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Cartan Calculus for Hopf Algebras and Quantum Groups *

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Abstract

A generalization of the differential geometry of forms and vector fields to the case of quantum Lie algebras is given. In an abstract formulation that incorporates many existing examples of differential geometry on quantum groups, we combine an exterior derivative, inner derivations, Lie derivatives, forms and functions all into one big algebra. In particular we find a generalized Cartan identity that holds on the whole quantum universal enveloping algebra of the left-invariant vector fields and implicit commutation relations for a left-invariant basis of 1-forms.

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1 Introduction

The question of how to endow a quantum group with a differential geometry has been studied extensively [1, 2, 3, 4, 5, 6]. Most of these approaches, however, are rather specific: many papers deal with the subject by considering the quantum group in question as defined by its R-matrix, and others limit themselves to particular cases. Here we shall attempt a more abstract formulation which depends primarily on the underlying Hopf algebraic structure; this will therefore be a generalization of many previously obtained results.

The approach we take starts with a Hopf algebra, which we identify with the (quantum) universal enveloping algebra (UEA) of some Lie algebra, taking its dual Hopf algebra to be the functions on the corresponding quantum group. We then construct a larger (non-Hopf) algebra which contains these as subalgebras combined by the "cross product" [7, 8, 9]. The differential geometry is then introduced by including in the algebra an exterior derivative, Lie derivatives, and inner derivations. (Much of the previous work in this area has emphasized the *actions* of these operators on functions and forms rather than treating them as elements in an extended algebra containing the cross product algebra and giving the appropriate commutation relations). Our approach is constructive in nature; this implies that not only we must treat each given Hopf algebra on a case-by-case basis, but that questions concerning uniqueness and even existence arise. These problems will be addressed in section 3.

2 Differential Geometry on Hopf Algebras

2.1 Hopf Algebras

A Hopf algebra $\mathcal{A}$ [10, 11, 12, 7] is an associative unital algebra $(\mathcal{A}, \cdot, +, k)$ over a field k, equipped with a coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, an antipode $S : \mathcal{A} \to \mathcal{A}$, and a counit $\epsilon : \mathcal{A} \to k$, satisfying $(\Delta \otimes id)\Delta(a) = (id \otimes \Delta)\Delta(a)$,

$(\epsilon \otimes id)\Delta(a) = (id \otimes \epsilon)\Delta(a) = a$, and $\cdot(S \otimes id)\Delta(a) = \cdot(id \otimes S)\Delta(a) = 1\epsilon(a)$, for all $a \in \mathcal{A}$. These axioms are dual to the axioms of an algebra. There are also a number of consistency conditions between the algebra and the coalgebra structure: $\Delta(ab) = \Delta(a)\Delta(b)$, $\epsilon(ab) = \epsilon(a)\epsilon(b)$, $S(ab) = S(b)S(a)$, $\Delta(S(a)) = \tau(S\otimes S)\Delta(a)$, where $\tau(a\otimes b) \equiv b\otimes a$ is the twist map, $\epsilon(S(a)) = \epsilon(a)$, and $\Delta(1) = 1\otimes 1$, $S(1) = 1$, $\epsilon(1) = 1_k$, for all $a, b \in \mathcal{A}$. We will often use Sweedler's [11] notation for the coproduct:

$$\begin{aligned} &\Delta(a) \equiv a_{(1)} \otimes a_{(2)} && \text{(summation is understood).} \\ &(\Delta \otimes id)\Delta(a) \equiv a_{(1)} \otimes a_{(2)} \otimes a_{(3)} && \textit{etc.} \end{aligned} \tag{1}$$

We call two Hopf algebras $\mathcal{U}$ and $\mathcal{A}$ dually paired if there exists a non-degenerate inner product $<\ ,\ >: \mathcal{U}\otimes\mathcal{A} \to k$, such that

$$< xy, a > \;=\; < x\otimes y, \Delta(a) > \equiv < x, a_{(1)} >< y, a_{(2)} >, \tag{2}$$

$$< x, ab > \;=\; < \Delta(x), a\otimes b > \equiv < x_{(1)}, a >< x_{(2)}, b >, \tag{3}$$

$$< S(x), a > \;=\; < x, S(a) >, \tag{4}$$

$$< x, 1 > \;=\; \epsilon(x), \qquad \text{and} \qquad < 1, a > = \epsilon(a), \tag{5}$$

for all $x, y \in \mathcal{U}$ and $a, b \in \mathcal{A}$, i.e. if the product of the first Hopf algebra induces the coproduct on the second and vice versa. Note that a Hopf algebra is in general non-cocommutative, i.e. $\tau \circ \Delta \neq \Delta$.

2.2 Cartan Calculus

The purpose of this article is to generalize the Cartan calculus of ordinary *commutative* differential geometry to the case of quantum Lie algebras. Before restricting ourselves to this particular case, we first consider an arbitrary Hopf algebra, and for this case we will introduce an exterior derivative $\mathbf{d}$, Lie derivatives $\pounds_x$ and inner derivations $\mathbf{i}_x$. The guideline for our generalization will be the classical Cartan identity

$$\pounds_x = \mathbf{i}_x \mathbf{d} + \mathbf{d}\mathbf{i}_x, \tag{6}$$

the Leibniz rule

$$\mathbf{d}(ab) = \mathbf{d}(a)b + a\mathbf{d}(b)^\dagger, \tag{7}$$

and the nilpotency of $\mathbf{d}$. In the following we will work with a Hopf algebra $\mathcal{U}$ which will be interpreted as an algebra of left-invariant differential operators or vector fields, and the dually paired Hopf algebra $\mathcal{A}$ which in this interpretation is the algebra of functions on which elements of $\mathcal{U}$ act via

$$x \triangleright a = a_{(1)} < x, a_{(2)} >, \tag{8}$$

where $x \in \mathcal{U}$ and $a \in \mathcal{A}$. The action of x on a pair of functions a, $b \in \mathcal{A}$ is given in terms of the coproduct by

$$x \triangleright ab = (x_{(1)} \triangleright a)(x_{(2)} \triangleright b). \tag{9}$$

This motivates the introduction of a product structure on the "cross product" algebra $\mathcal{A}\rtimes\mathcal{U}$ [8, 9] via the commutation relation

$$xa = a_{(1)} < x_{(1)}, a_{(2)} > x_{(2)}. \tag{10}$$

As in the classical case, the Lie derivative of a function is given by the action of the corresponding vector field, i.e.

$$\begin{aligned} \pounds_x(a) &= x \triangleright a = a_{(1)} < x, a_{(2)} >, \\ \pounds_x a &= a_{(1)} < x_{(1)}, a_{(2)} > \pounds_{x_{(2)}}. \end{aligned} \tag{11}$$

The Lie derivative along x of an element $y \in \mathcal{U}$ is given by the adjoint action in $\mathcal{U}$:

$$\pounds_x(y) = x \overset{\text{ad}}{\triangleright} y = x_{(1)} y S(x_{(2)}). \tag{12}$$

To find the action of $\mathbf{i}_x$ we can now attempt to use the Cartan identity (6)[‡]

$$\begin{aligned} x \triangleright a &= \pounds_x(a) \\ &= \mathbf{i}_x(\mathbf{d}a) + \mathbf{d}(\mathbf{i}_x a). \end{aligned} \tag{13}$$

[†]We use parentheses to delimit operations like $\mathbf{d}$, $\mathbf{i}_x$ and $\pounds$, e.g. $\mathbf{d}a = \mathbf{d}(a) + a\mathbf{d}$. However, if the limit of the operation is clear from the context, we will suppress the parentheses, e.g. $\mathbf{d}(\mathbf{i}_x\mathbf{d}a) \equiv \mathbf{d}(\mathbf{i}_x(\mathbf{d}(a)))$.

[‡]The idea is to use this identity as long as it is consistent and modify it when needed.

As the inner derivation $\boldsymbol{i}_x$ contracts 1-forms and is zero on 0-forms like a, we find

$$\boldsymbol{i}_x(\mathrm{d}a) = x \triangleright a = a_{(1)} < x, a_{(2)} > . \tag{14}$$

However this cannot be true for any $x \in \mathcal{U}$ because from the Leibniz rule for $\mathbf{d}$ we have $\mathbf{d}(1) = \mathbf{d}(1 \cdot 1) = \mathbf{d}(1)1 + 1\mathbf{d}(1) = 2\mathbf{d}(1)$ and any $\boldsymbol{i}_x$ that gives a non-zero result upon contracting $\mathbf{d}(1)$ will hence lead to a contradiction. From (14) we see that the troublemakers are $x \in \mathcal{U}$ with $\epsilon(x) \neq 0$. Noting that $\epsilon(x - 1\epsilon(x)) = 0$ we modify equation (14) to read

$$\boldsymbol{i}_x(\mathrm{d}a) = a_{(1)} < x - 1\epsilon(x), a_{(2)} >, \tag{15}$$

such that $\boldsymbol{i}_x(\mathrm{d}1) = 1 < x - 1\epsilon(x), 1 >= 0$. Without loss of generality we can now set

$$\mathrm{d}(1) \equiv 0 \quad \text{and} \quad \boldsymbol{i}_1 \equiv 0. \tag{16}$$

For $x \in \mathcal{U}$ with non-zero counit we also need to modify equation (6) to

$$\mathcal{L}_{x-1\epsilon(x)} = \boldsymbol{i}_x \mathbf{d} + \mathbf{d}\boldsymbol{i}_x, \tag{17}$$

or in view of (11) identifying $\mathcal{L}_1 \equiv 1$ and using the linearity of the Lie derivative

$$\mathcal{L}_x = \boldsymbol{i}_x \mathbf{d} + 1\epsilon(x) + \mathbf{d}\boldsymbol{i}_x \qquad \textit{(generalized Cartan identity)}. \tag{18}$$

Next consider for any form α

$$\begin{aligned} \mathcal{L}_x(\mathrm{d}\alpha) &= \mathrm{d}(\boldsymbol{i}_x \mathrm{d}\alpha) + \epsilon(x)\mathrm{d}(\alpha) + \boldsymbol{i}_x(\mathrm{dd}\alpha) \\ &= \mathrm{d}(\mathcal{L}_x \alpha) + 0, \end{aligned} \tag{19}$$

which shows that Lie derivatives commute with the exterior derivative:

$$\mathcal{L}_x \mathbf{d} = \mathbf{d} \mathcal{L}_x. \tag{20}$$

From this and (11) we find

$$\mathcal{L}_x \mathrm{d}(a) = \mathrm{d}(a_{(1)}) < x_{(1)}, a_{(2)} > \mathcal{L}_{x_{(2)}}. \tag{21}$$

To find the complete commutation relations of $\boldsymbol{i}_x$ with functions and forms rather than just its action on them we next compute the action of $\mathcal{L}_x$ on a product of functions $a, b \in \mathcal{A}$

$$\begin{aligned} \mathcal{L}_x(ab) &= \boldsymbol{i}_x \mathrm{d}(ab) + \epsilon(x)ab \\ &= \boldsymbol{i}_x(\mathrm{d}(a)b + a\mathrm{d}(b)) + \epsilon(x)ab \end{aligned} \tag{22}$$

and compare with equation (9). In place of an arbitrary $x \in \mathcal{U}$ let us for the moment specialize to the case of a set of left-invariant vector fields $\chi_i \in \mathcal{U}$, $i = 1 \dots n$, with zero counit and with coproduct

$$\Delta\chi_i = \chi_i \otimes 1 + O_i{}^j \otimes \chi_j; \qquad O_i{}^j \in \mathcal{U}. \tag{23}$$

A coproduct of this form is encountered in many know examples and as we shall see it is not hard to generalize the equations later. For this choice of vector fields we obtain

$$\begin{aligned} \boldsymbol{i}_{\chi_i} a &= (O_i{}^j \triangleright a)\, \boldsymbol{i}_{\chi_j} \\ &= \mathcal{L}_{O_i{}^j}(a)\, \boldsymbol{i}_{\chi_j}, \end{aligned} \tag{24}$$

if we assume that the commutation relation of $\boldsymbol{i}_{\chi_i}$ with $\mathbf{d}(a)$ is of the general form

$$\boldsymbol{i}_{\chi_i} \mathrm{d}(a) = \underbrace{\boldsymbol{i}_{\chi_i}(\mathrm{d}a)}_{\in \mathcal{A}} + \text{“braiding term”} \cdot \boldsymbol{i}_{\chi_?}. \tag{25}$$

A calculation of $\mathcal{L}_{\chi_i}(\mathrm{d}(a)\mathrm{d}(b))$ along similar lines gives in fact

$$\begin{aligned} \boldsymbol{i}_{\chi_i} \mathrm{d}(a) &= (\chi_i \triangleright a) - \mathrm{d}(O_i{}^j \triangleright a)\, \boldsymbol{i}_{\chi_j} \\ &= \boldsymbol{i}_{\chi_i}(\mathrm{d}a) - \mathcal{L}_{O_i{}^j}(\mathrm{d}a)\, \boldsymbol{i}_{\chi_j} \end{aligned} \tag{26}$$

and we propose for any p-form α:

$$\boldsymbol{i}_{\chi_i} \alpha = \boldsymbol{i}_{\chi_i}(\alpha) + (-1)^p \mathcal{L}_{O_i{}^j}(\alpha)\, \boldsymbol{i}_{\chi_j}. \tag{27}$$

Now we are ready to generalize to an arbitrary $x \in \mathcal{U}$ instead of χ_i. We observe that $\boldsymbol{i}_x = \boldsymbol{i}_{(x-1\epsilon(x))}$ because of $\boldsymbol{i}_1 = 0$ and the linearity of the inner derivation. The special coproduct given in (23) can now be replaced by

$$\begin{aligned} \Delta(x - 1\epsilon(x)) &= x_{(1)} \otimes x_{(2)} - 1\epsilon(x) \otimes 1 \\ &= (x - 1\epsilon(x)) \otimes 1 + x_{(1)} \otimes (x_{(2)} - 1\epsilon(x_{(2)})) \end{aligned} \tag{28}$$

leading to

$$\boldsymbol{i}_{(x-1\epsilon(x))}\alpha = \boldsymbol{i}_{(x-1\epsilon(x))}(\alpha) + (-1)^p \boldsymbol{\mathcal{L}}_{x_{(1)}}(\alpha)\, \boldsymbol{i}_{(x_{(2)}-1\epsilon(x_{(2)}))}, \tag{29}$$

which is equivalent to

$$\boldsymbol{i}_x\alpha = \boldsymbol{i}_x(\alpha) + (-1)^p \boldsymbol{\mathcal{L}}_{x_{(1)}}(\alpha)\, \boldsymbol{i}_{x_{(2)}}. \tag{30}$$

For a more direct argument we could also use the requirement that (11) and (18) be mutually consistent: The left-hand side of the second equation in (11) gives (using the Leibniz rule)

$$\boldsymbol{\mathcal{L}}_x a = \boldsymbol{i}_x\mathbf{d}(a) + \boldsymbol{i}_x a\mathbf{d} + \epsilon(x)a + \mathbf{d}\boldsymbol{i}_x a \tag{31}$$

and the right-hand side gives

$$\begin{aligned} & a_{(1)} < x_{(1)}, a_{(2)} > \boldsymbol{\mathcal{L}}_{x_{(2)}} = \\ & \quad a_{(1)} < x_{(1)}, a_{(2)} > \mathbf{d}\boldsymbol{i}_{x_{(2)}} \\ & \quad + a_{(1)} < x, a_{(2)} > + a_{(1)} < x_{(1)}, a_{(2)} > \boldsymbol{i}_{x_{(2)}}\mathbf{d}. \end{aligned} \tag{32}$$

Equating the two and using (11), (15), (19), and $\boldsymbol{i}_x(a) = 0$, we find

$$\boldsymbol{i}_x\mathbf{d}(a) - \boldsymbol{i}_x(\mathbf{d}a) + \boldsymbol{\mathcal{L}}_{x_{(1)}}(\mathbf{d}a)\, \boldsymbol{i}_{x_{(2)}} = -[\boldsymbol{i}_x a - \boldsymbol{i}_x(a) - \boldsymbol{\mathcal{L}}_{x_{(1)}}(a)\, \boldsymbol{i}_{x_{(2)}}, \mathbf{d}]_+. \tag{33}$$

Therefore, we propose equation (30) for any p-form α, so that both sides of the above relation vanish.

Missing in our list are commutation relations of Lie derivatives with vector fields and inner derivations. It was shown in [9] that the product in $\mathcal{U}$ can be expressed in terms of a right coaction $\Delta_{\mathcal{A}} : \mathcal{U} \to \mathcal{U} \otimes \mathcal{A}$, denoted $\Delta_{\mathcal{A}}(y) =$

$y^{(1)} \otimes y^{(2)'}$, such that $xy = y^{(1)} < x_{(1)}, y^{(2)'} > x_{(2)}$. In our context (12) this gives

$$\begin{aligned} \boldsymbol{\mathcal{L}}_x(y) &= x_{(1)} y S(x_{(2)}) = y^{(1)} < x, y^{(2)'} >, \quad (34) \\ \boldsymbol{\mathcal{L}}_x \boldsymbol{\mathcal{L}}_y &= \boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{L}}_{x_{(1)}}(y)} \boldsymbol{\mathcal{L}}_{x_{(2)}} = \boldsymbol{\mathcal{L}}_{y^{(1)}} < x_{(1)}, y^{(2)'} > \boldsymbol{\mathcal{L}}_{x_{(2)}}, \quad (35) \end{aligned}$$

and — using the Cartan identity —

$$\boldsymbol{\mathcal{L}}_x \boldsymbol{i}_y = \boldsymbol{i}_{\boldsymbol{\mathcal{L}}_{x_{(1)}}(y)} \boldsymbol{\mathcal{L}}_{x_{(2)}} = \boldsymbol{i}_{y^{(1)}} < x_{(1)}, y^{(2)'} > \boldsymbol{\mathcal{L}}_{x_{(2)}}. \tag{36}$$

Here is a summary of commutation relations valid on any form; $x, y \in \mathcal{U}$, $a \in \mathcal{A}$, α is a p-form and $v \in \mathcal{A} \rtimes \mathcal{U}$ is a vector field.

$$\begin{aligned} \boldsymbol{\mathcal{L}}_x a &= a_{(1)} < x_{(1)}, a_{(2)} > \boldsymbol{\mathcal{L}}_{x_{(2)}} & (37) \\ \boldsymbol{\mathcal{L}}_x \mathbf{d}(a) &= \mathbf{d}(a_{(1)}) < x_{(1)}, a_{(2)} > \boldsymbol{\mathcal{L}}_{x_{(2)}} & (38) \\ \boldsymbol{\mathcal{L}}_x \alpha &= \boldsymbol{\mathcal{L}}_{x_{(1)}}(\alpha)\, \boldsymbol{\mathcal{L}}_{x_{(2)}} & (39) \\ \boldsymbol{i}_x a &= a_{(1)} < x_{(1)}, a_{(2)} > \boldsymbol{i}_{x_{(2)}} & (40) \\ \boldsymbol{i}_x \mathbf{d}(a) &= a_{(1)} < x - 1\epsilon(x), a_{(2)} > - \mathbf{d}(a_{(1)}) < x_{(1)}, a_{(2)} > \boldsymbol{i}_{x_{(2)}} & (41) \\ \boldsymbol{i}_x \alpha &= \boldsymbol{i}_x(\alpha) + (-1)^p \boldsymbol{\mathcal{L}}_{x_{(1)}}(\alpha)\, \boldsymbol{i}_{x_{(2)}} & (42) \\ \mathbf{d}\alpha &= \mathbf{d}(\alpha) + (-1)^p \alpha \mathbf{d} & (43) \\ \mathbf{d}\mathbf{d}(\alpha) &= -(-1)^p \mathbf{d}(\alpha)\mathbf{d} & (44) \end{aligned}$$

$$\boldsymbol{\mathcal{L}}_x(v) = x_{(1)} v S(x_{(2)}) \tag{45}$$

$$\begin{aligned} \mathbf{d}^2 &= 0 & (46) \\ \mathbf{d}\boldsymbol{\mathcal{L}}_x &= \boldsymbol{\mathcal{L}}_x \mathbf{d} & (47) \\ \boldsymbol{\mathcal{L}}_x &= \mathbf{d}\boldsymbol{i}_x + 1\epsilon(x) + \boldsymbol{i}_x \mathbf{d} & (48) \\ \boldsymbol{\mathcal{L}}_x \boldsymbol{\mathcal{L}}_y &= \boldsymbol{\mathcal{L}}_{y^{(1)}} < x_{(1)}, y^{(2)'} > \boldsymbol{\mathcal{L}}_{x_{(2)}} & (49) \\ \boldsymbol{\mathcal{L}}_x \boldsymbol{i}_y &= \boldsymbol{i}_{y^{(1)}} < x_{(1)}, y^{(2)'} > \boldsymbol{\mathcal{L}}_{x_{(2)}} & (50) \end{aligned}$$

2.3 Maurer-Cartan Forms

The most general left-invariant 1-form can be written [1]

$$\omega_b := S(b_{(1)})\mathbf{d}(b_{(2)}) = -\mathbf{d}(Sb_{(1)})b_{(2)} \tag{51}$$

$$(\textit{left-invariance: } {}_{\mathcal{A}}\Delta(\omega_b) = S(b_{(2)})b_{(3)} \otimes S(b_{(1)})\mathbf{d}(b_{(4)}) = 1 \otimes \omega_b), \tag{52}$$

corresponding to a function $b \in \mathcal{A}$. If this function happens to be $t^i{}_k$, where $t \in M_m(\mathcal{A})$ is a matrix representation of $\mathcal{U}$ with $\Delta(t^i{}_k) = t^i{}_j \otimes t^j{}_k$ and $S(t) = t^{-1}$, we obtain the well-known Cartan-Maurer form $\omega_t = t^{-1}\mathbf{d}(t)$. Here is a nice formula for the exterior derivative of ω_b:

$$\begin{aligned}\mathbf{d}(\omega_b) &= \mathbf{d}(Sb_{(1)})\mathbf{d}(b_{(2)}) \\ &= \mathbf{d}(Sb_{(1)})b_{(2)}S(b_{(3)})\mathbf{d}(b_{(4)}) \\ &= -\omega_{b_{(1)}}\omega_{b_{(2)}}.\end{aligned} \tag{53}$$

The Lie derivative is

$$\begin{aligned}\pounds_x(\omega_b) &= \pounds_{x_{(1)}}(Sb_{(1)})\pounds_{x_{(2)}}(\mathbf{d}b_{(2)}) \\ &= < x_{(1)}, S(b_{(1)}) > S(b_{(2)})\mathbf{d}(b_{(3)}) < x_{(2)}, b_{(4)} > \\ &= \omega_{b_{(2)}} < x, S(b_{(1)})b_{(3)} > \\ &= < x_{(1)}, S(b_{(1)}) > \omega_{b_{(2)}} < x_{(2)}, b_{(3)} > .\end{aligned} \tag{54}$$

The contraction of left-invariant forms with $\boldsymbol{i}_x$ — i.e. by a *left-invariant* $x \in \mathcal{U}$ — gives a number in the field k rather than a function in $\mathcal{A}$ as was the case for $\mathbf{d}(a)$:

$$\begin{aligned}\boldsymbol{i}_x(\omega_b) &= \boldsymbol{i}_x(-\mathbf{d}(Sb_{(1)})b_{(2)}) \\ &= -\boldsymbol{i}_x(\mathbf{d}Sb_{(1)})b_{(2)} \\ &= -< x - 1\epsilon(x), S(b_{(1)}) > S(b_{(2)})b_{(3)} \\ &= -< x, S(b) > +\epsilon(x)\epsilon(b).\end{aligned} \tag{55}$$

As an exercise and to check consistency we will compute the same expression in a different way:

$$\begin{aligned}\boldsymbol{i}_x(\omega_b) &= \boldsymbol{i}_x(S(b_{(1)})\mathbf{d}(b_{(2)})) \\ &= < x_{(1)}, S(b_{(1)}) > S(b_{(2)})\boldsymbol{i}_{x_{(2)}}(\mathbf{d}b_{(2)}) \\ &= < x_{(1)}, S(b_{(1)}) > S(b_{(2)})b_{(3)} < x_{(2)} - 1\epsilon(x_{(2)}), b_{(4)} > \\ &= < x_{(1)}, S(b_{(1)}) >< x_{(2)} - 1\epsilon(x_{(2)}), b_{(2)} > \\ &= \epsilon(x)\epsilon(b) - < x, S(b) > .\end{aligned} \tag{56}$$

3 Quantum Lie Algebras

Now we turn our attention to the case where the Hopf algebras in question are quantum Lie algebras and functions on the quantum group corresponding to that algebra. We start with a Lie algebra $\mathbf{g}$; its *quantum universal enveloping algebra* $U_q\mathbf{g}$ is the Hopf algebra whose elements are polynomials in the generators of $\mathbf{g}$ modulo (deformed) commutation relations. Dually paired with $U_q\mathbf{g}$ is Fun($\mathbf{G}_q$), the Hopf algebra of functions on the quantum group $\mathbf{G}_q$.

In the following we would like to concentrate on a bicovariant basis of left-invariant vector fields [9] $\{\chi_i \in U_q\mathbf{g} | i = 1, \ldots, n\}$, i.e. the χ_is are left-invariant and close under right coaction:

$$\begin{aligned}{}_{\mathcal{A}}\Delta(\chi_i) &= 1 \otimes \chi_i, \\ \Delta_{\mathcal{A}}(\chi_i) &= \chi_j \otimes T^j{}_i,\end{aligned} \tag{57}$$

with $T \in M_n(\mathcal{A})$. The identification with $\mathbf{g}$ is made by requiring that, in the classical limit, the χ_is reduce to either a generator of $\mathbf{g}$ or a Casimir operator in $U(\mathbf{g})$. (Note that such a choice of basis might not be unique or even possible.) We interpret $\mathcal{T}_q$:=span$\{\chi_i\}$ as the quantum analog of a tangent bundle, and take $\mathcal{U}$ (see previous section) to be its UEA.

Dual to this basis of vector fields is a basis of 1-forms $\omega^i \equiv \omega_{b^i}$ corresponding to a set of functions $b^i \in \mathcal{A}$ that satisfy

$$\boldsymbol{i}_{\chi_i}(\omega^j) = -< \chi_i, S(b^j) >= \delta_i^j. \tag{58}$$

$T_q^* :=$span$\{\omega^j\}$ is to be interpreted as the quantum analog of the cotangent bundle. Using these particular 1-forms we can reexpress the exterior derivative on functions $a \in \mathcal{A}$ as

$$\mathbf{d}(a) = \omega^i(\chi_i \triangleright a) = \omega^i \pounds_{\chi_i}(a). \tag{59}$$

From the previous equation and the Leibniz rule for $\mathbf{d}$, we find

$$a\omega^i = \omega^j \pounds_{\Theta_j{}^i}(a), \tag{60}$$

where

$$\Theta_j{}^i = -\chi_{j(1)} < \chi_{j(2)}, S(b^i) > +\chi_j \epsilon(b^i). \tag{61}$$

Furthermore, the requirement that $\mathbf{d}$ be invariant under coactions gives an (often overlooked) additional condition on the b^i's:

$$\Delta_{\mathcal{A}}(\omega^i) = \omega^j \otimes S^{-1}(T^i{}_j) \tag{62}$$

and therefore

$$\Delta^{\mathrm{Ad}}(b^i) = b^j \otimes S^{-1}(T^i{}_j), \tag{63}$$

where $\Delta^{\mathrm{Ad}}(b) \equiv b_{(2)} \otimes S(b_{(1)})b_{(3)}$ and we have used that $\Delta_{\mathcal{A}}(\omega_b) = \omega_{b_{(2)}} \otimes S(b_{(1)})b_{(3)}$. If we assume for simplicity a coproduct of the standard form

$$\Delta(\chi_i) = \chi_i \otimes 1 + O_i{}^j \otimes \chi_j, \tag{64}$$

where $O_i{}^j \in U_q\mathbf{g}$ then from (54) and (55) we find commutation relations for $\boldsymbol{i}_{\chi_i}$ with ω^j

$$\begin{aligned} \boldsymbol{i}_{\chi_i}\omega^j &= \delta_i^j - \pounds_{O_i{}^k}(\omega^j)\boldsymbol{i}_{\chi_k} \\ &= \delta_i^j - \omega^m < O_i{}^k, S^{-1}(T^j{}_m) > \boldsymbol{i}_{\chi_k} \end{aligned} \tag{65}$$

which can be used to define the wedge product $\wedge$ as some kind of antisymmetrized tensor product as follows.[§] As in the classical case we make an ansatz for the product of two forms in terms of tensor products

$$\omega^i \wedge \omega^j = \omega^i \otimes \omega^j - \hat{\sigma}^{ij}{}_{mn}\omega^m \otimes \omega^n, \tag{66}$$

[§]So far we have suppressed the $\wedge$-symbol; to avoid confusion we will reinsert it in this paragraph.

with as yet unknown numerical constants $\hat{\sigma}^{ij}{}_{mn} \in k$, and define $\boldsymbol{i}_{\chi_i}$ to act on this product by contracting in the first tensor product space; i.e.

$$\boldsymbol{i}_{\chi_i}(\omega^j \wedge \omega^k) = \delta_i^j\omega^k - \hat{\sigma}^{jk}{}_{mn}\delta_i^m\omega^n. \tag{67}$$

But from (65) we already know how to compute this, namely

$$\begin{aligned} \boldsymbol{i}_{\chi_i}(\omega^j \wedge \omega^k) &= \delta_i^j\omega^k - \pounds_{O_i{}^\ell}(\omega^j)\delta_\ell^k \\ &= \delta_i^j\omega^k - \omega^m < O_i{}^k, S^{-1}(T^j{}_m) > \end{aligned} \tag{68}$$

and by comparison we find:

$$\hat{\sigma}^{ij}{}_{mn} =< O_m{}^j, S^{-1}(T^i{}_n) > \tag{69}$$

or

$$\begin{aligned} \omega^i \wedge \omega^j &= \omega^i \otimes \omega^j - < O_m{}^j, S^{-1}(T^i{}_n) > \omega^m \otimes \omega^n \\ &= (I - \hat{\sigma})^{ij}{}_{mn}\omega^m \otimes \omega^n \\ &= \omega^i \otimes \omega^j - \omega^k \otimes \pounds_{O_k{}^j}(\omega^i). \end{aligned} \tag{70}$$

These equations can be used to obtain the (anti)commutation relations between the ω^is; by using the characteristic equation for $\hat{\sigma}$, projection matrices orthogonal to the antisymmetrizer $I - \hat{\sigma}$ can be found, and these will annihilate $\omega^i \wedge \omega^j$. The resulting equations will determine how to commute the 1-forms. Using the same method we can also obtain a tensor product decomposition of products of inner derivations

$$\boldsymbol{i}_m \wedge \boldsymbol{i}_n = \boldsymbol{i}_m \otimes \boldsymbol{i}_n - \hat{\sigma}^{ij}{}_{mn}\boldsymbol{i}_i \otimes \boldsymbol{i}_j, \tag{71}$$

defined to act on 1-forms by contraction in the first tensor product space. This can again be used to find (anti)commutation relations for the $\boldsymbol{i}$s via projection matrices as mentioned above.

Remark: The tensor product decomposition of the wedge product is invariant under linear changes of the $\{\chi_i\}$ basis, but it does depend on our choice of quantum tangent bundle.

There is actually an operator W that recursively translates wedge products into the tensor product representation:

$$W : \Lambda_q^p \to \mathcal{T}_q^* \otimes \Lambda_q^{p-1}, \quad p \geq 1, \\ W(\alpha) = \omega^n \otimes \boldsymbol{i}_{\chi_n}(\alpha) \tag{72}$$

for any p-form α. E.g.:

$$\begin{aligned} \omega^n \otimes \boldsymbol{i}_{\chi_n}(\omega^j \wedge \omega^k) &= \omega^n \otimes (\delta_n^j \omega^k - \mathcal{L}_{O_n{}^m}(\omega^j)\delta_m^k) \\ &= \omega^j \otimes \omega^k - \omega^n \otimes \mathcal{L}_{O_n{}^k}(\omega^j). \end{aligned} \tag{73}$$

W is *not* limited to Hopf algebras with a coproduct of the standard form (64); any form of the coproduct is admissible. For example, the matrix $\hat{\sigma}$ will generalize to

$$\hat{\sigma}^{ij}_{mn} = - < \chi_m, S^{-1}(T^i{}_n)S(b^j) > + \epsilon(b^j) < \chi_m, S^{-1}(T^i{}_n) > . \tag{74}$$

This is a generalization of results in [1]. However examples of bicovariant vector fields with comultiplication more general than (64) have not been studied yet.

4 Conclusion

In this article we were able to define the actions of an exterior derivative, Lie derivatives and inner derivations on forms and non-commutative functions such that these objects satisfy a closed algebra, namely a generalized Cartan Calculus. Most of the relations that we derive require only a Hopf algebra structure. To be able to give commutation relations for inner derivations, forms, and forms with functions, however, we need to make reference to a (finite) bicovariant basis of left-invariant vector fields. Such a bicovariant basis permits the decomposition of wedge products into tensor products as well as a realization of the exterior derivative in terms of 1-forms and vector fields. It is interesting to observe that all the "braiding" was done by Lie derivatives like e.g. $\mathcal{L}_{O_i{}^j}$.

We focused on Lie derivatives and inner derivations along *left-invariant* vector fields i.e. elements of $\mathcal{U}$. This approach is both a generalization and a restriction of the undeformed theory: The classical case only involves derivatives along vector fields in the tangent bundle (e.g. $X \in$ span(g)) but allows functional coefficients i.e. the vector fields need not be left-invariant. In contrast to this we introduce derivatives along any element of the UEA. Noting that $1 + X$ and even e^X are such elements, the name "transport" instead of "derivative" might be more appropriate. An attempt to introduce derivatives along elements in $\mathcal{A} \rtimes \mathcal{U}$ leads us to the following set of equations:

$$\boldsymbol{i}_{f\chi} = f\boldsymbol{i}_\chi \tag{75}$$

$$\mathcal{L}_{f\chi} = \mathrm{d}f\boldsymbol{i}_\chi + f\boldsymbol{i}_\chi\mathrm{d} + \epsilon(\chi)\mathcal{L}_f \tag{76}$$

$$\mathcal{L}_{f\chi} = f\mathcal{L}_\chi + \mathrm{d}(f)\boldsymbol{i}_\chi + \epsilon(\chi)(\mathcal{L}_f - f) \tag{77}$$

$$\mathcal{L}_{f\chi}\mathrm{d} = \mathrm{d}\mathcal{L}_{f\chi} \tag{78}$$

The range of validity of these equations is rather limited; if, for instance, we allow χ to have non-zero counit then the aforementioned formulas seem to be only consistent when evaluated on a or $\mathrm{d}(a)$, where $a \in \mathcal{A}$ is an arbitrary function. If however we consider only χs with zero counit then the problematic term $\mathcal{L}_f$ drops out of all equations. Equation (77) becomes

$$\mathcal{L}_{f\chi} = f\mathcal{L}_\chi + \mathrm{d}(f)\boldsymbol{i}_\chi, \qquad \epsilon(\chi) = 0, \tag{79}$$

and can be used to define Lie derivatives recursively on any form. There does not seem to be a way to generalize (45), i.e. to introduce Lie derivatives of *vector fields* along *arbitrary* elements of $\mathcal{A} \rtimes \mathcal{U}$ in the quantum case. Exceptions will be discussed in [13].

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