m(\bar{P}_m - P_i)}{\phi_m \mu c_m a_m^2 B} = \frac{k_m t}{\phi_m \mu c_m a_m^2} - \frac{1}{\pi^2} + \frac{1}{\pi^2}\exp(-\pi^2 k_m t/\phi_m \mu c_m a_m^2) \ . \tag{27}$$

The Vermeulen equation cannot be solved in closed-form for the ramp-function boundary condition, but can be integrated numerically to yield the results plotted in Fig. 2. As was the case for the step-function boundary condition, the Vermeulen equation is more accurate than the Warren-Root equation in predicting the matrix block pressures. (Strictly speaking, in lumped-parameter formulations such as those embodied in the Warren-Root and Vermeulen equations, the boundary pressures enter directly into the differential equations as forcing functions, not as boundary conditions. However, in a physical sense, P_f is still the pressure at the *boundary* of the matrix block).

We have shown that the Vermeulen equation accurately predicts the matrix block pressures, over all ranges of time, for both the step-function and ramp-function boundary conditions. Furthermore, since the Vermeulen equation reduces to the Warren-Root equation for "large times", (*i.e.*, when $\bar{P}_m$ is close to P_f), it should be accurate in the quasi-steady-state regime. Moreover, we can also show, analytically, that the Vermeulen equation predicts the correct small-time exponent for $\bar{P}_m$, for quite general types of variations in P_f. To do this, assume first that for small times P_f varies as Bt^m, where B is some constant, and m is either an integer or a half-integer:

$$P_f = P_i + Bt^m \ . \tag{28}$$

For small times, $\bar{P}_m - P_i$ will be of higher order than $P_f - P_i$, and so eq. (19) can be integrated to yield the small-time approximation

$$\bar{P}_m = P_i + \left[\frac{k_m}{\phi_m \mu c_m a_m^2}\right]^2 \frac{\pi B}{\sqrt{(2m+1)}} t^{m+1/2} \ . \tag{29}$$

The exact small-time approximation can be found from the solution given by *Crank*

[1975, p. 34] for diffusion into a semi-infinite media, which applies to *any* geometry at sufficiently small times:

$$\bar{P}_m = P_i + \left[\frac{k_m}{\phi_m \mu c_m a_m^2}\right]^2 \frac{3B\,\Gamma(m+1)}{\Gamma(m+3/2)} t^{m+1/2}, \tag{30}$$

where $\Gamma(z)$ is the gamma function [*Ghez,* 1988, p. 118]. The Warren-Root equation (14) can be integrated in this case to give

$$\bar{P}_m = P_i + \left[\frac{k_m}{\phi_m \mu c_m a_m^2}\right] \frac{\pi^2 B}{m+1} t^{m+1}. \tag{31}$$

As was found in the specific cases of ramp-function (m=1) and step-function (which can be approximated by letting $m \to 0$) boundary pressures, the Warren-Root method predicts an exponent for the matrix pressure that is too high by 1/2, whereas the Vermeulen equation predicts the correct exponent. The numerical constant in eq. (29) is somewhat smaller than the exact value given by eq. (30), and depends on the coefficient m. In the limit as $m \to 0$, the constant is too low by 7%, and in the limit $m \to \infty$ it is too low by 26%. This second limit is found by using Stirling's approximation [*Ghez,* 1988, p. 119] to show that as $m \to \infty$, $\Gamma(m+1)/\Gamma(m+3/2) \approx 1/\sqrt{m}$. This slight error in the numerical constant, but with the correct exponent, leads to more accurate pressure predictions than does the Warren-Root method, which predicts entirely incorrect exponents for the time dependence.

As a further test of the Vermeulen equation, we consider the case where the fracture pressure P_f at first increases linearly with time, and then asymptotically levels off to some new value, according to

$$P_f = P_i + (P_o - P_i)\left[1 - \exp(-\beta\pi^2 k_m t/\phi_m \mu c_m a_m^2)\right] , \tag{32}$$

where β is some dimensionless constant. This type of variation in P_f approximates, to some extent, a diffusive pressure front such as might exist near a wellbore during injection or withdrawal of fluid. This is the first type of P_f variation we have discussed that includes its own intrinsic time scale, and hence the results cannot be plotted in dimensionless form without considering the value of β. This parameter reflects the rate at which pressure diffuses through the fracture system, relative to the rate at which it diffuses through a matrix block (since the other terms in the exponent in eq. (32) represent the natural time scale for matrix diffusion). Consider the case where $\beta = 100$, which corresponds to, very roughly speaking, a matrix block whose permeability is 100 times less than the effective permeability of the fracture continuum. Fig. 3 shows the average matrix block pressures that were computed from eqs. (14) and (19), as well as the exact solution for this boundary condition, which is [*Crank,* 1975, p. 92]:

$$\frac{\bar{P}_m - P_i}{P_o - P_i} = 1 + \frac{6\beta}{\pi^2}\sum_{n=1}^{\infty}\frac{\exp(-n^2\pi^2 k_m t/\phi_\mu c_m a_m^2)}{n^2(n^2-\beta)}$$

$$- \frac{3}{\pi^2\beta}\left[1 - \pi\sqrt{\beta}\cot(\pi\sqrt{\beta})\right]\exp(-\beta\pi^2 k_m t/\phi_m \mu c_m a_m^2) . \tag{33}$$

This expression has a removable singularity whenever β is equal to the square of an integer, such as when $\beta = 100$. The principal parts of the two singular terms, one arising from $\cot(10\pi)$ and the other from the $n = 10$ term in the summation, in fact cancel out. If we take the limit of eq. (33) as $\beta \to 100$, we arrive at an expression which is

more convenient for computational purposes:

$$\frac{\bar{P}_m - P_i}{P_o - P_i} = 1 + \frac{600}{\pi^2} \sum_{n \neq 10} \frac{\exp(-n^2\pi^2 k_m t / \phi_m c_m a_m^2)}{n^2(n^2 - 100)}$$

$$- \frac{6\exp(-100\pi^2 k_m t / \phi_m \mu c_m a_m^2)}{100\pi^2}, \tag{34}$$

where the summation is taken over all positive integers *except* $n = 10$. Fig. 3 shows that the Vermeulen equation is again far superior to the Warren-Root equation, in this case over almost the entire range of times. The Warren-Root solution does not become accurate until flow into the matrix block is about 90% complete. A similar comparison was also carried out for a fracture pressure that varies exponentially with the *square* of t, and the qualitative conclusions, as well as the curve shapes on a semi-log plot, are very similar to the case shown in Fig. 3.

Each of the examples discussed above showed that the Vermeulen equation accurately predicts the mean matrix pressures when the fracture pressures are known in advance as specified functions of time. (We have also verified that the Vermeulen equation is reasonably accurate for the case where the fracture pressure oscillates sinusoidally in time. This requires the insertion of absolute value signs in appropriate places in eq. (19), and is discussed in Appendix B.) We interpret these results as justifying the incorporation of the Vermeulen equation as the fracture/matrix flow interaction term in a numerical reservoir simulator. We discuss this incorporation below, and present a simulation for a test problem in which both $\bar{P}_m$ and P_f are computed, in a coupled manner, as part of the solution process.

Coupled Dual-Porosity Simulator

Numerical reservoir simulators used for single-continuum systems typically solve eq. (1) by discretizing the reservoir into a number of computational cells, and use some numerical scheme such as finite-differences [*Huyakorn and Pinder,* 1983], finite elements [*Pinder and Gray,* 1977], or integral finite-differences [*Edwards,* 1972; *Narasimhan and Witherspoon,* 1976], to reduce the partial differential equation to a set of algebraic equations. These algebraic equations are solved at each time-step, t_n, in order to yield the pressures in each cell at the next time-step, $t_{n+1} = t_n + \Delta t$. (We refer to the computational units as cells, in order to avoid confusion between "matrix blocks" and "computational grid-blocks"). Our approach is to assign to the computational cells those properties that correspond to the fractured continuum, averaged over a suitably-large REV. Fluid that enters or leaves the fracture system from the matrix blocks is then treated as a source/sink term, the magnitude of which is determined from eqs. (6,19). A certain number of matrix blocks will be associated with each computational cell, with physical properties $\{k_m, \phi_m, a_m, \text{and}\, c_m\}$ that must be entered as input. In general, the matrix properties are allowed to vary from one computational cell to the next. Each computational cell will have associated with it a new variable, $\bar{P}_m$, which represents the average matrix pressure in those matrix blocks that are contained in that cell.

We have implemented this approach into the TOUGH simulator [*Pruess,* 1987], an integral-finite-difference code that has been shown to accurately simulate three-dimensional, single-phase, isothermal flow processes such as those discussed in this paper (as well as non-isothermal and two-phase processes). TOUGH uses an implicit formulation in which the fluxes over the time interval $t \rightarrow t + \Delta t$ are calculated from Darcy's law in terms of the pressures at time $t + \Delta t$. This leads to a system of $3N$ algebraic equations that must be solved at each time-step, where N is the number of computational cells, and the factor 3 arises from the fact that TOUGH solves a mass

balance equation for water and air, as well as an energy balance equation, for each cell. The fracture/matrix interaction equation has been incorporated as an option in a subroutine which is normally used for sources/sinks that represent injection or withdrawal of fluid from a well, etc. We found that an explicit calculation of the fracture/matrix flux, in terms of the current values of $\bar{P}_m$ and P_f at time t, leads, as might be expected [see *Huyakorn and Pinder*, 1983, p. 351], to numerical instabilities for large time-steps. *Rossen* [1977] discusses this problem in the context of capillary-driven flow in two-phase oil/water matrix blocks. Hence we treat the calculation of the fracture/matrix flow implicitly, to avoid numerical instabilities. This requires the calculation of additional contributions, stemming from the fracture/matrix flow, to the terms in the coefficient matrix of the algebraic equations. The total number of algebraic equations to be solved, however, remains equal to three times the number of fracture elements.

As a test of the use of our modified dual-porosity code, consider the problem of linear one-dimensional flow from a boundary that is maintained at some pressure P_o, into a semi-infinite formation that is initially at pressure P_i. We have also tested the modified version of TOUGH on problems involving radial flow to a well, and under constant-flux boundary conditions. However, the problem discussed here seems to most clearly illustrate the different time regimes, and the effects of fracture/matrix flow, since a log-log plot of flux *vs.* time will exhibit straight-line segments. The boundary and initial conditions for this problem are

$$P_f(x_f, t=0) = \bar{P}_m(x_f, t=0) = P_i \ , \tag{35}$$

$$P_f(x_f = 0, t > 0) = P_o \ , \tag{36}$$

$$\lim_{x_f \to \infty} P_f(x_f, t) = P_i \ . \tag{37}$$

The results of the simulation using the new semi-analytical dual-porosity version of TOUGH, incorporating eq. (19) as the fluid coupling term, are presented in Fig. 4. The figure shows the flowrate from the inlet feeding the fractures, as a function of time. In the simulation, the permeabilities were taken as $k_f = 10^{-15}\,\mathrm{m}^2$ and $k_m = 10^{-18}\,\mathrm{m}^2$; the porosities as $\phi_f = 0.001$ and $\phi_m = 0.1$; and the matrix block radii as $a_m = 10\,\mathrm{m}$. The initial temperature was set at 20 °C, and the boundary and initial pressures were taken to be P_i = 10 MPa and P_o = 11 MPa. Under these conditions, the viscosity of water is roughly 0.001 Pa·s, the density is roughly 1000 kg/m^3, and the compressibility is roughly 4.5×10^{-10} / Pa, although the TOUGH code actually uses more accurate values that are computed at each temperature and pressure from empirically-derived equations of state. For simplicity, we assume that the rock is rigid, so that the compressibility term reflects only the compressibility of the water. The semi-infinite fracture continuum was broken up into fourteen computational cells, of length 1 m, 2 m, 4 m, etc. The total length of 16,383 m was sufficiently large so as to simulate a semi-infinite formation, for the duration of the process considered in this problem (10^8 s).

At small times, flow takes place primarily in the fractures, and the flux varies as $t^{-1/2}$, as expected for one-dimensional diffusion. As time progresses, the leakage of fluid into the matrix blocks has the effect of temporarily halting the decline of the flux into the system. According to the Warren-Root method, this leads to an intermediate-time regime in which the overall flux is essentially constant. This is illustrated in the curve labelled "MINC - 1 shell per matrix block", which is a numerical implementation of the Warren-Root model; a regime in which the flux is nearly constant is seen for $10^4 < t < 10^5$ s. Also shown are the results calculated using a fully discretized MINC-type approach, in which each spherical matrix block was broken up into ten nested shells. As the number of shells in the MINC simulation increases, the pressure gradients within the matrix blocks are more accurately simulated, and the overall flux

into the formation approaches that calculated with the new semi-analytical approach; this agreement serves as a verification of the new method. The Warren-Root method overestimates the time needed for flow into the matrix blocks to begin to appreciably influence the overall flowrate into the formation, since it underestimates the amount of matrix influx at small times. As seen clearly in Fig. 4 for $10^2 < t < 10^5$ s, the Warren-Root model gives inaccurate flowrates at intermediate times.

The results of the new method can also be compared to the asymptotic analytical solutions developed by *Nitao and Buscheck* [1991]. They treated the mathematically analogous problem of linearized capillary-driven flow into an unsaturated, fractured formation, and found the leading-order terms for the flux in each of the three time regimes. These three time regimes are physically analogous to the three regimes that have been found to exist for the problem of radial, constant-flowrate injection into a dual-porosity formation [*Streltsova,* 1983]. Following their procedure, we can develop the asymptotic solutions for the present saturated flow problem. At small times the solution corresponds to flow into the fracture network with the matrix blocks assumed to be impermeable, which leads to a mass flowrate of

$$q_{early} = \frac{C_1}{\sqrt{t}} = \rho_w \left[\frac{k_f \phi_f c_f}{\pi \mu t} \right]^{1/2} (P_i - P_o) = 3.78 \times 10^{-4} t^{-1/2} \text{ kg/m}^2\text{s} . \qquad (38)$$

In the late time regime, *Nitao and Buscheck* [1991] showed that the problem becomes asymptotically equivalent to one-dimensional flow into a porous formation whose permeability is essentially equal to k_f (since $k_m \ll k_f$), but whose porosity is equal to $\phi_f + \phi_m$. Since the compressibility terms (which reflect only c_{water}) in both the matrix and fractures are equal, the late-time flowrate is asymptotically given by

$$q_{late} = \frac{C_3}{\sqrt{t}} = \rho_w \left[\frac{k_f (\phi_f + \phi_m) c_f}{\pi \mu t} \right]^{1/2} (P_i - P_o) = 3.80 \times 10^{-3} t^{-1/2} \text{ kg/m}^2\text{s} . \qquad (39)$$

Nitao and Buscheck [1991, eq. (90)] also showed the existence of an intermediate time regime in which the flux drops off as $C_2 t^{-1/4}$. The constant C_2 was found to be equal to $2^{3/2} t_b^{-1/4} C_1$, where t_b is a certain characteristic time at which matrix leakage begins to affect the overall flux into the system. This time constant must be transformed into an analogous value appropriate for the present saturated flow problem. To do this, we first note that for their geometry of parallel fractures separated by a distance $2a$, the specific fracture/matrix surface area is given by $A/V = 1/a$, whereas for the present geometry of spherical matrix blocks of radius a_m we have $A/V = 3/a_m$ [*Zimmerman et al.*, 1990]. If we further identify the hydraulic diffusivity of the matrix blocks as $D_m = k_m/\phi_m \mu c_m$, we find that

$$t_b = \frac{4\pi\phi_m \mu c_m a_m^2}{9k_m}\left[\frac{\phi_f}{\phi_m}\right]^2, \tag{40}$$

which leads to

$$q_{interm} = 2^{3/2}\rho_w (P_o - P_i)\left[\frac{k_f \phi_f c_f}{\pi\mu}\right]^{1/2}\left[\frac{9k_m \phi_m}{4\pi\mu c_m \phi_f^2 a_m^2}\right]^{1/4} t^{-1/4}$$

$$= 2.14\times 10^{-4} t^{-1/4}\ \mathrm{kg/m^2s}. \tag{41}$$

The three asymptotic expressions given by eqs. (38,39,41) are compared in Fig. 5 to the results calculated by our modified version of the TOUGH code. The close agreement over all time scales is further validation of the accuracy of our new method.

The CPU time required for simulating a MINC-type problem with a code such as TOUGH grows linearly with the number of computational cells, since most of the computing effort consists of a Gaussian-elimination inversion of a relatively sparse matrix [see also *Gilman and Kazemi,* 1983]. In the example discussed above, the fully-discretized MINC-type simulation used 155 cells (14×11, plus one large "boundary" cell required to fix the boundary pressure at 11 MPa), whereas the semi-analytical method required only 15 cells. We would expect the ratio of CPU times for the two simulations to be about 155:15 = 10.3, for a savings of 90%, and in fact the time savings was 88%, the slight difference probably attributable to the need to calculate additional contributions to the coefficient matrix. A similar savings is achieved in the amount of computer memory required.

Conclusions

We have developed a new dual-porosity model for single-phase fluid flow in porous/fractured media. The model uses a nonlinear ordinary differential equation to calculate the fracture/matrix interaction term. This equation has been shown to be more accurate than the linear Warren-Root equation, for a wide variety of matrix block boundary conditions. This differential equation has been incorporated into the numerical simulator TOUGH, to serve as a source/sink term for the discretized fracture continuum. The accuracy of the modified semi-analytical code has been verified by comparison with simulations in which the matrix blocks are broken up into a number of concentric shells. For the test problems we have simulated, the modified TOUGH code yields results that are more accurate than can be achieved using as many as ten concentric shells in the matrix blocks. Since the matrix blocks no longer need to be discretized, the total number of computational cells required for a simulation decreases by a factor of about ten, leading to approximately a ten-fold increase in computational speed. This increase in computational speed, coupled with a decrease in the time and

effort required for mesh discretization, allows more efficient simulation of fluid flow problems in fractured/porous media.

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Appendix A

The analysis of the fracture/matrix interaction equation described above was performed for matrix blocks of spherical shape. This does not pose a loss of generality for the Warren-Root method, since block geometry enters the calculations only through the single parameter α in eqs. (7) and (8). This parameter has dimensions of area^{-1}, and is therefore inversely proportional to the square of some suitably-defined characteristic block size. If the blocks have a known simple shape, such as cubical, cylindrical, or slab-like, α can be approximated in the same manner as for the spherical block, which is by differentiating the most-slowly-decaying exponential term in the Fourier series solution for the step-function response, eliminating t, and comparing the resulting equation with the general form given in eq. (8). Using the Fourier-series solutions given by *Crank* [1975] and *Carslaw and Jaeger* [1959], we find, for example, that $\alpha = \pi^2/L^2$ for a thin slab of thickness L; $\alpha = 3\pi^2/L^2$ for a cubical block of length L; and $\alpha = z_1^2/a^2$ for a long cylinder of radius a, where $z_1 = 2.405$ is the first positive root of the Bessel function J_o. A discussion of the relationship between α and block geometry is given by *deSwaan* [1990], who derived slightly different values than those given above by forcing the Warren-Root normalized matrix pressure to reach 0.50 at the same value of t as does the exact normalized matrix pressure, under step-function boundary conditions. This approach has the effect of making the Warren-Root method accurate at some intermediate time regime, but causes the asymptotic quasi-steady-state flux to be off by some multiplicative constant. *Warren and Root* [1963] left α as an open parameter, but proposed the value $60/L^2$ for a cube of length L, which agrees with deSwaan's value.

The nonlinear flow interaction term given by eq. (19) was derived for a spherical matrix block, and so it is not immediately clear how to generalize it to other geometries. One approach would be to identify the terms π^2/a_m^2 on the right-hand side of eq. (19) with α of the Warren-Root equation, and substitute the "exact" Warren-

Root α values discussed above for cases of other block geometries. This would assure that eq. (19) is accurate in the quasi-steady-state regime. As a test of the accuracy of this approach, consider the problem of flow into a cubical matrix block of length L_m, under step-function boundary conditions analogous to eqs. (9) and (10). If we use the Vermeulen equation to predict $\bar{P}_m$ as a function of time, we arrive at the expression given in eq. (17), with π^2/a_m^2 replaced by $3\pi^2/L_m^2$. The exact expression for $P_m(x_m,t)$ given by *Carslaw and Jaeger* [1959, p. 184] can be averaged over the cube to yield

$$\frac{\bar{P}_m - P_i}{P_o - P_i} = 1 - \frac{512}{\pi^6} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\exp[-\pi^2 k_m (l^2+m^2+n^2)t/\phi_m \mu c_m L^2]}{l^2 m^2 n^2} , \qquad \text{(A1)}$$

where the summations take place over all *odd* positive integers. Fig. 6 shows that this procedure of adapting the Vermeulen equation to another geometry by using the Warren-Root α parameter in place of π^2/a_m^2 in eq. (19) is reasonably accurate. Due to the manner in which α was defined, the approximate solution is asymptotically exact for large times. At small times, the average matrix block pressure is too high by a multiplicative factor of $\sqrt{4/3}$, which is an error of only 15%. In general, there is no method of choosing α for non-spherical block shapes that will render the Vermeulen equation exactly correct in both the small-time and large-time limits; it will, however, yield the correct power-law exponents at short times, as is seen by comparing the slopes of the curves in Fig. 6.

The approach described above could be taken if one were studying the effect of matrix block shape on overall reservoir behavior. In simulations of a specific reservoir, the term π^2/a_m^2 in eq. (19) could also be left as an open parameter, with dimensions of area^{-1}, whose value is chosen so as to yield a good match between the predicted and observed pressure response in the fractures. This approach was taken by *Moench* [1984] in the context of the Warren-Root model.

Appendix B

Eq. (19) governing the mean pressure response in the matrix blocks is, as written, applicable only to processes in which P_f changes monotonically with time. For example, if P_f starts at P_i and increases with t, P_m will also increase, but will lag behind P_f. Hence $(P_f - P_i)^2$ will be greater than $(\bar{P}_m - P_i^2)$, and eq. (19) will correctly predict an increase in $\bar{P}_m$. Similarly, if P_f decreases with t, $\bar{P}_m$ will decrease but lag behind, and so the numerator on the right side of eq. (19) will be positive, the denominator will be negative, and the equation correctly predicts that $\bar{P}_m$ will decrease with t. Difficulties arise if the variation in P_f is not monotonic. Strictly speaking, the sign of $d\bar{P}_m/dt$ will depend only on the integral of $\partial P_m/\partial n$ over the outer boundary of the block (see eq. (3)). When using only the mean value $\bar{P}_m$, knowledge of $\partial P_m/\partial n$ is lost, so to speak. It seems plausible, however, that $d\bar{P}_m/dt$ should have the same sign as $P_f - \bar{P}_m$. Using the fact that $a^2 - b^2 = (a-b)(a+b)$, we can recast eq. (19) in a way that leaves it unchanged if both $P_f - P_i$ and $\bar{P}_m - P_i$ are of the same sign, and which assures that $d\bar{P}_m/dt$ has the same sign as $P_f - \bar{P}_m$:

$$\frac{d\bar{P}_m}{dt} = \frac{\pi^2 k_m}{2\phi_m \mu c_m a_m^2} \frac{(P_f - \bar{P}_m)(|P_f - P_i| + |\bar{P}_m - P_i|)}{|\bar{P}_m - P_i|} . \tag{B1}$$

To test this form of the equation, consider a case where P_f varies sinusoidally in time:

$$P_f(t) = P_i + P_o \sin\omega t . \tag{B2}$$

Due to the form of eq. (B1), we can set $P_i = 0$, without any loss of generality. An exact solution for this problem can be developed from the step-function response using

the convolution principle, which states that the pressure distribution in the sphere is given by [*Crank,* 1975, p. 91]

$$P_m(r,t) = \frac{-2D}{ar}\sum_{n=1}^{\infty}(-1)^n n\pi\sin(n\pi r/a)\int_0^t \exp[-Dn^2\pi^2(t-\tau)/a^2]P_f(\tau)d\tau, \quad \text{(B3)}$$

where $D = k/\phi\mu c$ is the hydraulic diffusivity, and for convenience we drop the subscript m on all physical parameters. If $P_f(t) = P_o \sin\omega t$, eq. (B3) can be written as

$$P_m(r,t) = \frac{-2DP_o}{ar}\sum_{n=1}^{\infty}(-1)^n n\pi\sin(n\pi r/a)\exp(-Dn^2\pi^2 t/a^2)$$

$$\times\int_0^t \exp(Dn^2\pi^2\tau/a^2)\sin\omega\tau d\tau. \quad \text{(B4)}$$

The integrals in eq. (B4), which we denote by I_n, can be evaluated to yield

$$I_n = \frac{\gamma_n \sin\omega t\, e^{\gamma_n t} - \omega\cos\omega t\, e^{\gamma_n t} + \omega}{\gamma_n^2 + \omega^2}, \quad \text{(B5)}$$

where $\gamma_n = Dn^2\pi^2/a^2$.

The mean pressure $\bar{P}_m(t)$ is found by integrating $P_m(r,t)$ through the sphere, and dividing by the sphere volume $4\pi a^3/3$. The average values of the space-dependent terms in eq. (B4) are found to be

$$\overline{\frac{1}{r}\sin(n\pi r/a)} = \frac{3}{a^3}\int_0^a r\sin(n\pi r/a)dr = \frac{3}{n\pi a}(-1)^{n+1}. \quad \text{(B6)}$$

Combining eqs. (B4-B6) leads to

$$\frac{\bar{P}_m(t)}{P_o} = 6\sum_{n=1}^{\infty} \frac{n^2\pi^2 \sin\omega t - \Omega\cos\omega t + \Omega\exp(-Dn^2\pi^2 t/a^2)}{\Omega^2 + n^4\pi^4}, \tag{B7}$$

where $\Omega = \omega a^2/D$. If we rewrite each pair of trigonometric terms as a single sinusoid with a nonzero phase angle, we arrive at

$$\frac{\bar{P}_m(t)}{P_o} = 6\sum_{n=1}^{\infty} \frac{\sin(\omega t + \delta_n)}{\sqrt{\Omega^2 + n^4\pi^4}} + \frac{\Omega\exp(-Dn^2\pi^2 t/a^2)}{\Omega^2 + n^4\pi^4}, \tag{B8}$$

where $\delta_n = arctan(-\Omega/n^2\pi^2)$. *Carslaw and Jaeger* [1959, p. 235] presented the solution to this problem, but expressed the oscillatory components of $P_m(r,t)$ in terms of a magnitude and a phase angle. This leads to terms that contain an awkward dependence on r, and which are not easily integrated to find $\bar{P}_m(t)$. The sinusoidal portions of eq. (B8) represent the "steady-state" part of the solution, whereas the exponential terms represent the transient portion.

The quasi-steady-state part of the mean pressure response depends on the dimensionless frequency $\Omega = \omega a^2/D$. Since the time needed for the sphere to equilibrate after a step-function increase in P_f is approximately a^2/D (see Fig. 1), and the time at which $P_o \sin\omega t$ stops increasing is $\pi/2\omega$, we see that Ω is approximately equal to the ratio of these two time scales. In order to test the applicability of the modified Vermeulen equation to sinusoidal boundary conditions, over a range of frequencies, we can integrate eq. (B1) numerically, using eq. (B2) for $P_f(t)$, and compare the results to the exact values given by eq. (B8). At low frequencies, $\Omega < 1$, eq. (B1) was found to be extremely accurate; the accuracy decreases as Ω increases. Fig. 7 shows $\bar{P}_m$ for $\Omega = 10$, as computed by eq. (B1) and eq. (B8); for comparison, $P_f(t)$ is also plotted.

The modified Vermeulen equation predicts the phase angle, which represents the lag between $\bar{P}_m$ and P_f, very accurately. The maximum magnitude of $\bar{P}_m$ is predicted within an error of about 9%. At a value of Ω as large as 100, the error in the maximum value of $\bar{P}_m$ reaches about 29%. For comparison, we note that for $\Omega = 100$, the Warren-Root equation underestimates the maximum value of $\bar{P}_m$ by 58%.

Figure Captions

Fig. 1. Normalized average matrix pressure for a spherical block subjected to a step-function increase in the pressure at its boundary, as given by the exact solution, the Vermeulen equation, and the Warren-Root equation. For comparison, the pressure at the boundary (*i.e.*, in the fractures) is also shown.

Fig. 2. Same as Fig. 1, for a ramp-function increase in the boundary pressure. The diffusion coefficient D is defined here to be $k_m/\phi_m \mu c_m$.

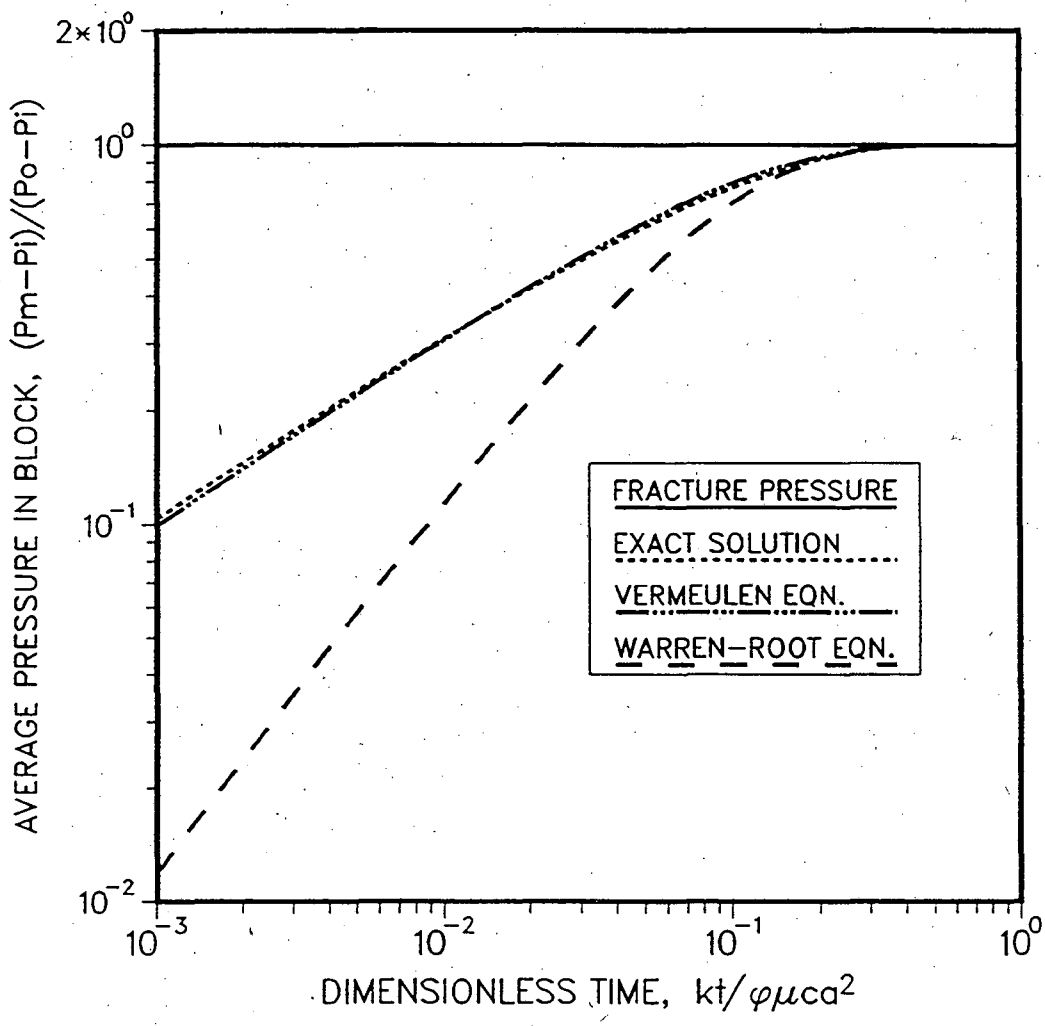
Fig. 3. Same as Fig. 1, for the case where the boundary pressure increases according to eq. (32), with $\beta = 100$.

Fig. 4. Total instantaneous flux for one-dimensional flow into a dual-porosity formation with constant boundary pressure. The meanings of the parameters, and their values, are discussed in the text. MINC simulations were carried out using the TOUGH code; "new method" simulation was carried out using TOUGH, with the modifications described in the text.

Fig. 5. Same problem as in Fig. 4, with results of new method compared to the three asymptotic expressions (eqs. (38,39,41)) found following the analysis of *Nitao and Buscheck* [1991].

Fig. 6. Same as Fig. 1, but for a cube of length L_m. In both the Warren-root equation (14) and the Vermeulen equation (19), the term a_m^2 has been replaced by $L_m^2/3$.

Fig. 7. Same as Fig. 1, but with the boundary pressure given by $P_f = P_i + P_o \sin\omega t$. The dimensionless frequency $\Omega = \omega a^2/D_m$ is taken to be 10.



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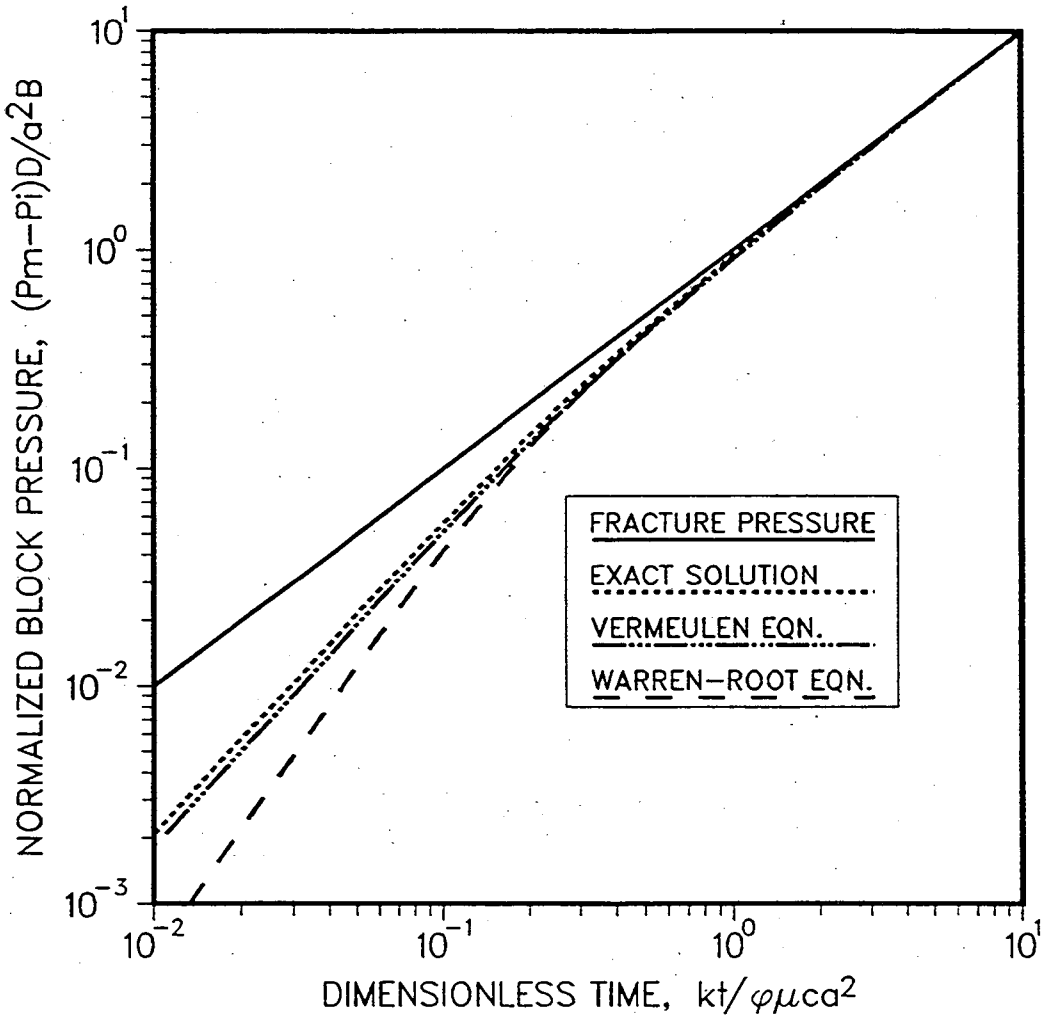

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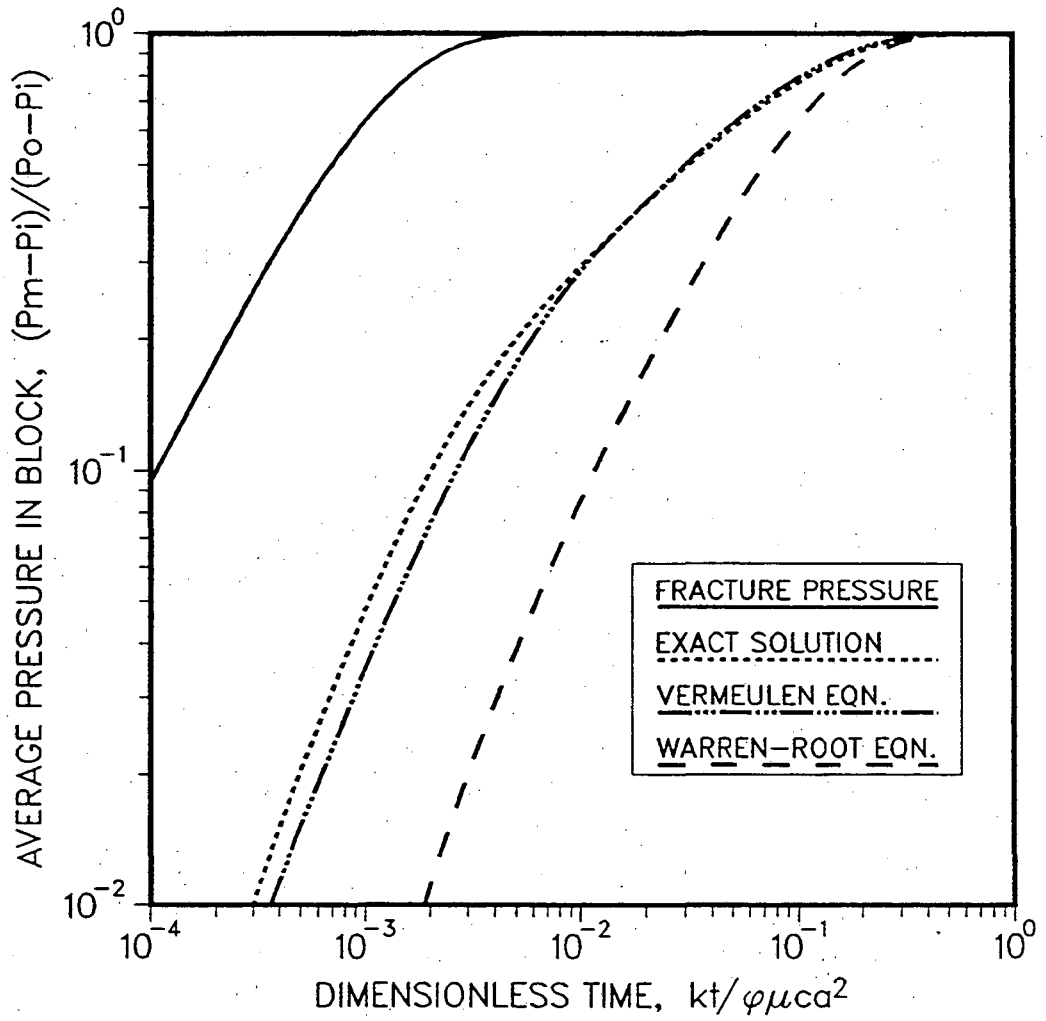

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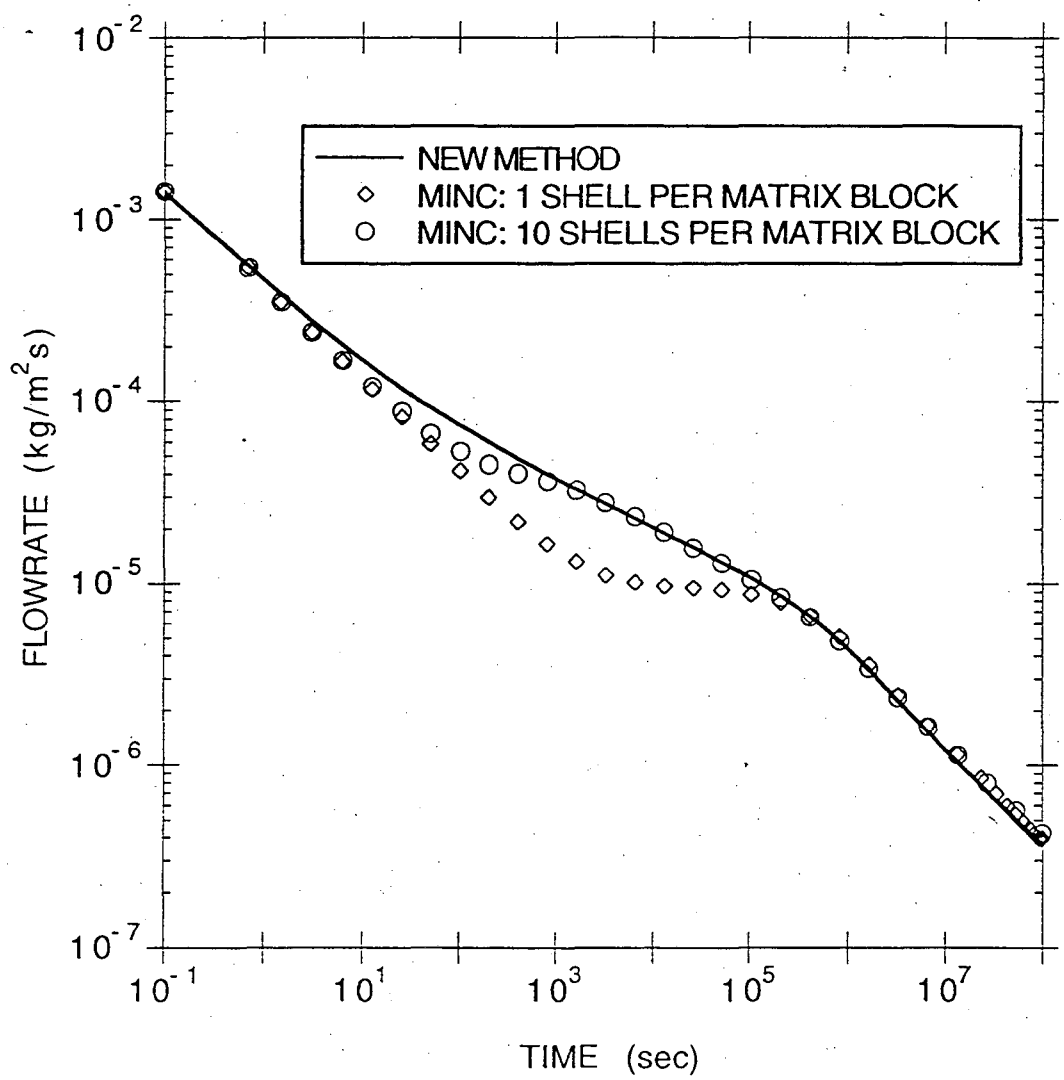

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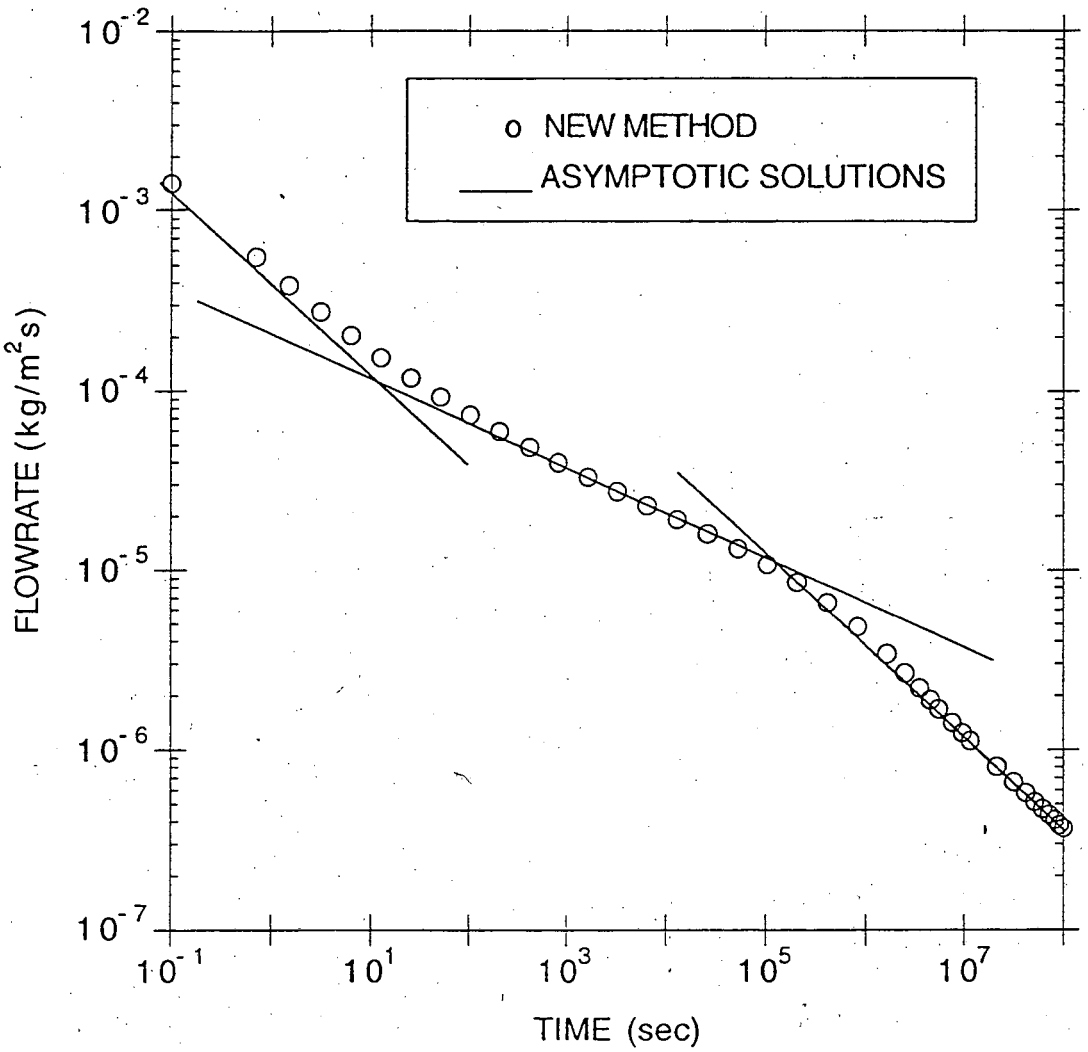

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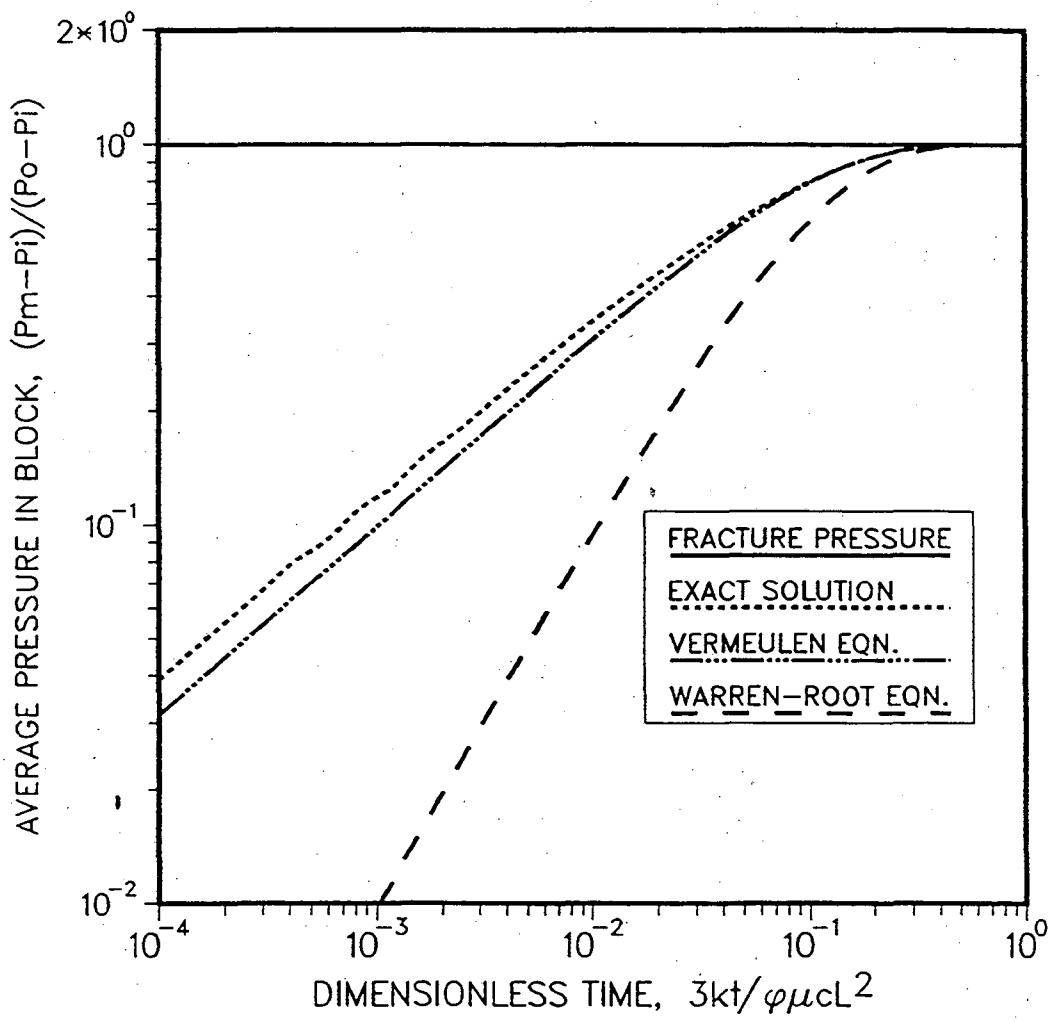

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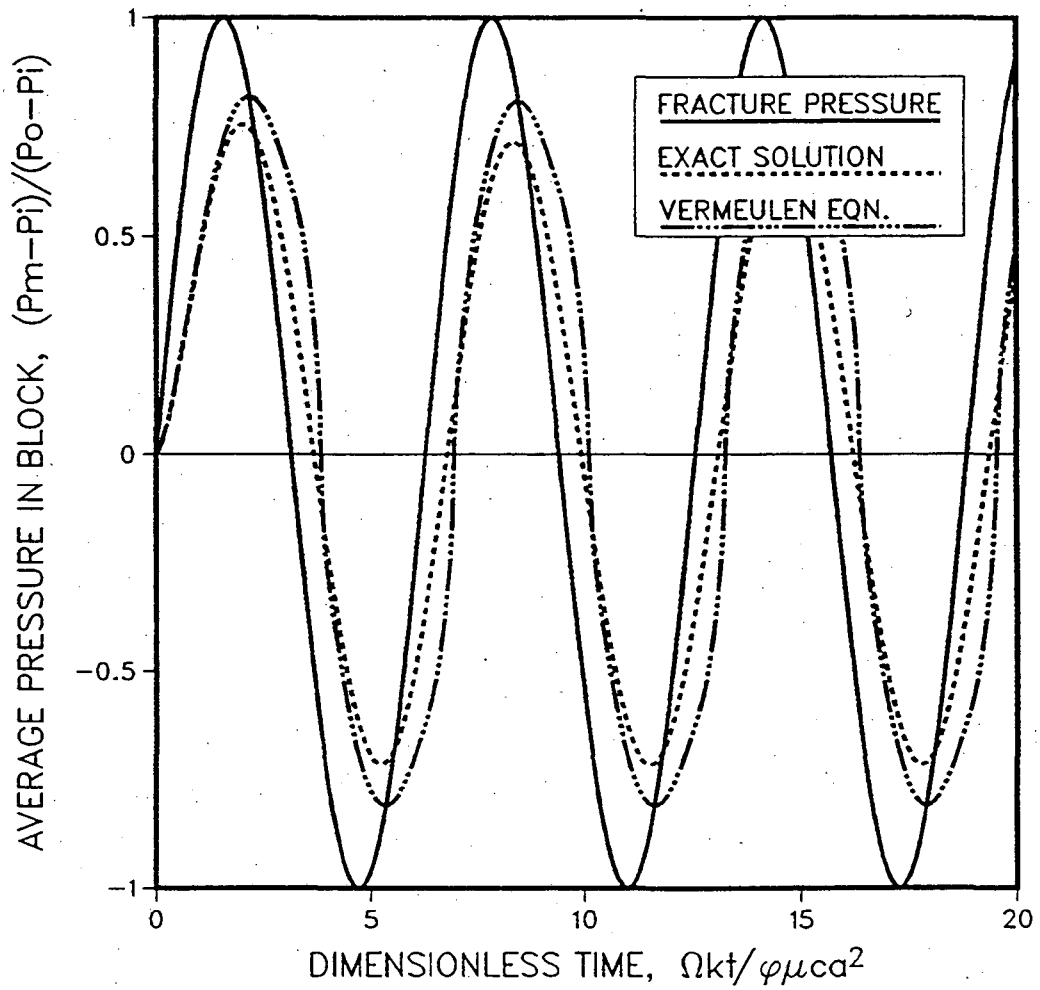

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