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Coadjoint Orbits and Conformal Field Theory

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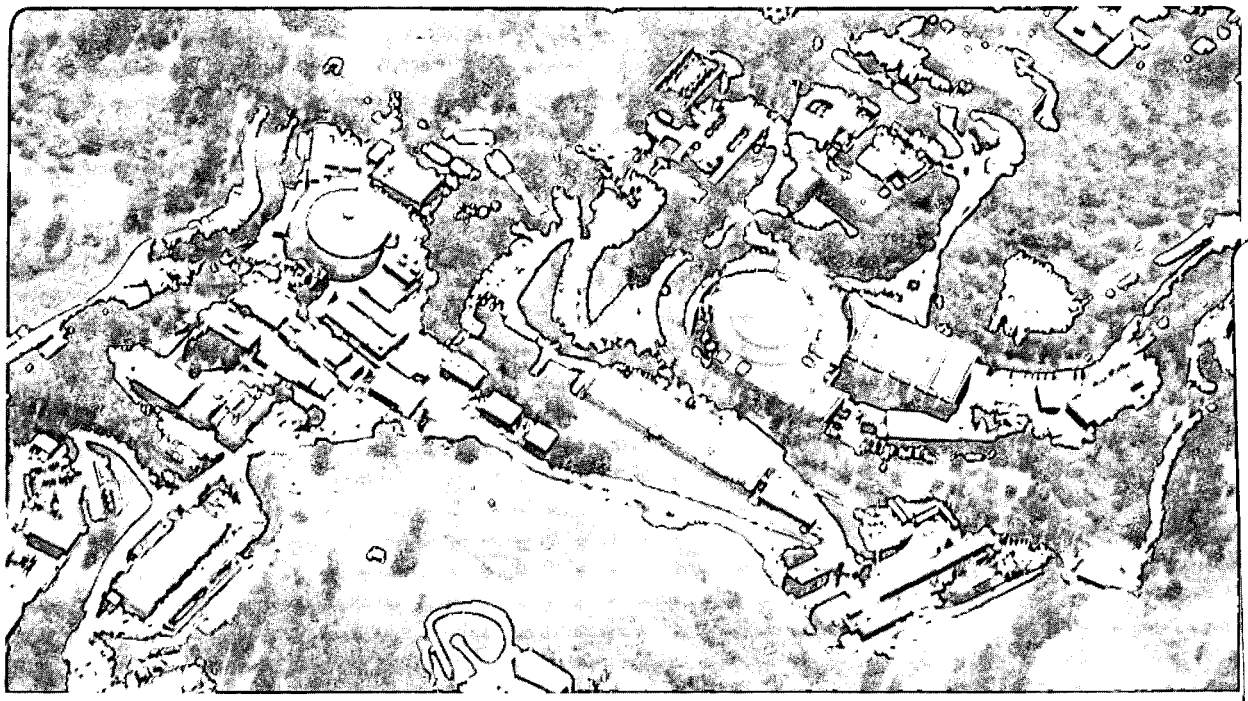

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Coadjoint Orbits and Conformal Field Theory*

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Chapter 1

Introduction

1.1 Results

This thesis is primarily a study of certain aspects of the geometric and algebraic structure of coadjoint orbit representations of infinite-dimensional Lie groups. The goal of this work is to use coadjoint orbit representations to construct conformal field theories, in a fashion analogous to the free-field constructions of conformal field theories.

The new results which are presented in this thesis are as follows: First, an explicit set of formulae are derived giving an algebraic realization of coadjoint orbit representations in terms of differential operators acting on a polynomial Fock space. These representations are equivalent to dual Verma module representations. Next, intertwiners are explicitly constructed which allow the construction of resolutions for irreducible representations using these Fock space realizations. Finally, vertex operators between these irreducible representations are explicitly constructed as chain maps between the resolutions; these vertex operators allow the construction of rational conformal field theories according to an algebraic prescription.

From the point of view presented in this thesis, the space of states associated with each primary field of a conformal field theory is described by a Hilbert space of holomorphic sections of a certain line bundle over a complex homogeneous space. The vertex operators are simply differential operators taking sections of one line bundle to sections of another line bundle. For conformal field theories with a simple Virasoro symmetry, there is a BRST complex of line bundles connected by differential operators for each primary state, and physical states are those states in the BRST cohomology of this complex. In the case

of conformal field theories with affine algebra symmetries such as the WZW model, the restriction to holomorphic sections of the line bundles automatically restricts to the space of physical states, so no BRST complex is necessary.

The infinite-dimensional groups which are studied in this thesis are the centrally extended loop groups $\widehat{L}G$ where G is a compact simple finite-dimensional Lie group, and the Virasoro group $\widehat{\mathrm{Diff}}S^1$. In the case of the loop groups, the coadjoint orbit construction is effectively equivalent to the Borel-Weil theory describing irreducible representations, and is already fairly well understood [53]. For the Virasoro group, on the other hand, there is not yet a complete understanding of the structure of coadjoint orbit representations; the results presented here are a modest step towards such an understanding. For all these infinite-dimensional groups, the explicit realization in terms of differential operators given here is new.

The construction of rational conformal field theories using resolutions and vertex operators on dual Verma module representations is algebraically very similar to the related construction using free field theories and Feigin-Fuchs or Wakimoto modules [27, 18]. The approach developed here has several advantages over the free field construction. Because of the geometric nature of the coadjoint orbit construction, the dual Verma module resolutions are more naturally geometrically motivated. Additionally, the resolutions in terms of coadjoint orbit representations are one-sided, and therefore avoid some of the complications associated with the two-sided resolutions arising from Feigin-Fuchs and Wakimoto modules. Finally, the known relationship between coadjoint orbits and actions for conformal field theories may indicate that the vertex operators described here naturally arise in some unifying geometric approach to conformal field theory. On the other hand, some of the advantages of the free field realizations are absent in the coadjoint orbit construction. The most significant of these features is the field theory interpretation of the free field construction. As yet, we have no analogous field theoretic interpretation for the coadjoint orbit representations, which presents a major obstacle to the physical interpretation of this construction. Also, the free field construction is based on well-studied techniques from string theory, which lead to powerful computational methods for the conformal field theory correlation functions. The construction presented here in terms of coadjoint orbit representations is not yet sufficiently well developed to allow the computation of correlation functions of any significant complexity.

1.2 Background

The method of coadjoint orbits was originated by Kirillov and Kostant twenty years ago [43]. This approach has proven to be a valuable tool in investigating geometrical aspects of the representation theory of Lie groups. The Kirillov-Kostant approach is essentially a generalization of the Borel-Weil theorem, which constructs irreducible unitary representations of a finite-dimensional compact semi-simple Lie group G as spaces of holomorphic sections of complex line bundles over the homogeneous space G/T, where T is a maximal subtorus of G. In the coadjoint orbit approach, one begins with a group G, with Lie algebra $\mathfrak{g}$. The group G has a natural coadjoint action on the dual space $\mathfrak{g}^*$. Choosing an element b in $\mathfrak{g}^*$, one considers the coadjoint orbit W_b of b in $\mathfrak{g}^*$. For any b, the space W_b has a natural symplectic form ω. For those b with the property that a complex line bundle $\mathcal{L}_b$ can be constructed over W_b with curvature form $i\omega$, one attempts to relate an appropriate space of sections of $\mathcal{L}_b$ to an irreducible unitary representation of G by using the technique of geometric quantization on the space W_b. For finite-dimensional compact semi-simple G, the representations produced by this construction are equivalent to those given by the Borel-Weil theory. The coadjoint orbit approach is particularly useful in the case of non-compact groups, where the Borel-Weil theory does not apply. It is possible to apply the Borel-Weil approach to certain infinite-dimensional groups such as the centrally extended loop groups $\widehat{L}G$ [53]. For other infinite-dimensional groups, such as the (orientation-preserving) diffeomorphism group of the circle $\mathrm{Diff}S^1$, and its central extension $\widehat{\mathrm{Diff}S^1}$, the Virasoro group, there are difficulties with applying even the more general coadjoint orbit theory. Many of the Virasoro coadjoint orbits do not admit a Kähler structure, so that it is difficult to geometrically quantize these spaces. Also, it is known that the Virasoro group has rather peculiar mathematical properties, such as the fact that the exponential map on the Lie algebra is neither onto nor 1-1 in the vicinity of the identity. Due to these difficulties, a full understanding of the coadjoint orbit representations for this group has not yet been attained, although there are some partial results in this direction [44, 47, 58, 64]. Achieving a full understanding of the geometry of the coadjoint orbit representations of the Virasoro group could be a valuable step in the general study of Virasoro representations and conformal field theory.

A more direct relationship between irreducible representations of infinite-dimensional Lie groups and conformal field theory (CFT) is embodied in the free field approach to CFT's.

The free field approach was fundamental in early developments in string theory [37]. It was eventually shown that the operators associated with free fields acting on a bosonic Fock space could be used to construct all irreducible unitary representations of the Virasoro algebra. The "Feigen-Fuchs" free field representations were described in the work of Dotsenko and Fateev [22] using a Coulomb gas-like free field theory with background charge. In this work, these representations were used to calculate correlation functions in Virasoro minimal models. The structure of the Fock space in the Feigen-Fuchs representations was originally described in [26]. It was subsequently shown by Felder [27] that the irreducible representations in the $c \leq 1$ discrete series could be described in terms of the Feigin-Fuchs representations using a BRST-type screening operator which had previously been introduced by Thorn [61]. A similar construction for conformal field theories with affine algebra symmetries has also been carried out [14, 33, 63]. The resulting free field representations of the affine algebras are known as "Wakimoto" modules. Recently, the free field approach to conformal field theory has been described in the algebraic language of modules and resolutions, and put into a systematic algebraic formalism. From this point of view, the algebraic structure of the Feigin-Fuchs representations is similar to that of a twisted Verma module. For a review of this approach, see [18].

1.3 Summary

The structure of this thesis is as follows. In Chapter 2, we review the coadjoint orbit description of irreducible representations of Lie groups, and construct an explicit algebraic realization of the Lie algebra of an arbitrary Lie group by taking local coordinates on a coadjoint orbit and performing a gauge fixing on the complex line bundle over that coadjoint orbit. This construction gives a set of representations of the Lie algebra which are expressed in terms of first-order differential operators acting on a polynomial Fock space. These representations are studied in detail for finite-dimensional Lie groups, loop groups, and the Virasoro group, and examples of explicit realizations are given for each of these types of groups. By considering global aspects of these representations, we show that in the cases of finite-dimensional groups and loop groups this construction gives rise to irreducible unitary representations, and that for such groups all irreducible unitary representations can be constructed using this method. In the case of the Virasoro group, these representations are reducible representations, with irreducible subspaces corresponding to the irreducible

unitary representations of the Virasoro group. In Chapter 3, we consider in more detail the module structure of these representations. We show that all the coadjoint orbit representations we have constructed are locally equivalent to dual Verma modules, and describe resolutions for irreducible representations in terms of explicitly defined formulae for intertwiners between coadjoint orbit representations. We go on in Chapter 4 to describe vertex operators which allow these irreducible representations to be combined into rational conformal field theories. In Chapter 5, we review our results, and discuss open questions and relationships between this research and other recent work. In particular, we describe briefly a recently active area of research in which coadjoint orbits are used to construct conformal field theory actions. We speculate that there may be an underlying connection between the structure of these actions and the conformal field theory formalism described in this thesis.

Chapter 2

Coadjoint Orbit Representations

In this chapter, the geometric and algebraic structures of coadjoint orbit representations are presented and analyzed. In Section 2.1, we describe the class of groups to which the analysis in this chapter is applicable and establish notation for Lie groups and algebras which will be used throughout this thesis. For each of the groups in this section, a description is given of a local coordinate system near the identity on a homogeneous quotient space of the group which will be later identified with a coadjoint orbit space. In Section 2.2, we review the geometric construction of coadjoint orbit representations as sections of line bundles $\mathcal{L}_b$ over the coadjoint orbit spaces associated with elements b in the dual of the Lie algebra. In Section 2.3, we prove a pair of general propositions in which sufficient conditions are given for a set of functions to describe a local gauge-fixed connection on the line bundle $\mathcal{L}_b$. These propositions are used in Section 2.4 to construct formulae for the coadjoint orbit representations of the group algebra in terms of differential operators on the space of locally holomorphic functions at the point b in the orbit space. In Section 2.5, we give examples of these realizations for the groups $SU(2), SU(3), \widehat{L}SU(2)$, and the Virasoro group. In Section 2.6, we discuss global questions about these representations; in particular we consider the question of which sections of the line bundle $\mathcal{L}_b$ are globally holomorphic, and we discuss the existence of Hermitian structures on the line bundles $\mathcal{L}_b$. Both of these questions are addressed from the point of view of the local formulae derived in the earlier sections.

2.1 Lie groups and algebras

There are three main types of Lie groups which, along with their associated Lie algebras, will be of interest in this thesis. The compact simple Lie groups are the most familiar and best understood examples in the theory of continuous groups and their representations. We will use these groups as a class of elementary examples and as a reference point with which to compare our results for infinite-dimensional groups. The centrally-extended loop groups, whose associated Lie algebras are affine algebras, are the second class of groups which we will study. Although not as well understood as the finite-dimensional simple Lie groups, these groups share many of the properties of groups in that class. The affine algebras are important in physics, where they appear as current algebras in two-dimensional field theories and provide a primary tool for the understanding of the algebraic structure of a large class of conformal field theories. Finally, we will study the representations of the Virasoro group, which is of central importance in the physics of conformal field theories. This group is even less understood from a geometric perspective than the loop groups – hopefully some of the work in this thesis will help to provide a basis for a deeper understanding of the geometric structure of the representations of this group.

In this section, we define these three types of Lie groups and develop the notation which will be used throughout the thesis. We also derive or state some important theorems about these groups which will be used in later developments. In particular, we give a description in terms of local coordinates of a natural quotient space of each Lie group by a maximal abelian subgroup. In the case of the Virasoro group, a rigorous proof of the validity of this coordinate system is not given; however, arguments are given for the plausibility of this coordinate system in a well-defined mathematical context. General references for the background material in this section are [1, 40, 41, 45, 51, 53].

2.1.1 Compact simple Lie groups

A finite-dimensional Lie group G is a manifold of finite dimension which has a group structure

$$\cdot : G \times G \to G. \tag{2.1}$$

In this subsection, we assume that all Lie groups are finite-dimensional. Associated with every Lie group is an algebraic structure on the tangent space $\mathfrak{g}$ to G at the identity. This

algebra, called the Lie algebra of G, is defined by the product

$$[\xi, \eta] = \lim_{s,t\to 0} \frac{1}{st}\{e^{s\xi}e^{t\eta}e^{-s\xi}e^{-t\eta} - 1\}, \tag{2.2}$$

where the exponential map $e : \mathfrak{g} \to G$ is defined by taking $e^{t\xi}$ to be a one-parameter subgroup of G with $de^{t\xi}/dt = \lim_{t\to 0} e^{t\xi}/t = \xi$. When G is finite-dimensional, it can be shown that this exponential map exists and is locally 1-1. The Lie algebra product is antisymmetric and satisfies the Jacobi identity

$$[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0. \tag{2.3}$$

A Lie group is compact if its underlying manifold is compact. A Lie group is complex if its manifold is a complex manifold and the group composition law is holomorphic. A Lie algebra is complex if the vector space is complex and the Lie algebra product is complex-bilinear. The Lie algebra associated with a complex Lie group is a complex Lie algebra. A complexification $G_{\mathbb{C}}$ of a Lie group G is a complex Lie group whose Lie algebra is the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ of the Lie algebra $\mathfrak{g}$ of G. When G is a compact group, such a complexification always exists [53].

A simple Lie group is a nonabelian Lie group with no continuous normal subgroups. It follows that a simple Lie algebra is a nonabelian Lie algebra with no nontrivial ideals. The compact simple Lie groups have been classified, and consist of the groups $SU(N)$, $SO(N)$, $Sp(N)$, the exceptional groups G_2, F_4, E_6, E_7, E_8, and groups which are related to any of these groups by taking a quotient through a finite normal subgroup [40]. For the remainder of this subsection, we assume that all Lie groups are compact and simple. Most of the results for simple groups can be easily generalized to the class of semisimple Lie groups, which are groups locally isomorphic to a product of simple groups (with no abelian factors); we will not bother, however, to explicitly discuss these generalizations, in order to keep the presentation relatively simple.

A representation of a Lie group G (Lie algebra $\mathfrak{g}$) is a linear action of G ($\mathfrak{g}$) on a vector space V. A representation is irreducible if V does not contain a nontrivial subspace W which is mapped into itself by G ($\mathfrak{g}$). It is a theorem that all irreducible representations of compact Lie groups are finite-dimensional [53].

The group G has a natural representation on $\mathfrak{g}$ known as the *adjoint action*; for each $g \in G$

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}, \tag{2.4}$$

where

$$\mathrm{Ad}_g : u \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\{g e^{tu} g^{-1} - 1\}. \tag{2.5}$$

There is a related *coadjoint action* of G on the dual space to $\mathfrak{g}$, $\mathfrak{g}^*$. The coadjoint action is denoted by Ad^*, and given by

$$\langle \mathrm{Ad}_g^* b, u\rangle = \langle b, \mathrm{Ad}_{g^{-1}} u\rangle, \quad \text{for } b \in \mathfrak{g}^*, u \in \mathfrak{g}. \tag{2.6}$$

The derivative of the adjoint action gives an action of $\mathfrak{g}$ on $\mathfrak{g}$, denoted by ad, where $\mathrm{ad}_u v = [u, v]$, for all $u, v \in \mathfrak{g}$. Similarly, the infinitesimal coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ is denoted ad^*, and is given by

$$\langle \mathrm{ad}_v^* b, u\rangle = \langle b, [u, v]\rangle, \quad \text{for } b \in \mathfrak{g}^*, u, v \in \mathfrak{g}. \tag{2.7}$$

In order to study the representations of a Lie group G and its algebra $\mathfrak{g}$, it is useful to describe a *root space decomposition* of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$. One first picks a maximal subtorus T of G. (A maximal subtorus of G is a continuous abelian subgroup of G which is not contained in a continuous abelian subgroup of higher dimension.) The complexification $\mathfrak{t}_{\mathbb{C}}$ of the Lie algebra of T is a *Cartan subalgebra* of $\mathfrak{g}_{\mathbb{C}}$. The dimension of T is defined to be the *rank* r of G. All maximal subtori of G are conjugate [1], so the rank is well-defined. The adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$ extends naturally to an action of $\mathfrak{g}$ on $\mathfrak{g}_{\mathbb{C}}$. Since the Lie algebra $\mathfrak{t}$ is abelian, the restriction of the adjoint action on $\mathfrak{g}_{\mathbb{C}}$ to $\mathfrak{t}$ is simultaneously diagonalizable, and we can write

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \tag{2.8}$$

where $\Phi \subset \mathfrak{t}_{\mathbb{C}}^*$ is the set of nonzero *roots* of G (which we will also refer to as roots of $\mathfrak{g}$), and

$$\mathfrak{g}_\alpha = \{u \in \mathfrak{g}_{\mathbb{C}} | [h, u] = \langle \alpha, h\rangle u \ \forall h \in \mathfrak{t}\}. \tag{2.9}$$

An analysis of this root space decomposition reveals that the root spaces $\mathfrak{g}_\alpha$ are one-dimensional and that $\alpha \in \Phi \Rightarrow -\alpha \in \Phi$. It is possible to choose elements $e_\alpha \in \mathfrak{g}_\alpha$ and $h_\alpha \in \mathfrak{t}_{\mathbb{C}}$ such that when $f_\alpha = e_{-\alpha}$, we have

$$\begin{aligned} [e_\alpha, f_\alpha] &= h_\alpha \\ [h_\alpha, e_\alpha] &= 2e_\alpha \\ [h_\alpha, f_\alpha] &= -2f_\alpha. \end{aligned} \tag{2.10}$$

In this basis, when $\alpha \neq -\beta$ and $[e_\alpha, e_\beta] \neq 0$, it follows that $[e_\alpha, e_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. It is furthermore possible [40] to choose a *base* Δ for Φ, where $\Delta \subset \Phi$ is defined to be a base when $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ gives a basis for the linear space $\mathfrak{t}^*_{\mathbb{C}}$ spanned by Φ, and each root $\beta \in \Phi$ has a (unique) decomposition

$$\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha, \tag{2.11}$$

with k_α integral and with all coefficients k_α being either nonnegative or nonpositive. The roots in Δ are called *simple* roots. If all the coefficients k_α in the decomposition (2.11) of a root β are nonnegative, β is defined to be a *positive* root, and we write $\beta \succ 0$. Similarly, if all coefficients are nonpositive, we define β to be a *negative* root. Because the root spaces are one-dimensional, we can directly associate generators e_α with roots; thus, we will often refer to generators as being simple or positive when the associated root has such a property. We denote by $\Phi_+ = \{\alpha \in \Phi | \alpha \succ 0\}$ the set of all positive roots. In physics terminology, positive and negative roots are often referred to as "annihilation" and "creation" operators on a representation space, respectively. The *root lattice* Λ is defined to be the sublattice of $\mathfrak{t}^*_{\mathbb{C}}$ spanned by the simple roots Δ.

We denote the generators associated with simple roots by $e_j = e_{\alpha_j}$ for $1 \leq j \leq r$; similarly, we define $f_j = e_{-\alpha_j}$, $h_j = h_{\alpha_j}$. The generators $\{e_j, h_j, f_j : 1 \leq j \leq r\}$ give rise to a complete basis for the Lie algebra $\mathfrak{g}_{\mathbb{C}}$, under the relations

$$\begin{aligned}
[e_i, f_i] &= h_i \\
[e_j, f_k] &= 0 \quad \text{when} \quad j \neq k \\
[h_i, e_j] &= A_{ij} e_j \\
[h_i, f_j] &= -A_{ij} f_j \\
(\text{ad } e_i)^{1-A_{ij}} e_j &= 0 \quad \text{when} \quad i \neq j \\
(\text{ad } f_i)^{1-A_{ij}} f_j &= 0 \quad \text{when} \quad i \neq j.
\end{aligned} \tag{2.12}$$

This basis for the Lie algebra is called a *Chevalley basis* for the algebra, and the matrix A is the *Cartan matrix* for the group G. Note that the algebra elements e_j associated with simple roots generate the algebra of positive roots in $\mathfrak{g}_{\mathbb{C}}$; similarly, the negatives of the simple roots, f_j, generate the algebra of negative roots. Both of these subalgebras are closed subalgebras of $\mathfrak{g}_{\mathbb{C}}$. Note also that the algebra elements h_j form a basis Θ for the Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$.

We now turn our attention to irreducible representations of simple groups. To begin with, it is a well-known result that all irreducible representations of simple groups admit unitary structures. That is, on any complex representation space V of a finite-dimensional compact simple group G, it is possible to construct a positive-definite inner product which is invariant under the action of G. The existence of such a unitary structure can be seen by taking an arbitrary positive-definite inner product on V, and averaging over G with respect to an invariant measure (such a measure, called a Haar measure, exists on all compact finite-dimensional groups). In most physical systems where group representations play an important role, the understanding of a unitary structure on the representation space is essential. It is customary for physicists, when dealing with unitarity group representations, to take a basis for the Lie algebra of the form iJ_a where J_a are Hermitian operators. In terms of this basis, the Lie algebra is written in terms of the *structure constants* ${f_{ab}}^c$ as

$$[J_a, J_b] = i{f_{ab}}^c J_c. \tag{2.13}$$

We will freely switch between mathematical and physical notations for Lie groups and representations in this thesis.

In any (complex) vector space V carrying a representation of G, it is possible to choose a basis with respect to which the Cartan algebra $\mathfrak{t}_{\mathbb{C}}$ is diagonal. The eigenvalues of $\mathfrak{t}_{\mathbb{C}}$ on a basis element are then described by a *weight* $w \in \mathfrak{t}_{\mathbb{C}}^*$. The weights w of a representation lie on the *weight lattice* $\Lambda_w \subset \mathfrak{t}_{\mathbb{C}}^*$, which contains the root lattice Λ as a sublattice. Clearly, acting on a basis element of weight w with a generator e_α gives a vector of weight $w + \alpha$ in V. A representation of G on a vector space V is defined to be *highest weight* if there exists a vector $v \in V$ which is annihilated by all positive roots; *i.e.*, $e_\alpha v = 0$ for all $\alpha \in \Phi_+$. For v to be a highest weight vector, it clearly suffices for all simple roots e_j to annihilate v. It follows from the fact that all irreducible representations of a compact Lie group are finite-dimensional, that all irreducible representations of such a group are highest weight representations.

Given a choice Δ of base for G, the associated Borel subalgebra $\mathfrak{b}^+ \subset \mathfrak{g}_{\mathbb{C}}$ is defined to be the subalgebra generated by the Cartan subalgebra and the positive roots,

$$\mathfrak{b}^+ = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha. \tag{2.14}$$

Similarly, the Borel subalgebra $\mathfrak{b}^- \subset \mathfrak{g}_{\mathbb{C}}$ is defined to be the subalgebra generated by the

Cartan subalgebra and the negative roots,

$$\mathfrak{b}^- = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \prec 0} \mathfrak{g}_\alpha. \tag{2.15}$$

The Borel subalgebra which is customarily used in representation theory is the positive subalgebra $\mathfrak{b}^+$. For the purposes of this thesis, however, we will find it more constructive to use the negative subalgebra $\mathfrak{b}^-$. It is a theorem that these two Borel subalgebras are conjugate [40]; therefore the analysis using $\mathfrak{b}^-$ is equivalent to that using $\mathfrak{b}^+$. (In fact, generally a Borel subalgebra is defined as a maximal solvable subalgebra; the theorem states that all Borel subalgebras are conjugate.) The Borel subgroup B^- of $G_{\mathbb{C}}$ is the subgroup whose Lie algebra is spanned by $\mathfrak{b}^-$. The space $G_{\mathbb{C}}/B^-$ is a compact Kähler homogeneous space for $G_{\mathbb{C}}$; as long as the group G is compact, we have $G_{\mathbb{C}}/B^- = G/T$ [53]. This space can be used to construct a general irreducible representation of G according to the Borel-Weil theory [17], which we now review briefly, following [53].

It is a general fact that every irreducible representation of G is uniquely determined by the associated representation of T on the highest weight state [1]. The representations of T are the one-dimensional representations on a space of fixed weight, and are simply the products of representations of the circle group $S^1 = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$,

$$e^{i\theta} \mapsto e^{ni\theta}, \quad n \in \mathbb{Z}. \tag{2.16}$$

Given a representation $\lambda : T \to S^1$ of T, it is possible to construct a line bundle $\mathcal{L}_\lambda$ over the homogeneous space G/T. This is done by considering the product bundle $G \times \mathbb{C}$, and modding out by the equivalence relation $\sim$ defined by

$$(g, z) \sim (gt, \lambda(t^{-1})z) \forall t \in T. \tag{2.17}$$

The line bundle is thus defined by

$$\mathcal{L}_\lambda = G \times \mathbb{C}/\sim. \tag{2.18}$$

An alternative description of this line bundle is given by extending the representation λ holomorphically to the Borel subgroup B^-. This gives a holomorphic representation of B^- taking values in $\mathbb{C} \setminus \{0\}$. The line bundle $\mathcal{L}_\lambda$ can then be defined by

$$\mathcal{L}_\lambda = G_{\mathbb{C}} \times \mathbb{C}/\sim, \tag{2.19}$$

where the equivalence relation (2.17) is extended to all elements $t \in B^-$. From this point of view, it follows immediately that $\mathcal{L}_\lambda$ is a homogeneous complex line bundle and that the group $G_\mathbb{C}$ has a natural action on the space $\mathcal{H}_\lambda$ of holomorphic sections of $\mathcal{L}_\lambda$. The Borel-Weil theorem states that the resulting representation of G on $\mathcal{H}_\lambda$ is an irreducible representation which reduces to the representation λ of T on a one-dimensional subspace. All irreducible representations of G can be obtained in this fashion; however, an irreducible representation of G may arise from several distinct representations of T which are equivalent under a Weyl group symmetry. In particular, however, choosing λ to be the representation of T associated with the highest weight subspace of a particular representation of G will reproduce that representation of G in the action on $\mathcal{H}_\lambda$.

Most of the analysis in this thesis will be done locally, in a coordinate patch around the identity of G. In order to perform local calculations, it will be useful to have a general system of coordinates in a neighborhood of the identity both on G and on G/T. It is clearly desirable to choose these coordinates to be holomorphic on G/T. Such a coordinate system is a familiar tool in physics; see e.g., [66, 6]. We can define a natural set of coordinates $\{z_\alpha, a_\alpha, \beta_\alpha | \alpha \in \Phi_+\}$ on $G_\mathbb{C}$ by writing an arbitrary element near the identity in the form

$$g = \exp\left[\sum_{\alpha\in\Phi_+} z_\alpha e_\alpha\right] \exp\left[\sum_{\alpha\in\Phi_+} a_\alpha e_{-\alpha}\right] \exp\left[\sum_{\alpha\in\Delta} \beta_\alpha h_\alpha\right], \tag{2.20}$$

Where h_α are the basis for the Cartan subalgebra $\mathfrak{t}_\mathbb{C}$ defined by (2.10). Clearly, z_α are a set of coordinates on $G_\mathbb{C}/B^-$, since modding out by the right action of B^- simply corresponds to dropping the coordinates a_α, β_α. It follows that z_α can also be taken as coordinates on G/T. This result can be made more explicit by observing that (2.20) can be constrained to be an element of G, which gives $a_\alpha(z,\bar{z})$ and $b_\alpha(z,\bar{z}) = \mathrm{Re}\ \beta_\alpha$ as functions of the coordinates $z_\alpha, \bar{z}_\alpha$. These functions are single-valued in a neighborhood of the identity, and can be calculated in a perturbative expansion about $z_\alpha = 0$ by applying the Baker-Campbell-Hausdorff (BCH) formula [43]

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\cdots}, \tag{2.21}$$

which expresses the product of two exponentiated elements of a Lie algebra in terms of a single exponentiated element of the algebra as a formal power series. (The ellipses in this formula denote third- and higher-order commutators between X and Y.) It can be seen that the coordinates z_α define a G-invariant complex structure on G/T by multiplying (2.20)

on the left by an arbitrary element $g' \in G$; expressing the result again in the form (2.20), one finds that the coordinates $z_\alpha(g'g)$ of the resulting point in G/T depend only on g' and the holomorphic coordinates $z_\alpha(g)$ and not on the antiholomorphic coordinates $\bar{z}_\alpha(g)$ [66]. Although this complex structure is defined locally, it can be extended to a global complex structure on G/T by translating under the left action of G.

We now conclude this subsection with a pair of examples of finite-dimensional simple Lie groups, $SU(2)$ and $SU(3)$, which we will use throughout this thesis as canonical examples with which to compare results from infinite-dimensional groups. We will follow the customary practice of using lower-case characters to denote Lie algebras, so that the algebras of $SU(2)$ and $SU(3)$ are written su(2), su(3).

First we consider the group $SU(2)$ of unitary 2×2 matrices. As usual (for physicists), we take the generators of the algebra $\mathfrak{g} = \mathrm{su}(2)$ to be $\{iJ_k : k = 1,2,3\}$, where $[J_j, J_k] = i\epsilon_{jkl}J_l$. The structure constants of su(2) are thus $f_{ab}{}^c = \epsilon_{abc}$. su(2) is a three-dimensional real vector space. Taking coordinates x_1, x_2, x_3 on $\mathfrak{g}$, an arbitrary element $u \in \mathfrak{g}$ can be written as $u = i\Sigma x_k J_k$. An arbitrary element g of G can be written as $g = \mathrm{e}^u$, where $u \in \mathfrak{g}$. We will take the maximal subtorus T to be $\{\mathrm{e}^{2i\theta J_3} : 0 \leq \theta \leq 2\pi\}$. A root space decomposition of $\mathfrak{g}_\mathbb{C}$ is given by defining the generators

$$J_\pm = J_1 \pm iJ_2. \tag{2.22}$$

These generators satisfy $[J_3, J_\pm] = \pm J_\pm$, and $[J_+, J_-] = 2J_3$. The basis elements e_α, h_α can be described in terms of these generators by $e_+ = J_+, e_- = f_+ = J_-$, and $h_+ = 2J_3$. We can now define $\Delta = \{J_+\}$ to be a base, so that J_+ is the unique positive root. The resulting Chevalley basis is $e_1 = e_+, f_1 = e_-, h_1 = h_+$, and the Cartan matrix is the 1×1 matrix (2).

The finite-dimensional irreducible representations of su(2) are representations with highest weight $j \in \mathbb{Z}/2$, corresponding to the eigenvalue of J_3 on the highest weight vector. The set of weights for the "spin" j representation is given by $\{j, j-1, \ldots, 1-j, -j\}$; to each weight there corresponds a single vector in the representation space V_j. To each finite-dimensional irreducible representation of su(2) there corresponds a representation of $SU(2)$. The weight space of a typical $SU(2)$ representation (spin 3/2) is shown in Figure 2.1.

With each irreducible representation j of $SU(2)$, there is an associated representation of the maximal subtorus T. The homogeneous space $SU(2)/T$ is S^2 with the usual complex structure. The bundle structure of this quotient space is exactly that of the well-known Hopf fibration $S^1 \to S^3 \to S^2$ [60]. Each representation j of T defines a complex

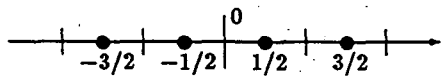


Figure 2.1: Spin 3/2 representation of $SU(2)$

line bundle $\mathcal{L}_j$ over S^2 by the Borel-Weil theory (2.19). The bundle associated with the representation j has a first Chern class with integral $2j$, and $2j+1$ linearly independent holomorphic sections which transform under the representation j of $SU(2)$. The geometric structure of this representation is equivalent to the coadjoint orbit construction for $SU(2)$ and will be discussed in more detail in Section 2.5.

We next consider the group $SU(3)$ of unitary 3×3 matrices. The generators of su(3) are customarily taken to be $\{J_a : 1 \leq a \leq 8\}$, which in the fundamental representation are related to the Hermitian Gell-Mann matrices $\{\lambda_a\}$ by $J_a = \lambda_a/2$. The Gell-Mann matrices are

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned} \tag{2.23}$$

The generators J_3, J_8 form a maximal abelian subalgebra of su(3) and can be taken to be the generators of a maximal subtorus

$$T = \{e^{2i\theta J_3} e^{2\sqrt{3}i\psi J_8} : 0 \leq \theta, \psi < 2\pi\}. \tag{2.24}$$

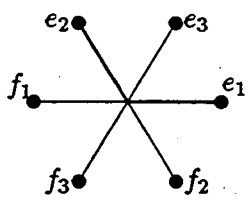


Figure 2.2: Roots of $SU(3)$; simple roots are bold.

A root space decomposition of $\mathfrak{g}_{\mathbb{C}}$ is given by the generators

$$\begin{array}{lllllllll} e_t & = & J_1 + iJ_2 & e_u & = & J_6 + iJ_7 & e_v & = & J_4 + iJ_5 \\ f_t & = & J_1 - iJ_2 & f_u & = & J_6 - iJ_7 & f_v & = & J_4 - iJ_5 \\ h_t & = & 2J_3 & h_u & = & (\sqrt{3}J_8 - J_3) & h_v & = & (\sqrt{3}J_8 + J_3), \end{array} \tag{2.25}$$

where the roots t, u, v are given by $t = (\langle t, J_3\rangle, \langle t, J_8\rangle) = (1, 0)$, $u = (-1/2, \sqrt{3}/2)$, $v = (1/2, \sqrt{3}/2)$. In terms of this basis, we can choose a base Δ for su(3) to be $\Delta = \{e_t, e_u\}$. The resulting Chevalley basis is given by

$$\begin{array}{lllllll} e_1 & = & e_t & & e_2 & = & e_u \\ f_1 & = & f_t & & f_2 & = & f_u \\ h_1 & = & h_t & & h_2 & = & h_u, \end{array} \tag{2.26}$$

and the Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{2.27}$$

In terms of this Chevalley basis, the remaining generators are $e_3 = [e_1, e_2] = e_v$ and $f_3 = [f_2, f_1] = f_v$. The generators are graphed according to their roots in Figure 2.2.

Irreducible representations of $SU(3)$ are labeled by the eigenvectors (p, q) of the highest weight vector with respect to the Cartan algebra elements h_1, h_2. The weights of a typical highest weight representation $(4, 1)$ of $SU(3)$ are shown in Figure 2.3. Note that there are in general multiple linearly independent vectors in the representation space with a given weight. As in the case of $SU(2)$, all the irreducible representations of $SU(3)$ can be realized via the Borel-Weil theory in terms of the left action of $SU(3)$ on spaces of holomorphic sections of line bundles over the homogeneous space $SU(3)/T$.

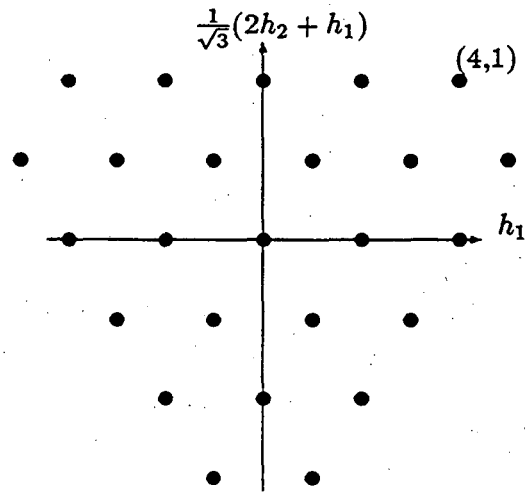


Figure 2.3: Representation (4,1) of $SU(3)$

2.1.2 Loop groups

In the previous subsection we described a class of finite-dimensional Lie groups. It is possible to develop a similar theory for infinite-dimensional Lie groups [51]. In general, an infinite-dimensional Lie group is a group whose manifold is an infinite-dimensional manifold modeled on a complete, locally convex topological vector space (a manifold is modeled on a vector space V when the coordinate charts give diffeomorphisms between neighborhoods in the manifold and open sets in V). When the group is modeled on a Banach space, there is a fairly well-developed theory, and several theorems are true which do not hold in a more general context. We will primarily be concerned with local algebraic properties of certain infinite-dimensional groups, so we will not develop the general theory of infinite-dimensional groups in any detail, but will simply quote results as necessary.

One of the main properties which we will need in the analysis of infinite-dimensional groups is the existence of a real analytic structure on the group. A sufficient condition for a group to have a real analytic structure is that the exponential map from the Lie algebra onto the Lie group is locally 1-1 and onto. The existence of a real analytic structure on a Lie group is in turn sufficient for the group to have the BCH property, which means that the Baker-Campbell-Hausdorff formula (2.21) is valid in a neighborhood of the identity [51]. Groups with this property are completely described in a neighborhood of the identity by the Lie algebra structure. Since we will be concentrating on analyzing the representations

of the algebra, this property is necessary to validate our analysis in a more global context.

It is possible to generalize the concepts of a complex structure and differential forms on a manifold to the infinite-dimensional case, provided that certain conditions are satisfied. These generalizations are possible for all the manifolds we consider here [53] .

The first class of infinite-dimensional groups which we will consider are the centrally extended loop groups $\widehat{L}G$. The loop group LG is defined for an arbitrary finite-dimensional Lie group G by the set of smooth maps from S^1 to G

$$LG = \{f : S^1 \to G | f \in C^\infty(S^1, G)\}. \tag{2.28}$$

The composition law in a loop group is defined pointwise by

$$(f \cdot g)(t) = f(t) \cdot g(t) \, \forall t \in S^1. \tag{2.29}$$

For simplicity, in this thesis we will restrict attention to loop groups LG where G is simple; as in the finite-dimensional case, an extension to general semisimple Lie groups is straightforward.

The Lie algebra $L\mathfrak{g}$ is simply the algebra of smooth maps from S^1 to the Lie algebra $\mathfrak{g}$. Again, the Lie algebra product of $L\mathfrak{g}$ is defined pointwise on S^1. It is not hard to show [51] that the loop groups LG admit real analytic structures, and thus have the BCH property (the argument is more generally valid for arbitrary map groups; *i.e.*, groups of maps from an arbitrary compact manifold M to G). The existence of a real analytic structure in this case follows from the fact that the exponential map

$$\exp : L\mathfrak{g} \to LG \tag{2.30}$$

is a local homeomorphism. If the Lie group G has a complexification $G_{\mathbb{C}}$, there is a natural complexification $LG_{\mathbb{C}}$ of the loop group LG, which gives a natural complexification $L\mathfrak{g}_{\mathbb{C}}$ of the Lie algebra. The Lie algebra $L\mathfrak{g}_{\mathbb{C}}$ has a natural basis $J_{a(n)}$ given by the set of Fourier modes relative to a basis J_a of $\mathfrak{g}_{\mathbb{C}}$ (actually, technically this is a basis for the related algebra of real-analytic maps from S^1 to $\mathfrak{g}$; in this thesis we will ignore this distinction, for details see [53]). The function from $S^1 \to \mathfrak{g}$ described by an element of this Fourier basis is

$$J_{a(n)}(\theta) = J_a e^{in\theta}. \tag{2.31}$$

If the structure constants of $\mathfrak{g}$ are given by $f_{ab}{}^c$, then the Lie algebra of the loop group is given by

$$[J_{a(n)}, J_{b(m)}] = i f_{ab}{}^c J_{c(n+m)}. \tag{2.32}$$

The loop groups LG appear in physics as the symmetry groups of certain conformal field theories (CFT's); see for example [34]. For example, the conformal field theory with n free real fermionic fields has a natural $LSO(n)$ symmetry, and the CFT with n free complex fermionic fields has $LSU(N)$ symmetry. These symmetries apply to the classical field theories. When the quantum field theories associated with these classical theories are constructed, the classical symmetry group acquires a central extension due to a quantum anomaly. For other conformal field theories, such as the WZW model, the classical symmetry group has already a central extension. Thus, we now discuss the modification to a loop group caused by a central extension.

A central extension $\widehat{G}$ of a group G is topologically a circle bundle over G. The algebraic structure of a central extension is described by the exact sequence

$$S^1 \to \widehat{G} \to G. \tag{2.33}$$

In a neighborhood of the identity, we can choose a coordinate system in which the circle bundle of the central extension is locally trivial. In such a coordinate system, an element of $\widehat{G}$ can be written as (g, z), $g \in G, z \in S^1$; the group product is given by

$$(g, z) \cdot (g', z') = (gg', c(g, g')zz'), \tag{2.34}$$

where $c(g, g') \in S^1$ is a group cocycle satisfying

$$c(g, g')c(gg', g'') = c(g, g'g'')c(g', g''). \tag{2.35}$$

The Lie algebra of a centrally extended group $\widehat{G}$ has a similar description as $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$. An element of $\widehat{\mathfrak{g}}$ is given by a pair (u, t) where $u \in \mathfrak{g}$ and $t \in \mathbb{R}$. The Lie algebra product is given by

$$[(u, t), (v, s)] = ([u, v], \omega(u, v)), \tag{2.36}$$

where ω is a skew-symmetric Lie algebra cocycle satisfying

$$\omega([u, v], w) + \omega([v, w], u) + \omega([w, u], v) = 0. \tag{2.37}$$

The cocycle conditions (2.35) and (2.37) follow directly from the condition of associativity on the group $\widehat{G}$ and the Jacobi identity on the algebra $\widehat{\mathfrak{g}}$, respectively.

It is a straightforward algebraic exercise to demonstrate that in terms of the Fourier basis $J_{a(n)}$, the most general central extension of $L\mathfrak{g}$ has a complexification with a Lie algebra described by

$$[J_{a(n)}, J_{b(m)}] = i f_{ab}{}^c J_{c(n+m)} + n\delta_{n,-m}\gamma(J_a, J_b)C, \tag{2.38}$$

where C is the (Hermitian) central generator and $\gamma(J_a, J_b)$ is a symmetric invariant bilinear in $\mathfrak{g}_{\mathbb{C}}$[53]. In particular, since G is simple, there is a unique symmetric invariant bilinear form in $\mathfrak{g}_{\mathbb{C}}$ (up to scalar multiplication). This is the *Killing form*

$$\eta_{ab} = f_{ad}{}^{c} f_{bc}{}^{d}. \tag{2.39}$$

It follows that $\gamma(J_a, J_b)$ is proportional to η_{ab}. By choosing a basis in which the structure constants $f_{ab}{}^{c}$ are completely antisymmetric, the Killing form can be put in diagonal form; often physicists write the algebra (2.38) in terms of the diagonal form δ_{ab}.

In order to extend the algebra (2.38) to a global central extension of LG, it is necessary to make a choice of γ such that the cocycle ω can be extended to a central extension of the group. A necessary and sufficient condition for this to be possible is that $\gamma(h_\alpha, h_\alpha)$ be an even integer for every root α of G. It is fairly easy to see that this condition is at least necessary. Consider the subalgebra of (2.38) generated by

$$\begin{aligned} I_\pm &= e_{\pm\alpha(0)}, \\ I_3 &= h_{\alpha(0)}/2 \end{aligned} \tag{2.40}$$

This subalgebra is isomorphic to the (complexified) Lie algebra su(2), which follows from $[I_3, I_\pm] = \pm I_\pm$ and $[I_+, I_-] = 2I_3$. The central terms vanish, since they are proportional to the mode number which in this case is 0. We now consider the subalgebra generated by

$$\begin{aligned} \tilde{I}_\pm &= e_{\mp\alpha(\pm 1)}, \\ \tilde{I}_3 &= \frac{xC}{4} - h_{\alpha(0)}/2, \end{aligned} \tag{2.41}$$

where $x = \gamma(h_\alpha, h_\alpha)$. This subalgebra is also an su(2) subalgebra, as can be seen from computing

$$\begin{aligned} \left[\tilde{I}_3, \tilde{I}_\pm\right] &= \pm\tilde{I}_\pm \\ \left[\tilde{I}_+, \tilde{I}_-\right] &= 2\tilde{I}_3 - \frac{xC}{2} + \gamma(f_\alpha, e_\alpha)C, \end{aligned} \tag{2.42}$$

and observing that the invariance of γ under G implies that

$$\begin{aligned} \gamma(f_\alpha, e_\alpha) &= \frac{1}{2}\gamma([f_\alpha, h_\alpha], e_\alpha) \\ &= \frac{1}{2}\gamma(h_\alpha, [e_\alpha, f_\alpha]) = \frac{x}{2}. \end{aligned} \tag{2.43}$$

From the subalgebra (2.40), we know that that $\exp(4\pi i I_3) = 1$ in the group LG. Similarly, for (2.38) to extend to a global group structure, we must have $\exp(4\pi i \tilde{I}_3) = 1$. From the definition of C as the generator of the central extension satisfying $\exp(2\pi i C) = 1$, it follows that $x \in 2\mathbb{Z}$.

It can be shown that the above condition is also sufficient to guarantee that the algebra (2.38) can be globally extended to a central extension $\widehat{L}G$ of the group LG [53]. The condition on the algebra can be rewritten as a global topological condition on the differential form associated with the cocycle ω; we will discuss the geometry of this condition further in Section 2.5.3. It is easy to see that there must be a minimum multiple of the Killing form which has the desired property. We will denote this minimum multiple by $g_{ab} = \gamma_m(J_a, J_b)$. In terms of g_{ab}, the centrally extended algebra (2.38) is written

$$[J_{a(n)}, J_{b(m)}] = i f_{ab}{}^c J_{c(n+m)} + kn\delta_{n,-m} g_{ab} C, \tag{2.44}$$

where k is integral.

At this point, it is necessary to discuss a difference between physical and mathematical notation for current algebras. In general, physicists take the convention that $C = 1$, and define the algebra (2.44) to be the current algebra, or (untwisted) affine algebra, on G at level k. Mathematicians, on the other hand, restrict attention to the case $k = 1$, and retain the description of C as a (commuting) algebra element. The group for $k = 1$ is referred to as the universal central extension of LG, and is written as $\widehat{L}G$. This extension is called universal because all other extensions can be realized as quotients of $\widehat{L}G$ by a finite cyclic group. We will primarily use the mathematical description here, because it allows for a simpler and more consistent description of the coadjoint orbit representations of centrally extended groups. We will, however, denote the eigenvalue of the operator C by k, so that the notation is still ostensibly equivalent to that used by physicists. Note that the centrally extended loop groups have locally 1-1 exponential maps from the Lie algebra, which follows directly from the fact that the groups without central extensions have this property. The exponential map is also locally surjective, so it follows that the groups $\widehat{L}G$ have the BCH property. (Note that in general, the exponential map is not globally surjective for loop groups.)

We will now describe the adjoint and coadjoint action of the centrally extended loop group $\widehat{L}G$ on the algebra $\widehat{L}\mathfrak{g}$. Denoting an arbitrary element of $\widehat{L}\mathfrak{g}$ by

$$(f, ia) = f(t) + iaC, \quad f : S^1 \to \mathfrak{g}, a \in \mathbb{R}, \tag{2.45}$$

we can describe the adjoint action of an element $g : S^1 \to G$ of LG on (f, ia) by

$$\mathrm{Ad}_g(f, ia) = \left(\mathrm{Ad}_g(f), ia - \frac{i}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \gamma_m(g^{-1}(\theta)g'(\theta), f(\theta))\right). \tag{2.46}$$

(The adjoint action of the central element S^1 is trivial.) To prove that (2.46) is the correct formula for the adjoint action, it suffices to demonstrate that taking g close to the identity reproduces (2.44) and that the action is indeed a group action in the sense that $\mathrm{Ad}_h\mathrm{Ad}_g = \mathrm{Ad}_{hg}$. The first of these conditions is easily verified; when

$$g = 1 + \epsilon J_{a(n)}, \tag{2.47}$$

and $f = J_{b(m)}$, (2.46) reduces to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\epsilon}|_{\epsilon=0}\mathrm{Ad}_g(f, ia) &= \left(if_{ab}{}^c J_{c(n+m)}, -\frac{i}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \gamma_m(inJ_{a(n)}, J_{b(m)})\right) \\ &= \left(if_{ab}{}^c J_{c(n+m)}, n\delta_{n,-m}g_{ab}\right). \end{aligned} \tag{2.48}$$

Although this calculation is performed in the complexified algebra, clearly the restriction to real linear combinations satisfies the same equation. Recall that we are using the mathematician's convention of $k = 1$. To demonstrate that the action is a group action, we write

$$\mathrm{Ad}_h\mathrm{Ad}_g(f, ia) = (\mathrm{Ad}_h\mathrm{Ad}_g(f), ia - x), \tag{2.49}$$

where

$$\begin{aligned} x &= \frac{i}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \left[\gamma_m(g^{-1}(\theta)g'(\theta), f(\theta)) + \gamma_m(h^{-1}(\theta)h'(\theta), \mathrm{Ad}_{g(\theta)}f(\theta))\right] \\ &= \frac{i}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \gamma_m(g^{-1}(\theta)g'(\theta) + g^{-1}(\theta)h^{-1}(\theta)h'(\theta)g(\theta), f(\theta)) \\ &= \frac{i}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \gamma_m((hg)^{-1}(\theta)(hg)'(\theta), f(\theta)), \end{aligned} \tag{2.50}$$

so

$$\mathrm{Ad}_h\mathrm{Ad}_g(f, ia) = \left(\mathrm{Ad}_{hg}(f), ia - \frac{i}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \gamma_m((hg)^{-1}(\theta)(hg)'(\theta), f(\theta))\right). \tag{2.51}$$

The dual space to $\widehat{L}\mathfrak{g}$ is $L(\mathfrak{g}^*) \oplus \mathbb{R}$. We can write elements of this space as pairs $(b, -it)$ where $b : S^1 \to \mathfrak{g}^*$ and $t \in \mathbb{R}$. The dual pairing between the extended loop algebra and dual space is given by

$$\langle (b, -it), (f, ia)\rangle = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \langle b(\theta), f(\theta)\rangle + at. \tag{2.52}$$

It is straightforward to verify that in order for this dual pairing to be preserved under the group action, the coadjoint action of an arbitrary element $g \in LG$ on $(b, -it)$ must be given by

$$\mathrm{Ad}_g^*(b, -it) = (\mathrm{Ad}_g^* b + t(g'g^{-1})^*, -it), \tag{2.53}$$

where we have defined a dual map $*$ from $\mathfrak{g}$ to $\mathfrak{g}^*$ such that

$$\langle g^*, f\rangle = \gamma_m(g, f), \quad \forall f, g \in \mathfrak{g}. \tag{2.54}$$

This map is well-defined since the Killing metric is nondegenerate for simple G. From (2.53) we can calculate the coadjoint action of the algebra; we find that

$$\langle \mathrm{ad}_u^*(b, -it), (f, ia)\rangle = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \ \left[\langle b(\theta), [f(\theta), u(\theta)]\rangle + t\gamma_m(u'(\theta), f(\theta))\right], \tag{2.55}$$

in agreement with (2.44).

Just as for finite-dimensional groups, we will find it useful to define simple roots for the centrally extended algebra $\widehat{L}\mathfrak{g}$. If G is a simple finite-dimensional group of rank r, and we denote the highest root of $\mathfrak{g}$ by ψ, then we define the generators associated with the $r+1$ simple roots of $\widehat{L}\mathfrak{g}$ to be

$$e_j = \begin{cases} e_{j(0)}, & 1 \le j \le r \\ e_{-\psi(1)}, & j = r+1 \end{cases} \tag{2.56}$$

The space of roots for $\widehat{L}\mathfrak{g}$ is naturally described by $\mathbb{Z}\times\Phi$, where the integral parameter is the mode number and Φ is the root space of $\mathfrak{g}$. (Often, an auxiliary operator generating rotations about S^1 is introduced, for which the mode number is the eigenvalue; this formalism has the advantage of keeping the interpretation of a root as an element of the dual space to the Cartan subalgebra, where the dual of C is 0 and the new Cartan subalgebra is $\mathbb{R}\otimes\mathbb{R}C\otimes\mathfrak{t}_{\mathbb{C}(0)}$. We will forgo the extra complications in notation which would be involved in this formalism, and we will simply take the mode number as an extra parameter in the root space.) In this space, we define a positive root $(n, \alpha) \succ 0$ to be any root which can be written as a sum of the simple roots with nonnegative coefficients. For a general representation, positive weights can be defined similarly. We have then a natural decomposition of the algebra $\widehat{L}\mathfrak{g}_{\mathbb{C}}$,

$$\widehat{L}\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}(0)} \oplus \mathbb{C}C \oplus \bigoplus_{(n,\alpha)\succ 0} \mathfrak{g}_{\alpha(n)} \oplus \bigoplus_{(n,\alpha)\prec 0} \mathfrak{g}_{\alpha(n)}. \tag{2.57}$$

Note that the central element has weight $(0,0)$ in the adjoint representation. Note also, that the adjoint representation is not highest weight. Just as in (2.56), we can define the rest of a Chevalley basis for $\widehat{L}\mathfrak{g}_{\mathbb{C}}$ by

$$f_j = \begin{cases} f_{j(0)}, & 1 \leq j \leq r \\ e_{\psi(-1)}, & j = r+1 \end{cases} \tag{2.58}$$

and

$$h_j = \begin{cases} h_{j(0)}, & 1 \leq j \leq r \\ \frac{1}{2}\gamma_m(h_\psi, h_\psi)C - h_{\psi(0)}, & j = r+1 \end{cases} \tag{2.59}$$

Using this basis, the Cartan relations (2.12) are satisfied, with a singular Cartan matrix A. Using the Cartan matrix approach, a general class of affine algebras can be defined which contains in particular the centrally extended current algebras described here [41].

The main result we will need about simple roots is that just as for finite-dimensional groups, the simple roots of $\widehat{L}\mathfrak{g}$ generate the subalgebra S of positive roots. This result can easily be seen, as follows. Certainly, if $\alpha \in \Phi_+$ is a positive root of $\mathfrak{g}$ then $e_{\alpha(0)}$ is in S, since e_α is in the subalgebra of positive roots of $\mathfrak{g}$, which is generated by the simple roots of $\mathfrak{g}$. The root $e_{-\psi(1)}$ is a simple root of $\widehat{L}\mathfrak{g}$, and is thus in S automatically. Since the roots of $\mathfrak{g}$ form an irreducible representation of $\mathfrak{g}$ (the adjoint) with lowest weight $-\psi$, we can reach any root $e_{\beta(1)}$ by commuting positive roots $e_{\alpha(0)}$ with $e_{-\psi(1)}$. Since $\psi \neq 0$, there is an element $h = -h_\psi/2$ of the Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ which satisfies $[h, e_{-\psi}] = e_{-\psi}$. The first Fourier mode of this operator, $h_{(1)}$, is an element of S from the above argument, so $[h_{(1)}, e_{-\psi(1)}] = e_{-\psi(2)} \in S$. Continuing in this fashion, we see that all generators associated with positive roots are in S. Similarly, the negatives of the simple roots generate the algebra of negative roots, again as in the finite-dimensional case.

We now review briefly the representation theory for centrally extended loop groups and their algebras. In the case of finite-dimensional groups, we had the result that all irreducible representations are highest weight representations. This is not the case for the groups $\widehat{L}G$. We shall restrict attention, however, to a class of irreducible representations which are of physical interest; all the representations we shall consider are highest weight. To begin with, we will restrict attention to representations on vector spaces V which can be written as direct sums of weight spaces

$$V = \bigoplus_{(n,\psi)\in\mathbb{Z}\times\mathfrak{t}_{\mathbb{C}}^*} V_{\psi(n)} \tag{2.60}$$

Not all representations are of this form, since some representation spaces do not admit an action of the rotation group S^1 which commutes with the constant generators $J_{\alpha(0)}$. We will further restrict to the class of representations for which the decomposition (2.60) contains only weights (n, ψ) with $n < m$ for some integer m. These representations are referred to as "positive energy" representations, because the mode number $m - n$ can be related to an energy operator of a physical system. From now on, unless otherwise noted we will assume that all representations of $\widehat{L}G$ and its algebra are positive energy representations which admit the decomposition (2.60). As an example of a representation which does not fit in this category, consider the adjoint representation of $\widehat{L}G$ on $\widehat{L}\mathfrak{g}$, which has both positive and negative mode numbers of arbitrary magnitude. It can be shown that all irreducible representations in the category of interest admit unitary structures [53]. We shall discuss these unitary structures further in Section 2.6.

With the above restriction, the representation theory of the groups $\widehat{L}G$ can be described in a very similar fashion to that of the finite-dimensional simple groups. In fact [53], it can be shown that all irreducible representations of the algebra $\widehat{L}\mathfrak{g}$ of the desired type are highest weight representations. Note that we have defined highest weight representations with respect to the choice of simple roots given in (2.56). In some of the literature, these representations are referred to as "antidominant" representations, due to a different convention for positive and negative roots. The fact that all irreducible representations are highest weight can be seen fairly easily by showing that the subspace of highest mode number n must admit an irreducible representation of the subalgebra $\mathfrak{g}_{(0)}$ of $L\mathfrak{g}$ generated by the 0 modes $J_{a(0)}$; since this representation is irreducible, it must be finite-dimensional, and therefore must contain a highest weight vector which is annihilated by all simple roots of $\widehat{L}G$.

Thus, we can describe every representation of $\widehat{L}\mathfrak{g}$ by an integer k (the eigenvalue of C, also called the level of the representation) and a highest weight vector v, with weight (n, λ). Since the mode number n of the highest weight vector can be shifted by an arbitrary integer, we will take $n = 0$ for all representations. Finally, by examining $SU(2)$ subgroups of $\widehat{L}G$, it is possible to show that k and λ must satisfy an additional relation for a highest weight representation of $\widehat{L}\mathfrak{g}$ to extend to a representation of $\widehat{L}G$. From (2.40) and (2.41), we know that the eigenvalues of the operators $2I_3$ and $2\tilde{I}_3$ defined in those equations must be nonnegative integers when acting on the highest weight state v. It follows that for all

roots α, we have

$$0 \leq \langle \lambda, h_\alpha \rangle \leq \frac{1}{2} k \gamma_m(h_\alpha, h_\alpha). \tag{2.61}$$

(Recall that the weight λ is naturally in the dual space $\mathfrak{t}^*_{\mathbb{C}}$.) It can also be shown [53] that for any k, λ satisfying the constraint (2.61) an irreducible representation of $\widehat{L}G$ exists. Note that the condition (2.61) indicates that for a given level k, there are a finite number of distinct irreducible representations of $\widehat{L}G$.

The Borel-Weil theory which is used to describe irreducible representations of finite-dimensional compact groups can be generalized to cover the groups $\widehat{L}G$; however, the apparatus used to implement the construction in the infinite-dimensional case is considerably more sophisticated [53]. This construction is again closely related to the coadjoint orbit technique which we will describe later in this thesis. As in the finite-dimensional case, one can construct a homogeneous space

$$\widehat{L}G/(T_{(0)} \times S^1) = \widehat{L}G_{\mathbb{C}}/B^-, \tag{2.62}$$

where $T_{(0)}$ is the zero mode restriction of a maximal subtorus of G, S^1 is the central extension, and B^- is a Borel subgroup formed just as in the finite-dimensional case from the group whose Lie algebra contains the negative roots of $\widehat{L}\mathfrak{g}$ and the zero modes of a Cartan subalgebra (and also C). Because $\widehat{L}G$ has an analytic structure, we can put a coordinate system on this quotient space just as in (2.20) (with an infinite number of coordinates $\{z_{\alpha(n)} : (n, \alpha) \succ 0\}$). We will use such a coordinate system when describing coadjoint orbit representations of loop groups in Section 2.5.3.

Finally, we will give a simple example of a representations of a loop group. Consider the group $\widehat{L}SU(2)$. For $SU(2)$, weights are given by a single integer $2j = \langle \lambda, h_+ \rangle = \langle \lambda, 2J_3 \rangle$. The inner product γ_m satisfies $\gamma_m(h_+, h_+) = 2$ From (2.61) it follows that at level k, the allowed highest weights λ satisfy $j \leq \frac{k}{2}$. Thus, for $k = 1$, the only allowed representations of $\widehat{L}SU(2)$ have $j = 0$ and $j = 1/2$. Consider the representation with $j = 0$. For each n, we have an su(2) subalgebra of the current algebra generated by

$$\begin{aligned} I_{\pm(n)} &= e_{\mp(\pm n)}, \\ I_{3(n)} &= \frac{nC}{2} - h_{+(0)}/2. \end{aligned} \tag{2.63}$$

The eigenvalue of $I_{3(n)}$ on v is $n/2$, so the operator $I^{n+1}_{-(n)}$ must annihilate v. Based on this

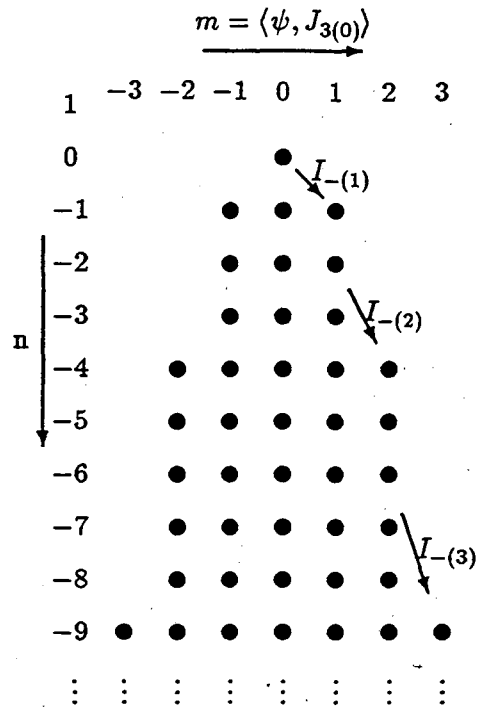


Figure 2.4: Weights of representation $(j,k) = (0,1)$ of $\widehat{L}SU(2)$

observation, we graph the weights of the representation $(j,k) = (0,1)$ in Figure 2.4[1]. In general, highest weight representations of $\widehat{L}G$ have this general structure. The subspace with mode number 0 carries an irreducible representation of the group G, and the subspaces with fixed mode number contain a finite number of irreducible representations of G, with highest weight vectors whose weights increase approximately as $\sqrt{n}$. We will discuss the detailed structure of these representations further later in this thesis.

2.1.3 The Virasoro group

We will now consider our final example, the Virasoro group. The Virasoro group is the universal central extension of the group DiffS^1 of smooth diffeomorphisms of the circle S^1. Unlike the centrally extended loop groups, the Virasoro group does not admit a locally 1-1 or onto exponential map from its Lie algebra. We shall, however, avoid this difficulty by analyzing a more well-behaved group which has an equivalent algebraic structure.

[1]Note that the weights of this representation (called the fundamental representation) are ostensibly given in [53] (Figure 3, Chapter 9; pp. 180). The weights in that figure are different from those shown here. By examining the character formula from [53] (Chapter 14; pp. 282), it is clear that the weights given here are correct and those in the reference are incorrect. Apparently, the error in [53] is that the inner product should be defined (in their notation) by $||\mu||^2 = \mu^2/2$, not by $||\mu||^2 = \mu^2$.

The group $\mathrm{Diff}S^1$ is defined to be the group of all smooth orientation-preserving diffeomorphisms f of the circle S^1,

$$\mathrm{Diff}S^1 = \{f : S^1 \xrightarrow{1:1} S^1 |\ f \in C^\infty(S^1, S^1)\}. \tag{2.64}$$

The group multiplication is defined by the composition of maps, so $(g \cdot f)(\theta) = f(g(\theta))$. The Lie algebra of this group is the algebra $\mathrm{Vect}S^1$ of smooth vector fields on S^1. Writing a vector field f as a function $f : S^1 \to \mathbb{R}$, the Lie algebra product is given by

$$[f(\theta), g(\theta)] = f(\theta)g'(\theta) - g(\theta)f'(\theta). \tag{2.65}$$

Just as for central extensions of loop groups, it is possible to find a single central extension $\widehat{\mathrm{Diff}}S^1$ of $\mathrm{Diff}S^1$ which is universal in the sense that all other central extensions can be realized by taking a quotient by a finite cyclic group. This universal central extension is the Virasoro group. We denote its algebra by $\widehat{\mathrm{Vect}}S^1$. Elements of $\widehat{\mathrm{Vect}}S^1$ are of the form $(f, -ia)$, with $f(\theta)\partial/\partial\theta$ a vector field on S^1 and $a \in \mathbb{R}$. The commutation relation between elements of $\widehat{\mathrm{Vect}}S^1$ is given by

$$[(f, -ia_1), (g, -ia_2)] = \left(fg' - gf', \frac{i}{48\pi}\int_0^{2\pi}(f(\theta)g'''(\theta) - g(\theta)f'''(\theta))\mathrm{d}\theta\right). \tag{2.66}$$

Defining the (complex) vector fields $l_n = ie^{in\theta}\partial/\partial\theta$ in $\mathrm{Vect}S^1$, we can define the usual Virasoro generators by

$$\begin{aligned} L_n &= (l_n, 0); \text{ for } n \neq 0, \\ L_0 &= (l_0, \frac{1}{24}), \\ C &= (0, 1). \end{aligned} \tag{2.67}$$

The commutation relations then take the standard form

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m,-n}, \\ [C, L_n] &= 0. \end{aligned} \tag{2.68}$$

The Virasoro algebra is defined to be the complex Lie algebra spanned by the generators (2.67). As in the case of current algebras, it is traditional for physicists to treat the operator C as a c-number c, whereas mathematicians leave C in operator form and take c to be its eigenvalue in a specific representation. For the purpose of the coadjoint orbit description of representations, we will adhere to the latter convention.

We will now briefly describe the adjoint and coadjoint action of $\widehat{\mathrm{Vect}}S^1$ on itself and its dual space. The (smooth) dual space to $\widehat{\mathrm{Vect}}S^1$ consists of pairs (b, it), with $b(\theta)d\theta^2$ a quadratic differential on S^1, and $t \in \mathbb{R}$. The dual pairing between (b, it) and an element $(f, -ia) \in \widehat{\mathrm{Vect}}S^1$ is given by

$$\langle (b, it), (f, -ia) \rangle = \int_0^{2\pi} b(\theta) f(\theta) d\theta + at. \tag{2.69}$$

For this pairing to be invariant under the action of the algebra $\widehat{\mathrm{Vect}}S^1$, (b, it) must transform under the coadjoint action by

$$\mathrm{Ad}^*_{(f,-ia)}(b, it) = (2bf' + b'f - \frac{t}{24\pi} f''', 0). \tag{2.70}$$

Using these equations, it is possible to derive formulae describing the coadjoint action Ad^* of the group $\widehat{\mathrm{Diff}}S^1$ on the dual space $\widehat{\mathrm{Vect}}S^{1\,*}$; however, we will not need these formulae since the orbits of the coadjoint action with which we will be concerned are topologically trivial (see Section 2.5.4).

Unlike the centrally extended loop groups, the Virasoro group cannot be described by the formalism of simple roots and the Cartan matrix. In many ways, the Virasoro group is structurally similar to finite-dimensional non-compact groups, while the loop groups are more similar to compact groups. Nonetheless, it is possible to make a definition of positive and negative weights and roots for the Virasoro algebra. The Virasoro algebra has a natural decomposition

$$\widehat{\mathrm{Diff}}S^1 = (\mathbb{C}C \oplus \mathbb{C}L_0) \oplus V_+ \oplus V_- \tag{2.71}$$

where

$$V_+ = \bigoplus_{n>0} \mathbb{C}L_n \tag{2.72}$$

and

$$V_- = \bigoplus_{n>0} \mathbb{C}L_{-n} \tag{2.73}$$

are defined to be the spaces of positive and negative roots of $\widehat{\mathrm{Diff}}S^1$. (Technically, these are the spaces of generators associated with the positive and negative roots; we will simplify notation by referring to these generators as themselves being the roots. This will not lead to confusion, since the roots and generators of this algebra are in a 1-1 correspondence, just as is the case for compact groups.) The operators C and L_0 form a maximal commuting subalgebra of the Virasoro algebra. On any irreducible representation space V of $\widehat{\mathrm{Diff}}S^1$,

the eigenvalue of C is a constant c, and the representation space can be decomposed into subspaces with distinct eigenvalues of L_0

$$V = \bigoplus_{n \in \mathbb{Z}} V_n \tag{2.74}$$

where $L_0 v = nv$ for all $v \in V_n$. We take n to be the weight of the space V_n; according to the definition (2.72) of positive roots, a weight n is "positive" when $n < 0$ (this unfortunate terminology arises from the merging of well-established conventions in mathematics and physics for highest weight representations and Virasoro generators).

At this point, one might be tempted to define L_1 to be the single simple root of $\widehat{\mathrm{Diff}}S^1$. Unfortunately, however, the generators $\{L_0, L_{\pm 1}\}$ form a closed subalgebra of the Virasoro algebra, isomorphic to the algebra of $SL(2, \mathbb{C})$. Thus, it is impossible to define the Virasoro algebra from this subalgebra with any choice of Cartan matrix. It is impossible to choose more than one simple root for the Virasoro algebra, since then roots could not be uniquely written as a linear combination of simple roots. For our purposes, however, the only result about simple roots which will be used in the following development is the fact that the simple roots generate the subalgebra of positive roots. To this end, we can select the pair of roots L_1, L_2 of the Virasoro algebra, and denote these roots to be "quasi-simple" roots. These roots do indeed generate the algebra of positive roots, which can be seen inductively by observing that $[L_1, L_n]$ is nonzero and proportional to L_{n+1} for $n \geq 2$. In the general analysis in the subsequent sections of this chapter, whenever we are discussing a general Lie group and its simple roots, we include under consideration the Virasoro group and its quasi-simple roots L_1, L_2.

In discussing representations of the Virasoro group, we will again restrict attention to a category of representations of particular physical interest. This category of representations consists of highest weight representations which admit a unitary structure. Since some highest weight representations do not admit unitary structures, this means that we will restrict attention to representations with a certain set of values for the highest weight.

A highest weight representation of the Virasoro algebra $\widehat{\mathrm{Vect}}S^1$ is simply a representation which admits a decomposition (2.74) such that for some weight h the space V_h is one-dimensional, and all spaces V_m with $m < h$ are zero-dimensional (recall the convention of positivity for the Virasoro roots). For all values of h and c (c being the eigenvalue of C), there exists a unique irreducible highest weight representation of the Virasoro algebra. These representations do not all admit unitary structures, and cannot all be integrated to

representations of the Virasoro group $\widehat{\mathrm{Diff}}S^1$. A theorem by Goodman and Wallach, however, states that all unitary representations of the Virasoro algebra can be integrated to a continuous unitary representation of the Virasoro group [36]. This theorem is based on a conjecture due to Kac [42]. Since we are actually only interested in unitary representations, it will suffice to categorize representations of $\widehat{\mathrm{Vect}}S^1$ which admit unitary structures.

The set of values (h, c) for which the Virasoro algebra admits an irreducible unitary highest weight representation is a well-known result to physicists. We will simply quote the result here, although in Section 2.6 we will briefly discuss the formalism used for its proof. The result is that unitary representations break up into two categories. In the first category are all pairs (h, c) with $c \geq 1$ and $h \geq 0$. The second category consists of the so-called "discrete series" of representations, which have

$$c = 1 - \frac{6}{m(m+1)}, \tag{2.75}$$

and

$$h = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}, \tag{2.76}$$

for integers m, p and q satisfying $m > 2$ and $1 \leq q \leq p \leq m-1$.

We conclude this subsection with a description of the geometry of the Virasoro group and a set of coordinates on a quotient space analogous to the homogeneous spaces G/T for compact finite-dimensional groups.

As mentioned above, the Virasoro group has a poorly behaved exponential map, which keeps us from defining coordinates in a neighborhood of the origin as we did in the cases of compact groups and loop groups using (2.20). The failure of the exponential map to be 1-1 and onto follows from two important features of this group, which we will now describe, following Milnor [51]. To begin with, we note that because the extension of the diffeomorphism group $\mathrm{Diff}S^1$ is central, the failure of the exponential map must also occur for the unextended group. Thus, we will focus attention here on the unextended group.

The first aspect of the full diffeomorphism group which gives problems for the exponential map is the inclusion of diffeomorphisms which have no fixed-points. From the existence of such diffeomorphisms in the group, we can prove that the exponential map is neither locally 1-1 or onto. To see that the map is not locally 1-1, consider the diffeomorphism $\phi : S^1 \to S^1$ given by $\phi(\theta) = \theta + 2\pi/n$ where n is an integer. For any vector field $f \in \mathrm{Vect}S^1$ which is nowhere zero ($f(\theta) \neq 0\ \forall\theta : 0 \leq \theta < 2\pi$) and which is periodic

with frequency n ($f(\theta + 2\pi/n) = f(\theta)$), there must exist a real number $t \in \mathbb{R}$ such that

$$\phi(\theta) = \mathrm{e}^{tf}(\theta). \tag{2.77}$$

This follows because $\mathrm{e}^{tf}(\theta)$ satisfies

$$\frac{\partial}{\partial s}\mathrm{e}^{sf}(\theta) = f(\mathrm{e}^{sf}(\theta)), \tag{2.78}$$

so

$$\mathrm{e}^{tf}(\theta) = \theta + \int_0^t \mathrm{d}s f(\mathrm{e}^{sf}(\theta)), \tag{2.79}$$

and when f is periodic with frequency n it suffices for

$$\int_0^t \mathrm{d}s f(\mathrm{e}^{sf}(\theta)) = \frac{2\pi}{n} \tag{2.80}$$

to hold for any value of θ to guarantee that the condition (2.80) holds for all values of θ. It follows that the periodic diffeomorphism ϕ lies on the one-parameter families corresponding to the trajectories under the exponential map of an infinite number of elements of the Lie algebra; thus, the exponential map is ∞-1 at each point ϕ. Since by choosing n large, we can make the diffeomorphism ϕ arbitrarily small, the exponential map is certainly not locally 1-1.

In a similar fashion, we can use the existence of diffeomorphisms without fixed-points to show that the exponential map is not surjective. Consider the diffeomorphism

$$\phi(\theta) = \theta + \frac{2\pi}{n} + \epsilon \sin^2(n\theta), \tag{2.81}$$

where $\epsilon \ll 1/n$. The diffeomorphism ϕ^n clearly has fixed-points at $\theta = \pi k/n$ for integral k, however no other points are fixed under ϕ^n. Because ϕ has no fixed points, if $\phi = \mathrm{e}^f$ for some vector field f it follows that f can have no zeros. But by the same argument as above, if f has no zeros and $\mathrm{e}^{nf}(0) = 0$ then $\mathrm{e}^{nf}(\theta) = \theta$ for all θ. It follows that ϕ is not in the image of the exponential map. Since ϕ can be taken arbitrarily small by taking n arbitrarily large and ϵ arbitrarily small, it follows that the exponential map on $\mathrm{Vect}\,S^1$ is not locally surjective onto $\mathrm{Diff}\,S^1$. (Actually, here we must be careful about our definition of local; technically, $\mathrm{Diff}\,S^1$ as a group is modeled on a Frechet space – see below. In fact, however, it is still true that any neighborhood of the identity in $\mathrm{Diff}\,S^1$ contains some diffeomorphism of the form of ϕ.)

We see then, that the existence of diffeomorphisms without fixed-points in the diffeomorphism group makes the exponential map from the algebra of vector fields rather

poorly behaved. In fact, still more problems arise from the inclusion of non-analytic vector fields; this is the second problem referred to above. As an example, take f to be a vector field which vanishes with all derivatives at $\theta = 0$, but which is nonzero in a region $\pi/2 < \theta < 3\pi/2$. The constant vector field $I = -il_0 = 1$ can be exponentiated to form the rotation diffeomorphism R_ψ satisfying $R_\psi(\theta) = \theta + \psi$. If the Lie group Diff$S^1$ had a well-behaved real analytic structure, the adjoint action of the group on the algebra which gives

$$(R_\psi f R_{-\psi})(\theta) = f(\theta + \psi) \tag{2.82}$$

could be described by an exponentiation of the adjoint action of the algebra on itself

$$[I, f] = \mathrm{ad}_I f = f'. \tag{2.83}$$

However, since f and all its derivatives vanish at $\theta = 0$ it follows that

$$e^{\psi\, \mathrm{ad}_I} f(0) = 0 \;\; \forall \psi, \tag{2.84}$$

which contradicts (2.82). Thus, the analytic structure of DiffS^1 is further disrupted by the existence of non-analytic diffeomorphisms.

In order to accomplish the coadjoint orbit construction of representations of the Virasoro group, we will need to have coordinates on a homogeneous space of the form DiffS^1/S^1. Although by the arguments above, such a coordinate system cannot directly be constructed using a formula like (2.20), it is possible to proceed formally as though such a formula were applicable. This approach has been successfully used in the past, and leads to correct results for the curvature and other properties of the space DiffS^1/S^1 [66] (the curvature result was first obtained by Bowick and Rajeev using a generalization of a method of Freed [20, 30]). We will make some attempt to justify the use of this type of coordinate system here, by considering a closely related group which is more nicely behaved. Finally, however, the justification for the formulas we will derive is that they successfully give algebraic representations of the form expected from the coadjoint orbit construction. A more rigorous demonstration of the validity of the methods used here is left as a project for further research.

We now sketch an argument for the validity of formula (2.20) on a group with an algebra isomorphic to the Virasoro algebra. To begin with, the homogeneous space DiffS^1/S^1 is the space of the infinite-dimensional manifold of the group Diff$_0S^1$ of diffeomorphisms of S^1 which have a fixed point at $\theta = 0$. By restricting to this group, the problems associated

with diffeomorphisms without fixed-points are removed. The only obvious difficulty which remains is the existence of non-analytic diffeomorphisms. For our purposes, and as far as most physicists are concerned, we can restrict to the subgroup of $\mathrm{Diff}_0 S^1$ of real-analytic diffeomorphisms, which we denote D_0. This group has none of the problems described above for its exponential map; however, it is still not clear whether this group admits a real-analytic structure. We will now show that a closely related group with an isomorphic algebra can be given a real-analytic structure; we will denote this group by V, and we will show that $D_0 \subset \mathrm{V}$. Although this result does not prove conclusively that D_0 admits a real-analytic structure, it indicates the plausibility of this assertion. For those purists unwilling to accept this statement without a more rigorous proof, the discussion in the rest of this chapter should be taken to apply to the formal group V, with the homogeneous space $\mathrm{Diff} S^1/S^1$ being the formal quotient of V through the action of L_0.

We define the group V to be the set of all formal power series in x with leading term proportional to x and leading coefficient positive,

$$\mathrm{V} = \{f = f_1 x + f_2 x^2 + \cdots \in x\mathbb{R}[[x]] : f_1 > 0\}. \tag{2.85}$$

We can define a topology on the space of V by treating V as a subspace of the vector space $A = x\mathbb{R}[[x]]$. We take a local base around $0 \in A$ for the topology of A to be given by the countable family of open sets

$$B(n,m) = \{f \in \mathrm{V} : f_i < \frac{1}{n} \forall i \leq m\}, \tag{2.86}$$

where n and m are integers. The topology on V is then that induced by the embedding in A. Since the topology thus defined on V has a countable, convex local basis, it follows that V has a metric invariant under linear translations on the vector space [56]. It can be shown that V is complete in this metric, so V is a Frechet space. It is also possible to show that A has no bounded neighborhood of 0 with the resulting metric, so V cannot be a Banach space. The technical details of what type of space V is modeled on will not concern us here; however, it is significant that V is modeled on a Frechet space, since the essential problem here is to show that a group modeled on a Frechet space can have a real-analytic structure.

Having defined a topology on V, we can now define a group structure given by composition. Given elements $f = f_1 x + f_2 x^2 + \cdots$ and $g = g_1 x + g_2 x^2 + \cdots$, we define $g \cdot f \in \mathrm{V}$ by

$$(g \cdot f)(x) = f(g(x)) = c_1 x + c_2 x^2 + \cdots, \tag{2.87}$$

where

$$c_k = \sum f_s g_{l_1} g_{l_2} \cdots g_{l_s}, \tag{2.88}$$

with the sum taken over all $s, l_1, \ldots l_s > 0$ with $k = l_1 + \cdots + l_s$.

The inverse can be computed; if $g \cdot f = 1$, then

$$\begin{aligned} f_1 &= 1/g_1 \\ f_k &= -g_1^{-k} \left[\sum_{s<k} f_s g_{l_1} \cdots g_{l_s} \right] \quad k > 2, \end{aligned} \tag{2.89}$$

with the sum taken over the same range as before, however with the additional restriction that $s < k$.

From these explicit expressions, it is easy to verify that multiplication and inverses are continuous in the chosen topology, so V is a Lie group modeled on a Frechet space. Furthermore, we can define a smooth "square root" operation in V. If $f = g^2$, then we have

$$\begin{aligned} g_1 &= (f_1)^{1/2} \\ g_k &= (f_k - \sum_{1<s<k} g_s g_{l_1} \cdots g_{l_s})/(g_1 + g_1^k) \quad k > 2, \end{aligned} \tag{2.90}$$

where the sum is again as before, except now we restrict $1 < s < k$.

Now, we can prove the further claim that the exponential map from the Lie algebra of V to V is 1-1 and onto (globally). The Lie algebra of V is exactly the space A, with the Lie bracket of two vector fields $u = u_1 x + u_2 x^2 + \cdots$ and $v = v_1 x + v_2 x^2 + \cdots$ given by

$$[u, v] = w = w_1 x + w_2 x^2 + \cdots, \tag{2.91}$$

with

$$\begin{aligned} w_1 &= 0 \\ w_k &= \sum_{j=1}^{k} j(u_{k-j+1} v_j - u_j v_{k-j+1}). \end{aligned} \tag{2.92}$$

The Lie bracket operation on the algebra is clearly smooth in the same topology we had before.

The proof that $\exp : A \to \mathrm{V}$ is 1-1 and onto essentially follows from iterating the square root map above. If we define $f^{(n)}$ by $f^{(0)} = f$, $f^{(n-1)} = f^{(n)} \cdot f^{(n)}$, then by iterating

the square root map we can show that the limits

$$\begin{aligned} g_1 &= \lim_{n\to\infty}\left(2^n(f_1^{(n)}-1)\right) \\ g_k &= \lim_{n\to\infty}\left(2^n f_k^{(n)}\right) \quad k>2 \end{aligned} \tag{2.93}$$

exist. We then define $g = g_1x + g_2x^2 + \cdots$, and have $e^g = f$. Since each function f has a unique "logarithm" g, the exponential map is 1-1 and surjective.

We have thus shown that the Lie group V, which is modeled on a Frechet space, has a 1-1 and onto exponential map. The Lie group D_0 can be identified with the subgroup of V consisting of power series which are a) convergent with all derivatives for all x, b) periodic in x in the sense that $f(x+2\pi) = f(x)+2\pi$, and c) monotonically increasing in x ($f'(x) > 0\ \forall x$). The topology on D_0 induced by this embedding is different from the usual topology, however the group operation is smooth under both definitions. In order to prove that the exponential map on D_0 is 1-1 and onto, it would suffice to demonstrate that the square root operation defined above for V is closed on D_0. In fact, we assert that it suffices to show that if $f \in D_0 \subset$ V and $f = g\cdot g$ in V then condition (a) holds for g. That is, if g is convergent for all x conditions (b) and (c) on g follow from the same conditions on f. A brief outline of this argument will now be given to conclude this subsection.

Assume that f and g are as above, with $f = g^2$ in V and f satisfying conditions (a), (b), (c), and with g satisfying condition (a). It follows from the fact that g satisfies the differential equation $g'(g(x)) = f'(x)/g'(x)$, that if g has a bounded continuous derivative, then g is monotone increasing. It is then fairly easy to see that the following conditions must hold on g:

$$\begin{aligned} f(x) > g(x) > x, &\quad \text{when} \quad f(x) > x \\ f(x) = g(x) = x, &\quad \text{when} \quad f(x) = x \\ f(x) < g(x) < x, &\quad \text{when} \quad f(x) < x. \end{aligned} \tag{2.94}$$

To see that these conditions must hold, assume that x_0, x_1 are consecutive fixed-points of g (and therefore also of f). If $g(x) > x$ for $x_0 < x < x_1$ then $g(g(x)) = f(x) > g(x) > x$ since $g(x) < x_1$, which follows from the monotonicity of g. The other conditions can be proven in a similar fashion. From these inequalities, we see that $G(x) = g(x+2\pi) - 2\pi$ is also a solution to $G(G(x)) = f(x)$. If the solution g is unique, which it is in V, then $g = G$, and g is periodic. We have thus shown everything except that the square root of an element

of D_0 must be convergent. The exponential map on D_0 is thus at least 1-1; however, the surjectivity of this map has not been shown rigorously. To be certain of working with a group with the BCH property, one must consider V; in the remainder of this thesis, however, we will ignore the details of this question and simply calculate with the Virasoro group and algebra as though it were well-behaved.

2.2 Coadjoint orbit construction

This section contains a description of the geometry of coadjoint orbit representations. The construction of coadjoint orbit representations is originally due to Kirillov and Kostant [43]. The discussion here is similar to the presentations in [43, 64]; however, the notation is slightly different; in particular, some difficulties with signs are dealt with here in an internally consistent fashion.

As described in Section 1.1, given a Lie group G with algebra $\mathfrak{g}$, G acts on the dual space $\mathfrak{g}^*$ by the coadjoint action (2.7). For any $b \in \mathfrak{g}^*$, one can consider its orbit W_b in $\mathfrak{g}^*$ under the coadjoint action of G. It turns out that W_b admits a natural symplectic structure, which may be defined as follows: There is a natural association between elements of $\mathfrak{g}$ and tangent vectors to W_b at b. Given an element $u \in \mathfrak{g}$, we define $\tilde{u}(b) \in T_bW_b$ to be the tangent vector to W_b at b associated with $\mathrm{ad}^*_u b$. (Note that $\tilde{u}(b) = 0$ when u is in the stabilizer of b; *i.e.*, when $\mathrm{ad}^*_u b = 0$.) We can define a 2-form ω on W_b by

$$\omega(\tilde{u}(b), \tilde{v}(b)) = \langle b, [u, v] \rangle. \tag{2.95}$$

It can be verified that this 2-form is well-defined, closed, G-invariant, and nondegenerate, and thus defines a G-invariant symplectic structure on W_b. ω also gives a Poisson bracket structure to the space of functions on W_b. In component notation, the Poisson bracket of two functions f and g is given by

$$\{f, g\} = \omega^{ij}(\partial_i f)(\partial_j g), \tag{2.96}$$

where ω^{ij} are the components of ω^{-1}. Every function f on W_b generates a Hamiltonian vector field v_f on W_b, defined by

$$v_f^i = \omega^{ij}(\partial_j f). \tag{2.97}$$

For any $u \in \mathfrak{g}$, there is a function Φ_u on W_b which generates the Hamiltonian vector field

$\tilde{u}$. This function is given by

$$\Phi_u(b) = -\langle b, u\rangle. \tag{2.98}$$

To see that Φ_u generates the vector field $\tilde{u}$, we use the fact that for any $v \in \mathfrak{g}$,

$$\tilde{v}^j \partial_j \Phi_u(b) = -\langle b, [u, v]\rangle = \omega_{jk}\tilde{v}^j \tilde{u}^k. \tag{2.99}$$

Since the vector fields $\tilde{v}$ span the tangent space to W_b at b, we have

$$\partial_j \Phi_u(b) = \omega_{jk}\tilde{u}^k(b), \tag{2.100}$$

so

$$\omega^{ij}\partial_j \Phi_u(b) = \tilde{u}^i. \tag{2.101}$$

The functions Φ_u also satisfy the equation

$$\{\Phi_u, \Phi_v\} = \Phi_{[u,v]}, \tag{2.102}$$

since

$$\begin{aligned} \{\Phi_u, \Phi_v\} &= \omega^{ij}(\partial_i \Phi_u)(\partial_j \Phi_v) = \omega^{ij}(\omega_{ik}\tilde{u}^k)(\omega_{jl}\tilde{v}^l) \\ &= \omega_{ik}\tilde{u}^k\tilde{v}^i = \langle b, [v, u]\rangle = \Phi_{[u,v]}. \end{aligned} \tag{2.103}$$

In order to construct representations of G using the coadjoint orbit W_b, it is now necessary to quantize the manifold W_b according to the technique of geometric quantization [65, 59]. The first step in this procedure is to construct a complex line bundle $\mathcal{L}_b$ over W_b with curvature form $i\omega$. This is known as "prequantization". For this step to be possible, it is necessary that $\frac{\omega}{2\pi}$ be an integral cohomology class (*i.e.*, that the integral of ω over any closed 2-surface in W_b be an integral multiple of 2π.) If such a line bundle $\mathcal{L}_b$ exists, then there is a natural homomorphism ϕ from the Lie algebra $\mathfrak{g}$ to the space of first-order differential operators on sections of $\mathcal{L}_b$, given by

$$\phi : u \mapsto \hat{u} = -\nabla_{\tilde{u}} + i\Phi_u, \tag{2.104}$$

where $\nabla_{\tilde{u}}$ is the covariant derivative in $\mathcal{L}_b$ in the direction $\tilde{u}$. Explicitly, written in component notation in a local coordinate chart,

$$\hat{u} = -\tilde{u}^i(b)(\partial_i + A_i(b)) + i\Phi_u(b), \tag{2.105}$$

where A_i is a connection on $\mathcal{L}_b$ satisfying $\partial_i A_j - \partial_j A_i = i\omega_{ij}$. To verify that ϕ is a homomorphism, we must check that

$$[\hat{u}, \hat{v}] = \widehat{[u, v]}. \tag{2.106}$$

We define ξ_u to be the differential operator corresponding to the vector field $-\tilde{u}$; *i.e.*, $\xi_u = -\tilde{u}^i \partial_i$, and we define $A_u = \tilde{u}^i A_i$. With these definitions,

$$\hat{u} = \xi_u - A_u + i\Phi_u. \tag{2.107}$$

One can easily calculate

$$[\xi_u, \xi_v] = \xi_{[u,v]}, \tag{2.108}$$

and

$$\xi_u \Phi_v(b) = \Phi_{[u,v]}(b). \tag{2.109}$$

One also finds that

$$\begin{aligned} \xi_u A_v - \xi_v A_u &= \widehat{[u, v]}^i A_i - \tilde{u}^i \tilde{v}^j (\partial_i A_j - \partial_j A_i) \\ &= A_{[u,v]} + i\Phi_{[u,v]}. \end{aligned} \tag{2.110}$$

Note that since the vectors $\tilde{u}$ span the tangent space to W_b at each point, Equation 2.110, along with the conditions that A_u is linear in u and that $A_u(b) = 0$ when $\mathrm{ad}_u^* b = 0$, could have been taken as the definition of a connection A_u associated with the derivative operators ξ_u. It is now trivial to compute the commutator

$$\begin{aligned} [\hat{u}, \hat{v}] &= [\xi_u - A_u + i\Phi_u, \xi_v - A_v + i\Phi_v] \\ &= \xi_{[u,v]} - A_{[u,v]} + i\Phi_{[u,v]} \\ &= \widehat{[u, v]}. \end{aligned} \tag{2.111}$$

Thus ϕ is a homomorphism, so we have determined that ϕ gives a representation of $\mathfrak{g}$ on the space of smooth sections of $\mathcal{L}_b$. Unfortunately, this representation is in general much too large to be irreducible; this is where the second stage of geometric quantization enters, which involves choosing a "polarization". We will only be concerned here with a specific type of polarization, the Kähler polarization. In general, choosing a polarization restricts the space of allowed smooth sections of $\mathcal{L}_b$ to a subspace containing only those sections which satisfy some local first-order differential equations. A Kähler polarization of W_b exists when W_b admits a G-invariant Kähler structure with ω as the associated $(1,1)$-form. This condition

is equivalent to the condition that W_b admits a G-invariant complex structure with respect to which ω is a $(1,1)$-form; *i.e.*, the only nonvanishing terms in ω have one holomorphic and one antiholomorphic index. In general, if W_b does not admit a Kähler polarization, and is not equivalent to a cotangent bundle, there is no standard way to find a polarization, and carrying out the geometric quantization program becomes extremely difficult. In case W_b does admit a Kähler polarization, we can restrict the space of allowed sections of $\mathcal{L}_b$ to the space $\mathcal{H}_b$ of holomorphic sections. When $\mathcal{L}_b$ has a Hermitian metric, then we can further restrict $\mathcal{H}_b$ to be the Hilbert space of square-integrable holomorphic sections of $\mathcal{L}_b$. According to the general principles of Kirillov and Kostant, the action of G on $\mathcal{H}_b$ should give an irreducible unitary representation of G for every b such that $\mathcal{H}_b$ can be constructed. This principle holds fairly well for compact semi-simple finite-dimensional groups, and even for loop groups; however, it does not seem to hold in complete generality. Some of the representations of $\widehat{\mathrm{Diff}}S^1$ constructed this fashion are nonunitary, and some are reducible, as we demonstrate below. In the case of finite-dimensional compact simple Lie groups, this construction is equivalent to the Borel-Weil construction outlined in Section 1.1, and the Kähler structure compatible with ω is equivalent to the complex structure defined in (2.20) for the homogeneous space G/T.

2.3 Gauge fixing

In this section, we prove several assertions which will simplify the process of explicitly constructing the coadjoint orbit representations in local coordinates. If one attempts to use Equation 2.107 to construct explicit formulae for the operators $\hat{u}$ as differential operators on $\mathcal{H}_b$, one encounters several obstacles. First, it is necessary to calculate the functions Φ_u in local coordinates. Second, one must find an explicit formula for a connection A_u which satisfies (2.110). Finding these expressions in terms of a local set of holomorphic coordinates is in general a somewhat nontrivial problem. Note, however, that the operator $\hat{u}$ can be written as

$$\hat{u} = \xi_u + f_u, \tag{2.112}$$

where ξ_u is the first-order differential operator defined above, and f_u is a function of the local coordinates satisfying

$$\xi_u f_v - \xi_v f_u = f_{[u,v]}. \tag{2.113}$$

We will find it easiest to construct explicit expressions for the operators $\hat{u}$ by finding directly a set of functions f_u which satisfy (2.113), and which correspond to the representation in question. We find these functions f_u by making a simplifying assumption which amounts to choosing a simple gauge for the connection A_u. To ensure that the set of f_u's we construct in this fashion are equivalent to those we would get from (2.110) by a specific choice of gauge, we will need the following two propositions.

Proposition 1 *Given a coadjoint orbit W_b of a group G, with $\mathcal{L}_b$ a complex line bundle over W_b with curvature $i\omega$, and with ξ_u and Φ_u defined as above, on a coordinate chart corresponding to a local trivialization of $\mathcal{L}_b$, if a set of functions f_u on W_b are linear in $u \in \mathfrak{g}$, and satisfy the conditions*

(*i*) $\xi_u f_v - \xi_v f_u = f_{[u,v]}$,

(*ii*) $f_u(b) = i\Phi_u(b)$ *when* $\mathrm{ad}_u^* b = 0$,

then the operators $\hat{u} = \xi_u + f_u$ are equal to the operators $\hat{u}$ from Equation 2.107 for some choice of connection A_u on $\mathcal{L}_b$ satisfying (2.110).

Proof. To prove this proposition, it will suffice to show that the functions $A'_u(b) = -f_u(b) + i\Phi_u(b)$ satisfy (2.110), are linear in u, and are zero when $\mathrm{ad}_u^* b = 0$. The last two conditions follow immediately from the definition of f_u and assumption (*ii*). To see that A'_u satisfies (2.110) is a simple calculation:

$$\begin{aligned} \xi_u A'_v - \xi_v A'_u &= \xi_u(-f_v + i\Phi_v) - \xi_v(-f_u + i\Phi_u) \\ &= -f_{[u,v]} + 2i\Phi_{[u,v]} \\ &= A'_{[u,v]} + i\Phi_{[u,v]}. \end{aligned} \tag{2.114}$$

Thus, A'_u is a valid connection on $\mathcal{L}_b$, and the proposition is proven. □

Proposition 2 *With the same premises as Proposition 1, when G is path connected the condition* (ii) *can be replaced by the weaker condition*

(*ii'*) *For some point $b_0 \in W_b$, $f_u(b_0) = i\Phi_u(b_0)$ for all u such that $\mathrm{ad}_u^* b_0 = 0$,*

and the result of proposition 1 still holds.

Proof. We need to prove that when G is path connected, condition (*ii'*) implies condition (*ii*). Assume $\mathrm{ad}_u^* b = 0$ for some $u \in \mathfrak{g}, b \in W_b$. Since $b_0 \in W_b$, for some $g \in G$ we have

$b = \mathrm{Ad}_g^* b_0$. If u stabilizes b, then $u_0 = \mathrm{Ad}_{g^{-1}} u$ must stabilize b_0. But then we have

$$\langle b, u\rangle = \langle \mathrm{Ad}_g^* b_0, \mathrm{Ad}_g u_0\rangle = \langle b_0, u_0\rangle, \tag{2.115}$$

so $\Phi_u(b) = \Phi_{u_0}(b_0)$. It remains to be shown that $f_u(b) = f_{u_0}(b_0)$. Since G is path connected, we have a path $g(t)$ in G with $g(0) = 1$ and $g(1) = g$. We claim that

$$\frac{d}{dt} f_{u(t)}(b(t)) = 0, \tag{2.116}$$

where $u(t) = \mathrm{Ad}_{g(t)} u_0$, and $b(t) = \mathrm{Ad}_{g(t)}^* b_0$. Defining

$$v(t) = \frac{dg(t)}{dt} g^{-1}(t) \in \mathfrak{g}, \tag{2.117}$$

we have

$$\frac{d}{dt} b(t) = \mathrm{ad}_v^* b(t), \tag{2.118}$$

and

$$\frac{d}{dt} u(t) = \mathrm{ad}_v u(t). \tag{2.119}$$

It follows that

$$\begin{aligned} \frac{d}{dt} f_{u(t)}(b(t)) &= -\xi_v f_{u(t)}(b(t)) + f_{[v,u(t)]}(b(t)) \\ &= -\xi_v f_{u(t)}(b(t)) + \xi_{u(t)} f_v(b(t)) + f_{[v,u(t)]}(b(t)) \\ &= 0, \end{aligned} \tag{2.120}$$

where we have used the fact that $\tilde{u}(t)(b(t)) = 0$. Thus, we have shown that

$$f_u(b) = f_{u_0}(b_0) = i\Phi_{u_0}(b_0) = i\Phi_u(b). \tag{2.121}$$

Since u and b were an arbitrary solution of $\mathrm{ad}_u^* b = 0$, we have proven that condition (ii') implies condition (ii), and thus the proposition is proven.□

2.4 Local formulae

In this section, we derive a set of general formulae for the local realization of a coadjoint orbit representation in a neighborhood of a point b in the coadjoint orbit W_b. This is done by first computing an exact local formula for the vector fields ξ_u and then performing a gauge-fixing to derive a general expression for a set of functions f_u satisfying the conditions

of Proposition 2. For finite-dimensional compact groups, the resulting formulae give a representation of the Lie algebra $\mathfrak{g}$ in terms of first-order differential operators acting on a ring R of polynomials in a finite number of complex variables. When the group G is an infinite-dimensional group, such as a centrally extended loop group or the Virasoro group, the realizations are again in terms of first-order differential operators acting on a ring R of polynomials in a set of complex variables; however, the set of complex variables becomes infinite, and the differential operators are described by infinite series, with only a finite number of terms acting nontrivially on any fixed polynomial in R. One important feature of these explicit realizations of Lie algebras in terms of differential operators is that the differential operators associated with raising operators $J_\alpha, \alpha \in \Phi_+$ in the Lie algebra are independent of which representation of the algebra is being realized. The independence of these operators from the highest weight of the relevant representation will be used in the following chapters to simplify formulae for conformal field theory observables calculated using these representations.

The first step in finding a local realization of the coadjoint orbit representations is to choose a set of coordinates on the orbit space W_b in the vicinity of b. In Section 2.1 we described such a set of coordinates on quotient spaces of the form G/T where T is a maximal abelian subgroup of G, for all types of groups under consideration in this thesis. A given coadjoint orbit space W_b is homeomorphic to a quotient space of G of the form G/S where S is the stabilizing subgroup of b in G. (The stabilizer of a point p in a space V which carries an action of the group G is the subgroup of G which leaves p invariant.) All the coadjoint orbit spaces which we will consider here arise from elements $b \in \mathfrak{g}^*$ whose stabilizer is a maximal abelian subgroup T; thus, these coadjoint orbits are homeomorphic to quotient spaces of the form G/T. We will describe these coadjoint orbits more explicitly for specific Lie groups in Section 2.5. In general, however, we can use the coordinates $\{z_\alpha : \alpha \in \Phi_+\}$ associated with the positive roots of a group G, which are defined through (2.20), as complex coordinates on the coadjoint orbit spaces of interest.

Once we have a set of coordinates on W_b, we can use explicit formulae for the coadjoint action to calculate the value of the symplectic form ω associated with the coadjoint orbit W_b in terms of the local coordinates z_α. As discussed above, the form $\frac{\omega}{2\pi}$ must be an integral cohomology class in order for the coadjoint orbit to admit a holomorphic line bundle $\mathcal{L}_b$ with curvature $i\omega$. For all the groups with which we are concerned in this thesis, there is a set of coadjoint orbits which have a symplectic form ω satisfying this condition. These

coadjoint orbits have the additional property that with respect to the complex structure described by the coordinates (2.20), the symplectic form ω is a $(1,1)$ form. Thus, on all these coadjoint orbits we can take a Kähler polarization of the space of sections of $\mathcal{L}_b$ by restricting to the space $\mathcal{H}_b$ of holomorphic sections. Locally, the holomorphic sections of $\mathcal{L}_b$ are described around the point b by polynomials in the variables z_α. The action (2.104) of the Lie algebra on the space of holomorphic sections reduces to a representation of $\mathfrak{g}$ in terms of first-order differential operators $\hat{u}$ on the space of these polynomials. The remainder of this section is devoted to the derivation of a general formula for these differential operators.

For the rest of this section, we assume that a specific group G has been chosen, and that a particular coadjoint orbit W_b satisfying the necessary conditions for quantization has also been selected. We will assume that the group G has a set of simple roots which generate the subalgebra associated with positive roots; as described in 2.1.3, when G is the Virasoro group, the set of simple roots actually refers to the set $\{L_1, L_2\}$ of quasi-simple roots, which also generates the subalgebra of positive roots. We will use a combination of physical and mathematical notation for the generators of the algebra $\mathfrak{g}_{\mathbb{C}}$; we write all generators in the form J_a, where a can either be a root $a = \alpha \in \Phi$, or an element $a = h \in \Theta$ of the basis $\Theta = \{h_\alpha | \alpha \in \Delta\}$ of the Cartan subalgebra. When J_a is in the Cartan subalgebra, we write $a = h \approx 0$, since a corresponds to a weight of 0. We use the physics notation ${f_{ab}}^c$ for algebra structure constants, so that

$$[J_a, J_b] = i{f_{ab}}^c J_c. \tag{2.122}$$

In particular, in this notation we have

$$i{f_{h\alpha}}^\alpha = \langle \alpha, h \rangle \quad \text{for } \alpha \in \Phi, h \in \Theta. \tag{2.123}$$

All the equations in this section could be rewritten in terms of a Chevalley basis and Cartan matrix; however, this does not seem to simplify the form of the results.

2.4.1 Vector fields

The first step in constructing the operators

$$\hat{J}_a = \xi_a + f_a \tag{2.124}$$

which implement the action (2.104) of $\mathfrak{g}$ on $\mathcal{H}_b$ is to calculate the vector fields

$$\xi_a = -\tilde{u}_a^\alpha \partial / \partial z_\alpha \tag{2.125}$$

associated with the coadjoint action $\mathrm{ad}^*_{J_a}$ of J_a on W_b. Actually, these are vector fields in the complexification of the tangent space to W_b, since the generators J_a are in $\mathfrak{g}_{\mathbb{C}}$ and not necessarily in $\mathfrak{g}$. The vector fields in the actual tangent space are constructed by taking the complex linear combinations of J_a in $\mathfrak{g}$; since the left action of G on W_b leaves the complex structure invariant, we are only interested in the holomorphic parts of the vector fields ξ_a. We are constructing operators which will act on polynomials in the holomorphic variables z_α, so by taking the holomorphic vector fields (2.125) we actually will construct a representation of $\mathfrak{g}_{\mathbb{C}}$ on the space of polynomials in $\mathcal{H}_b$.

The holomorphic vector fields ξ_a can be calculated by multiplying an arbitrary element of the form (2.20) on the left by the group element $\exp(\epsilon J_a)$. The components $\tilde{u}^\alpha_a$ appear as the order ϵ shifts to the holomorphic coordinates z_α, and can be derived by writing the product $\exp(\epsilon J_a)\exp(\sum z_\alpha J_\alpha)$ in the form

$$\mathrm{e}^{\epsilon J_a}\exp(\sum_{\alpha\in\Phi_+} z_\alpha J_\alpha) = \exp\left[\sum_{\alpha\in\Phi_+}(z_\alpha+\epsilon\tilde{u}^\alpha_a)J_\alpha\right] f(\{J_a|a\preceq 0\}) + \mathcal{O}(\epsilon^2) \tag{2.126}$$

to first order in ϵ, where f is some function of the generators J_a corresponding to negative roots and the Cartan subalgebra. To explicitly compute these vector fields, we use the infinitesimal forms of the BCH theorem,

$$\mathrm{e}^{\epsilon X}\mathrm{e}^{Y} = \exp(Y+\epsilon\sum_{k\geq 0}\frac{B_k}{k!}(\mathrm{ad}_Y)^k X) + \mathcal{O}(\epsilon^2), \tag{2.127}$$

$$\mathrm{e}^{Y+\epsilon Z+\epsilon X} = \exp\left(Y+\epsilon Z-\epsilon\sum_{k\geq 1}\frac{B_k}{k!}(-\mathrm{ad}_Y)^k X\right)\ \mathrm{e}^{\epsilon X} + \mathcal{O}(\epsilon^2), \tag{2.128}$$

and

$$\mathrm{e}^{\epsilon X}\mathrm{e}^{Y} = \mathrm{e}^{Y+\epsilon[X,Y]}\mathrm{e}^{\epsilon X} + \mathcal{O}(\epsilon^2), \tag{2.129}$$

where B_k is the kth Bernoulli number (Table 2.2). The first of these equations can be derived by writing

$$g(\epsilon) = \mathrm{e}^{\epsilon X}\mathrm{e}^{Y} = \mathrm{e}^{Y+\epsilon Z} + \mathcal{O}(\epsilon^2) \tag{2.130}$$

and defining

$$\begin{aligned} g &= g(0) = \mathrm{e}^{Y}, \\ \delta g &= \frac{\partial}{\partial\epsilon}|_{\epsilon=0} g(\epsilon) = X\mathrm{e}^{Y}. \end{aligned} \tag{2.131}$$

From these definitions it follows that

$$g^{-1}\delta g = \mathrm{e}^{-\mathrm{ad}_Y} X. \tag{2.132}$$

But writing

$$g_t(\epsilon) = \mathrm{e}^{t(Y+\epsilon Z)}, \tag{2.133}$$

we have

$$\frac{\partial}{\partial t}(\delta g_t) = Z g_t + Y \delta g_t. \tag{2.134}$$

It follows that

$$\frac{\partial}{\partial t}(g_t^{-1}\delta g_t) = g_t^{-1} Z g_t, \tag{2.135}$$

and thus that

$$g^{-1}\delta g = \int_0^1 \mathrm{d}t\, \mathrm{e}^{-t\,\mathrm{ad}_Y} Z = \left(\frac{1-\mathrm{e}^{-\,\mathrm{ad}_Y}}{\mathrm{ad}_Y}\right) Z. \tag{2.136}$$

From this expression and (2.132), it follows by formally manipulating the power series in ad_Y that

$$Z = \left(\frac{\mathrm{ad}_Y}{\mathrm{e}^{\,\mathrm{ad}_Y}-1}\right) X = \sum_{k=0}^{\infty} \frac{B_k}{k!} (\mathrm{ad}_Y)^k X, \tag{2.137}$$

which gives (2.127). Equations (2.128) and (2.129) follow immediately from the same type of argument.

We now give the general expression for the vector field components which arise from performing the calculation (2.126) using (2.127), (2.128) and (2.129).

Proposition 3 *The components $\tilde{u}_a^\alpha$ of the vector fields ξ_a are given by*

$$\tilde{u}_a^\alpha = -\sum_{k\geq 0, A_a(k,\alpha)} \beta_{k,\lambda} C_a(a_1,\ldots,a_k) z_{a_1}\ldots z_{a_k}, \tag{2.138}$$

where λ is the minimum integer such that $a + a_1 + a_2 + \ldots + a_\lambda \succ 0$ ($\lambda = 0$ when $a \succ 0$),

$$A_a(k,\alpha) = \{(a_1,a_2,\ldots,a_k) : a_1,\ldots,a_k \succ 0, a + a_1 + a_2 + \ldots + a_k = \alpha\}, \tag{2.139}$$

$$C_a(a_1,\ldots,a_k) = i^k \sum_{b_1,\ldots b_{k-1}} f_{a_1 a}{}^{b_1} f_{a_2 b_1}{}^{b_2} \ldots f_{a_k b_{k-1}}{}^{\alpha}, \tag{2.140}$$

and

$$\beta_{k,\lambda} = (-1)^{k+1} \sum_{l=0}^{k-\lambda} \frac{B_l}{l!\,(k-l)!}. \tag{2.141}$$

(B_l is the lth Bernoulli number, as in (2.127) and (2.153). The Bernoulli numbers and values of $\beta_{k,\lambda}$ are tabulated for small values of the subscripts l, k, λ in tables 2.1, 2.2 at the end of this section.)

Proof. We begin by noting the identities

$$(\mathrm{ad}_{(\sum z_\alpha J_\alpha)})^k J_a = \sum_{a_1,\ldots,a_k \succ 0} C_a(a_1,\ldots,a_k) z_{a_1}\ldots z_{a_k} J_{a+a_1+\ldots+a_k}, \tag{2.142}$$

and

$$C_a(a_1,\ldots,a_s) C_{a+a_1+\ldots+a_s}(a_{s+1},\ldots,a_k) = C_a(a_1,\ldots,a_k). \tag{2.143}$$

Applying Equation 2.127 to $\exp(\epsilon J_a)\exp(\sum z_\alpha J_\alpha)$, we have

$$\exp(\epsilon J_a)\exp(\sum_{\alpha\in\Phi_+} z_\alpha J_\alpha) \sim \tag{2.144}$$
$$\exp\left(\sum_{\alpha\in\Phi_+} z_\alpha J_\alpha + \epsilon \sum_{k\geq 0, a_1,\ldots,a_k\succ 0} \frac{B_k}{k!} C_a(a_1,\ldots,a_k) z_{a_1}\ldots z_{a_k} J_{a+a_1+\ldots+a_k}\right),$$

where by $x \sim y$ it is meant that $x = yf + \mathcal{O}(\epsilon^2)$, with f some function of the J_a's with $a \preceq 0$. Dividing the terms in the exponential into generators J_a with $a \succ 0$ and $a \preceq 0$, this can be rewritten as

$$\exp(\epsilon J_a)\exp(\sum_{\alpha\in\Phi_+} z_\alpha J_\alpha) \sim \tag{2.145}$$
$$\exp\left(\sum_{\alpha\in\Phi_+} z_\alpha J_\alpha - \epsilon \sum_{k\geq 0, A_a^+(k)} \beta_{k,\lambda}^{(0)} C_a(a_1,\ldots,a_k) z_{a_1}\ldots z_{a_k} J_{a+a_1+\ldots+a_k}\right.$$
$$\left. + \epsilon \sum_{l_1\geq 0, A_a^-(l_1)} \frac{B_{l_1}}{l_1!} C_a(a_1,\ldots,a_{l_1}) z_{a_1}\ldots z_{a_{l_1}} J_{a+a_1+\ldots+a_{l_1}}\right),$$

where $\beta_{k,\lambda}^{(t)}$ is defined by

$$\beta_{k,\lambda}^{(t)} = -\frac{B_k}{k!} - \sum_{t\geq s>0, 0\leq l_1<l_2<\ldots<l_s<\lambda} (-1)^{s+k-l_1} \frac{B_{l_1}}{l_1!}\frac{B_{l_2-l_1}}{(l_2-l_1)!}\cdots\frac{B_{l_s-l_{s-1}}}{(l_s-l_{s-1})!}\frac{B_{k-l_s}}{(k-l_s)!}, \tag{2.146}$$

and the sets $A_a^\pm(k)$ are defined by

$$A_a^+(k) = \{(a_1,a_2,\ldots,a_k) : a_1,\ldots,a_k \succ 0, a+a_1+a_2+\ldots+a_k \succ 0\}, \tag{2.147}$$

and

$$A_a^-(k) = \{(a_1,a_2,\ldots,a_k) : a_1,\ldots,a_k \succ 0, a+a_1+a_2+\ldots+a_k \preceq 0\}. \tag{2.148}$$

Applying (2.128) and (2.143) to Equation 2.145 t times, we get

$$\exp(\epsilon J_a)\exp(\sum_{\alpha\in\Phi_+} z_\alpha J_\alpha) \sim$$

$$\exp\left(\sum_{\alpha\in\Phi_+} z_\alpha J_\alpha - \epsilon \sum_{k\geq 0, A_a^+(k)} \beta_{k,\lambda}^{(t)} C_a(a_1,\ldots,a_k) z_{a_1}\ldots z_{a_k} J_{a+a_1+\ldots+a_k}\right. \tag{2.149}$$

$$\left. +\epsilon \sum_{0\leq l_1<\ldots<l_t<l, A_a^-(l)} (-1)^{t+l-l_1} \frac{B_{l_1}}{l_1!}\frac{B_{l_2-l_1}}{(l_2-l_1)!}\cdots\frac{B_{l-l_t}}{(l-l_t)!} C_a(a_1,\ldots,a_l) z_{a_1}\ldots z_{a_l} J_{a+a_1+\ldots+a_l}\right).$$

Since $\beta_{k,\lambda}^{(t)} = \beta_{k,\lambda}^{(\infty)}$ for $t \geq \lambda$, to all orders in z we have

$$\exp(\epsilon J_a)\exp(\sum_{\alpha\in\Phi_+} z_\alpha J_\alpha) \sim \tag{2.150}$$

$$\exp\left(\sum_{\alpha\in\Phi_+} z_\alpha J_\alpha - \epsilon \sum_{k\geq 0, A_a^+(k)} \beta_{k,\lambda}^{(\infty)} C_a(a_1,\ldots,a_k) z_{a_1}\ldots z_{a_k} J_{a+a_1+\ldots+a_k}\right),$$

We will now show that $\beta_{k,\lambda}^{(\infty)} = \beta_{k,\lambda}$. Using the fact that $B_{2k+1} = 0$ for $k > 0$, it is not hard to determine that

$$\begin{aligned}
\beta_{k,0}^{(\infty)} &= -\frac{B_k}{k!}, \\
\beta_{k,1}^{(\infty)} &= \delta_{k,1}, \\
\beta_{k,\lambda}^{(\infty)} &= \sum_{s\geq 0, 1<l_1<\ldots<l_s<\lambda} (-1)^{s+k} \frac{B_{l_1-1}}{(l_1-1)!}\frac{B_{l_2-l_1}}{(l_2-l_1)!}\cdots\frac{B_{l_s-l_{s-1}}}{(l_s-l_{s-1})!}\frac{B_{k-l_s}}{(k-l_s)!}, \quad \text{for } \lambda > 1.
\end{aligned} \tag{2.151}$$

When $\lambda > 1$, we can write a generating function for $\beta_{k,\lambda}^{(\infty)}$ by

$$\sum_{k\geq\lambda>1} \beta_{k,\lambda}^{(\infty)} y^{k-\lambda} x^k = \sum_{l>m\geq 0} y^m \frac{B_l}{l!}(-x)^{l+1} \sum_{s\geq 0}(1-\phi(-x))^s, \tag{2.152}$$

where

$$\phi(x) = \frac{x}{e^x - 1} = \sum_{n\geq 0} \frac{B_n}{n!} x^n. \tag{2.153}$$

From (2.152), it follows that

$$\beta_{k,\lambda}^{(\infty)} = (-1)^k \sum_{l=k-\lambda+1}^{k-1} \frac{B_l}{l!\,(k-l)!} = (-1)^{k+1} \sum_{l=0}^{k-\lambda} \frac{B_l}{l!\,(k-l)!}, \tag{2.154}$$

where we have used the fact that

$$\sum_{l=0}^{k-1} B_l \binom{k}{l} = 0, \quad \text{for } k > 1. \tag{2.155}$$

From (2.151) and (2.155), it is also easy to verify that $\beta_{k,0}^{(\infty)} = \beta_{k,0}$ and $\beta_{k,1}^{(\infty)} = \beta_{k,1}$. Thus, for all k and λ, we have shown that $\beta_{k,\lambda}^{(\infty)} = \beta_{k,\lambda}$, and Proposition 3 is proven.□

The result of Proposition 3 can be restated in terms of the vector fields ξ_a.

Corollary 1 *The vector fields ξ_a are given by*

$$\xi_a = \sum_{k\geq 0, A_a^+} \beta_{k,\lambda} C_a(a_1,\ldots,a_k) z_{a_1}\ldots z_{a_k} \frac{\partial}{\partial_{z_{a+a_1+\cdots+a_k}}}, \tag{2.156}$$

where λ, C_a, β, and A_a^+ are defined as above.

When $a \succeq 0$, we can use (2.151) to simplify the formulae for ξ_a to

$$\xi_a = \sum_{k\geq 0, a_1,\ldots,a_k \succ 0} -\frac{B_k}{k!} C_a(a_1,\ldots,a_k) z_{a_1}\ldots z_{a_k} \frac{\partial}{\partial z_{a+a_1+\ldots+a_k}} \quad \text{for } a \succ 0, \tag{2.157}$$

and

$$\xi_h = \sum_{\alpha\succ 0} -\langle \alpha, h\rangle z_\alpha \frac{\partial}{\partial z_\alpha} \quad \text{for } h \in \Theta. \tag{2.158}$$

Note that (2.158) could also have been obtained more directly through (2.129).)

2.4.2 Gauge fixing

Now that we have explicit formulae for the vector fields ξ_a, in order to construct the operators $\hat{J}_a$ it will suffice by the results of the previous section to find a set of functions f_u which are linear in the parameter $u \in \mathfrak{g}_{\mathbb{C}}$ and which satisfy conditions (*i*) from Proposition 1 and (*ii'*) from Proposition 2. We will explicitly construct these functions by making an appropriate choice of gauge; before discussing the gauge-fixing procedure, however, it will be useful to discuss some general features of the vector fields ξ_a calculated in the previous subsection.

The main observation to be made is that the ring $R = \mathbb{C}[\{z_\alpha\}]$ of polynomials in the variables z_α has a natural grading, according to the weights $w \in \Lambda$ in the root lattice of the group G. Recall that the root lattice $\Lambda \subset \mathfrak{t}_{\mathbb{C}}^*$ is the lattice generated by the simple roots $\alpha \in \Delta$. We can associate to each variable z_α a degree

$$\deg z_\alpha = \alpha \in \Lambda. \tag{2.159}$$

By giving the unit element 1 a degree

$$\deg 1 = 0, \tag{2.160}$$

and by defining the degree of a product of monomials recursively by

$$\deg(fg) = \deg f + \deg g, \tag{2.161}$$

we define the degree for all monomials $f = z_{\alpha_1} \cdots z_{\alpha_i}$ to be $\alpha_1 + \cdots + \alpha_i$. The ring R thus admits a Λ-grading

$$R = \bigoplus_{w \in \Lambda} R_w, \tag{2.162}$$

where

$$R_w = \{f \in R : \deg(f) = w\}, \tag{2.163}$$

and where

$$R_w R_y \subseteq R_{w+y}. \tag{2.164}$$

We define a polynomial $f \in R$ to be *quasi-homogeneous* when f has a definite degree $\deg f \in \Lambda$.

Considering the action of the operators ξ_a on the space R of polynomials in the z_α, we find that

$$\xi_a R_w \subseteq R_{w-a} \quad \text{when } a \not\succ 0, \tag{2.165}$$

and

$$\xi_h f = -\langle w, h \rangle f \quad \text{for } f \in R_w, h \in \Theta. \tag{2.166}$$

Thus, the action of the vector field part of the operators $\hat{J}_a$ on R has a very natural structure with respect to the grading.

We now prove a proposition which describes further the action of the vector fields ξ_a on R; this proposition will be used in this section to calculate the gauge-fixed functions f_a, and will also be used in Chapter 3 where we describe in more detail the $\mathfrak{g}_{\mathbb{C}}$-module structure of $\mathcal{H}_b$.

Proposition 4 *The unit element 1 in R, which we denote by $|\,\rangle$, is the unique function in R (up to scalar multiplication) which is annihilated by ξ_a for all $a \succ 0$; i.e., $|\,\rangle$ is the unique highest weight state in the module R under the action of the ξ_a's.*

Proof. Assume there is another function $\phi \in R$ which is annihilated by ξ_a for all $a \succ 0$. Since R is graded, ϕ can be written as a sum of quasi-homogeneous functions,

$$\phi = \sum_{w \succeq 0} \phi_w, \quad \phi_w \in R_w. \tag{2.167}$$

Take w to be a minimal nonzero weight with $\phi_w \neq 0$ (w is a minimal weight satisfying this condition as long as there does not exist a nonzero weight $y \in \Lambda$ with $w \succ y$ and $\phi_y \neq 0$). Now, let α be a maximal root such that some term in ϕ_w contains a factor of z_α. ϕ_w can now be written in the form

$$\phi_w = \sum_{m \geq 0} z_\alpha^m g_{w-m\alpha}^{(m)}(\{z_\beta : \beta \not\succeq \alpha\}), \tag{2.168}$$

where for each $m \geq 0$, $g_{w-m\alpha}^{(m)}$ is a quasi-homogeneous polynomial of degree $w - m\alpha$ in the set of variables z_β with $\beta \not\succeq \alpha$. Since for $a \succ 0$, all terms in ξ_a except the leading term $-\partial/\partial z_a$ contain derivatives $\partial/\partial z_\beta$ with $\beta \succ \alpha$, we can compute

$$\xi_\alpha \phi_w = -\sum_{m>0} m z_\alpha^{m-1} g_{w-m\alpha}^{(m)}(\{\beta : \beta \not\succeq \alpha\}). \tag{2.169}$$

For this expression to be zero, all the functions $g_{w-m\alpha}^{(m)}$ would have to be zero for $m > 0$. But then ϕ_w would not contain any terms with a factor of z_α, contradicting our assumption. Thus, the only states in R annihilated by all ξ_a with $a \succ 0$ are the constant functions in R_0.□

Note that the proof of this proposition also indicates that there are not even any formal power series in $\mathbb{C}[[\{z_\alpha\}]]$ other than $|\,\rangle$ which are annihilated by all $\xi_a, a \succ 0$. We will mention this point again in connection with Virasoro representations.

The choice of gauge we use to construct the functions f_a is a gauge in which the structure of the action (2.165), (2.166) of the operators ξ_a on R is preserved for the operators $\hat{J}_a$; that is, we will find a gauge in which

$$\hat{J}_a R_w \subseteq R_{w-a} \tag{2.170}$$

for all generators $\hat{J}_a$ ($w - a = w$ when $a \in \Theta$). A priori, it is not clear that such a gauge choice is possible. We will use (2.170) as an Ansatz for the functions $f_a = \hat{J}_a - \xi_a$. We now demonstrate that there is a unique set of functions f_a satisfying this Ansatz and also conditions (*i*), (*ii'*), and that therefore these functions f_a correspond to a particular choice of gauge by Proposition 2.

The first consequence of the Ansatz is that the functions f_a all lie in the ring R, and furthermore that $f_a \in R_{-a}$. Specifically, this means that

$$f_a = 0, \quad \text{for } a \succ 0,$$

$$\begin{aligned} f_h &\in \mathbb{C}, \quad \text{for } h \in \Theta, \\ f_{-a} &\in R_a, \quad \text{for } a \succ 0. \end{aligned} \tag{2.171}$$

We have chosen the element b in the coadjoint orbit W_b to be such that the stabilizer of b under the coadjoint action of $\mathfrak{g}_{\mathbb{C}}$ is precisely $\mathfrak{t}_{\mathbb{C}}$. In the coordinates z_α, b is the point at which all coordinates are 0. By condition (*ii'*), it follows that the functions f_h are given at the point $b = 0$ by

$$f_h(0) = i\Phi_h(b) = -i\langle b, h\rangle. \tag{2.172}$$

Since the functions f_h are constant functions in R, it follows that

$$f_h = -i\langle b, h\rangle. \tag{2.173}$$

We will now use (2.173) and the consequences (2.171) of the Ansatz to construct functions $f_{-a} \in R_a$ for all $a \succ 0$ which satisfy (*i*). For these functions, the condition (*i*) is given by

$$\xi_a f_c - \xi_c f_a = i f_{ac}{}^d f_d. \tag{2.174}$$

For particular choices of a and c, this equation reduces to the equations

$$\xi_h f_{-a} = -\langle a, h\rangle f_{-a}, \quad \forall a \succ 0, h \in \Theta \tag{2.175}$$

$$\xi_a f_{-a} = \sum_{h\in\Theta} i f_{a(-a)}{}^h f_h, \quad \forall a \succ 0 \tag{2.176}$$

$$\xi_a f_{-c} = 0, \quad \forall a, c \succ 0 \text{ where } c - a \not\succeq 0 \tag{2.177}$$

$$\xi_a f_{-c} = i f_{a(-c)}{}^{a-c} f_{a-c}, \quad \forall a, c \succ 0 \text{ where } c - a \succ 0 \tag{2.178}$$

$$\xi_{-a} f_{-c} - \xi_{-c} f_{-a} = i f_{(-a)(-c)}{}^{-a-c} f_{-a-c}, \quad \forall a, c \succ 0. \tag{2.179}$$

From (2.166) and (2.171), it follows immediately that (2.175) and (2.177) must hold. We can use (2.176) to calculate the functions $f_{-\alpha}$ for simple α. For a simple root α, the space R_α is one-dimensional and consists of all functions of the form $f = xz_\alpha$ with $x \in \mathbb{C}$. The vector field operator ξ_α has a leading term $-\partial/\partial z_\alpha$, with all other terms containing derivatives $\partial/\partial z_\beta$ with $\beta \succ \alpha$. Thus, if we write

$$f_{-\alpha} = xz_\alpha, \tag{2.180}$$

then

$$\xi_\alpha f_{-\alpha} = -x = f_{h_\alpha} = -i\langle b, h_\alpha\rangle. \tag{2.181}$$

It follows that

$$f_{-\alpha} = i\langle b, h_\alpha\rangle z_\alpha, \quad \forall \alpha \in \Delta. \tag{2.182}$$

(Note: in the case of the Virasoro algebra, this equation does not give the function f_{-2} associated with the quasi-simple root $\hat{L}_2$; this special case is discussed in the next section. The rest of the analysis in this subsection, however, is still correct for the Virasoro algebra after one correctly fixes the function f_{-2}.)

Now that we have expressions for f_a where a is either a simple root or an element of the Cartan subalgebra, we can proceed to construct the remaining functions f_a inductively, using (2.179). Such a construction is possible because the set of simple roots always generates the algebra of positive roots. We will now proceed to prove that this construction gives rise to a unique and well-defined set of functions f_a for each $a \succ 0$, which satisfy (2.175)-(2.179).

Proposition 5 *The functions $f_a \in R$ defined by the recursive formulae*

$$\begin{aligned} f_a &= 0, \quad a \succ 0, \\ f_h &= -i\langle b, h\rangle, \quad h \in \Theta \\ f_{-\alpha} &= i\langle b, h_\alpha\rangle z_\alpha, \quad \alpha \in \Delta, \\ f_{a+c} &= \frac{-i}{f_{ac}{}^{a+c}}\left[\xi_a f_c - \xi_c f_a\right], \quad a, c \prec 0 \end{aligned} \tag{2.183}$$

are well-defined, and satisfy (2.174).

Proof. In order to prove this proposition, it is clearly necessary to demonstrate that for a fixed root $d \prec 0$, the last equation in (2.183) gives a well-defined function for $f_d = f_{a+(d-a)}$, independent of the choice of a satisfying $0 \succ a \succ d$. The fact that all functions f_d are defined by this equation follows again from the fact that the simple roots generate the algebra of all positive roots. Because the functions f_a defined through (2.183) clearly all satisfy (2.171), we know that (2.175) and (2.177) are satisfied by these functions. What we must therefore show in order to guarantee that the remaining conditions (2.176), (2.178), and (2.179) hold and that the functions f_a are well-defined, is that for every $a \succ 0$, there is a unique function $\tilde{f}_{-a} \in R_a$ which satisfies the conditions

$$\xi_a \tilde{f}_{-a} = \sum_{h\in\Theta} i f_{a(-a)}{}^h f_h, \tag{2.184}$$

$$\xi_c \tilde{f}_{-a} = i f_{c(-a)}{}^{c-a} \tilde{f}_{c-a}, \quad \forall c : a \succ c \succ 0, \tag{2.185}$$

$$\tilde{f}_{-a} = \frac{-i}{f_{(c-a)(-c)}{}^{-a}}\left[\xi_{c-a}\tilde{f}_{-c} - \xi_{-c}\tilde{f}_{c-a}\right], \quad \forall c: \ a \succ c \succ 0. \tag{2.186}$$

We will prove (2.184)-(2.186) by induction on a. As a basis for the induction, we take the simple roots $a = \alpha \in \Delta$. We have defined the functions $f_{-\alpha}$ for simple roots precisely such that (2.184) holds when $a \in \Delta$. In this case, there are no roots c satisfying $a \succ c \succ 0$, so (2.185) and (2.186) are vacuously satisfied. We now proceed by induction, taking as the induction hypothesis the assertion that for all roots a' satisfying $a \succ a' \succ 0$, conditions (2.184)-(2.186) are satisfied. In order to guarantee that these conditions are also satisfied for the root a, we claim that it will suffice to demonstrate that

$$\xi_c \frac{-i}{f_{(d-a)(-d)}{}^{-a}}\left[\xi_{d-a}f_{-d} - \xi_{-d}f_{d-a}\right] = \sum_e i f_{c(-a)}{}^e f_e, \tag{2.187}$$
$$\forall d, c: \ a \succ d \succ 0, c \succ 0.$$

(where we have written $\tilde{f}_{-a'}$ as $f_{-a'}$ when $a' \prec a$.) That this condition implies conditions (2.184) and (2.185) follows immediately when the function f_{-a} is well-defined. However, because the right hand side of (2.187) is independent of d, it also follows from this condition that

$$\xi_c\left[\frac{-i}{f_{(d-a)(-d)}{}^{-a}}\left(\xi_{d-a}f_{-d} - \xi_{-d}f_{d-a}\right) - \frac{-i}{f_{(d'-a)(-d')}{}^{-a}}\left(\xi_{d'-a}f_{-d'} - \xi_{-d'}f_{d'-a}\right)\right] = 0$$
$$\forall d, d', c: \ a \succ d, d' \succ 0, c \succ 0. \tag{2.188}$$

But by Proposition 4 this implies that the quantity in brackets is an element of $\mathbb{C}$ for fixed values of d, d'. Since this quantity is an element of R_a, and $a \succ 0$, the quantity must be 0. But this would imply that f_{-a} is well-defined, and thus that (2.186) is satisfied for a. Therefore, it will be sufficient to prove (2.187) in order to have a proof of the proposition.

We shall now demonstrate that (2.187) is satisfied, given the induction hypothesis. This is essentially a straightforward algebraic computation. Using the induction hypothesis and the Jacobi identity, we have

$$\begin{aligned}
&\xi_c \frac{-i}{f_{(d-a)(-d)}{}^{-a}}\left[\xi_{d-a}f_{-d} - \xi_{-d}f_{d-a}\right] \qquad (2.189)\\
&= \frac{-i}{f_{(d-a)(-d)}{}^{-a}}\left[i\sum_e f_{c(d-a)}{}^e \xi_e f_{-d} + \xi_{d-a}\xi_c f_{-d} - i\sum_e f_{c(-d)}{}^e \xi_e f_{d-a} - \xi_{-d}\xi_c f_{d-a}\right]\\
&= \frac{1}{f_{(d-a)(-d)}{}^{-a}}\sum_e\left[f_{c(d-a)}{}^e \xi_e f_{-d} + f_{c(-d)}{}^e \xi_{d-a} f_e - f_{c(-d)}{}^e \xi_e f_{d-a} - f_{c(d-a)}{}^e \xi_{-d} f_e\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{f_{(d-a)(-d)}{}^{-a}} \sum_e \left[f_{c(d-a)}{}^e (\xi_e f_{-d} - \xi_{-d} f_e) + f_{c(-d)}{}^e (\xi_{d-a} f_e - \xi_e f_{d-a}) \right] \\
&= \frac{i}{f_{(d-a)(-d)}{}^{-a}} \sum_{e,g} \left[f_{c(d-a)}{}^e f_{e(-d)}{}^g f_g + f_{c(-d)}{}^e f_{(d-a)(e)}{}^g f_g \right] \\
&= \frac{i}{f_{(d-a)(-d)}{}^{-a}} \sum_{e,g} \left[f_{(d-a)(-d)}{}^e f_{ce}{}^g f_g \right] \\
&= i f_{c(-a)}{}^g f_g.
\end{aligned}
$$

We have thus shown by induction that the condition (2.187) is satisfied for all $a \succ 0$, so the proposition is proven.□

We now have an explicit set of formulae which define the functions f_a in the gauge $f_a \in R_{-a}$.

2.4.3 General formula

We can now state the complete result of this section.

Proposition 6 *An explicit representation of the algebra $\mathfrak{g}$ on the space R of polynomials in the variables $\{z_\alpha : \alpha \in \Phi_+\}$ is given by the operators*

$$\hat{J}_a = \xi_a + f_a, \tag{2.190}$$

where ξ_a are the first-order differential operators given by (2.156), and f_a are the functions defined by (2.183). On the space of holomorphic sections $\mathcal{H}_b$ of $\mathcal{L}_b$, this representation corresponds to the coadjoint orbit representation in the particular gauge $f_a \in R_{-a}$.□

Because the operators ξ_a are independent of which coadjoint orbit W_b is used, and the functions f_a are 0 for $a \prec 0$, it follows immediately from this proposition that the operators $\hat{J}_a$ with $a \prec 0$ are independent of the chosen coadjoint orbit, and thus independent of the highest weight of the associated representation. This feature of the coadjoint orbit representations will be exploited in Section 4, where it is used to simplify formulae for certain correlation functions of vertex operators in conformal field theories.

Note also that the explicit representations described in this section can be defined for any highest weight representation of the Lie algebra, whether or not it is integrable to a representation of the Lie group. Thus, the set of formulae given here are actually somewhat more general than what we need for analysis of the coadjoint orbit representations.

$\lambda\backslash k$	0	1	2	3	4	5	6	7	8	9
0	-1	$\frac{1}{2}$	$-\frac{1}{12}$	0	$\frac{1}{720}$	0	$-\frac{1}{30240}$	0	$\frac{1}{1209600}$	0
1	0	1	0	0	0	0	0	0	0	0
2	0	0	$-\frac{1}{2}$	$-\frac{1}{12}$	0	$\frac{1}{720}$	0	$-\frac{1}{30240}$	0	$\frac{1}{1209600}$
3	0	0	0	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{720}$	$-\frac{1}{1440}$	$-\frac{1}{30240}$	$\frac{1}{60480}$	$\frac{1}{1209600}$
4	0	0	0	0	$-\frac{1}{24}$	$-\frac{1}{80}$	$-\frac{1}{1440}$	$\frac{1}{5040}$	$\frac{1}{60480}$	$-\frac{17}{3628800}$
5	0	0	0	0	0	$\frac{1}{120}$	$\frac{1}{360}$	$\frac{1}{5040}$	$-\frac{1}{24192}$	$-\frac{17}{3628800}$
6	0	0	0	0	0	0	$-\frac{1}{720}$	$-\frac{1}{2016}$	$-\frac{1}{24192}$	$\frac{1}{145152}$
7	0	0	0	0	0	0	0	$\frac{1}{5040}$	$\frac{1}{13440}$	$\frac{1}{145152}$
8	0	0	0	0	0	0	0	0	$-\frac{1}{40320}$	$-\frac{1}{103680}$
9	0	0	0	0	0	0	0	0	0	$\frac{1}{362880}$

Table 2.1: Values of $\beta_{k,\lambda}$

In the following sections and chapters we will give explicit examples of these representations and study their properties further. We give here tables of the Bernoulli numbers (Table 2.2) and the constants $\beta_{k,\lambda}$ which appear in (2.156) (Table 2.1) for convenience in explicit calculations.

l	0	1	2	3	4	5	6	7	8	9	10
B_l	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$

Table 2.2: Bernoulli numbers B_l

2.5 Examples of coadjoint orbit representations

In this section, we calculate some explicit examples of the coadjoint orbit representations described in the previous sections. We give the formulae in local coordinates for the coadjoint orbit representations of the simple groups $SU(2)$ and $SU(3)$, the loop group $\widehat{L}SU(2)$, and the Virasoro group. The coadjoint orbit description of the representations of the groups $SU(2), SU(3)$ and $\widehat{L}SU(2)$ are already understood from a global perspective (in these cases, the coadjoint orbit approach is equivalent to that given by the Borel-Weil theory); however, an explicit realization for $\widehat{L}SU(2)$ such as that given here has not previously been described. In the case of the Virasoro algebra, the representations arising from the coadjoint orbit description have not been previously analyzed in a complete fashion, since the Borel-Weil theory does not hold for the Virasoro group. In this case, the explicit formulae given here provide a useful tool for understanding previously unstudied aspects of these representations.

2.5.1 $SU(2)$

We begin with the group $G = SU(2)$. As described in Section 2.1.1, the Lie algebra $\mathfrak{g} = \mathrm{su}(2)$ is a 3-dimensional vector space spanned by the generators iJ_k, $k \in \{1,2,3\}$. The effect of the adjoint action of iJ_k is to generate a rotation around the x_k-axis in $\mathfrak{g}$. The dual space $\mathfrak{g}^*$ is also a 3-dimensional vector space. We choose a set of coordinates b_k in this space such that the vector

$$\mathbf{b} = (b_1, b_2, b_3) \in \mathfrak{g}^* \tag{2.191}$$

has coordinates

$$b_k = \langle \mathbf{b}, iJ_k \rangle. \tag{2.192}$$

In terms of these coordinates, the coadjoint action of iJ_k generates a rotation around the b_k-axis in $\mathfrak{g}^*$. Given a vector $\mathbf{b} = (b_1, b_2, b_3) \in \mathfrak{g}^*$, the coadjoint orbit of $\mathbf{b}$ is given by

$$W_b = \{\mathbf{b}' \in \mathfrak{g}^* : |\mathbf{b}'|^2 = b^2\}, \tag{2.193}$$

which is just the 2-sphere in $\mathfrak{g}^*$ of radius $b = |\mathbf{b}|$. We will now explicitly calculate the 2-form ω on W_b. We choose a canonical element $\mathbf{b} = (0, 0, -b) \in W_b$. To calculate ω at the point $\mathbf{b}$, we need only find the explicit correspondence between elements of $\mathfrak{g}$ and $T_{\mathbf{b}}W_b$. Under

the Lie algebra coadjoint action, we have

$$\begin{aligned} \mathrm{ad}^*_{iJ_1}\mathbf{b} &= (0,-b,0), \\ \mathrm{ad}^*_{iJ_2}\mathbf{b} &= (b,0,0), \\ \mathrm{ad}^*_{iJ_3}\mathbf{b} &= (0,0,0). \end{aligned} \tag{2.194}$$

It follows that

$$\omega_{12}(\mathbf{b}) = \langle \mathbf{b}, [\frac{iJ_2}{b}, -\frac{iJ_1}{b}]\rangle = \frac{1}{b}. \tag{2.195}$$

Since ω is G-invariant, it is easy to see that ω is defined globally on W_b by

$$\omega_{ij}(\mathbf{b}) = -\frac{1}{b^2}\epsilon_{ijk}b_k. \tag{2.196}$$

In order for $\omega/2\pi$ to be an integral form, we must have $\int_{W_b}\omega/2\pi = 2b \in \mathbb{Z}$, so b must be a half-integer. Thus, whenever $b \in \mathbb{Z}/2$, we can construct a line bundle $\mathcal{L}_b$ over W_b with curvature form $i\omega$.

Because the stabilizer of $\mathbf{b}$ under the coadjoint action is precisely the maximal subtorus $T = \{e^{2i\theta J_3} : 0 \le \theta \le 2\pi\}$, the coadjoint orbit space W_b is homeomorphic to the space G/T. Using the results of Section (2.1.1), we have a simple complex coordinate system on this space. As in (2.20), given any complex number z, it is possible to find functions $\alpha(z,\bar{z})$ and $\beta(z,\bar{z})$, with $\beta(z,\bar{z})$ real, such that

$$e^{zJ_+}e^{a(z,\bar{z})J_-}e^{\beta(z,\bar{z})J_3} \in G. \tag{2.197}$$

The functions $a(z,\bar{z})$ and $\beta(z,\bar{z})$ can be calculated explicitly by working in the fundamental representation of $SU(2)$; one finds that

$$\begin{aligned} a(z,\bar{z}) &= \frac{-\bar{z}}{1+|z|^2}, \\ \beta(z,\bar{z}) &= \ln(1+|z|^2). \end{aligned} \tag{2.198}$$

By the general argument given for (2.20), z gives a natural G-invariant complex structure to the space W_b. In fact, W_b is just the space S^2, and z is just the usual complex coordinate on S^2 given by projection from the south pole onto $\mathbb{C}$, which is naturally invariant under the rotations generated by $SU(2)$.

We can now rewrite the symplectic form ω in terms of the complex coordinate z. We begin by rewriting the differentials dz, $d\bar{z}$ in terms of the original coordinates b_k. At $\mathbf{b}$,

we have

$$\begin{aligned} dz &= \frac{1}{2b}(db_1 - idb_2), \\ d\bar{z} &= \frac{1}{2b}(db_1 + idb_2). \end{aligned} \tag{2.199}$$

It is now possible to express ω at **b** in terms of the $z, \bar{z}$ coordinates; one finds that

$$\begin{aligned} \omega_{\bar{z}z} &= -\omega_{z\bar{z}} = 2bi, \\ \omega_{zz} &= -\omega_{\bar{z}\bar{z}} = 0. \end{aligned} \tag{2.200}$$

Thus, ω is indeed a (1,1)-form, and along with the G-invariant complex structure given by z, defines a Kähler structure on W_b. We can therefore restrict attention to the space $\mathcal{H}_b$ of holomorphic sections of $\mathcal{L}_b$. Since $\frac{-\omega}{2\pi}$ is the first Chern class of $\mathcal{L}_b$, $\mathcal{L}_b$ is a holomorphic line bundle over $W_b = S^2$ of degree $2b$. For $b \geq 0$ it is a simple result of the Riemann-Roch theorem that $\mathcal{L}_b$ admits exactly $2b+1$ linearly independent holomorphic sections (see for example Griffiths and Harris [38]). The $2b+1$ holomorphic sections can be represented in the vicinity of the origin $z = \bar{z} = 0$ by the holomorphic monomials, $1, z, z^2, \ldots, z^{2b}$. $\mathcal{L}_b$ also has a natural Hermitian metric, which we will discuss further in the next section. Note that when $b < 0$, the coadjoint orbit W_b is the same as the coadjoint orbit corresponding to $-b$. However, the complex structure on this space defined by (2.197) has the opposite orientation to that defined for $-b$. The holomorphic line bundle thus defined has negative degree, and thus no holomorphic sections. We will discuss the representations associated with these bundles again briefly in 2.6; for now, we simply restrict attention to the coadjoint orbits and associated line bundles defined for $b \geq 0$.

It is also important to observe that when $b = 0$, the coadjoint orbit becomes singular and the above analysis breaks down. On the level of the coadjoint orbit, what happens is that for this exceptional value of b, the stabilizing subgroup increases in size from T to the entire group $SU(2)$. Thus, in this case the coadjoint orbit is a single point. In this case constructing a line bundle, polarization, and representation over this space is trivial and leads immediately to the trivial representation. However, for the other groups which we consider in this thesis, there are analogous singular coadjoint orbits; particularly in the cases involving infinite-dimensional groups, discovering a polarization on these coadjoint orbits is a difficult and in some cases unsolved problem. Thus, we will use a slight variation on the usual coadjoint orbit method to construct representations for such singular values of the dual space variable.

The essential observation is that when $b = 0$; that is, when the usual coadjoint orbit is singular, we can simply take the usual space G/T as the base space upon which to construct a line bundle leading to the appropriate representation of G. We use the same holomorphic coordinates on $W_0 = G/T = S^2$ as above, and continue to use $i\omega$ as a curvature form where ω is defined through (2.200). Although ω is degenerate and thus no longer a symplectic form on the manifold when $b = 0$, the form is still a nonsingular G-invariant (1,1) form on S^2, and gives rise to a Kähler structure. The rest of the analysis here, from the construction of a line bundle over W_b with curvature $i\omega$ to the final local formulae for a representation of $\mathfrak{g}$ on a polynomial space, can be carried out directly on this "alternative" orbit space. However, the resulting line bundle $\mathcal{L}_b$ will now be flat in certain directions (all directions in the case of $SU(2)$). We shall see that for all the groups considered here, this general procedure gives rise to representations for exceptional dual space parameters similar in structure to the more generic coadjoint orbit representations. For the remainder of this section when we refer to a general coadjoint orbit W_b, we include the alternative orbit space $W_0 = S^2$.

We have shown that we have a Kähler polarization for $\mathcal{H}_b$. We can now proceed to apply the rest of the results of the previous section to the group $SU(2)$. The result (2.190) states that there exist operators

$$\hat{J}_a = \xi_a + f_a \tag{2.201}$$

for $a = 3, \pm$, which act on $\mathcal{H}_b$ to give a representation of $\mathfrak{g}_{\mathbb{C}}$, where the operators ξ_a and f_a are given by (2.156) and (2.183) respectively.

We will first describe the calculation of ξ_a in this case. The operators ξ_+, ξ_3 can easily be computed from the simplified versions of the vector field formula (2.157) and (2.158) respectively. Since J_+ is the only positive root for $\mathfrak{g}$, we have

$$\xi_+ = -\partial/\partial z. \tag{2.202}$$

From $\langle J_+, J_3 \rangle = 1$, we have

$$\xi_3 = -z\frac{\partial}{\partial z}. \tag{2.203}$$

For the remaining vector field ξ_-, it follows from (2.156) that

$$\xi_- = \beta_{2,2} C_-(+,+) z^2 \frac{\partial}{\partial z}. \tag{2.204}$$

Computing $C_-(+,+) = -2$, and looking up $\beta_{2,2} = -1/2$ in Table 2.1, we have

$$\xi_- = z^2 \frac{\partial}{\partial z}. \tag{2.205}$$

It is easy to verify that these vector field operators satisfy the correct commutation relations $[\xi_3, \xi_\pm] = \pm\xi_\pm$, $[\xi_+, \xi_-] = 2\xi_3$.

We can now calculate the functions f_a. The nonzero functions are f_3 and f_-; these functions are immediately given by (2.183), and are

$$\begin{aligned} f_3 &= -i\langle \mathbf{b}, J_3\rangle = b, \\ f_- &= i\langle \mathbf{b}, 2J_3\rangle z = -2bz. \end{aligned} \tag{2.206}$$

Combining these results together, we have

$$\begin{aligned} \hat{J}_3 &= -z\frac{\partial}{\partial z} + b, \\ \hat{J}_+ &= -\frac{\partial}{\partial z}, \\ \hat{J}_- &= z^2\frac{\partial}{\partial z} - 2bz. \end{aligned} \tag{2.207}$$

Again, it is easy to verify that these operators satisfy the correct commutation relations $[\hat{J}_3, \hat{J}_\pm] = \pm\hat{J}_\pm$, $[\hat{J}_+, \hat{J}_-] = 2\hat{J}_3$.

The operators (2.207) give an explicit realization of the algebra $\mathrm{su}(2)_{\mathbb{C}}$ on the space $R = \mathbb{C}[z]$ of polynomials in the variable z. The Hilbert space $\mathcal{H}_b \subset R$ consists of all polynomials in z of order $\leq 2b$, and carries an irreducible representation of the algebra. A basis for the Hilbert space $\mathcal{H}_b$ is given by the vectors

$$|m\rangle = z^m, \quad 0 \leq m \leq 2b. \tag{2.208}$$

In terms of this basis, the operators $\hat{J}_a$ act according to

$$\begin{aligned} \hat{J}_3|m\rangle &= (b-m)|m\rangle \\ \hat{J}_+|m\rangle &= -m|m-1\rangle \\ \hat{J}_-|m\rangle &= (m-2b)|m+1\rangle. \end{aligned} \tag{2.209}$$

Note that the state $|0\rangle = |\,\rangle$ is annihilated by $\hat{J}_+$ and has weight b with respect to $\hat{J}_3$, and the state $|2b\rangle$ is annihilated by $\hat{J}_-$; all of these conditions are necessary in order for $\mathcal{H}_b$ to form an irreducible representation of the algebra. In Section 2.6 we will describe in more detail the condition of global holomorphicity on sections and the existence of a Hermitian metric on this Hilbert space; in the following chapters we will also discuss the general module structure of these representations.

2.5.2 $SU(3)$

We now consider the case $G = SU(3)$. The Lie algebra $\mathfrak{g} = \mathrm{su}(3)$ is an 8-dimensional vector space spanned by the generators $iJ_a = i\lambda_a/2, a \in \{1, 2, \ldots, 8\}$ defined through (2.23). As in the case of $SU(2)$, we can choose a set of coordinates b_k in the dual space $\mathfrak{g}^*$ such that the vector

$$\mathbf{b} = (b_1, \ldots, b_8) \in \mathfrak{g}^* \tag{2.210}$$

has coordinates

$$b_k = \langle \mathbf{b}, iJ_k \rangle. \tag{2.211}$$

We now choose a canonical element $\mathbf{b} \in \mathfrak{g}^*$ with

$$\begin{aligned} b_3 &= -p/2 \\ b_8 &= -\frac{1}{\sqrt{3}}(q + p/2) \\ b_k &= 0, \ \ k \notin \{3, 8\}. \end{aligned} \tag{2.212}$$

In this subsection, we will denote the generators of $\mathfrak{g}$ by

$$\begin{aligned} J_{+\alpha} &= e_\alpha, \\ J_{-\alpha} &= f_\alpha, \\ J_\alpha &= \frac{1}{2} h_\alpha, \end{aligned} \tag{2.213}$$

where $\alpha \in \{t, u, v\}$ (note that $J_v = J_t + J_u$). With this notation, the nonzero structure constants of the algebra are given by

$$\begin{aligned} i f_{\pm t \pm u}{}^{\pm v} &= -i f_{\pm u \pm t}{}^{\pm v} = \pm 1 \\ i f_{\pm \alpha \mp \alpha}{}^{\alpha} &= \pm 2 \\ i f_{\alpha \pm \beta}{}^{\pm \beta} &= -i f_{\pm \beta \alpha}{}^{\pm \beta} = \pm \langle \beta, J_\alpha \rangle \\ i f_{\pm u \mp v}{}^{\mp t} &= -i f_{\mp v \pm u}{}^{\mp t} = \pm 1 \\ i f_{\pm t \mp v}{}^{\mp u} &= -i f_{\mp v \pm t}{}^{\mp u} = \mp 1. \end{aligned} \tag{2.214}$$

For generic values of p, q the only generators which stabilize $\mathbf{b}$ under the coadjoint action are J_u and J_t. In this case, the (continuous) stabilizing subgroup is the maximal subtorus T of $SU(3)$, and the coadjoint orbit space is equivalent to the homogeneous space $SU(3)/T$. There are exceptional values of the parameters p and q (for example, $p = 0$ or $q = 0$) for

which the stabilizing subgroup of $\mathbf{b}$ is larger than T. In this case, the actual coadjoint orbit space becomes a smaller quotient space of $SU(3)$. As in the case of an exceptional orbit for $SU(2)$ described above, rather than dealing directly with these smaller coadjoint orbit spaces, we use the alternative coadjoint orbit space $W_b = SU(3)/T$ for all values of p, q and carry out the analysis using the consequent degenerate symplectic form ω.

We will use the local complex coordinates t, u, v on $W_b = G/T$ defined by

$$\exp\left[tJ_{+t} + uJ_{+u} + vJ_{+v}\right] \exp\left[\sum_{\alpha\in\Phi_+} a_\alpha J_{-\alpha}\right] \exp\left[\sum_{\alpha\in\Delta} \beta_\alpha J_\alpha\right] \in G, \tag{2.215}$$

where a_α and β_α are functions of $t, u, v, \bar{t}, \bar{u}, \bar{v}$, with β_α real. In terms of these coordinates, it is easy to calculate that ω is a (1,1) form at the point $\mathbf{b}$, with nonzero components

$$\begin{aligned} \omega_{t\bar{t}} &= \langle \mathbf{b}, [J_{+t}, -J_{-t}] \rangle = -ip \\ \omega_{u\bar{u}} &= \langle \mathbf{b}, [J_{+u}, -J_{-u}] \rangle = -iq \\ \omega_{v\bar{v}} &= \langle \mathbf{b}, [J_{+v}, -J_{-v}] \rangle = -i(p+q). \end{aligned} \tag{2.216}$$

Again, note that in the exceptional cases when $p = 0$ or $q = 0$, this form is degenerate. Because the complex structure and ω are both G-invariant, it follows that ω is a (1,1) form everywhere on W_b, and thus that we can choose a Kähler polarization when $\omega/2\pi$ is an integral cohomology class. By using the analysis for the $SU(2)$ case in the previous subsection, necessary conditions for the integrality condition on ω can be derived. Essentially, these conditions follow from taking all subgroups S of G which are equivalent to $SU(2)$, and which have the property that the stabilizer T of $\mathbf{b}$ in S is a maximal subtorus (homeomorphic to S^1) of S. In fact, because we are taking the orbit space to be $SU(3)/T$ regardless of the size of the stabilizer of $\mathbf{b}$, we can also take $SU(2)$ subgroups which completely stabilize $\mathbf{b}$. For each $SU(2)$ subgroup containing at least a T stabilizer, the coadjoint orbit contains a (possibly trivial) second homology class S/T. The integral of ω over this homology class must be an integral multiple of 2π; by contracting $\mathbf{b}$ with the generator of T this implies an integrality condition analogous to that derived for $SU(2)$. For the specific group $SU(3)$ with which we are currently concerned, these necessary conditions for integrality can be derived by taking the two $SU(2)$ subgroups generated by the algebra elements $\{iJ_t, iJ_1, iJ_2\}$ and $\{iJ_u, iJ_6, iJ_7\}$, with stabilizing subgroups generated by iJ_t and iJ_u. The consequent integrality condition is that p and q must be integral. It is possible to show that this condition is also sufficient for $\omega/2\pi$ to be an integral cohomology class. For the rest of this subsection,

we restrict to coadjoint orbits which satisfy this integrality condition. Just as we restricted to $b \geq 0$ for $SU(2)$ to guarantee that the line bundle $\mathcal{L}_b$ was of nonnegative degree, we will also restrict here to line bundles with $p, q \geq 0$. Negative values for these parameters again correspond to a different choice of complex structure on the same coadjoint orbit, as was the case for $SU(2)$; the resulting line bundles admit no globally holomorphic sections.

We will now proceed to construct the operators

$$\hat{J}_a = \xi_a + f_a. \tag{2.217}$$

As in the previous subsection, we begin by calculating the vector fields ξ_a. From (2.158), we can calculate

$$\begin{aligned} \xi_t &= -t\frac{\partial}{\partial t} - \frac{1}{2}v\frac{\partial}{\partial v} + \frac{1}{2}u\frac{\partial}{\partial u} \\ \xi_u &= -u\frac{\partial}{\partial u} - \frac{1}{2}v\frac{\partial}{\partial v} + \frac{1}{2}t\frac{\partial}{\partial t}. \end{aligned} \tag{2.218}$$

From (2.157), we calculate

$$\begin{aligned} \xi_{+t} &= -\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial v} \\ \xi_{+u} &= -\frac{\partial}{\partial u} + \frac{1}{2}t\frac{\partial}{\partial v} \\ \xi_{+v} &= -\frac{\partial}{\partial v}. \end{aligned} \tag{2.219}$$

Finally, from the general equation (2.156), we calculate the remaining vector fields

$$\begin{aligned} \xi_{-t} &= t^2\frac{\partial}{\partial t} + (-v - \frac{1}{2}tu)\frac{\partial}{\partial u} + (\frac{1}{2}tv - \frac{1}{4}t^2u)\frac{\partial}{\partial v} \\ \xi_{-u} &= (v - \frac{1}{2}tu)\frac{\partial}{\partial t} + u^2\frac{\partial}{\partial u} + (\frac{1}{2}uv + \frac{1}{4}tu^2)\frac{\partial}{\partial v} \\ \xi_{-v} &= (tv - \frac{1}{2}t^2u)\frac{\partial}{\partial t} + (uv + \frac{1}{2}tu^2)\frac{\partial}{\partial u} + (v^2 + \frac{1}{4}t^2u^2)\frac{\partial}{\partial v}. \end{aligned} \tag{2.220}$$

It can be verified that these vector fields satisfy the commutation relations $[\xi_a, \xi_b] = if_{ab}{}^c\xi_c$, with the structure constants (2.214).

We can now proceed to calculate the functions f_a from (2.183). The nonzero functions are f_t, f_u, and $f_{-\alpha}$, $\alpha \in \{t, u, v\}$. The functions associated with the Cartan subalgebra are given by

$$\begin{aligned} f_t &= -i\langle \mathbf{b}, J_t\rangle = p/2 \\ f_u &= -i\langle \mathbf{b}, J_u\rangle = q/2. \end{aligned} \tag{2.221}$$

The functions associated with the simple roots are given by

$$\begin{aligned} f_{-t} &= i\langle \mathbf{b}, 2J_t\rangle t = -pt \\ f_{-u} &= i\langle \mathbf{b}, 2J_u\rangle u = -qu. \end{aligned} \tag{2.222}$$

We can compute the final function f_{-v} from the recursive definition in (2.183), which gives

$$f_{-v} = [\xi_{-u} f_{-t} - \xi_{-t} f_{-u}] = -v(p+q) + \frac{1}{2}tu(p-q). \tag{2.223}$$

Again, it is straightforward to verify that $\xi_a f_b - \xi_b f_a = i f_{ab}{}^c f_c$.

We can now write the expressions for the complete operators $\hat{J}_a$ as first-order differential operators on the space $R = \mathbb{C}[t, u, v]$ of polynomials in the variables t, u, v.

$$\begin{aligned}
\hat{J}_t &= -t\frac{\partial}{\partial t} - \frac{1}{2}v\frac{\partial}{\partial v} + \frac{1}{2}u\frac{\partial}{\partial u} + p/2 \\
\hat{J}_u &= -u\frac{\partial}{\partial u} - \frac{1}{2}v\frac{\partial}{\partial v} + \frac{1}{2}t\frac{\partial}{\partial t} + q/2. \\
\hat{J}_{+t} &= -\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial v} \\
\hat{J}_{+u} &= -\frac{\partial}{\partial u} + \frac{1}{2}t\frac{\partial}{\partial v} \\
\hat{J}_{+v} &= -\frac{\partial}{\partial v}. \\
\hat{J}_{-t} &= t^2\frac{\partial}{\partial t} + (-v - \frac{1}{2}tu)\frac{\partial}{\partial u} + (\frac{1}{2}tv - \frac{1}{4}t^2u)\frac{\partial}{\partial v} - pt \\
\hat{J}_{-u} &= (v - \frac{1}{2}tu)\frac{\partial}{\partial t} + u^2\frac{\partial}{\partial u} + (\frac{1}{2}uv + \frac{1}{4}tu^2)\frac{\partial}{\partial v} - qu \\
\hat{J}_{-v} &= (tv - \frac{1}{2}t^2u)\frac{\partial}{\partial t} + (uv + \frac{1}{2}tu^2)\frac{\partial}{\partial u} + (v^2 + \frac{1}{4}t^2u^2)\frac{\partial}{\partial v} - v(p+q) + \frac{1}{2}tu(p-q).
\end{aligned} \tag{2.224}$$

We will discuss the module structure of these representations in later sections. For now, we consider a simple example of an $SU(3)$ coadjoint orbit representation, when $p = q = 1$. This choice of values for (p, q) corresponds to the adjoint representation of $SU(3)$. The irreducible representation of $\mathrm{su}(3)_{\mathbb{C}}$ on R which contains the highest weight vector $|\rangle$ is a representation on an 8-dimensional subspace $\mathcal{H}_b$ of R. A basis for $\mathcal{H}_b$ is graphed in Figure 2.5, according to the weights of the states. Note that there are two linearly independent basis states with weight (0,0). It is a simple calculation to verify that the operators (2.224) act on the basis of $\mathcal{H}_b$ according to the usual description of the adjoint representation. Note that we have not normalized this basis for $\mathcal{H}_b$; a Hermitian structure for this Hilbert space is discussed in Section 2.6.

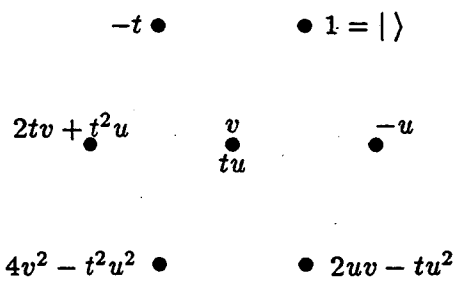


Figure 2.5: Polynomial basis for adjoint representation of $SU(3)$.

2.5.3 $\widehat{L}SU(2)$

We now consider the example of the centrally extended loop group $\widehat{L}SU(2)$. We denote the Fourier generators of this group by $J_{a(n)}$ where $a \in \{3, \pm\}$ and $n \in \mathbb{Z}$; the central generator will be denoted by $J_c = C$ for uniformity of notation. We will also abbreviate $J_{3(n)}$ to simply $J_{(n)}$ The dual space $L(\mathfrak{g}^*) \oplus \mathbb{R}$ to the Lie algebra of a general loop group was described in 2.1.2. In this case, an element $(b, -it)$ of the dual space is defined by a function

$$b : S^1 \to \mathrm{su}(2)^* \tag{2.225}$$

and a real number $t \in \mathbb{R}$. We will concern ourselves here only with the elements of the dual space where b is a smooth function. The dual pairing between the algebra $\widehat{L}\mathfrak{g}$ and the dual space is given by (2.52), and the coadjoint action of $\widehat{L}G$ on the dual space is given by (2.53). A fairly straightforward analysis of the coadjoint orbits of a general loop group leads to the result [53] that the coadjoint orbits are in 1-1 correspondence with pairs $(c, -ik)$ where c is a conjugacy class in G and $k \in \mathbb{R}$. In the case we are concerned with here, namely $\widehat{L}SU(2)$, we can choose a representative of each coadjoint orbit to be described by a pair $(\mathbf{b}, -ik)$ with $\mathbf{b}$ being a constant function taking a value in $\mathfrak{g}^*$ with coordinates $(0, 0, -b)$, where we use the same coordinates b_j for $\mathfrak{g}^*$ as in Subsection 2.5.1. We will take the dual elements of this type to be the canonical elements of the coadjoint orbits. We denote a fixed coadjoint orbit of this type by $W_{b,k}$.

From the expression (2.55) for the coadjoint action of the algebra, we can calculate the change in $(\mathbf{b}, -ik)$ under the action of the generator $J_{a(n)}$. Calculating the components of this shift, we have

$$\langle \mathrm{ad}_{J_{a(n)}}(\mathbf{b}, -ik), (J_{c(m)}, 0)\rangle = \delta_{n,-m}\left[ib(if_{ac}{}^3) + inkg_{ac}\right]. \tag{2.226}$$

From this equation, it is straightforward to verify that when $2b/k \notin \mathbb{Z}$, the stabilizer of $(\mathbf{b}, -ik)$ under the coadjoint action of $\widehat{L}SU(2)$ is generated by the elements $J_{(0)}$ and C, and is given by $T_{(0)} \times S^1$. Thus, the coadjoint orbits associated with these dual space elements are homeomorphic to $\widehat{L}G/(T \times S^1)$. For the exceptional values of b, k where $2b = nk$ for some $n \in \mathbb{Z}$, the coadjoint orbit spaces are smaller. As in the previous subsections, we simply take $\widehat{L}G/(T \times S^1)$ to be the alternative orbit space in these exceptional cases and carry out the analysis in a uniform fashion.

As mentioned in Section 2.1.2, the expression (2.20) can be used to construct a holomorphic coordinate system on $W_{b,k} = \widehat{L}G/(T \times S^1)$, with an infinite set of coordinates $\{z_{(n,\alpha)} | (n, \alpha) \succ 0\}$. Explicitly, we will use the coordinates $z_{a(n)}$ where $a \in \{\pm, 3\}$, $n \in \mathbb{Z}$ and either $n > 0$ or $(n, \alpha) = (0, +)$. As with the generators, we will write $z_{(n)} = z_{3(n)}$.

In these coordinates, we can proceed to calculate the components of the symplectic form ω. We find that ω is a (1,1) form at the point $(\mathbf{b}, -ik)$, with nonzero components

$$\begin{aligned} \omega_{z_{a(n)}\bar{z}_{a(n)}} &= \langle (\mathbf{b}, -ik), [J_{a(n)}, -J_{-a(-n)}] \rangle \\ &= -ib(if_{a(-a)}{}^3) + ikng_{a(-a)}. \end{aligned} \tag{2.227}$$

In particular, we have

$$\begin{aligned} \omega_{z_{\pm(n)}\bar{z}_{\pm(n)}} &= \mp 2ib + ink \\ \omega_{z_{(n)}\bar{z}_{(n)}} &= ink/2. \end{aligned} \tag{2.228}$$

This symplectic form (degenerate for exceptional values of b, k) is a G-invariant (1,1) form on $W_{b,k}$; thus, as usual, we can choose a Kähler polarization when $\omega/2\pi$ is an integral cohomology class. As in the previous subsection, we can use $SU(2)$ subgroups of $\widehat{L}SU(2)$ to give necessary conditions on b and k for the integrality condition to be satisfied. This analysis is parallel to that carried out in Section 2.1.2 to determine the integrality condition for a central extension of LG. Clearly, the $SU(2)$ subgroup given by the constant loops $SU(2)_{(0)}$ is a subgroup of $\widehat{L}SU(2)$ with a subgroup stabilizing $(\mathbf{b}, -ik)$ given by $T_{(0)}$. The generators of this subgroup are the generators I_a given by (2.40). The integrality condition associated with this subgroup is that

$$\langle (\mathbf{b}, -ik), iI_3 \rangle = -b \in \mathbb{Z}/2. \tag{2.229}$$

So we have the condition that $2b \in \mathbb{Z}$. Similarly, we find that the subalgebra (2.41) also generates an $SU(2)$ subgroup with stabilizing subgroup T. The real generators of this

subgroup are given by the functions

$$\begin{aligned}\tilde{I}_1(\theta) &= \frac{1}{2}(\tilde{I}_+(\theta)+\tilde{I}_-(\theta)) = \cos\theta J_1 + \sin\theta J_2 \\ \tilde{I}_2(\theta) &= \sin\theta J_1 - \cos\theta J_2 \\ \tilde{I}_3(\theta) &= C/2 - J_3.\end{aligned} \tag{2.230}$$

The integrality condition associated with this $SU(2)$ subgroup is that

$$k - 2b \in \mathbb{Z}. \tag{2.231}$$

As previously, in order to have holomorphic sections in our line bundles, we must furthermore restrict to values where $k \geq 2b \geq 0$. Thus, for all integers k and half-integers b satisfying $k \geq 2b \geq 0$, we have a coadjoint orbit $W_{b,k}$ which admits a Kähler polarization, over which we can choose a line bundle $\mathcal{L}_b$ with curvature form $i\omega$.

We can now proceed to construct the operators

$$\hat{J}_{a(n)} = \xi_{a(n)} + f_{a(n)} \tag{2.232}$$

and

$$\hat{C} = \hat{J}_c = \xi_c + f_c \tag{2.233}$$

These operators act on the space R of polynomials in the infinite set of variables $z_{a(n)}$. This space is graded according to $\mathbb{Z}\oplus\mathbb{Z}$, with the variables $z_{\pm(n)}$ having degrees $(n,\pm 1)$ and $z_{(n)}$ having degree $(n,0)$. From (2.158) we can calculate the vector fields corresponding to the generators in the stabilizer

$$\begin{aligned}\xi_{(0)} &= \sum_{n\geq 0} -z_{+(n)}\frac{\partial}{\partial z_{+(n)}} + \sum_{n>0} z_{-(n)}\frac{\partial}{\partial z_{-(n)}} \\ \xi_c &= 0.\end{aligned} \tag{2.234}$$

The vector field $\xi_{(0)}$ contains an infinite number of terms, but its action on any fixed polynomial in R only involves a finite number of these terms. This is a characteristic property of the generators in the polynomial coadjoint orbit representations for infinite-dimensional algebras. The vector field ξ_c is 0 because C is central in the algebra.

We can now proceed to calculate the vector fields $\xi_{a(n)}$ for $(n,a) \succ 0$ using (2.157), and the remaining vector fields using (2.156). It is difficult to write down these vector fields in closed-form notation; again, there are an infinite number of terms in each vector field,

only a finite number of which contribute to the action on a specific polynomial in R. The leading terms in the vector field operators $\xi_{a(n)}$ corresponding to positive roots are

$$\begin{aligned}
\xi_{+(n)} &= -\frac{\partial}{\partial z_{+(n)}} + \frac{1}{2}\sum_{m>0} z_{(m)}\frac{\partial}{\partial z_{+(m+n)}} - \sum_{m>0} z_{-(m)}\frac{\partial}{\partial z_{(m+n)}} + \cdots \quad (n \geq 0) \\
\xi_{-(n)} &= -\frac{\partial}{\partial z_{-(n)}} - \frac{1}{2}\sum_{m>0} z_{(m)}\frac{\partial}{\partial z_{-(m+n)}} - \sum_{m\geq 0} z_{+(m)}\frac{\partial}{\partial z_{(m+n)}} + \cdots \quad (n > 0) \\
\xi_{(n)} &= -\frac{\partial}{\partial z_{(n)}} + \frac{1}{2}\sum_{m>0} z_{-(m)}\frac{\partial}{\partial z_{-(m+n)}} - \sum_{m\geq 0} z_{+(m)}\frac{\partial}{\partial z_{+(m+n)}} + \cdots \quad (n > 0).
\end{aligned} \tag{2.235}$$

We can calculate the functions $f_{a(n)}$ using (2.183). The functions associated with the generators in the stabilizer are given by

$$\begin{aligned}
f_{(0)} &= b, \\
f_c &= k.
\end{aligned} \tag{2.236}$$

In general, the functions $f_{\pm(n)}$ are of degree $(-n, \mp 1)$, and the function $f_{(n)}$ is of degree $(-n, 0)$. The simple roots of $\widehat{L}SU(2)$ are associated with the generators $J_{+(0)}$ and $J_{-(1)}$. The associated functions are

$$\begin{aligned}
f_{-(0)} &= -2bz_{+(0)} \\
f_{+(-1)} &= (2b-k)z_{-(1)}.
\end{aligned} \tag{2.237}$$

All the remaining functions $f_{a(n)}$ can be calculated using the recursive equation (2.183). Explicitly, we have

$$\begin{aligned}
f_{+(-n-1)} &= \xi_{(-1)}f_{+(-n)} - \xi_{+(-n)}f_{(-1)} \quad (n > 0) \\
f_{-(-n)} &= \xi_{-(0)}f_{(-n)} - \xi_{(-n)}f_{-(0)} \quad (n > 0) \\
f_{(-n)} &= \frac{1}{2}\left[\xi_{+(-n)}f_{-(0)} - \xi_{-(0)}f_{+(-n)}\right] \quad (n > 0).
\end{aligned} \tag{2.238}$$

We can combine these functions with the vector fields described above to get the complete set of operators $\hat{J}_{a(n)}$, $\hat{C}$ which realize the algebra $\widehat{L}\mathrm{su}(2)$ on the space R of polynomials. Explicitly writing all the terms in these operators which have nonzero matrix elements between states of degrees with $n \leq 2$, we have

$$\begin{aligned}
\hat{C} &= k \\
\hat{J}_{-(2)} &= -\frac{\partial}{\partial z_{-(2)}} + z_{+(0)}\frac{\partial}{\partial z_{(2)}} + \frac{1}{6}z_{+(0)}^2\frac{\partial}{\partial z_{+(2)}} + \mathcal{D}_3
\end{aligned}$$

$$
\begin{aligned}
\hat{J}_{(2)} &= -\frac{\partial}{\partial z_{(2)}} - \frac{1}{2} z_{+(0)} \frac{\partial}{\partial z_{+(2)}} + \mathcal{D}_3 \\
\hat{J}_{+(2)} &= -\frac{\partial}{\partial z_{+(2)}} + \mathcal{D}_3 \\
\hat{J}_{-(1)} &= -\frac{\partial}{\partial z_{-(1)}} + z_{+(0)} \frac{\partial}{\partial z_{(1)}} + \frac{1}{6} z_{+(0)}^2 \frac{\partial}{\partial z_{+(1)}} - \left(\frac{1}{6} z_{+(0)} z_{-(1)} + \frac{1}{2} z_{(1)}\right) \frac{\partial}{\partial z_{-(2)}} \\
&\quad + \left(\frac{1}{6} z_{+(0)} z_{(1)} + z_{+(1)}\right) \frac{\partial}{\partial z_{(2)}} + \left(\frac{1}{3} z_{+(0)} z_{+(1)} - \frac{1}{90} z_{+(0)}^3 z_{-(1)}\right) \frac{\partial}{\partial z_{+(2)}} + \mathcal{D}_3 \\
\hat{J}_{(1)} &= -\frac{\partial}{\partial z_{(1)}} - \frac{1}{2} z_{+(0)} \frac{\partial}{\partial z_{+(1)}} + \frac{1}{2} z_{-(1)} \frac{\partial}{\partial z_{-(2)}} - \frac{1}{3} z_{+(0)} z_{-(1)} \frac{\partial}{\partial z_{(2)}} \\
&\quad + \left(\frac{1}{12} z_{+(0)} z_{(1)} - \frac{1}{2} z_{+(1)}\right) \frac{\partial}{\partial z_{+(2)}} + \mathcal{D}_3 \\
\hat{J}_{+(1)} &= -\frac{\partial}{\partial z_{+(1)}} - z_{-(1)} \frac{\partial}{\partial z_{(2)}} + \left(\frac{1}{2} z_{(1)} - \frac{1}{6} z_{+(0)} z_{-(1)}\right) \frac{\partial}{\partial z_{+(2)}} + \mathcal{D}_3 \\
\hat{J}_{-(0)} &= -2b z_{+(0)} + z_{+(0)}^2 \frac{\partial}{\partial z_{+(0)}} - \left(z_{+(0)} z_{-(1)} + z_{(1)}\right) \frac{\partial}{\partial z_{-(1)}} \\
&\quad + (2 z_{+(1)} - \frac{1}{3} 2 z_{+(0)}^2 z_{-(1)}) \frac{\partial}{\partial z_{(1)}} + \left(\frac{1}{6} z_{+(0)}^2 z_{(1)} + z_{+(0)} z_{+(1)}\right) \frac{\partial}{\partial z_{+(1)}} \\
&\quad + \left(\frac{1}{6} z_{+(0)} z_{-(1)} z_{(1)} - z_{+(0)} z_{-(2)} - z_{(2)}\right) \frac{\partial}{\partial z_{-(2)}} \\
&\quad + \left(\frac{2}{45} z_{+(0)}^3 z_{-(1)}^2 - \frac{2}{3} z_{+(0)} z_{-(1)} z_{+(1)} - \frac{2}{3} z_{+(0)}^2 z_{-(2)} + 2 z_{+(2)}\right) \frac{\partial}{\partial z_{(2)}} \\
&\quad + \left(\frac{1}{6} z_{+(0)} z_{(1)} z_{+(1)} - \frac{1}{90} z_{+(0)}^3 z_{-(1)} z_{(1)} + \frac{1}{6} z_{+(0)}^2 z_{(2)} + z_{+(0)} z_{+(2)}\right) \frac{\partial}{\partial z_{+(2)}} + \mathcal{D}_3 \\
\hat{J}_{(0)} &= b - z_{+(0)} \frac{\partial}{\partial z_{+(0)}} + z_{-(1)} \frac{\partial}{\partial z_{-(1)}} \\
&\quad - z_{+(1)} \frac{\partial}{\partial z_{+(1)}} + z_{-(2)} \frac{\partial}{\partial z_{-(2)}} - z_{+(2)} \frac{\partial}{\partial z_{+(2)}} + \mathcal{D}_3 \qquad (2.239) \\
\hat{J}_{+(0)} &= -\frac{\partial}{\partial z_{+(0)}} - z_{-(1)} \frac{\partial}{\partial z_{(1)}} + \left(\frac{1}{2} z_{(1)} - \frac{1}{6} z_{+(0)} z_{-(1)}\right) \frac{\partial}{\partial z_{+(1)}} \\
&\quad + \frac{1}{6} z_{-(1)}^2 \frac{\partial}{\partial z_{-(2)}} + \left(\frac{1}{6} z_{-(1)} z_{(1)} - z_{-(2)}\right) \frac{\partial}{\partial z_{(2)}} \\
&\quad + \left(\frac{1}{90} z_{+(0)}^2 z_{-(1)}^2 - \frac{1}{12} z_{(1)}^2 - \frac{1}{6} z_{-(1)} z_{+(1)} - \frac{1}{6} z_{+(0)} z_{-(2)} + \frac{1}{2} z_{(2)}\right) \frac{\partial}{\partial z_{+(2)}} + \mathcal{D}_3 \\
\hat{J}_{-(-1)} &= \frac{k - 4b}{3} z_{+(0)}^2 z_{-(1)} + \frac{k - 2b}{2} z_{+(0)} z_{(1)} - (k + 2b) z_{+(1)} \\
&\quad + \left(\frac{1}{3} z_{+(0)}^3 z_{-(1)} + 2 z_{+(0)} z_{+(1)}\right) \frac{\partial}{\partial z_{+(0)}} \\
&\quad - \left(\frac{1}{3} z_{+(0)}^2 z_{-(1)}^2 + \frac{2}{3} z_{+(0)} z_{-(1)} z_{(1)}\right. \\
&\qquad \left. + \frac{1}{2} z_{(1)}^2 + z_{-(1)} z_{+(1)} + z_{+(0)} z_{-(2)} + z_{(2)}\right) \frac{\partial}{\partial z_{-(1)}}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(2z_{+(2)}-\frac{2}{5}z^3_{+(0)}z^2_{-(1)}-\frac{1}{3}z^2_{+(0)}z_{-(1)}z_{(1)}\right.\\
&\qquad\left.-\frac{1}{6}z_{+(0)}z^2_{(1)}-\frac{1}{3}z_{+(0)}z_{-(1)}z_{+(1)}+z_{(1)}z_{+(1)}+\frac{1}{3}z^2_{+(0)}z_{-(2)}\right)\frac{\partial}{\partial z_{(1)}}\\
&+\left(\frac{1}{45}z^4_{+(0)}z^2_{-(1)}+\frac{1}{9}z^3_{+(0)}z_{-(1)}z_{(1)}+\frac{1}{6}z^2_{+(0)}z_{-(1)}z_{+(1)}\right.\\
&\qquad\left.+\frac{1}{2}z_{+(0)}z_{(1)}z_{+(1)}+z^2_{+(1)}-\frac{1}{6}z^3_{+(0)}z_{-(2)}-\frac{1}{3}z^2_{+(0)}z_{(2)}+z_{+(0)}z_{+(2)}\right)\frac{\partial}{\partial z_{+(1)}}\\
&+\mathcal{O}(z_{-(3)})
\end{aligned}
$$

$$
\begin{aligned}
\hat{J}_{(-1)} &= \frac{2b-k}{2}z_{+(0)}z_{-(1)}-\frac{k}{2}z_{(1)})+\left(\frac{1}{2}z_{+(0)}z_{(1)}-\frac{2}{3}z^2_{+(0)}z_{-(1)}-z_{+(1)}\right)\frac{\partial}{\partial z_{+(0)}}\\
&+\left(\frac{2}{3}z_{+(0)}z^2_{-(1)}+\frac{1}{2}z_{-(1)}z_{(1)}+z_{-(2)}\right)\frac{\partial}{\partial z_{-(1)}}\\
&+\left(\frac{2}{3}z^2_{+(0)}z^2_{-(1)}-z_{-(1)}z_{+(1)}-z_{+(0)}z_{-(2)}\right)\frac{\partial}{\partial z_{(1)}}\\
&+\left(\frac{1}{12}z_{+(0)}z^2_{(1)}-\frac{1}{45}z^3_{+(0)}z^2_{-(1)}-\frac{1}{6}z^2_{+(0)}z_{-(1)}z_{(1)}\right.\\
&\qquad\left.-\frac{1}{2}z_{+(0)}z_{-(1)}z_{+(1)}+\frac{1}{6}z^2_{+(0)}z_{-(2)}+\frac{1}{2}z_{+(0)}z_{(2)}-z_{+(2)}\right)\frac{\partial}{\partial z_{+(1)}}+\mathcal{O}(z_{-(3)})\\
\hat{J}_{+(-1)} &= (2b-k)z_{-(1)}+\left(z_{(1)}-z_{+(0)}z_{-(1)}\right)\frac{\partial}{\partial z_{+(0)}}+z^2_{-(1)}\frac{\partial}{\partial z_{-(1)}}\\
&+\left(\frac{2}{3}z_{+(0)}z^2_{-(1)}-2z_{-(2)}\right)\frac{\partial}{\partial z_{(1)}}\\
&+\left(z_{(2)}-\frac{1}{6}z_{+(0)}z_{-(1)}z_{(1)}-z_{-(1)}z_{+(1)}\right)\frac{\partial}{\partial z_{+(1)}}+\mathcal{O}(z_{-(3)})\\
\hat{J}_{-(-2)} &= \frac{k-4b}{15}z^3_{+(0)}z^2_{-(1)}+\frac{k-2b}{6}z^2_{+(0)}z_{-(1)}z_{(1)}+\frac{k-2b}{6}z_{+(0)}z^2_{(1)}\\
&+\frac{k-8b}{3}z_{+(0)}z_{-(1)}z_{+(1)}-bz_{(1)}z_{+(1)}+\frac{2k-4b}{3}z^2_{+(0)}z_{-(2)}\\
&+(k-b)z_{+(0)})z_{(2)}-(2k+2b)z_{+(2)}\\
&+\left(\frac{2}{45}z^4_{+(0)}z^2_{-(1)}+\frac{1}{12}z^2_{+(0)}z^2_{(1)}\right.\\
&\qquad\left.+z^2_{+(0)}z_{-(1)}z_{+(1)}+z^2_{+(1)}+\frac{1}{3}z^3_{+(0)}z_{-(2)}+2z_{+(0)}z_{+(2)}\right)\frac{\partial}{\partial z_{+(0)}}+\mathcal{O}(z_{-(3)})\\
\hat{J}_{(-2)} &= \frac{4b-k}{6}z^2_{+(0)}z^2_{-(1)}-\frac{k}{6}z_{+(0)}z_{-(1)}z_{(1)}+2bz_{-(1)}z_{+(1)}\\
&+(2b-k)z_{+(0)}z_{-(2)}-kz_{(2)}\\
&+\left(\frac{1}{6}z^2_{+(0)}z_{-(1)}z_{(1)}-\frac{2}{15}z^3_{+(0)}z^2_{-(1)}-\frac{1}{6}z_{+(0)}z^2_{(1)}-z_{+(2)}\right.\\
&\qquad\left.-\frac{4}{3}z_{+(0)}z_{-(1)}z_{+(1)}+\frac{1}{2}z_{(1)}z_{+(1)}-\frac{2}{3}z^2_{+(0)}z_{-(2)}+\frac{1}{2}z_{+(0)}z_{(2)}\right)\frac{\partial}{\partial z_{+(0)}}\\
&+\mathcal{O}(z_{-(3)})
\end{aligned}
$$

$$\begin{aligned}\hat{J}_{+(-2)} &= \frac{4b-k}{3} z_{+(0)} z_{-(1)}^2 - b z_{-(1)} z_{(1)} + (2b-2k) z_{-(2)} \\ &+ \left(\frac{2}{3} z_{+(0)} z_{-(1)} z_{(1)} - \frac{1}{3} z_{+(0)}^2 z_{-(1)}^2 \right. \\ &\left. - \frac{1}{2} z_{(1)}^2 - z_{-(1)} z_{+(1)} - z_{+(0)} z_{-(2)} + z_{(2)} \right) \frac{\partial}{\partial z_{+(0)}} + \mathcal{O}(z_{-(3)}),\end{aligned}$$

where by $\mathcal{D}_3$ we denote terms containing derivatives $\partial/\partial z_{\alpha(n)}$ with $n \geq 3$, and by $\mathcal{O}(z_{-(3)})$ we denote terms containing quasi-homogeneous polynomials of degree $(n, w) \succeq (3, -1)$.

As an example, we consider the coadjoint orbit with $(b, k) = (0, 1)$. In the representation of the algebra on R, the highest weight state $|\,\rangle$ clearly must satisfy

$$\hat{J}_{(0)}|\,\rangle = b|\,\rangle, \tag{2.240}$$

and

$$\hat{C}|\,\rangle = k|\,\rangle. \tag{2.241}$$

From this, it follows that the irreducible representation of the algebra which contains $|\,\rangle$ must be exactly the irreducible representation with $(j, k) = (0, 1)$ which was described in Section 2.1.2 and whose weights are graphed in Figure 2.4. As a check on this equivalence, we can verify that the generator $\hat{J}_{+(-1)}$ acting twice on $|\,\rangle$ gives 0. Indeed,

$$\hat{J}_{+(-1)}|\,\rangle = (2b-k) z_{-(1)} = -z_{-(1)}, \tag{2.242}$$

and

$$\hat{J}_{+(-1)}^2|\,\rangle = (2b-k) z_{-(1)}^2 + (2b-k)^2 z_{-(1)}^2 = 0. \tag{2.243}$$

We will study the structure of the coadjoint orbit loop group representations further in the following sections.

2.5.4 Virasoro group

Finally, we discuss the coadjoint orbit representations of the Virasoro group. As described in Subsection 2.1.3, the generators of $\widehat{\mathrm{Vect}} S^1{}_{\mathbb{C}}$ are L_n and C. The equation describing the coadjoint action of the Virasoro algebra $\widehat{\mathrm{Vect}} S^1$ on the dual space $\widehat{\mathrm{Vect}} S^{1*}$ is given by (2.70). By computing the stabilizer of a general dual element (b, it), it is possible to completely classify the coadjoint orbits of $\widehat{\mathrm{Diff}} S^1$ [58, 47]. A clear review of this analysis in the general case is given in [64]. We will only be concerned here with the simplest case, in which the orbit contains an element (b, ic) with $b(\theta) = b$ a constant function. We will

refer to this orbit as $W_{b,c}$. In this case, the stabilizer in $\widehat{\mathrm{Vect}}S^1$ of the point (b, ic) is given by all elements $(f, -ia)$ with $f(\theta)$ satisfying

$$\frac{c}{24\pi} f''' = 2bf'. \tag{2.244}$$

When $-48\pi\frac{b}{c}$ is not the square of an integer n, the only solution to this equation with period 2π is $f(\theta) = 1$. In this case, the stabilizer of (b, ic) is the subgroup generated by L_0 and C, so the space $W_{b,c}$ is equivalent to the space $\mathrm{Diff}S^1/S^1$. For the exceptional values of b, c, the generators $\mathrm{L}_{\pm n}$ are also stabilizers of (b, ic). Thus, the coadjoint orbits $W_{-cn^2/48\pi,c}$ are given by the spaces $\mathrm{Diff}S^1/SL^{(n)}(2,\mathbb{R})$, where $SL^{(n)}(2,\mathbb{R})$ is generated by the elements $l_0, l_{\pm n}$ in $\mathrm{Vect}S^1$. As we have done for all the other groups, we will take the alternative orbit space to be $W_{b,c} = \mathrm{Diff}S^1/S^1$ even for these exceptional values of b, c, and we will proceed with the analysis using the degenerate symplectic form ω on this space.

From now on, we will consider a fixed orbit $W_{b,c}$, of the $\mathrm{Diff}S^1/S^1$ type. We can now follow the identical procedure to that used for all the examples in the previous subsections to construct representations of $\widehat{\mathrm{Vect}}S^1$ on a polynomial space. In Subsection (2.1.3), we gave an argument justifying the use of (2.20) in constructing a holomorphic coordinate system on the quotient space $\mathrm{Diff}S^1/S^1$. Explicitly, we have the formal result that given a countable set of variables $z = \{z_1, z_2, \ldots\}$, there exist unique functions $\mu_n(z,\bar{z}), \rho(z,\bar{z}), \gamma(z,\bar{z})$, expressed as formal power series in the $z_i's$, such that $\gamma(z,\bar{z})$ is real and

$$\exp(\sum_{n>0} z_n L_n)\exp(\sum_{n>0} \mu_n(z,\bar{z})L_{-n})\exp(\rho(z,\bar{z})L_0)\exp(\gamma(z,\bar{z})C) \in \mathrm{Diff}_0 S^1. \tag{2.245}$$

The variables z_n thus form a set of holomorphic coordinates on $\mathrm{Diff}S^1/S^1$ which give a G-invariant complex structure.

We can now compute the components of the symplectic form ω with respect to this coordinate system. As usual, ω is a (1,1)-form; the nonzero components are given by

$$\omega_{\bar{z}_m, z_n} = -\omega_{z_n, \bar{z}_m} = i\delta_{m,n}(4\pi mb + \frac{c}{12}m^3). \tag{2.246}$$

Note again that when $b = -\frac{cn^2}{48\pi}$, the 2-form ω is degenerate.

We now make the observation that $\mathrm{Diff}S^1/S^1$ is a contractible space. To see this, we use the identification of $\mathrm{Diff}S^1/S^1$ with the group $\mathrm{Diff}_0 S^1$ of orientation-preserving diffeomorphisms of S^1 which fix the point 1. Viewing an element of $\mathrm{Diff}_0 S^1$ as a monotonically

increasing function $f : \mathbb{R} \to \mathbb{R}$ with the properties $f(0) = 0$ and $f(x + 2\pi) = 2\pi + f(x)$, we can explicitly give a retraction of $\mathrm{Diff}_0 S^1$ to a point by defining the one-parameter family of functions $f_t(x) = (1-t)f(x) + tx$, for each $f \in \mathrm{Diff}_0 S^1$, $t \in [0,1]$. Since $\mathrm{Diff}\, S^1/S^1$ is a contractible space, all 2-cycles are homologous to the null 2-cycle, so that $\int_a \omega = 0$ for any 2-cycle a. Thus, for any b, c, we can construct a line bundle $\mathcal{L}_{b,c}$ over $W_{b,c}$ with curvature $i\omega$. (Note that the second cohomology of $\mathrm{Diff}\, S^1/S^1$ is nontrivial if one restricts to forms invariant under $\mathrm{Diff}\, S^1$, however this should not affect the construction of $\mathcal{L}_b$; it does however imply that $\mathcal{L}_b$ will not have a global $\mathrm{Diff}\, S^1$-invariant connection.) We can use the Kähler structure defined by ω to restrict to the space $\mathcal{H}_b$ of holomorphic sections of $\mathcal{L}_{b,c}$, which can be identified with the ring $R = \mathbb{C}[z_1, z_2, \ldots]$ of polynomials in the variables z_i due to the topologically trivial nature of the orbit space (actually, for this group all power series can be considered to be holomorphic sections; see Section 2.6.1). We now proceed to construct the operators

$$\hat{L}_n = \xi_n + f_n \tag{2.247}$$

and

$$\hat{C} = \xi_c + f_c \tag{2.248}$$

as differential operators on R. For this group, the ring R has an integer grading with $\deg(z_n) = n$. As usual, the operators $\hat{L}_n$ act on the subspace R_m of degree m by

$$\hat{L}_n R_m \subset R_{m-n}. \tag{2.249}$$

From (2.158) we calculate the vector fields corresponding to the generators L_0, C;

$$\begin{aligned} \xi_0 &= \sum_{k>0} k z_k \frac{\partial}{\partial z_k}, \\ \xi_c &= 0. \end{aligned} \tag{2.250}$$

The remaining vector fields are difficult to write in closed-form; the leading terms are given by

$$\xi_n = -\frac{\partial}{\partial z_n} + \frac{1}{2}\sum_{m>0}(m-n) z_m \frac{\partial}{\partial z_{n+m}} + \ldots, \quad \text{for } n > 0, \tag{2.251}$$

and

$$\xi_{-n} = \sum_{m>n}(m+n) z_m \frac{\partial}{\partial z_{m-n}} + \ldots, \quad \text{for } n > 0. \tag{2.252}$$

Just as was the case for $\widehat{L}SU(2)$, although these operators are all expressed as an infinite sum of terms each of which is a first-order differential operator, the action of a fixed vector

field operator on a fixed polynomial in R can be computed using only a finite number of these terms.

To calculate the functions f_{-n} and f_c using (2.183), we begin as usual by calculating the functions for the stabilizing generators. Writing the quantity $2\pi b + c/24$ as

$$h = 2\pi b + c/24, \tag{2.253}$$

we have

$$\begin{aligned} f_c &= -i\langle (b, ic), C\rangle = c \\ f_0 &= -i\langle (b, ic), L_0\rangle = 2\pi b + \frac{c}{24} = h. \end{aligned} \tag{2.254}$$

For the simple root L_1, we have

$$f_{-1} = i\langle (b, ic), 2L_0\rangle z_1 = -2hz_1. \tag{2.255}$$

The quasi-simple root L_2, however, is a special case. It cannot be computed inductively, and does not follow from the equation in (2.183) for simple roots. However, we can write a pair of equations which f_{-2} must satisfy, by using the condition (2.174). In this case, that condition implies that

$$\xi_m f_n - \xi_n f_m = (m-n) f_{m+n} + \frac{c}{12}\delta_{m,-n}(m^3 - m). \tag{2.256}$$

It follows that we must have

$$\begin{aligned} \xi_2 f_{-2} &= -\frac{\partial}{\partial z_2} f_{-2} = 4f_0 + \frac{c}{2} = 4h + \frac{c}{2}, \\ \xi_1 f_{-2} &= -\frac{\partial}{\partial z_1} f_{-2} = 3f_{-1} = -6hz_1. \end{aligned} \tag{2.257}$$

Since f_{-2} is a linear combination of z_1^2 and z_2, these two equations determine both coefficients, so that

$$f_{-2} = -(4h + \frac{c}{2})z_2 + 3hz_1^2. \tag{2.258}$$

From here, we can continue the analysis according to the usual prescription, treating the quasi-simple roots L_1, L_2 as simple roots. The remaining functions f_{-n} can be calculated inductively by

$$f_{-n} = \frac{1}{n-2}(\xi_{-1} f_{1-n} - \xi_{1-n} f_{-1}), \text{ for } n > 2. \tag{2.259}$$

We now give explicit formulae for the terms in the operators $\hat{L}_n$ which have nonzero matrix elements between polynomials in R of degree ≤ 3. These terms are given by

$$\begin{aligned}
\hat{L}_3 &= -\frac{\partial}{\partial z_3} + \mathcal{D}_4, \\
\hat{L}_2 &= -\frac{\partial}{\partial z_2} - \frac{1}{2} z_1 \frac{\partial}{\partial z_3} + \mathcal{D}_4, \\
\hat{L}_1 &= -\frac{\partial}{\partial z_1} + \frac{1}{2} z_2 \frac{\partial}{\partial z_3} + \mathcal{D}_4, \\
\hat{L}_0 &= h + z_1 \frac{\partial}{\partial z_1} + 2 z_2 \frac{\partial}{\partial z_2} + 3 z_3 \frac{\partial}{\partial z_3} + \mathcal{D}_4, \\
\hat{L}_{-1} &= -2hz_1 + (3z_2 - z_1^2)\frac{\partial}{\partial z_1} + (4z_3 - 2z_1 z_2)\frac{\partial}{\partial z_2} + \mathcal{O}(z_4), \\
\hat{L}_{-2} &= -(4h + \frac{c}{2})z_2 + 3hz_1^2 + (5z_3 - \frac{13}{2} z_1 z_2 + z_1^3)\frac{\partial}{\partial z_1} + \mathcal{O}(z_4), \\
\hat{L}_{-3} &= -(6h + 2c)z_3 + (13h + c)z_1 z_2 - 4hz_1^3 + \mathcal{O}(z_4),
\end{aligned} \tag{2.260}$$

where $\mathcal{D}_4$ denotes terms containing derivatives $\partial/\partial z_k$ with $k \geq 4$, and $\mathcal{O}(z_4)$ denotes terms containing quasi-homogeneous polynomials of degree at least 4.

We will discuss in more detail the structure of these representations in later sections.

2.6 Global properties

For most of the rest of this thesis, we will concentrate on properties of the coadjoint orbit representations which can be described in terms of local algebraic properties of the representations. These properties are completely captured by the explicit formulae for operators on spaces of polynomials described in the previous sections of this chapter. In this section, however, we will consider some global properties of the coadjoint orbit representations. We will discuss these properties from the point of view of the local algebraic representations, and will give results in a form which can be represented in terms of local algebraic formulae. In Subsection 2.6.1 we consider the question of which polynomials (or power series) in the local holomorphic variables z_α can be extended to global holomorphic sections of the line bundle $\mathcal{L}_b$. In Subsection 2.6.2 we discuss the question of whether the holomorphic line bundle $\mathcal{L}_b$ admits a Hermitian structure, and the related question of whether the representation of the group G on the space $\mathcal{H}_b$ of holomorphic sections is a unitary representation.

2.6.1 Global holomorphicity

In the previous section, we derived formulae which described for various groups G an action of the Lie algebra $\mathfrak{g}$ on the ring of polynomials in the variables z_α corresponding to positive roots of G. We argued that locally, this ring corresponds to the space of holomorphic sections of the line bundle $\mathcal{L}_b$ over the coadjoint orbit space W_b. In this description, we have ignored the question of which of these polynomials actually corresponds to a globally holomorphic section of $\mathcal{L}_b$. In this subsection, we describe which of the polynomials in R actually extends to a global holomorphic section. We begin with the simple case of $SU(2)$, where the coadjoint orbit space is simply S^2 with the usual single complex coordinate z. We showed in Subsection 2.5.1 that the line bundle $\mathcal{L}_b$ corresponding to any half-integral value of b was precisely the unique holomorphic line bundle over S^2 of degree $2b$. When $b \geq 0$, this bundle admits precisely $2b+1$ holomorphic sections, which correspond to the local functions $1, z, \ldots, z^{2b}$ in R. In order to derive analogous constraints on holomorphic functions for other groups, we will find it useful to see how this result can be arrived at purely by local considerations.

In order to analyze which functions of z are holomorphic globally, we must begin by choosing a second coordinate chart on S^2 which covers the point $z = \infty$. The standard choice is the chart $w \in \mathbb{C}$ with the transition homomorphism $w = 1/z$. In terms of these variables, the partial derivative operators $\partial/\partial z, \partial/\partial w$ are related by

$$\frac{\partial}{\partial z} = -w^2 \frac{\partial}{\partial w}. \tag{2.261}$$

Given a line bundle $\mathcal{L}$ over S^2 with a connection described by local gauge choices as A_z, A_w in the two coordinate charts, the connections are similarly related by

$$A_z = -w^2 A_w. \tag{2.262}$$

For the gauge choice used for the local formulae (2.207), the connection terms $A_a, a = 3, \pm$ can be written in terms of A_z, and are given by

$$\begin{aligned} A_3 &= -zA_z \\ A_+ &= -A_z \\ A_- &= z^2 A_z. \end{aligned} \tag{2.263}$$

The connection $A_a = -f_a + i\Phi_a$ can be explicitly calculated by evaluating

$$\Phi_a(z) = -\langle \mathrm{Ad}^*_{g(z,\psi)}\mathbf{b}, J_a\rangle \tag{2.264}$$

$$= -\langle \mathrm{b}, e^{-a(z,\bar{z})J_-} e^{-zJ_+} J_a e^{zJ_+} e^{a(z,\bar{z})J_-} \rangle.$$

One finds that

$$\begin{aligned} A_3 &= 2bza(z,\bar{z}), \\ A_+ &= 2ba(z,\bar{z}), \\ A_- &= -2bz^2 a(z,\bar{z}). \end{aligned} \tag{2.265}$$

Thus, we have

$$A_z = \frac{-2b\bar{z}}{1+|z|^2}, \tag{2.266}$$

and therefore in this gauge,

$$A_w = \frac{2bz|z|^2}{1+|z|^2} = \frac{2b}{w(1+|w|^2)}. \tag{2.267}$$

In the coordinate space w, we must find a choice of gauge where this connection is nonsingular. If we perform the gauge transformation on sections given by

$$\phi \to w^n \phi, \tag{2.268}$$

then $A_w \to A_w - \frac{n}{w}$. Thus, when we choose $n = 2b$, we have

$$A_w = \frac{2b}{w}\left(\frac{1}{1+|w|^2} - 1\right) = \frac{-2b\bar{w}}{1+|w|^2}, \tag{2.269}$$

which is not only nonsingular, but is of precisely the form of (2.266). Under the gauge transformation (2.268) with $n = 2b$, the monomial z^m is described in the w chart by the monomial

$$z^m \to z^{m-2b} = w^{2b-m}. \tag{2.270}$$

It follows that for this choice of gauge, the functions in $R = \mathbb{C}[z]$ which extend to globally holomorphic sections of $\mathcal{L}_b$ are precisely the functions z^m with $m \leq 2b$, just as stated above. For $b < 0$, this indicates that there are no holomorphic sections and that the associated representation of G is 0-dimensional. For $b = 0$, this result states that only the constant function is a global holomorphic section. This is exactly the same result we would have gotten had we taken the correct pointlike coadjoint orbit space in this case, rather than using the alternative orbit space S^2 with a flat bundle.

The result we have derived for $SU(2)$ can be restated in the following fashion: The only polynomials in R which can extend to globally holomorphic sections of $\mathcal{L}_b$ are those

which lie in the orbit of the unique highest weight state $|\,\rangle$. (For those coadjoint orbits for which $|\,\rangle$ is not globally holomorphic, there are no globally holomorphic sections.) This follows for $SU(2)$ because for a given choice of b, the monomial z^{2b} is annihilated by the operator $\hat{J}_-$. We claim that in fact, for all compact simple groups G this formulation of the global holomorphicity condition is correct not only for the group G but also for $\widehat{L}G$. An immediate consequence of this result is that for this set of groups, the representation on the space of holomorphic sections is an irreducible representation.

We can prove the general condition for any G by using the above analysis for $SU(2)$ to give the necessary conditions for local functions in R to extend to holomorphic sections of the line bundle over S^2 given by the restriction of $\mathcal{L}_b$ to various $SU(2)$ subgroups of G. This analysis is similar to the method used in the previous section to ascertain the quantization condition on the coadjoint orbit parameters using $SU(2)$ subgroups. Basically, the point is that for every $SU(2)$ subgroup S of G containing a subgroup $T = S^1$ or $SU(2)$ which stabilizes the canonical element b of the coadjoint orbit, there is a condition on the polynomials in z_α which is necessary to ensure that the polynomial can be extended to a globally holomorphic section. This condition is simply that for each such subgroup S, a polynomial $\phi \in R$ must lie in an irreducible representation of the su(2) algebra given by the Lie algebra of S. For finite-dimensional compact simple G, all of the positive roots of G are in the complexification of such an su(2) subalgebra. The same condition is true for $\widehat{L}G$. It follows then, that for all these groups the polynomial ϕ can only be extended to a globally holomorphic section if it lies in the orbit of $|\,\rangle$.

We will demonstrate this argument explicitly for $SU(3)$; the general argument follows in an analogous fashion. For $SU(3)$, let us choose a general polynomial $\phi \in R = \mathbb{C}[t, u, v]$. From the $SU(2)$ subgroup generated by $\{iJ_t, iJ_1, iJ_2\}$, we see that ϕ must be in an irreducible representation of the corresponding subalgebra, so that applying $\hat{J}_{+t}$ to ϕ some integral number of times, we arrive at a function

$$\phi' = \hat{J}^n_{+t}\phi \tag{2.271}$$

which satisfies $\hat{J}_{+t}\phi' = 0$. We now consider the $SU(2)$ subgroup generated by $\{iJ_v, iJ_4, iJ_5\}$. Again, we can apply $\hat{J}_{+v}$ some number of times to get a function ϕ'' which is annihilated by $\hat{J}_{+v}$. Because $[\hat{J}_{+t}, \hat{J}_{+v}] = 0$, the function ϕ'' is also still annihilated by $\hat{J}_{+t}$. Finally, we take the $SU(2)$ subgroup generated by $\{iJ_u, iJ_6, iJ_7\}$. Again, we apply $\hat{J}_{+u}$ some number of times to get a function $\psi \in R$ which is annihilated by $\hat{J}_{+u}$. From the commutation

relations, we again verify that ψ is annihilated also by $\hat{J}_{+v}$ and $\hat{J}_{+t}$. Thus, $\psi = |\,\rangle$. Since each of the transformations we performed by repeated application of annihilation operators was carried out inside an irreducible representation of the relevant $SU(2)$ subalgebra, these operations are all invertible, and it follows that ϕ can be reached from $|\,\rangle$ by the application of the operators $\hat{J}_{-\alpha}$, and thus that ϕ is in the orbit of $|\,\rangle$ under the action of the full algebra on R.

This analysis can immediately be generalized to an arbitrary group G with a finite number of simple roots which lie in complexifications of appropriate $SU(2)$ subgroup of G, simply by applying the simple roots one at a time. Thus, we have shown that for a function to be extensible to a global section, it is necessary for the function to lie in the orbit of $|\,\rangle$ for the groups G and $\widehat{L}G$. Furthermore, if there exists any globally holomorphic section whatsoever, this result implies that $|\,\rangle$ is such a section, and that the set of holomorphic sections must be precisely those sections in the orbit of $|\,\rangle$ since the space of holomorphic sections must be invariant under the action of G. We have thus shown that for all these groups, the representation of the algebra (and therefore of the group) on the space of holomorphic sections of $\mathcal{L}_b$ is irreducible.

Let us now consider the case of the Virasoro group and algebra. In this case, as we observed in Section 2.5.4 the coadjoint orbits are topologically trivial, so there is absolutely no constraint on functions of the variables z_n necessary for functions to be holomorphic. Thus, in this case not only are all functions in $R = \mathbb{C}[\{z_n\}]$ holomorphic, but even all power series in $\mathbb{C}[[\{z_n\}]]$ with infinite radii of convergence must be considered as holomorphic functions on $\mathcal{L}_{b,c}$. We will not concern ourselves with the general power series here; for the purposes of this thesis we restrict attention to the polynomials in $R = \mathbb{C}[\{z_n\}]$. (In any case, we must extract irreducible representations of the Virasoro group by the resolutions which are constructed in the next chapter; the infinite power series' do not contribute to the cohomology of these resolutions and so vanish from consideration at that point.) The difference between this group and the loop groups essentially lies in the fact that although the (complex) Virasoro algebra contains many $\mathrm{su}(2)_{\mathbb{C}}$ subalgebras, these subalgebras do not correspond to compact $SU(2)$ subgroups of the Virasoro group, but rather to noncompact $SL(2,\mathbb{R})$ subgroups, as demonstrated in Section 2.5.4.

Finally, we conclude this section with a brief remark on other approaches to the question of global holomorphicity. For finite-dimensional compact groups, an argument is

often given (see for instance [53]) for the irreducibility of the representation on the space of holomorphic sections, which follows by demonstrating that there is a unique highest weight state in this representation. In the case of compact, finite-dimensional groups, this result follows because the space of holomorphic sections is known to be finite-dimensional, and thus can be written as a direct sum of irreducible representations. For infinite-dimensional groups, this argument breaks down because the space of holomorphic sections is infinite-dimensional. For example, the space R of locally holomorphic polynomials for the group $\widehat{L}SU(2)$ carries a representation of the algebra with a single highest weight state; this representation is, however, not irreducible. Although the expected conclusion holds nevertheless for this group, in the case of the Virasoro group not only does the argument break down, but the conclusion is also incorrect.

2.6.2 Hermitian structures

We will now discuss the question of whether for a given group G and coadjoint orbit W_b, the line bundle $\mathcal{L}_b$ admits a Hermitian structure. In general, a complex line bundle $\mathcal{L}$ over a complex manifold $\mathcal{M}$ is defined to have a Hermitian structure when $\mathcal{M}$ admits a measure $\mathrm{d}\mu$ and $\mathcal{L}$ admits an inner product which when integrated over $\mathcal{M}$ with the measure $\mathrm{d}\mu$ gives a positive-definite inner product on the space of holomorphic sections of $\mathcal{L}_b$. Written in terms of a local trivialization of $\mathcal{L}_b$, the inner product is described by a real function $h : \mathcal{M} \to \mathbb{R}$; the inner product between two holomorphic sections ϕ, ψ of $\mathcal{L}_b$ is then given by

$$\langle \phi, \psi \rangle = \int_{\mathcal{M}} \mathrm{d}\mu \, \mathrm{e}^h \phi^* \psi. \tag{2.272}$$

In terms of such a local trivialization, there is a unique connection A, the *metric connection*, compatible with both the Hermitian metric and the complex structure on $\mathcal{M}$; written in a local set of holomorphic coordinates z_i, this connection is given by $A_{\bar{z}_i} = 0$, $A_{z_i} = \partial h / \partial z_i$ [38].

As an example of such a Hermitian structure, let us consider again the line bundle $\mathcal{L}_b$ over S^2 of degree $2b$. In terms of the coordinate z on S^2, the rotationally invariant measure on S^2 is given by

$$d\mu = \frac{2i}{(1+|z|^2)^2} dz d\bar{z}. \tag{2.273}$$

Associated with this measure, there is a natural Hermitian metric on $\mathcal{L}_b$, described by the

function

$$e^{h(z,\bar{z})} = \frac{1}{(1+|z|^2)^{2b}}. \tag{2.274}$$

Combining these factors, the inner product $\langle , \rangle$ on $\mathcal{H}_b$ is given by

$$\langle \phi, \psi \rangle = \int_{S^2} d\mu \, e^h \phi^* \psi = \int_{S^2} \frac{2idzd\bar{z}}{(1+|z|^2)^{2b+2}} \phi^*(z)\psi(z), \tag{2.275}$$

where ϕ and ψ are arbitrary holomorphic sections of $\mathcal{L}_b$. Performing this integral explicitly, one finds that

$$\langle z^l, z^m \rangle = \delta_{l,m} 4\pi \left[(2b+1) \begin{pmatrix} 2b \\ l \end{pmatrix} \right]^{-1}. \tag{2.276}$$

This inner product on $\mathcal{H}_b$ is proportional to the usual inner product on unitary irreducible representation spaces of $SU(2)$. The metric connection associated with this choice of Hermitian structure is precisely the connection A_z calculated in the previous subsection,

$$A_z = \frac{\partial h(z,\bar{z})}{\partial z} = \frac{-2b\bar{z}}{1+|z|^2}. \tag{2.277}$$

In a similar fashion, it is possible to construct explicitly a Hermitian structure on the line bundles $\mathcal{L}_b$ associated with coadjoint orbits of all finite-dimensional simple compact groups G. Unfortunately, such a construction does not seem to be possible when the group under consideration is infinite-dimensional. As an example, we consider the Virasoro group. If we wish to construct a Hermitian metric $\exp(H)$ on a coadjoint orbit $\mathcal{L}_{b,c}$ of the Virasoro group in the local coordinates defined in Section (2.5.4), we can begin by assuming that the connection we have defined by our gauge-fixing procedure is the associated metric connection, as was the case for $SU(2)$. We can then explicitly calculate H as a formal power series in the z's. We have

$$\begin{aligned} A_0 &= -f_0 + i\Phi_0 \\ &= -h - i\langle (b_0, ic), \exp(-\sum_{n>0} \mu_n L_{-n}) \exp(-\sum_{n>0} z_n L_n) L_0 \\ &\qquad \times \exp(\sum_{n>0} z_n L_n) \exp(\sum_{n>0} \mu_n L_{-n}) \rangle \\ &= \sum_{k,l>0;\{n,m\}} \frac{(-1)^{k+l}}{k!\,l!} z_{n_1} \ldots z_{n_k} \mu_{m_1} \ldots \mu_{m_l} \\ &\qquad \times C_0(n_1, \ldots, n_k, -m_1, \ldots, -m_l)(h + \frac{c}{24}(m_l^2 - 1)), \end{aligned} \tag{2.278}$$

where the sum is taken over all $n_1, \ldots, n_k, m_1, \ldots, m_l > 0$ satisfying $n_1 + \ldots + n_k = m_1 + \ldots + m_l$. If we assume $A_{\bar{z}_n} = 0$ and $A_{z_n} = \partial H / \partial z_n$, then we have

$$A_0 = -\sum_{n>0} n z_n A_{z_n}, \tag{2.279}$$

and

$$H = -\xi_0^{-1} A_0, \tag{2.280}$$

up to a constant. We will take this function H as a candidate for the Hermitian metric on $\mathcal{L}_{b,c}$. The first few terms in a power series expansion of H are given by

$$\begin{aligned} H = &-\sum_{n>0} 2n(h + \frac{c}{24}(n^2-1))|z_n|^2 \\ &+ \sum_{n,m>0} \left[(m^2+4mn+n^2)h + \frac{c}{24}(m^4+2m^3n+2mn^3+n^4-m^2-4mn-n^2)\right] \\ &\times(\frac{z_m z_n \bar{z}_{m+n} + \bar{z}_m \bar{z}_n z_{n+m}}{2}) + \mathcal{O}(z^4). \end{aligned} \tag{2.281}$$

Note that H is expected to be real, in order to be a Hermitian metric.

To have a complete description of a unitary structure on R, it would now be necessary to find an invariant metric on $\mathrm{Diff}\, S^1/S^1$. Unfortunately, it is unclear whether such a metric can be found. Attempting to describe such a metric as a formal power series gives rise to an expression with divergent coefficients. The matter is complicated by the fact that the adjoint representation of the Virasoro algebra is not a highest weight representation. It seems that some kind of regularization scheme may be necessary to construct such a metric in a sensible fashion. We can, however, get some information about when such a unitary structure is likely to be possible directly from (2.281). If we take only the first term in (2.281), and approximate the metric with a Gaussian, we see that for $h \ll 0$ or $c \ll 0$, the metric diverges badly, and we will certainly not find a unitary structure. When $h, c \gg 0$, a sensible inner product on R can be found, at least in perturbation theory, by taking a product of Gaussian integrals. Using the Hermitian metric (2.281) to compute anything nonperturbative, however, would be a difficult proposition. Further progress in this direction will be impossible until some sort of a regularized invariant metric on $\mathrm{Diff}\, S^1/S^1$ can be described explicitly.

In the case of loop groups $\widehat{L}G$, a similar problem arises. For these groups also, no invariant measure is known on the quotient space LG/T; furthermore, it is not known whether such a measure can exist [53].

The main reason that we are interested in Hermitian structures on the line bundles $\mathcal{L}_b$ is that when a line bundle admits a Hermitian structure, the corresponding representation of the group on the space of holomorphic sections is naturally unitary. Although we are unable to construct explicitly the desired Hermitian structures for infinite-dimensional groups, we can still investigate the question of whether the associated representations admit unitary structures. A simple way of constructing a unitary structure on a highest weight representation is to simply define the inner product $\langle 1,1\rangle = \langle\ |\ \rangle = 1$, and to assume that the creation and annihilation operators are Hermitian conjugates under $\hat{J}^{\dagger}_{-\alpha} = \hat{J}_{\alpha}$. In fact, this is the usual procedure used to construct unitary group representations from Verma modules (which will be discussed in more detail in the following chapter). The result of this construction is that the unitary structure is well-defined for all functions $\phi \in R$ which lie in the orbit of the highest weight state $|\ \rangle$. For loop groups, this means that the complete representation on the space $\mathcal{H}_b$ of holomorphic sections actually admits a unitary structure. For the Virasoro group, this result is slightly more subtle. We devote the remainder of this section to a more detailed description of this construction in the case of the Virasoro group.

The above construction of a unitary structure is closely related to the essential point in the proof of which irreducible Virasoro representations admit unitary structures. The usual approach to the unitarity proof (see for example [34]) is to consider all possible polynomials in the creation operators $\hat{L}_{-n}$ acting on a highest weight state $|\rangle$, and to use the above criteria to define an inner product on this space. (The space is a Verma module for the Virasoro algebra, which is discussed in more detail in the next section.) The resulting inner product, known as the Shapovalov form, gives a zero norm to some state in the Verma module precisely when the determinant of the inner product matrix on some level vanishes. By studying the changes of sign in this "Kac determinant" under variations of h and c, it was shown by Friedan, Qui, and Shenker [31] that for $c < 1$, a representation can only admit a unitary structure when when conditions (2.75) and (2.76) are satisfied. The existence of a unitary representation for all values of h and c satisfying these conditions was shown in [35]. From the point of view of the coadjoint orbit representations on polynomial spaces R, the vanishing of the Kac determinant at level n corresponds precisely to the existence of a function in R_n which does not lie in the orbit of $|\ \rangle$ (from the point of view of the next chapter, such a function is associated with a cosingular vector). In general, it is impossible to extend the unitary structure defined on the irreducible representation associated with the orbit of $|\ \rangle$ to these extra states. To see an explicit example of this problem, we can

compute explicitly the lowest degree states arising from the action of the Virasoro algebra on $|\,\rangle$. Using a basis for R of monomials in the variables z_i, where $f \in R$ is represented by the state $|f\rangle$, we have

$$\begin{aligned} \hat{L}_{-1}|\,\rangle &= -2h|z_1\rangle, \\ \hat{L}_{-1}^2|\,\rangle &= -6h|z_2\rangle + (4h^2+2h)|z_1^2\rangle, \\ \hat{L}_{-2}|\,\rangle &= -(4h+\frac{c}{2})|z_2\rangle + 3h|z_1^2\rangle. \end{aligned} \tag{2.282}$$

We expect that for those values of h, c with a vanishing Kac determinant at level 2, we will find a linear dependence between the states generated by combinations of raising operators of degree 2. From the above expressions for $\hat{L}_{-1}^2|\,\rangle$ and $\hat{L}_{-2}|\,\rangle$, it is easy to see that these states are linearly dependent when

$$h = \frac{5-c\pm\sqrt{(c-1)(c-25)}}{16}. \tag{2.283}$$

This is precisely the condition for the Kac determinant to vanish at level 2. For example, when $m=3, p=2, q=1$, and $h=c=1/2$, Equation 2.283 is satisfied, and we have a reducible representation on R, with an extra state $|z_1^2\rangle$ in R_2 which is not in the orbit of $|\,\rangle$. In this case we can compute

$$\langle z_1|z_1\rangle = \langle\,|\hat{L}_1\hat{L}_{-1}|\,\rangle = 2h = 1 \tag{2.284}$$

$$\langle -\frac{9}{4}z_2 + \frac{3}{2}z_1^2| -\frac{9}{4}z_2 + \frac{3}{2}z_1^2\rangle = \langle\,|\hat{L}_2\hat{L}_{-2}|\,\rangle = 9/4.$$

If we attempt to extend this inner product to R_2, we get

$$\begin{aligned} \langle 3z_2 - 2z_1^2|z_1^2\rangle &= \langle z_1|\hat{L}_1|z_1^2\rangle = -2 \\ &= -\frac{4}{3}\langle\,|\hat{L}_2|z_1^2\rangle = 0. \end{aligned} \tag{2.285}$$

Thus, the inner product on the orbit space cannot be extended to one on R.

We have shown, then, that for compact simple G and loop groups $\widehat{L}G$, the representation on the space of holomorphic sections of an appropriate line bundle $\mathcal{L}_b$ is irreducible and admits a unitary structure, in agreement with the results of the Borel-Weil theory. The same result holds for the coadjoint orbits associated with unitary representations of the Virasoro group where the entire space R lies in the orbit of $|\,\rangle$. In the case of those coadjoint orbits of the Virasoro group corresponding to the discrete series of unitary representations, however, there are states which correspond to globally holomorphic sections in R lying

outside the orbit of $|\rangle$. The unitary structure defined in this section does not lead to a well-defined inner product for these extra states, and thus the complete representation of the Virasoro group does not admit a unitary structure in these cases. However, the irreducible representation given by the orbit of $|\,\rangle$ does admit a unitary structure, which is just the usual unitary structure associated with this irreducible representation in the discrete series.

Chapter 3

Modules and Resolutions

In this chapter, we define a variety of types of modules for an arbitrary Lie algebra $\mathfrak{g}$, and describe how these modules can be combined to form any irreducible representation of $\mathfrak{g}$ by constructing a chain complex whose cohomology is precisely the desired irreducible representation space. In Section 3.1, we define the concept of a $\mathfrak{g}$-module, and discuss some particular types of modules, including Verma modules, dual Verma modules, and twisted Verma modules. We describe the singular vector structure of these modules. In Section 3.2, we discuss Fock space realizations of representations; these realizations are essentially equivalent to $\mathfrak{g}$-modules on spaces of polynomials where the action of the algebra is described in terms of differential operators. We observe that the coadjoint orbit representations described in the previous chapter are locally equivalent to dual Verma module representations, and we discuss the Feigin-Fuchs and Wakimoto (free field) representations of the Virasoro and affine algebras, which are similar to twisted Verma modules of finite-dimensional algebras. In Section 3.3, we define the notions of screening operators and intertwiners between $\mathfrak{g}$-modules, and give explicit formulae for the screening operators and intertwiners between coadjoint orbit representations. We find that the coadjoint orbit screening operators, like the associated coadjoint orbit raising operators, are independent of the highest weight of the module under consideration. In Section 3.4, we describe how intertwiners can be used to form a chain complex of $\mathfrak{g}$-modules whose cohomology is nonzero at a single position in the chain, and is given by an irreducible representation of the algebra $\mathfrak{g}$. The explicit formulae for coadjoint orbit intertwiners allow us to use these resolutions to construct irreducible representations using the coadjoint orbit representations even in the case of the Virasoro algebra, where the coadjoint orbit representations correspond precisely to dual

Verma modules.

For general background references on the algebraic constructions in this section, see [18, 39].

3.1 $\mathfrak{g}$-modules

A powerful approach in the study of representation theory is given by the algebraic formalism of modules. Given a Lie algebra $\mathfrak{g}$, every vector space V which admits a representation of $\mathfrak{g}$ is defined to be a $\mathfrak{g}$-module. (More generally, given a ring $\mathcal{R}$, an $\mathcal{R}$-module is defined to be an abelian group A along with a ring homomorphism from $\mathcal{R}$ to the ring of endomorphisms from A into itself. In this case, the vector space V has the abelian group structure given by vector space addition, and the ring $\mathcal{R}$ is the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ described below.) We will restrict attention here to modules which admit a grading of the natural type for the algebra $\mathfrak{g}$, as described in Section 2.1. There are several special types of $\mathfrak{g}$-modules which are of particular interest to us here. A useful structure in describing these special modules is the *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is defined to be the ring of all polynomials (over $\mathbb{R}$) in the generators of the algebra $\mathfrak{g}$, modulo the relations defined by interpreting the Lie algebra product as a commutator via the equation

$$x[u,v]y \sim xuvy - xvuy, \quad \forall x, y \in \mathcal{U}(\mathfrak{g}), u, v \in \mathfrak{g}. \tag{3.1}$$

A $\mathfrak{g}$-module V is said to be free over $\mathcal{U}(\mathfrak{h})$ on a vector $v \in V$ when each vector $w \in V$ has a unique representation as $w = X_w \cdot v$ for some $X_w \in \mathcal{U}(\mathfrak{h})$.

One of the most important types of modules in the study of representation theory is the Verma module [40]. A Verma module for the algebra $\mathfrak{g}$ is a module V containing a vector v, such that V is free over $\mathcal{U}(\mathfrak{g}_-)$ on v, where $\mathfrak{g}_-$ is the subalgebra of $\mathfrak{g}_{\mathbb{C}}$ generated by the negative roots in $\mathfrak{g}_{\mathbb{C}}$. It is an immediate consequence of this definition that v is a highest weight state in V. For a fixed weight $\lambda \in \Lambda_w$, the Verma module V_λ with a highest weight vector v_λ of weight λ is unique. A natural basis for this Verma module is given by the states $X \cdot v$, where X is a member of a canonical basis for $\mathcal{U}(\mathfrak{g}_-)$. From the definition, it is clear that the Verma module V_λ has the property that the entire module is in the orbit of the state v_λ under the action of the algebra $\mathfrak{g}$. This is a characteristic feature of Verma modules.

We can now define a dual space to V_λ, which we denote by

$$V_\lambda^* = \mathrm{Hom}(V_\lambda, \mathbb{R}). \tag{3.2}$$

For any algebra $\mathfrak{g}$ which has a polarization into positive roots $\alpha \succ 0$ and negative roots $\alpha \prec 0$, we can define a natural dual action of $\mathfrak{g}$ on V_λ^* by the equation

$$\langle e_\alpha d, w\rangle = \langle d, e_{-\alpha} w\rangle, \quad \forall d \in V_\lambda^*, w \in V_\lambda. \tag{3.3}$$

(For generators ih in $\mathfrak{t}$, we simply take $\langle hd, w\rangle = \langle d, hw\rangle$.) The module V_λ^* is a dual Verma module. Note that all the groups considered in this thesis admit the necessary type of polarization to construct modules of this type. In V_λ^*, there is a particular state v_λ^* which is defined by

$$\langle v_\lambda^*, w\rangle = \begin{cases} 1, & w = v_\lambda \\ 0, & w \neq v_\lambda \end{cases} \tag{3.4}$$

It is clear that v_λ^* is a highest weight state in the dual Verma module. Furthermore, it follows from the above definition that there are no other highest weight states in this module besides multiples of this state, since every other state in the dual space must have a nonzero contraction with some state of the form $e_\alpha w$ with $\alpha \prec 0$ and $w \in V_\lambda$. In fact, the existence of a single highest weight vector is a characteristic feature of dual Verma modules. Another way of defining dual Verma modules is through the characteristic property of being cofree over the enveloping algebra $\mathcal{U}(\mathfrak{g}_-)$ [18].

In addition to the Verma and dual Verma modules defined above, there are other types of modules of interest. We have already discussed irreducible highest weight representations, which have both the features that the associated module has a single highest weight vector, and that the entire module is in the orbit of the highest weight state under the action of $\mathfrak{g}$. There are also modules, called "twisted" Verma modules, which have neither of these properties. These modules can be defined by the property of being free over $\mathcal{U}(\mathfrak{m})$ and cofree over $\mathcal{U}(\mathfrak{m}')$, where $\mathfrak{m}$ and $\mathfrak{m}'$ are subalgebras of $\mathfrak{g}_-$ such that

$$\mathfrak{g}_- = \mathfrak{m} \oplus \mathfrak{m}'. \tag{3.5}$$

Twisted Verma modules were first introduced by Feigin and Frenkel [24].

In order to analyze these various types of modules and their properties, it is useful to define certain vectors in a $\mathfrak{g}$-module to be *singular*. Following Feigin and Fuchs, we define a vector $v \in V$ to be singular when $\mathfrak{g}_+ \cdot v = 0$, and cosingular when $v \notin \mathfrak{g}_- V$. Intuitively,

when a vector v is singular it is a highest weight vector for a submodule $W \subset V$ given by acting on v with $\mathcal{U}(\mathfrak{g}_-)$. In order to construct an irreducible representation containing the highest weight vector of V, it is clearly necessary to mod out by any such submodule W generated by a singular vector. Similarly, a cosingular vector is one which generates a submodule $X \subset V$ which lies outside the orbit of the highest weight vector of V under the action of $\mathcal{U}(\mathfrak{g})$. Any irreducible representation containing the highest weight vector of V must be modded out by any such submodule X generated by a cosingular vector. Clearly, a particular cosingular vector is only defined modulo $\mathfrak{g}_- V$. We will frequently make use of a choice of specific cosingular vectors from each equivalence class thereof, and will often simply refer to such vectors as "the" cosingular vectors of a particular weight. From the above definitions, it follows that a module V contains a singular vector of weight λ precisely when the dual module V contains a cosingular vector of the same weight. Clearly, a Verma module contains only singular vectors and no cosingular vectors, and a dual Verma module contains only cosingular vectors and no singular vectors. A useful graphical description of modules can be given by drawing a point for each singular or cosingular vector, and then drawing an arrow from the point associated with a vector v to the point associated with a vector w whenever the submodule $\mathcal{U}(\mathfrak{g})v$ contains the vector w. The arrow relation is assumed to be transitive, so that when a sequence of vectors are connected by arrows such as $v \to w \to u$, we omit the arrow between the vectors v and u in the diagram. In this diagrammatic notation, a Verma module has the property that all arrows lead away from the highest weight state. The dual of a module has the same graphical structure as the original module, however all arrows are reversed. In particular, for the dual Verma module the graph has the property that all arrows lead toward the highest weight vector. Twisted Verma modules are characterized by having singular or cosingular vectors with the same weights as the singular vectors in the Verma module, but where the graphical representation contains a different set of arrows from the graph representing the Verma module.

As examples, in Figures 3.1a and 3.1b, we give the graphs associated with the Verma modules and dual Verma modules of the algebras su(2) and su(3). The singular vector structure of Virasoro modules was first fully studied by Feigin and Fuchs [26]. In general, when the Virasoro Verma module associated with a given highest weight is reducible (contains a singular vector), the associated representation is said to be degenerate. For values of h and c corresponding to irreducible unitary representations, the Verma modules fall into several categories. When $c > 1$ and $h > 0$, the resulting Verma module has no

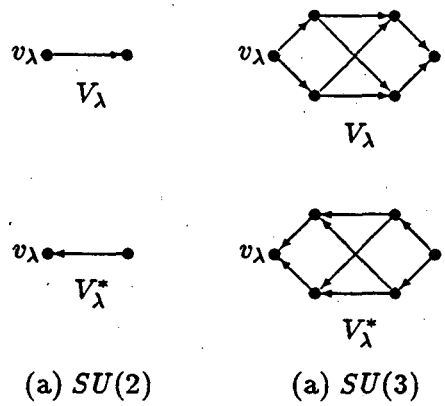


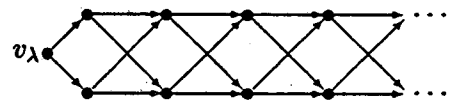
Figure 3.1: Graphs of singular vectors for $SU(2)$ and $SU(3)$ representations

singular vectors, and is therefore always nondegenerate. When $c > 1$ and $h = 0$, the resulting Verma module has a single singular vector which is always at level 1, and the resulting graph looks like the graph for an $SU(2)$ Verma module in Figure 3.1a. When $c = 1$ and $h = m^2/4$ for an integer m, the Verma module contains an infinite number of singular vectors; the resulting graph is shown in Figure 3.2a. The remaining unitary representations with $c = 1$ correspond to Verma modules with no singular vectors and are nondegenerate. The final category of Verma modules for unitary Virasoro representations corresponds to the discrete series of representations with c, h satisfying (2.75) and (2.76). The Verma modules corresponding to these representations again have an infinite number of singular vectors. The graph of the singular vector structure for these representations is shown in figure Figure 3.2b. The graph associated with the Feigin-Fuchs representations of the Virasoro algebra is shown in Figure 3.2c; these representations will be discussed further in the next section. Note that the graph corresponding to the Feigin-Fuchs modules remains fixed under the duality transformation. Similar graphs can be constructed describing the singular vector structure of the affine algebras corresponding to the loop groups $\widehat{LG}$ [24].

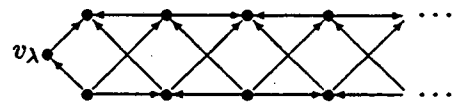
3.2 Fock space realizations

In this section we describe a particular class of $\mathfrak{g}$-modules which can be described in terms of bosonic Fock spaces. In order to describe representations on bosonic Fock spaces, we begin by defining a simple class of algebras, known as Heisenberg algebras. For any set S, we define the Heisenberg algebra on the set S to be the Lie algebra with generators 1

(a) Verma module V_λ for $c = 1$, $h = m^2/4$



(b) Verma module V_λ for discrete unitary series



(c) Feigin-Fuchs Virasoro module $\tilde{V}_\lambda$

Figure 3.2: Verma module and Feigin-Fuchs Virasoro representations

and $\{e_a, f_a | a \in S\}$ satisfying the commutation relations

$$\begin{aligned} [e_a, f_b] &= \delta_{ab} \cdot 1 \\ [e_a, e_b] &= [f_a, f_b] = 0 \\ [1, e_a] &= [1, f_a] = 0. \end{aligned} \tag{3.6}$$

For any set S, this algebra has a unique highest weight irreducible representation. The carrier space of this representation is the bosonic Fock space on S, which has a natural basis labeled by the occupation numbers N_a of the independent bosonic fields; these occupation numbers are the eigenvalues of the operators

$$n_a = f_a e_a. \tag{3.7}$$

There is a simple realization of the Heisenberg algebra and bosonic Fock space on any set S in terms of differential operators acting on a ring of polynomials. If we identify the Fock space on S with the polynomial ring $R_S = \mathbb{C}[\{z_a | a \in S\}]$, the Heisenberg algebra on S has a representation through the identifications

$$\begin{aligned} e_a &\rightarrow \partial/\partial z_a \\ f_a &\rightarrow z_a. \end{aligned} \tag{3.8}$$

Thus, all representations of algebras in terms of differential operators acting on a space of polynomials can be naturally rewritten in the language of bosonic Fock spaces and vice

versa.

There is a natural action of the Heisenberg algebra on S on the dual space R_S^*. This action is defined through the Fock space adjoint operation $e_a^\dagger = f_a$, so that for any $g \in R_S^*, h \in R_S$ we have

$$\begin{aligned} \langle f_a g, h\rangle &= \langle g, e_a h\rangle \\ \langle e_a g, h\rangle &= \langle g, f_a h\rangle. \end{aligned} \tag{3.9}$$

The space R_S^* carries a highest weight irreducible representation of the Heisenberg algebra, and therefore can be canonically identified with R_S.

In terms of this bosonic Fock space language, it is clear that the coadjoint orbit representations defined in Chapter 2 for an arbitrary Lie group G in terms of differential operators on the space of polynomials in the variables z_α with $\alpha \in \Phi_+$ are realizations on the bosonic Fock space over Φ_+. By taking the Fock space adjoint of these representations, where $e_\alpha^\dagger \to e_{-\alpha}$, we can construct the duals to the coadjoint orbit representations. For example, the dual to the coadjoint orbit representation of $SU(3)$ described in (2.224) is defined by the operators $\tilde{J}_a$, where

$$\begin{aligned}
\tilde{J}_t &= -t\frac{\partial}{\partial t} - \frac{1}{2}v\frac{\partial}{\partial v} + \frac{1}{2}u\frac{\partial}{\partial u} + p/2 \\
\tilde{J}_u &= -u\frac{\partial}{\partial u} - \frac{1}{2}v\frac{\partial}{\partial v} + \frac{1}{2}t\frac{\partial}{\partial t} + q/2. \\
\tilde{J}_{-t} &= -t - \frac{1}{2}v\frac{\partial}{\partial u} \\
\tilde{J}_{-u} &= -u + \frac{1}{2}v\frac{\partial}{\partial t} \\
\tilde{J}_{-v} &= -v. \\
\tilde{J}_{+t} &= t\frac{\partial^2}{\partial t^2} + u(-\frac{\partial}{\partial v} - \frac{1}{2}\frac{\partial}{\partial t}\frac{\partial}{\partial u}) + v(\frac{1}{2}\frac{\partial}{\partial t}\frac{\partial}{\partial v} - \frac{1}{4}\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial u}) - p\frac{\partial}{\partial t} \\
\tilde{J}_{+u} &= t(\frac{\partial}{\partial v} - \frac{1}{2}\frac{\partial}{\partial t}\frac{\partial}{\partial u}) + u\frac{\partial^2}{\partial u^2} + v(\frac{1}{2}\frac{\partial}{\partial u}\frac{\partial}{\partial v} + \frac{1}{4}\frac{\partial}{\partial t}\frac{\partial^2}{\partial u^2}) - q\frac{\partial}{\partial u} \\
\tilde{J}_{+v} &= t(\frac{\partial}{\partial t}\frac{\partial}{\partial v} - \frac{1}{2}\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial u}) + u(\frac{\partial}{\partial u}\frac{\partial}{\partial v} + \frac{1}{2}\frac{\partial}{\partial t}\frac{\partial^2}{\partial u^2}) + v(\frac{\partial^2}{\partial v^2} + \frac{1}{4}\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial u^2}) \\
&\quad -(p+q)\frac{\partial}{\partial v} + \frac{(p-q)}{2}\frac{\partial}{\partial t}\frac{\partial}{\partial u}.
\end{aligned} \tag{3.10}$$

In the analysis of the coadjoint orbit representations in Chapter 2, we observed that for a general group G, the generator $\hat{J}_\alpha$ corresponding to a positive root $\alpha \in \Phi_+$ has a leading term given by $-\partial/\partial z_\alpha$, with all other terms being of the form $\phi\, \partial/\partial z_\beta$ with $\beta \succ \alpha$ and with

ϕ being a polynomial in the variables z_α with no constant terms. It follows directly that in the dual representation, the generator $\tilde{J}_{-\alpha}$ corresponding to a negative root ($\alpha \in \Phi_+$) has a leading term $-z_\alpha$ with all other terms being differential operators of degree ≥ 1, which have at least as many derivatives as variables z_α in each term. From this observation, we can see that the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_-)$ must act freely on the dual space to the coadjoint orbit representation, since for any element X of this enveloping algebra there is at least one term with a maximal number n of factors (we minimize n over all representations of the element in question), and this term must contribute a characteristic term with n factors to the polynomial $X|\,\rangle$ which cannot be canceled by any terms in X with less than n factors. By a similar argument, we see that the entire Fock space is in the orbit of $|\,\rangle$ under the action of $\mathfrak{g}$. Thus, it follows that the dual representation to the coadjoint orbit representation is a Verma module representation, and therefore that the coadjoint orbit representation is a dual Verma module representation for any group G. The result that coadjoint orbits give rise to dual Verma module resolutions is already a well-known fact, at least for those groups where the coadjoint orbit construction is equivalent to the Borel-Weil theory. This property is perhaps somewhat simpler to understand using the explicit representations described here, however.

It is important to note that in an explicit Fock space realization of a particular representation of an algebra $\mathfrak{g}$, the expression of the generators in terms of the Heisenberg algebra will generally depend upon the highest weight of the particular representation being realized. The coadjoint orbit representations described in the previous chapter have the unusual feature that the form of all the raising operators is independent of the highest weight of the representation. This property of the coadjoint orbit representations will simplify certain calculations in Chapter 4.

We conclude this section with a brief discussion of bosonic realizations of the Feigin-Fuchs and twisted Verma module representations. We have seen above that the coadjoint orbit representation and dual coadjoint orbit representation of any group G correspond to dual Verma module and Verma module representations respectively, and can be realized in terms of bosonic Fock space representations. Similarly, we can realize twisted Verma module representations on a bosonic Fock space. In general [18, 24], a twisted Verma module realization can be constructed by beginning with a Verma module or dual Verma module realization on a bosonic Fock space, and performing a Bogoliubov transformation on the Heisenberg algebra (3.6) which leaves the form of the algebra unchanged, but requires

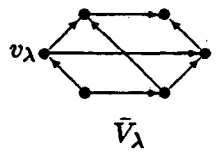


Figure 3.3: Twisted Verma module representation for $SU(3)$

a shift in the ground state $|\,\rangle$. In general, such a Bogoliubov transformation corresponds to a Weyl symmetry transformation on the Lie algebra. As an example of this construction, consider the Verma module representation of $SU(3)$ (3.10). Performing the Bogoliubov transformation $t \to -\partial/\partial t$, $\partial/\partial t \to t$, the form of this algebra is unchanged. However, the ground state $|\,\rangle$ is now annihilated by a different set of operators, so to fix the eigenvalues of the diagonal operators correctly we must replace (p, q) by the vector $(-p-2, q+p+1)$. This gives a new Fock space realization of the algebra of $SU(3)$ in terms of operators $\bar{J}_a$, where

$$
\begin{aligned}
\bar{J}_t &= t\frac{\partial}{\partial t} - \frac{1}{2}v\frac{\partial}{\partial v} + \frac{1}{2}u\frac{\partial}{\partial u} - p/2 \\
\bar{J}_u &= -u\frac{\partial}{\partial u} - \frac{1}{2}v\frac{\partial}{\partial v} - \frac{1}{2}t\frac{\partial}{\partial t} + (q+p)/2. \\
\bar{J}_{-t} &= \frac{\partial}{\partial t} - \frac{1}{2}v\frac{\partial}{\partial u} \\
\bar{J}_{-u} &= -u + \frac{1}{2}vt \\
\bar{J}_{-v} &= -v. \\
\bar{J}_{+t} &= -t^2\frac{\partial}{\partial t} - (\frac{1}{2}tu + \frac{1}{4}t^2v)\frac{\partial}{\partial u} + (\frac{1}{2}tv - u)\frac{\partial}{\partial v} + pt \\
\bar{J}_{+u} &= -\frac{\partial}{\partial t}\frac{\partial}{\partial v} + \frac{1}{2}t\frac{\partial}{\partial t}\frac{\partial}{\partial u} + (u + \frac{1}{4}tv)\frac{\partial^2}{\partial u^2} + \frac{1}{2}v\frac{\partial}{\partial u}\frac{\partial}{\partial v} - (q+p+\frac{1}{2})\frac{\partial}{\partial u} \\
\bar{J}_{+v} &= -t\frac{\partial}{\partial t}\frac{\partial}{\partial v} + \frac{1}{2}t^2\frac{\partial}{\partial t}\frac{\partial}{\partial u} + u\frac{\partial}{\partial u}\frac{\partial}{\partial v} + (\frac{1}{2}tu + \frac{1}{4}t^2v)\frac{\partial^2}{\partial u^2} + v\frac{\partial^2}{\partial v^2} \\
&\quad -q\frac{\partial}{\partial v} - \frac{(2p+q+1)}{2}t\frac{\partial}{\partial u}.
\end{aligned}
\tag{3.11}
$$

This representation of the $SU(3)$ algebra is a twisted Verma module representation $\bar{V}_\lambda$. The Fock space module is free on the state $|\,\rangle = 1$ over the subalgebra generated by $\bar{J}_{-u}$ and $\bar{J}_{-v}$, and cofree over the subalgebra generated by $\bar{J}_{-t}$. The graph of the singular vector structure for this representation is shown in Figure 3.3. Note that the graphical structure of this twisted Verma module is identical to that of the Verma and dual Verma modules;

however, the position of the highest weight vector has changed.

For any finite-dimensional Lie group, any twisted Verma module representation of the corresponding Lie algebra on a polynomial Fock space can similarly be constructed by performing a Bogoliubov transformation corresponding to some Weyl transformation on the Lie algebra. Even the dual Verma module can be constructed in this way by a Bogoliubov transformation on the original Verma module; note, however, that the consequent explicit realization on the bosonic Fock space is different from that given by the duality transformation described above. In the case of infinite-dimensional Lie groups, there are additional modules similar to the twisted Verma modules which cannot be constructed via a Bogoliubov transformation. For example, since the irreducible representations of the loop groups and Virasoro group do not contain lowest weight states, the dual Verma module representation cannot be achieved by a Bogoliubov transformation from the Verma module representation, unlike the case of finite-dimensional Lie groups.

Among the modules for infinite-dimensional groups which cannot be reached by Bogoliubov transformations from the Verma module representation, one class of representations is of particular interest. These are the free field realizations, which have been studied intensively by physicists since the early days of string theory (see [18] and references therein). In the case of the Virasoro algebra, these representations were first studied by Feigin and Fuchs [26] and were later described in terms of free fields by Dotsenko and Fateev [22]. In the Feigin-Fuchs representations, there is a set of operators $\{a_n : n \in \mathbb{Z}\}$, corresponding to the modes of a free bosonic field. These operators satisfy the Heisenberg-type algebra

$$[a_n, a_m] = 2n\delta_{n,-m}. \tag{3.12}$$

The operators a_n act on a bosonic Fock space with a vacuum $|\,\rangle$ satisfying $a_n|\,\rangle = 0$ for $n > 0$ and $a_0|\,\rangle = 2\alpha|\,\rangle$. The Virasoro generators appear as modes of the stress-energy tensor, and are written in terms of the a's as [27]

$$\begin{aligned} L_n &= \frac{1}{4}\sum_{k=-\infty}^{\infty} a_{n-k}a_k - \alpha_0(n+1)a_n, \quad \text{for } n \neq 0, \\ L_0 &= \frac{1}{2}\sum_{k=1}^{\infty} a_{-k}a_k + \frac{1}{4}a_0^2 - \alpha_0 a_0. \end{aligned} \tag{3.13}$$

These generators satisfy a Virasoro algebra with

$$h = \alpha(\alpha - 2\alpha_0), \quad c = 1 - 24\alpha_0^2. \tag{3.14}$$

We can rewrite this representation in terms of the polynomial realization of the bosonic Fock space by writing

$$\begin{aligned} a_n &= 2n\frac{\partial}{\partial z_n}, \quad \text{for } n > 0, \\ a_0 &= 2\alpha, \\ a_{-n} &= z_n, \quad \text{for } n > 0. \end{aligned} \tag{3.15}$$

In this notation, the Virasoro generators are

$$\begin{aligned} L_n &= \sum_{k=1}^{n-1} k(n-k)\frac{\partial}{\partial z_k}\frac{\partial}{\partial z_{n-k}} \\ &\qquad +2n(\alpha - \alpha_0(n+1))\frac{\partial}{\partial z_n} + \sum_{k=n+1}^{\infty} k z_{k-n}\frac{\partial}{\partial z_k}, \quad \text{for } n > 0 \\ L_0 &= \alpha(\alpha - 2\alpha_0) + \sum_{k=1}^{\infty} k z_k \frac{\partial}{\partial z_k}, \\ L_{-n} &= (\alpha + \alpha_0(n-1))z_n + \frac{1}{4}\sum_{k=1}^{n-1} z_k z_{n-k} + \sum_{k=1}^{\infty} k z_{k+n}\frac{\partial}{\partial z_k}, \quad \text{for } n > 0. \end{aligned} \tag{3.16}$$

The module structure of this representation is precisely described by the graph in Figure 3.2c, and corresponds to a module similar to a twisted Verma module, but which cannot be realized by a Bogoliubov transformation from the Verma or dual Verma modules. Note that in these Feigin-Fuchs representations, the form of all the generators is dependent upon both the parameters h and c of the representation through the parameters α and α_0, unlike the coadjoint orbit representations of the Virasoro algebra, where only the generators $\hat{L}_n$ for $n \leq 0$ depend upon h and c.

For the affine algebras corresponding to loop groups $\widehat{L}G$, there is an analogous class of free field representations which are similar to twisted Verma modules but again cannot be constructed from the Verma or dual Verma module by a finite Weyl transformation. These representations are the Wakimoto representations, which are constructed by taking the normal ordered product of the currents associated with free fields having $\widehat{L}G$ symmetry. An equivalent mathematical description of the construction of these representations is given by taking representations of the finite-dimensional group G and replacing all Heisenberg operators by free fields. The normal ordering must be enforced to give the correct affine algebra. This process is known as "affinization". For a review of this construction of Wakimoto representations, see [18, 11].

3.3 Intertwiners

In this section we derive explicit formulae for intertwining operators between the dual Verma modules realized through the coadjoint orbit construction. Generally, an intertwining operator, or "intertwiner" between two $\mathfrak{g}$-modules V and V' is an element of the group $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(V, V')$ of $\mathfrak{g}$-module homomorphisms between V and V'. A $\mathfrak{g}$-module homomorphism is a linear map ϕ between modules which intertwines with the action of $\mathfrak{g}$ in the sense that

$$u\phi x = \phi u x, \quad \forall x \in V, u \in \mathfrak{g}. \tag{3.17}$$

An analysis of the construction of intertwiners in a general algebraic context is given in [18]. In that work, it is proven that for two Verma modules $V_\lambda, V_{\lambda'}$, there is a 1-1 correspondence between intertwiners in $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(V_\lambda, V_{\lambda'})$ and singular vectors in $V_{\lambda'}$ of weight λ. For each singular vector v in $V_{\lambda'}$ of weight λ, the resulting $\mathfrak{g}$-module homomorphism ϕ satisfies $\phi|\,\rangle = v$ (here $|\,\rangle$ denotes the highest weight state in V_λ; we will denote the highest weight state in $V_{\lambda'}$ by $|\,\rangle'$). This characterization of intertwiners for Verma modules is derived by first considering elements of the group of $\mathfrak{g}_-$-homomorphisms $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g}_-)}(V_\lambda, V_{\lambda'})$. Since the orbit of $|\,\rangle$ under the action of the enveloping algebra $\mathcal{U}(\mathfrak{g}_-)$ contains the entire Verma module, such a homomorphism ϕ is determined uniquely by the value of $\phi|\,\rangle$ in $V_{\lambda'}$, and each element $x \in V_{\lambda'}$ determines a homomorphism of this type by setting $\phi|\,\rangle = x$. The condition that ϕ intertwine correctly with the Cartan subalgebra $\mathfrak{t}$ requires that the weight of the state $\phi|\,\rangle$ be exactly λ. This leads to the general definition of a set of *screening operators* s_α, which are $\mathfrak{g}_-$-homomorphisms mapping

$$s_\alpha : V_\lambda \to V_{\lambda+\alpha} \tag{3.18}$$

Explicitly, given an element $X \in \mathcal{U}(\mathfrak{g}_-)$, corresponding to a state $x = X|\,\rangle$ in the Verma module V_λ, the screening operator s_α acts on x by $s_\alpha x = X J_{-\alpha}|\,\rangle'$, where $|\,\rangle'$ is the highest weight state in the Verma module $V_{\lambda+\alpha}$. In general, all $(\mathfrak{g}_- \oplus \mathfrak{t})$-module homomorphisms between Verma modules are polynomials in the screening operators s_α. The screening operators generate an algebra isomorphic to $\mathfrak{g}_-$. The condition that an operator ϕ described by a polynomial in the screening operators intertwines with the remaining $\mathfrak{g}$ generators simply implies that the state $\phi|\,\rangle$ is annihilated by all the generators in $\mathfrak{g}_+$, which is equivalent to the assertion that $\phi|\,\rangle$ is a singular vector in $V_{\lambda'}$, giving the remaining part of the above result.

From this algebraic characterization of intertwining operators on Verma modules, it is straightforward to consider the dual action of the intertwiners, which gives intertwining operators on dual Verma modules. Clearly, for every intertwining operator ϕ between Verma modules V_λ and $V_{\lambda'}$, there is a corresponding intertwiner ϕ^* from the dual Verma module $V^*_{\lambda'}$ to V^*_λ, defined by

$$\langle \phi^* x, w\rangle = \langle x, \phi w\rangle, \quad \forall w \in V_\lambda, x \in V^*_{\lambda'}. \tag{3.19}$$

This correspondence is clearly 1-1, since the same argument can be applied on the dual space. From the result for Verma modules, we can derive the corresponding result for dual Verma modules, which is that the intertwiners in $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(V^*_{\lambda'}, V^*_\lambda)$ are in 1-1 correspondence with cosingular vectors in $V^*_{\lambda'}$ of weight λ, and that the intertwiner ϕ corresponding to a particular cosingular vector $v \in V^*_{\lambda'}$ satisfies $\phi v = |\,\rangle$ where $|\,\rangle$ is the highest weight vector in V^*_λ. This intertwiner must also satisfy the condition that $\phi w = 0$ for any vector $w \in \mathfrak{g}_- V^*_{\lambda'}$ of weight λ. By taking the dual of the screening operators s_α, we get a set of screening operators $\tilde{s}_\alpha = s^\dagger_\alpha$ for the dual Verma module. These screening operators are maps

$$\tilde{s}_\alpha : V^*_\lambda \to V^*_{\lambda-\alpha} \tag{3.20}$$

which intertwine with the algebra $\mathfrak{g}_+$ of generators corresponding to positive roots acting on the dual Verma modules. In fact, every intertwiner on a dual Verma module can be found by constructing an appropriate polynomial in the screening operators; thus, in order to characterize the intertwining operators in a particular realization, it will suffice to describe the screening operators in that realization.

We now proceed to give explicit formulae for the screening operators in the coadjoint orbit dual Verma module representations. For an arbitrary group G, we begin by defining a set of differential operators D_α for all $\alpha \in \Phi_+$ according to

$$D_\alpha = \sum_{\beta\in\Phi_+|\alpha+\beta\in\Phi_+} i f_{\alpha\beta}{}^{\alpha+\beta} z_\beta \frac{\partial}{\partial z_{\alpha+\beta}}. \tag{3.21}$$

These operators act on the ring $R = \mathbb{C}[\{z_\alpha\}]$ of locally holomorphic functions as first order differential operators.

Proposition 7 *The operators $\tilde{s}_\alpha$ defined by*

$$\tilde{s}_\alpha = \hat{J}_\alpha + D_\alpha \tag{3.22}$$

commute with the operators $\hat{J}_\beta$ with $\beta \in \Phi_+$, and form a set of screening operators for the dual Verma modules.

Proof. Because the only nonzero Bernoulli number B_l with l odd is $B_1 = -1/2$, using (2.157) we can write these operators explicitly as

$$\tilde{s}_a = \sum_{k\geq 0, a_1,\ldots,a_k \succ 0} -\frac{B_k(-1)^k}{k!} C_a(a_1,\ldots,a_k) z_{a_1}\ldots z_{a_k} \frac{\partial}{\partial z_{a+a_1+\ldots+a_k}} \quad \text{for } a \succ 0. \tag{3.23}$$

We now consider the effect of right multiplication of a general element (2.20) in the group by an infinitesimal positive generator on the right. By taking the formal multiplication rule

$$\exp(\sum_{\alpha\in\Phi_+} z_\alpha J_\alpha)e^{\epsilon J_a} = \exp\left[\sum_{\alpha\in\Phi_+}(z_\alpha + \epsilon\tilde{v}_a^\alpha)J_\alpha\right] f(\{J_a | a \preceq 0\}) + \mathcal{O}(\epsilon^2), \tag{3.24}$$

we define a set of vector field operators $S_a = -\tilde{v}_a^\alpha \partial/\partial z_\alpha$ which must commute with the operators ξ_a. By repeating the analysis in the proof of Proposition 3 for these vector fields, we find that the operators S_α are precisely equal to $\tilde{s}_\alpha$ for $\alpha \in \Phi_+$. Thus, we have proven the assertion that

$$[\tilde{s}_\alpha, \hat{J}_\beta] = 0, \quad \forall \alpha, \beta \in \Phi_+. \tag{3.25}$$

Since the operator $\tilde{s}_\alpha$ lowers the degree of a polynomial in R by precisely α, it follows that these operators are valid screening operators for the dual Verma modules □.

It is a consequence of this explicit formula that the screening operators are independent of the highest weight λ of the module on which they act, just as are the raising operators in the algebra, $\hat{J}_a$ with $a \succ 0$.

A similar approach to the one taken here was used by Awata, Tsuchiya, and Yamada in [11] to construct the Wakimoto representations of affine algebras; in their work, the result was achieved by using a construction similar to this one for finite-dimensional G and performing an affinization to get realizations of the Wakimoto representations.

As an example of an intertwiner, we consider again the coadjoint orbit dual Verma module representations of $SU(2)$ given by (2.207). For these representations, there is a single screening operator $\tilde{s}$, given by

$$\tilde{s} = \hat{J}_+ = -\frac{\partial}{\partial z}. \tag{3.26}$$

This screening operator obviously commutes with $\hat{J}_+$, and raises the weight of a state by 1, as measured by the eigenvalue of $\hat{J}_3$ in a particular module. Since the screening operator goes from a dual Verma module V_b^* to another dual Verma module V_{b-1}^* with a new highest weight state, we see that $\tilde{s}$ intertwines correctly with $\hat{J}_3$. It remains to construct an intertwiner

using $\tilde{s}$. The dual Verma module V_b^* has a single cosingular state of weight $-b-1$, which is represented in the coadjoint orbit realization by the polynomial $z^{2b+1} = \alpha \hat{J}_-^{2b+1}|\,\rangle$ where α is a proportionality constant. Thus, we expect that we may construct an intertwiner

$$\phi : V_b^* \to V_{-b-1}^*, \tag{3.27}$$

by taking $\phi = \tilde{s}^{2b+1}$. We have already verified that $\tilde{s}$ intertwines correctly with all operators in $\mathfrak{g}$ except for $\hat{J}_-$. We can now compute directly the result of applying $\hat{J}_-$ before and after the intertwiner ϕ to verify that ϕ intertwines correctly with the entire algebra. Acting on an arbitrary monomial z^n, we have

$$\hat{J}_- \phi z^n = \phi \hat{J}_- z^n = \frac{(n+1)!}{(n-2b-1)!}. \tag{3.28}$$

It follows that ϕ is an intertwining operator, as claimed above. Note that the two operators $\hat{J}_-$ in (3.28) have different realizations as differential operators, since they are acting on different dual Verma modules. In general, we will use a parenthesized superscript to denote the highest weight of the module on which a coadjoint orbit generator is acting when there is a possibility of confusion. Thus, we have

$$\hat{J}_-^{(b)} = z^2 \partial z - 2bz, \tag{3.29}$$

and (3.28) states that

$$\hat{J}_-^{(-b-1)} \phi z^n = \phi \hat{J}_-^{(b)} z^n. \tag{3.30}$$

As a second example of an intertwiner consider the coadjoint orbit dual Verma module representations of $SU(3)$ given by (2.224). For these representations, the screening operators are given by

$$\begin{aligned} \tilde{s}_{+t} &= -\frac{\partial}{\partial t} + \frac{1}{2} u \frac{\partial}{\partial v} \\ \tilde{s}_{+u} &= -\frac{\partial}{\partial u} - \frac{1}{2} t \frac{\partial}{\partial v} \\ \tilde{s}_{+v} &= -\frac{\partial}{\partial v}. \end{aligned} \tag{3.31}$$

A typical representation is the adjoint representation with $(p,q) = (1,1)$. The associated dual Verma module $V_{(1,1)}^*$ contains cosingular vectors with weights $(3,-3)$, $(-3,3)$, $(1,-5)$, $(-5,1)$ and $(-3,-3)$. A typical intertwining operator is the operator

$$\phi : V_{(1,1)}^* \to V_{(3,-3)}^* \tag{3.32}$$

given by

$$\phi \cdot X \cdot v = \tilde{s}^2_{+u} X \cdot v = X|\,\rangle, \tag{3.33}$$

where $|\,\rangle$ is the highest weight state in the dual Verma module $V^*_{(3,-3)}$, v is the cosingular vector of weight $(3,-3)$ in $V_{(1,1)}$, and X is an arbitrary element of $\mathcal{U}(\mathfrak{g}_-)$. In the explicit coadjoint orbit realization above, v is given by the polynomial $u^2/2$ (the unique vector with this weight in the dual Verma module). That this operator ϕ is an intertwining operator from $V^*_{(1,1)}$ to $V^*_{(3,-3)}$ follows from the fact that $\tilde{s}^2_{+u}(u^2/2) = 1 = |\,\rangle$.

As a final example of an intertwining operator, we consider again the simplest example of the discrete series of Virasoro representations, with $m = 3, p = 2, q = 1$, and $h = c = 1/2$. As was pointed out in Section 2.6.2, the dual Verma module associated with this representation has a cosingular vector at level 2. Thus, we expect to be able to construct an intertwiner ϕ mapping

$$\phi : V^*_{1/2} \to V^*_{5/2}, \tag{3.34}$$

where V^*_h refers to the Virasoro dual Verma module with highest weight h and central charge $c = 1/2$. This intertwiner is a polynomial of degree 2 in the screening operators $\tilde{s}_n = \hat{L}_n + D_n$, and therefore can be written in the form $\phi = a\tilde{s}_1^2 + b\tilde{s}_2$. Since the intertwining operator must annihilate the state

$$\hat{L}^2_{-1}|\,\rangle = \frac{4}{3}\hat{L}_2|\,\rangle = -3z_2 + 2z_1^2, \tag{3.35}$$

we determine that $4a - 3b = 0$. Fixing the normalization by choosing z_1^2 as the cosingular vector, we have $a = 1/2$, so the intertwining operator is given by

$$\phi = \frac{1}{2}\tilde{s}_1^2 + \frac{2}{3}\tilde{s}_2 = \frac{1}{2}\frac{\partial^2}{\partial z_1^2} + \frac{2}{3}\frac{\partial}{\partial z_2} + \frac{1}{3}z_1\frac{\partial}{\partial z_3} + \mathcal{D}_4. \tag{3.36}$$

In Chapter 4 we will review the intertwining operators for the Feigin-Fuchs Virasoro representations, which were first explicitly described by Felder [27]. Because of the method of construction of these intertwiners, explicit calculations of their action on particular states are rather more involved than in the coadjoint orbit realizations.

3.4 Resolutions

We now proceed to use the intertwining operators constructed in the previous section to define algebraically a chain complex whose cohomology is given by a single irreducible

representation of an algebra $\mathfrak{g}$. The resulting resolution is essentially the dual to the well-known BGG (Bernstein-Gelfand-Gelfand) resolution for finite-dimensional groups [15]. The explicit formulae for intertwining operators give us an explicit realization of this resolution. The analysis in this section is similar to that of [18] where the primary consideration was the understanding of twisted type Verma modules such as the Feigin-Fuchs and Wakimoto representations. The primary difference between this work and the analogous construction for free fields is the possibility of using single-sided resolutions; in the free field approach, one is forced into using two-sided resolutions, which are mathematically much more difficult objects to handle. From a mathematical point of view, the advantage of describing irreducible representations through a resolution of Fock space representations is that most properties of representations are much easier to calculate for Fock space representations than for irreducible representations. (An example is the character of a representation, which has a fairly straightforward expression for the Verma module of most groups, but which is more complicated for the irreducible representations.) Generally, a useful approach to computing such properties of irreducible representations is to compute the property for Fock modules and to then combine these results with the resolution to arrive at the result for an irreducible representation. From the physical point of view, the resolution is essentially equivalent to the BRST formalism for calculating properties of a physical Hilbert space defined through a BRST complex of simpler Fock module-type spaces. In the case of the free field approach to conformal field theory, this approach was first made explicit algebraically by Felder [27].

We begin by briefly reviewing the formalism of resolutions. A chain complex (M_i, d_i) of $\mathfrak{g}$-modules is defined to be a sequence of modules

$$\cdots \xrightarrow{d_{-2}} M_{-1} \xrightarrow{d_{-1}} M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} \cdots \tag{3.37}$$

where $d_i \in \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(M_i, M_{i+1})$ is a set of $\mathfrak{g}$-module homomorphisms satisfying

$$d_{i+1} \cdot d_i = 0. \tag{3.38}$$

A chain complex can terminate on either or both ends with a trivial module (0). The cohomology of the above chain complex is defined as usual to be

$$H^i = \frac{\mathrm{Ker}\ d_i}{\mathrm{Im}\ d_{i-1}}. \tag{3.39}$$

If L is an irreducible representation of $\mathfrak{g}$, the above complex is said to be a resolution of L

when

$$H^i = \begin{cases} L, & i = 0 \\ 0, & i \neq 0 \end{cases} \tag{3.40}$$

A resolution based on a complex which terminates with either $M_1 = 0$ or $M_{-1} = 0$ is said to be a one-sided resolution. Otherwise, the resolution is two-sided. In physical language, d is said to be a BRST operator, and the cohomology spaces H^i are the BRST cohomology spaces, which contain all the physical states in a given representation. The states not in the cohomology H^0 are called "ghost" states.

The classic example of a resolution is the BGG resolution for irreducible representations of finite-dimensional semi-simple groups G [15]. This resolution is constructed directly from the Verma modules of G. Given a highest weight λ, we can assign an integer $i(v)$ to each singular vector v in the Verma module V_λ by setting $i(v_\lambda) = 0$ for the highest weight state and $i(v) = i(v') + 1$ whenever the graph of singular vectors contains an arrow $v' \to v$. This ordering is equivalent to the "Bruhat" ordering on the Weyl group of G. We can then construct a complex by setting

$$M_j = \bigoplus_{v|i(v)=-j} V_{\lambda(v)}, \tag{3.41}$$

where $\lambda(v)$ is the weight of the singular vector v. The spaces M_j are thus $\mathfrak{g}$-modules. The $\mathfrak{g}$-module homomorphisms d are constructed by combining the intertwining operators $\phi : V_{\lambda(v)} \to V_{\lambda(v')}$ for all singular states $v' \to v$. It turns out that the signs on these intertwining operators can always be chosen in such a way that the resulting is a chain complex, by enforcing the condition that $d_i \cdot d_{i-1} = 0$.

As an example of the BGG resolution, consider again a general Verma module for a highest weight representation of $SU(3)$. The graph of the singular vectors $v_1, \ldots, v_5$ is shown in Figure 3.4. The signs on the arrows indicate the signs used in the construction of the chain complex. The $\mathfrak{g}$-modules in the resulting chain complex are

$$\begin{aligned} M_0 &= V_\lambda \\ M_{-1} &= V_{\lambda(v_1)} \oplus V_{\lambda(v_2)} \\ M_{-2} &= V_{\lambda(v_3)} \oplus V_{\lambda(v_4)} \\ M_{-3} &= V_{\lambda(v_5)}, \end{aligned} \tag{3.42}$$

and the operators are given by

$$d_{-1} = \phi_{1,0} + \phi_{2,0}$$

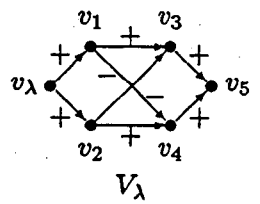


Figure 3.4: Singular vectors in $SU(3)$ Verma module

$$\begin{aligned} d_{-2} &= (\phi_{3,1} - \phi_{4,1}, -\phi_{3,2} + \phi_{4,2}) \\ d_{-3} &= (\phi_{5,3}, \phi_{5,4}) \end{aligned} \tag{3.43}$$

where we denote by $\phi_{i,j}$ the intertwining operator from $V_{\lambda(v_i)}$ to $V_{\lambda(v_j)}$ ($v_0 = v_\lambda$). Because the intertwining operators satisfy the relation

$$\phi_{j,k} \cdot \phi_{i,j} = \phi_{j',k} \cdot \phi_{i,j'}, \tag{3.44}$$

it follows that the operator d in the chain complex satisfies (3.38). That the cohomology of this complex indeed gives a resolution of the irreducible representation with highest weight λ is easy to verify by simple diagram chasing.

Just as the BGG resolution gives a chain complex whose cohomology is an irreducible representation of a finite-dimensional group, one can construct analogous resolutions in terms of Verma modules for the Virasoro and loop group irreducible representations. For the Virasoro algebra, the relevant resolutions were essentially described by Feigin and Fuchs [26]; for the affine algebras arising from loop groups, the analogous result was first obtained by Garland and Lepowsky [32]. A proof of the existence of such a resolution in terms of Verma modules for infinite-dimensional groups is also given in [54]; in [55], these resolutions are used to calculate the characters of irreducible representations of the Virasoro algebra.

Our interest here is in achieving an explicit realization of the dual of the chain complex of Verma modules. Because the resulting resolution is simply the dual of the resolution described in the above works, the existence of such a resolution in terms of dual Verma modules follows immediately. By combining the intertwining operators described in the previous section, we find for any group G among the groups under consideration in this thesis, and for any highest weight λ, a one-sided resolution of the irreducible representation with highest weight λ,

$$0 \to V_\lambda^* \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} \cdots. \tag{3.45}$$

As an example of a dual Verma module resolution, consider once more the coadjoint orbit dual Verma module realizations of $SU(2)$. As discussed in the previous section, when $b \geq 0$ and $b \in \mathbb{Z}/2$, the dual Verma module V_b^* contains a single cosingular state with weight $-b-1$. The dual Verma module V_{-b-1}^* contains no singular or cosingular states. Thus, using the simple intertwiner $\phi = \tilde{s}^{2b+1} = (-\partial/\partial z)^{2b+1}$, we have a resolution

$$0 \to V_b^* \xrightarrow{\phi} V_{-b-1}^* \to 0 \tag{3.46}$$

where Im $\phi = V_{-b-1}^*$, and Ker $\phi = I_b$ is the irreducible representation of $SU(2)$ with highest weight b.

For the Feigin-Fuchs and Wakimoto representations of the Virasoro and loop group algebras, it is believed that a similar resolution of highest weight irreducible representations can be constructed in terms of similar intertwining operators [24]. In the case of Feigin-Fuchs modules, this result was essentially proven by Felder [27]. For the Wakimoto representations of $\widehat{L}SU(2)$ and $\widehat{L}SU(3)$, the existence of such a resolution has also been proven [14, 25]. In general, however, because the resulting resolutions are two-sided, the usual methods of homological algebra which are used to prove these results for one-sided resolutions must be generalized to deal with this situation. In [18], some of the specific results regarding these two-sided resolutions are given or conjectured, and arguments are given for the correctness of the conjectures. Some of the outstanding problems regarding these two-sided resolutions were successfully dealt with in [24], but the mathematical formalism necessary to resolve these difficulties is significantly more complicated than in the case of one-sided resolutions. This is one of the reasons why the approach used in this thesis, which concentrates on dual Verma modules and one-sided resolutions, is perhaps a simpler approach to the algebraic construction of conformal field theories.

Finally, we reiterate the point that in the case of affine algebras arising from loop groups, in a global geometric setting the algebraic structure of resolutions is not necessary to extract irreducible highest weight representations from coadjoint orbit representations, as the constraint of global holomorphicity already restricts to a Hilbert space corresponding to an irreducible representation. In the case of the Virasoro algebra, however, the resolution described in this section is necessary in order to extract irreducible representations from the larger, reducible, dual Verma module representations which arise from coadjoint orbits.

Chapter 4

Conformal Field Theories

In this chapter we describe a method for the construction of conformal field theories using the explicit resolutions of irreducible representations through dual Verma modules described in the previous chapter. Because we do not yet have a purely field-theoretic description of the coadjoint orbit representations, we must proceed according to the algebraic approach to conformal field theory, which was recently developed in order to give a formal algebraic structure to the theory and to characterize conformal field theories in terms of free field representations. This algebraic approach to conformal field theory is well-described in the papers by Tsuchiya and Kanie [62] and Bouwknegt et. al. [18].

The essential element in the algebraic construction of a conformal field theory from a set of irreducible representations of an infinite-dimensional symmetry algebra $\mathcal{A}$, is the construction of *vertex operators* between the irreducible representations. In the case of finite-dimensional groups, a vertex operator is essentially just described by the set of Clebsch-Gordan coefficients between three irreducible representations of the group. In the infinite-dimensional case, vertex operators have a similar structure; however, they have the effect of combining two representations, one of which is highest weight and the other of which is not highest weight, into a third highest weight representation.

A conformal field theory is specified algebraically by giving an infinite-dimensional symmetry algebra $\mathcal{A}$, a set of highest weight irreducible representations of $\mathcal{A}$, and a set of vertex operators connecting the representations. In general, the symmetry algebra $\mathcal{A}$ contains holomorphic and antiholomorphic parts, and the vertex operators factorize into "chiral vertex operators" corresponding to the holomorphic and antiholomorphic parts of the theory. In this discussion, we will simplify by only considering the holomorphic part of

the theory; our vertex operators are therefore what are usually referred to in the physics literature as chiral vertex operators. The simplest class of conformal field theories are the *rational* conformal field theories, which contain a finite set of irreducible representations of $\mathcal{A}$. Each representation is generally associated with a *primary field* of the theory and a corresponding subset of the vertex operators. The *fusion algebra* of the theory is an algebra whose elements are the primary fields of the theory, and whose product is defined by the action of the set of nonzero vertex operators associated with one primary field on the irreducible module associated with a second primary field. The highest weight state of a particular representation with exceptional symmetry is generally singled out as the physical vacuum $|\Omega\rangle$; the physical observables of the theory (on the sphere) are then given by acting on the vacuum with a product of vertex operators and computing the component of the vacuum in the resulting state. Such an observable is referred to as a vacuum expectation value or correlation function, and is written in the form

$$\langle\Omega|\Phi_1(u_1,z_1)\dots\Phi_k(u_k,z_k)|\Omega\rangle, \tag{4.1}$$

where Φ_i are vertex operators which are parameterized by complex coordinates z_i on the sphere and enough extra parameters u_i to specify a particular vector in the representation of $\mathcal{A}$ (not highest weight) associated with Φ_i.

In Section 4.1, we define vertex operators for finite-dimensional algebras. We show that these vertex operators can be written in terms of the explicit screening operators for dual Verma modules introduced in the previous chapter, and give a simple example for the algebra of $SU(2)$. In Section 4.2, we define vertex operators for the Virasoro algebra, and show that all vertex operators between degenerate Virasoro representations satisfy a certain set of algebraic conditions. Furthermore, we show that any vertex operator which has a realization as a chain map between dual Verma module resolutions can be completely characterized by its matrix elements. In this section, we also give a simple class of examples of vertex operators whose action on the vacuum can be computed in terms of the Virasoro generator $\hat{L}_{-1}$. In Section 4.3, we review the explicit construction of vertex operators for the Feigin-Fuchs modules due to Felder, and the related construction of intertwining operators for these modules. In Section 4.4 we describe the explicit construction of vertex operators for the coadjoint orbit dual Verma modules. We discover a class of vertex operators whose action on any state can be represented entirely in terms of the Virasoro generators. The existence of these operators is due to the independence in form of raising operators from

the highest weight of the module on which they act in the coadjoint orbit realizations. Finally, in Section 4.5 we review several approaches to calculating the correlation functions of conformal field theories. In principle, the coadjoint orbit vertex operators can be used to calculate an arbitrary correlation function for a conformal field theory on any Riemann surface; in practice, however, a better description of these vertex operators is necessary to make any such calculation a practical enterprise.

For the purposes of the discussion in this chapter, we will restrict attention to conformal field theories whose symmetry algebra $\mathcal{A}$ is the Virasoro algebra. In such a conformal field theory, a fixed value of the central charge c is chosen for all fields in the theory. The physical vacuum $|\Omega\rangle$ is defined to be the highest weight state in the representation with $h = 0$. The class of conformal field theories corresponding to $c < 1$ contain only a finite number of primary fields, corresponding to the discrete series of Virasoro representations with h satisfying (2.76). These rational conformal field theories are called *minimal models*. A systematic definition and study of Virasoro conformal field theories was first performed by Belavin, Polyakov, and Zamolodchikov (BPZ); the point of view taken in that work [13] and in much of the subsequent literature (see for instance [34]) is based on treating correlation functions as the primary objects, and analyzing their properties using differential equations which they can be shown to satisfy. In the algebraic approach to conformal field theory, the operators themselves are taken to be the primary objects, and the correlation functions are viewed as algebraic constructs using these operators. We will primarily adhere to the latter point of view; however, we will discuss some of the parallels between structures in the algebraic formulation and their counterparts in terms of correlation functions. Much of the content of this chapter, particularly Section 4.2, is essentially a recasting of the original BPZ analysis of conformal field theory into the more algebraic framework of operators, modules, and resolutions, where the differential equations satisfied by correlation functions are expressed as algebraic equations satisfied by vertex operators.

Most of the results in this chapter can be generalized in a straightforward fashion using the preceding results in this thesis to construct conformal field theories whose symmetries are affine algebras, such as the WZW model, in a purely algebraic fashion from dual Verma module resolutions. From a global geometric point of view, however, as mentioned in the previous section, the conformal field theories with affine algebra symmetries can be directly constructed from the coadjoint orbit representations without the necessity of introducing resolutions since the constraint of global holomorphicity automatically picks out

the physical states in the theory. We discuss briefly the possibility of extending the results in this thesis to conformal field theories with other, extended symmetry algebras $\mathcal{A}$ in the last chapter.

4.1 Finite-dimensional vertex operators

Although vertex operators are primarily of interest for their properties relative to conformal field theories and infinite-dimensional algebras, it is possible to define analogous objects for finite-dimensional groups [18]. Because these finite-dimensional vertex operators are structurally similar to the usual infinite-dimensional vertex operators, but are mathematically much simpler, it is interesting to study them as toy models before proceeding to the more complicated objects.

A vertex operator $\mathcal{V}$ for a finite-dimensional group G can be defined for any three irreducible representations I_{λ_1}, I_{λ_2}, and I_{λ_3} of G with highest weights λ_1, λ_2, and λ_3, to be a set of operators

$$\mathcal{V}_u : I_{\lambda_2} \to I_{\lambda_3} \quad \forall u \in I_{\lambda_1} \tag{4.2}$$

which transform under $\mathfrak{g}$ according to

$$[x, \mathcal{V}_u] = \mathcal{V}_{xu} \quad \forall x \in \mathfrak{g} \tag{4.3}$$

Such a vertex operator is called a vertex operator of weight λ_1. By the Wigner-Eckart theorem, a matrix element of a vertex operator of this form is necessarily proportional to the corresponding Clebsch-Gordan coefficient.

Since we are concerned with irreducible representations which arise from dual Verma module resolutions, the next problem which arises is that of characterizing a vertex operator in terms of resolutions of irreducible representations through chain complexes. Generally, if the irreducible representations I_{λ_2} and I_{λ_2} are given through resolutions (M_i, d_i) and (M_i', d_i'), a bosonic realization of the vertex operator $\mathcal{V}_u$ is a set of operators

$$\mathcal{V}_u^i : M_i \to M_i' \tag{4.4}$$

which satisfy (4.3) and

$$d_i' \cdot \mathcal{V}_u^i = \mathcal{V}_u^{i+1} \cdot d_i, \tag{4.5}$$

and where $\mathcal{V}_u^0$ agrees with $\mathcal{V}_u$ on the cohomology of the complex. The constraint (4.5) is the statement that $\mathcal{V}_u^i$ is a map of chain complexes for every $u \in I_{\lambda_1}$, and is equivalent to the condition that the following diagram be commutative

$$\begin{tikzcd}
\cdots \arrow[r, "d_{-2}"] & M_{-1} \arrow[r, "d_{-1}"] \arrow[d, "\mathcal{V}_u^{-1}"] & M_0 \arrow[r, "d_0"] \arrow[d, "\mathcal{V}_u^0"] & M_1 \arrow[r, "d_1"] \arrow[d, "\mathcal{V}_u^1"] & \cdots \\
\cdots \arrow[r, "d'_{-2}"'] & M'_{-1} \arrow[r, "d'_{-1}"'] & M'_0 \arrow[r, "d'_0"'] & M'_1 \arrow[r, "d'_1"'] & \cdots
\end{tikzcd}$$

A chain map $\mathcal{V}^i$ which induces a trivial map in cohomology is called a trivial chain map. Some straightforward diagram chasing suffices to verify that any chain map induces a well-defined map in the cohomology, and that every trivial chain map can be written in the form

$$\mathcal{V}^i = d'_{i-1}t^i + t^{i+1}d_i \tag{4.6}$$

for some set of maps

$$t^i : M_i \to M'_{i-1}. \tag{4.7}$$

It follows that every two bosonic realizations $\mathcal{V}_u^i$ and $\bar{\mathcal{V}}_u^i$ of the same vertex operator $\mathcal{V}_u$ differ by operators of the form

$$\mathcal{V}_u^i - \bar{\mathcal{V}}_u^i = d'_{i-1}t_u^i + t_u^{i+1}d_i, \tag{4.8}$$

where for each $u \in I_{\lambda_1}$ the operators t_u^i satisfy (4.7), and furthermore we have

$$[x, t_u^i] = t_{xu}^i \quad \forall x \in \mathfrak{g}. \tag{4.9}$$

Since we are primarily interested in bosonic realizations of vertex operators in terms of chain maps, we will generally not distinguish in notation between vertex operators and their realizations. Vertex operators which are realized as chain maps are frequently referred to as "screened vertex operators".

As mentioned above, the matrix elements of vertex operators of a finite-dimensional group G are proportional to Clebsch-Gordan coefficients. It is fairly straightforward to prove that a nontrivial vertex operator of weight I_{λ_1} between representations I_{λ_2} and I_{λ_3} exists precisely when the tensor product rules of G are satisfied, so that the irreducible representation I_{λ_3} occurs in the tensor product representation $I_{\lambda_1} \otimes I_{\lambda_2}$ [18].

For finite-dimensional groups, it is possible to explicitly construct bosonic realizations of vertex operators as chain maps of dual Verma module resolutions using the screening operators defined in Section 3.3. In fact, since I_{λ_1} is a highest weight representation of G with a highest weight state v, we have

$$[x, \mathcal{V}_v^i] = 0, \quad \forall x \in \mathfrak{g}_+. \tag{4.10}$$

It follows that $\mathcal{V}_v^i$, the component of the vertex operator associated with the highest weight state, can be written as a polynomial in the screening operators for every i, along with an appropriate change in weight by λ_1. For $\mathcal{V}_v^0$, the degree μ of the polynomial in the screening operators must be such that $\lambda_2 + \lambda_1 - \mu = \lambda_3$; the degree of the polynomial for each value of i can similarly be calculated. Thus, for example, we must have a description of $\mathcal{V}_v^0$ as a polynomial in screening operators $\tilde{s}_\alpha$ of degree

$$\mu = \lambda_1 + \lambda_2 - \lambda_3. \tag{4.11}$$

Note that we must have $\lambda_3 \prec \lambda_1 + \lambda_2$ for such an operator to exist, in agreement with the Wigner-Eckart theorem. The remaining components of the vertex operator can be calculated using the relations (4.3).

As a simple example of a screened vertex operator for a finite-dimensional group G, we now give an explicit bosonic realization of an $SU(2)$ vertex operator of weight j acting on the irreducible representation with weight k and taking values in the irreducible representation with weight k'. As described above, the irreducible representation of $SU(2)$ of weight k has a resolution in terms of dual Verma modules by the chain complex

$$0 \to V_k^* \xrightarrow{\tilde{s}^{2k+1}} V_{-k-1}^* \to 0 \tag{4.12}$$

The highest weight component $\mathcal{V}_j$ of the screened vertex operator in question is given by two operators $\mathcal{V}_j^0 : V_k^* \to V_{k'}^*$ and $\mathcal{V}_j^1 : V_{-k-1}^* \to V_{-k'-1}^*$, as in the following diagram

$$\begin{tikzcd}
0 \arrow[r] & V_k^* \arrow[r, "\tilde{s}^{2k+1}"] \arrow[d, "\mathcal{V}_j^0"] & V_{-k-1}^* \arrow[r] \arrow[d, "\mathcal{V}_j^1"] & 0 \\
0 \arrow[r] & V_{k'}^* \arrow[r, "\tilde{s}^{2k'+1}"'] & V_{-k'-1}^* \arrow[r] & 0
\end{tikzcd}$$

From the above argument, $\mathcal{V}_j^0$ and $\mathcal{V}_j^1$ must be proportional to $\tilde{s}^{j+k-k'}$ and $\tilde{s}^{j+k'-k}$ respectively. In order to make the above diagram commute, we must choose the same proportionality constant for both operators. Note that the form of this vertex operator depends explicitly upon the weights k and k' of the initial and final modules. This is a general property of vertex operators defined for a particular Fock space realization.

4.2 Virasoro vertex operators

The vertex operators associated with the Virasoro algebra are defined in a similar fashion to the finite-dimensional vertex operators defined in the previous section. The main distinction is that the representation which defines the weight of the vertex operator is not a highest weight representation. Within a single conformal field theory, the central charge c does not vary between irreducible representations of the Virasoro algebra, so we shall keep c fixed to a constant value, and use the single parameter h for the highest weight of a Virasoro module throughout the rest of this chapter. Given two irreducible highest weight representations of the Virasoro algebra $I_k, I_{k'}$, with highest weights k and k' respectively, a vertex operator of weight h is defined to be an operator-valued function of a single complex variable z,

$$\Phi(z) : I_k \to I_{k'}, \tag{4.13}$$

which satisfies the relations

$$[L_n, \Phi(z)] = \left(z^{n+1} \frac{\partial}{\partial z} + (n+1)hz^n \right) \Phi(z). \tag{4.14}$$

In the physics literature, a vertex operator is often defined to be an operator of this type which has a nontrivial action on all irreducible representations occurring in a given theory. We will adhere to the convention that a vertex operator is associated with a fixed set of three weights h, k, and k', following [18].

To better study the structure of vertex operators, it is convenient to expand a fixed vertex operator $\Phi(z)$ in a power series expansion in z,

$$\Phi(z) = z^x \sum_{n \in \mathbb{Z}} \phi_{-h-x+n} z^{-n}, \tag{4.15}$$

where $x = k' - h - k$. Note that the vertex operator is not single valued unless $x \in \mathbb{Z}$. The modes ϕ_m of the vertex operator satisfy the commutation relations with the Virasoro

algebra

$$[L_n, \phi_m] = (nh - n - m)\phi_{n+m}. \tag{4.16}$$

Thus, the modes of the vertex operator carry a fairly simple representation of the Virasoro algebra. This representation, however, is not highest weight; the weight of the mode ϕ_{-h-x+n} is precisely $h + x - n$ which is unbounded in both directions as $n \in \mathbb{Z}$.

Just as we did for finite-dimensional vertex operators, we can define a bosonic realization of a vertex operator as a chain map between resolutions of the irreducible representations $I_k, I_{k'}$ in terms of Fock modules, where the chain map satisfies the correct commutation relations with the Virasoro generators. A vertex operator is again defined to be trivial when its action on the cohomology is trivial, and two vertex operator realizations which differ by a trivial chain map are considered to be equivalent. Again, equivalent vertex operator realizations $\phi_n^i, \bar{\phi}_n^i$ must differ by a chain map of the form $d'_{i-1}t_n^i + t_n^{i+1}d_i$, where $[L_n, t_m^i] = (nh - n - m)t_{n+m}^i$.

We are now interested in answering the question of which values of k, k', and h corresponding to unitary representations of the Virasoro group admit a vertex operator of weight h from I_k to $I_{k'}$. Moreover, we would like to know whether such a vertex operator, when it exists as a map between irreducible representation spaces, can be realized by a chain map between resolutions. Because the representation of the vertex operator itself is not highest weight, we cannot use the methods of the previous section. That is, there does not exist a component of the vertex operator which commutes with all raising operators L_n, $n > 0$, so we cannot describe the vertex operators of the Virasoro algebra in terms of the screening operators $\tilde{s}_n$. This complication makes the explicit description of screened vertex operators rather difficult. Nonetheless, for those vertex operators which do exist and have realizations as chain maps, we can give a set of equations which uniquely characterize the action of the vertex operator on the cohomology of the resolutions.

Unfortunately, we do not have a closed form expression for the realizations of the existing vertex operators on the coadjoint orbit dual Verma modules, nor do we have a proof of which vertex operators have realizations as chain maps. In the case where the resolutions are trivial and the dual Verma modules V_k^* and $V_{k'}^*$ contain no cosingular vectors, we can prove the existence of a vertex operator of arbitrary weight h. Because the resolutions of these modules are trivial, it follows immediately that these vertex operators have realizations as chain maps; we can characterize these vertex operators uniquely by calculating their

matrix elements. When the resolutions are not trivial, the existence of cosingular vectors in V_k^* and singular vectors in $V_{k'}$ give a set of algebraic constraints on the values of h, k, k' for which vertex operators can exist. In physical language, these are constraints on the fusion algebra of the associated CFT. Whenever this set of constraints is satisfied, there exists a vertex operator between the appropriate irreducible Virasoro representations. We believe that every such vertex operator has a realization as a chain map between dual Verma module resolutions, although we do not have a rigorous proof of this assertion. We can, however, show that if such a vertex operator exists, it is unique up to a scaling factor and the addition of cohomologically trivial vertex operators. Furthermore, we can calculate the action of such a vertex operator on the cohomology of the complex, which uniquely determines the action of the vertex operator on the irreducible representation and provides the essential information needed to compute (genus 0) correlation functions in a conformal field theory. We will now proceed to demonstrate the existence of vertex operators between dual Verma modules with no cosingular vectors, and to describe the constraints on the vertex operators between nontrivial resolutions of dual Verma modules.

In order to have a sensible algebraic description of Virasoro vertex operators in terms of matrix elements, we find it useful to discuss briefly several simple points about operators on bosonic Fock spaces. Given a Heisenberg algebra on a set $S = \{\alpha \in \Phi_+ \subset \Lambda\}$ acting on a Fock space F with a Λ-grading, where Φ_+ is the space of positive roots and Λ is the root lattice for a Lie algebra $\mathfrak{g}$, an operator

$$O : F \to F \tag{4.17}$$

is uniquely defined by the matrix elements

$$\langle w|O|v\rangle, \quad |v\rangle \in F, \langle w| \in F^*. \tag{4.18}$$

The number of elements $v \in F$ with weight λ is given by the Λ-partition function $\pi_\Lambda(\lambda)$, which is the number of distinct ways in which λ can be constructed as a sum of elements in Φ_+. In the particular case at hand, the Virasoro group, the Fock space is the space $R = \mathbb{C}[\{z_n : n > 0\}]$ with a $\mathbb{Z}$-grading. The dimension of the space R_n of degree n is just the integer partition function $\pi(n)$. The canonical basis for R is the space of monomials in the variables z_n. An operator $O : R \to R$ can be described either by giving the complete set of matrix elements (4.18), or by writing the operator in the form of a differential operator

$$O = \sum_{n,m} \sum_{\pi_n, \pi_m} \omega(\pi_n, \pi_m)(\prod_i z_{n_i})(\prod_j \frac{\partial}{\partial z_{m_j}}), \tag{4.19}$$

where π_n and π_m are summed over all partitions $\{n_i\}$ and $\{m_j\}$ with $n_1 + \cdots + n_a = n$ and $m_1 + \cdots + m_b = m$. The description of an operator by a set of coefficients ω is equivalent to giving the complete set of matrix elements for that operator. However, these two ways of writing the same operator are related in a rather nontrivial fashion. For instance, the operator P on the bosonic Fock space $\mathbb{C}[z]$ which projects onto the highest weight state $|\,\rangle = 1$ has all matrix elements equal to 0 except for the element $\langle\,|P|\,\rangle = 1$. This operator has the differential operator form

$$P = :\exp(-z\frac{\partial}{\partial z}): = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} z^i \frac{\partial^i}{\partial z^i}, \tag{4.20}$$

where by the dots around the second expression we indicate the normal ordering procedure, by which all annihilation operators $\partial/\partial z$ are moved to the right of creation operators z. Up to this point in this thesis, we have primarily described operators in terms of their description as differential operators; in the following, we will find it more useful to describe screened vertex operators in terms of their matrix elements.

The Virasoro vertex operators which we wish to construct are chain maps $\Phi(z)$ between resolutions (M_i, d_i) and (M_i', d_i') of the irreducible representation spaces I_k and $I_{k'}$ of the form

$$\begin{tikzcd}
0 \arrow[r] & V_k^* \arrow[r, "d_0"] \arrow[d, "\Phi^0(z)"] & M_1 \arrow[r, "d_1"] \arrow[d, "\Phi^1(z)"] & M_2 \arrow[r, "d_2"] \arrow[d, "\Phi^2(z)"] & \cdots \\
0 \arrow[r] & V_{k'}^* \arrow[r, "d_0'"'] & M_1' \arrow[r, "d_1'"'] & M_2' \arrow[r, "d_2'"'] & \cdots
\end{tikzcd}$$

We will find it useful to define a particular set of states in V_k^* and $V_{k'}^{**} = V_{k'}$. Associated with each partition $\pi_n = \{n_i | \sum n_i = n\}$, we define the operators

$$\begin{aligned} L_{\pi_n} &= \prod_i L_{n_i} \\ L_{-\pi_m} &= \prod_i L_{-m_i}, \end{aligned} \tag{4.21}$$

and the associated states

$$\begin{aligned} |\pi_m\rangle &= L_{-\pi_m}|\,\rangle \in V_k^* \\ \langle\pi_n| &= \langle\,|L_{\pi_n} \in V_{k'}. \end{aligned} \tag{4.22}$$

Because $V_{k'}$ is always a Verma module, containing no cosingular vectors, the set of states $\langle\pi_n|$ are always a linearly independent basis for $V_{k'}$.

We now consider the simplest case of a Virasoro vertex operator, where the dual Verma modules V_k^* and $V_{k'}^*$ contain no cosingular vectors and are therefore isomorphic to the irreducible representations I_k and $I_{k'}$. In this case, the corresponding resolutions are all trivial, with $M_1 = M_1' = 0$. Because there are no singular or cosingular vectors in the modules M_0, M_0', the states defined through (4.22) form complete bases for these Fock modules. As discussed above, to characterize the vertex operator $\Phi(z)$ it will suffice to give all the matrix elements with respect to these bases. By counting weights, we see that the only mode of the vertex operator which can contribute to a matrix element between states $\langle\pi_n|$ and $|\pi_m\rangle$ is $\phi_{-h-x+m-n}$, where $x = k' - h - k$, so that

$$\langle\pi_n|\Phi(z)|\pi_m\rangle = z^{x+n-m}\langle\pi_n|\phi_{-h-x+m-n}|\pi_m\rangle. \tag{4.23}$$

The only nontrivial matrix element between the highest weight states is then

$$\langle\,|\phi_{-h-x}|\,\rangle. \tag{4.24}$$

We will choose the normalization of the vertex operator to be such that this matrix element is equal to 1. Just as the Virasoro commutation relations are used to construct the Shapovalov form on the Verma module, the Virasoro relations coupled with (4.16) can be used to relate an arbitrary matrix element (4.23) to the canonical element (4.24). That this procedure defines each matrix element in a well-defined fashion, and that the resulting operator satisfies (4.16) follow from the condition that (4.16) defines a representation of the Virasoro algebra, which is equivalent to the Jacobi-type equation

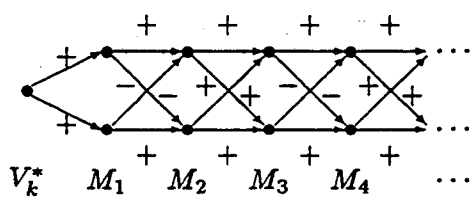
$$\begin{aligned}
&[L_n,[L_m,\phi_p]] + [L_m,[\phi_p,L_n]] + [\phi_p,[L_n,L_m]] \\
&= (m(h-1)-p)(n(h-1)-m-p) - (n(h-1)-p)(m(h-1)-n-p) \\
&\qquad +(m-n)((n+m)(h-1)-p) \\
&= 0.
\end{aligned} \tag{4.25}$$

Thus, we have shown that when the dual Verma modules V_k^* and $V_{k'}^*$ contain no cosingular vectors, there exists a unique vertex operator $\Phi(z)$ of any weight h. In addition, the matrix elements (4.23) completely characterize $\Phi(z)$. The resulting realization of $\Phi(z)$ as a differential operator on a coadjoint orbit representation can be explicitly calculated by choosing

a monomial basis for R and relating this basis through a nonsingular linear transformation to the basis $|\pi_n\rangle$.

We now consider the case where the dual Verma modules V_k^* and $V_{k'}^*$ contain cosingular vectors and the resolutions (M_i, d_i) and (M_i', d_i') are nontrivial. As the most general case, we assume that both weights k and k' are associated with representations whose Verma modules have the singular vector structure graphed in Figure 3.2b, and we consider the resulting dual Verma module resolutions. For any of the unitary representations with a simpler singular vector structure the resulting resolution is equivalent to taking this resolution and simplifying by setting to 0 the modules which correspond to singular vectors which do not exist in the simpler representation.

A choice of signs on the intertwining operators which allows us to construct the dual Verma module resolutions is as follows



In this resolution, each module $M_i, i > 0$ contains a direct sum of two dual Verma modules $V_{h(v_i)}^*, V_{h(w_i)}^*$ associated with cosingular vectors v_i, w_i in V_k^* with weights $h(v_i), h(w_i)$. The arrows are (surjective) intertwining operators with the given signs.

From the discussion in Section 3.3 on the intertwining operators between dual Verma modules, we know that the kernels of the operators d_0 and d_0' are precisely the orbits $\mathcal{U}(\mathfrak{g})|\,\rangle$ of the highest weight vectors in the modules V_k^*, $V_{k'}^*$. For every cosingular vector $|v\rangle$ in $V_{k'}^*$ at level n, there exists a singular vector $\langle s|$ in $V_{k'}$, which can be written in the form

$$\langle s| = \sum_{\pi_n} c_{\pi_n} \langle \pi_n|, \tag{4.26}$$

where c_{π_n} are real coefficients. Such a singular vector is annihilated by all lowering operators,

$$\langle s|L_{-n} = 0 \quad \forall n > 0. \tag{4.27}$$

It follows that

$$\langle s|\pi_n\rangle = 0 \quad \forall |\pi_n\rangle \in V_{k'}^*, n > 0, \tag{4.28}$$

and therefore that for any vector $|w\rangle \in V_{k'}^*$ satisfying $\langle s|w\rangle \neq 0$, we have

$$d_0'|w\rangle \neq 0. \tag{4.29}$$

A consequence of this condition is that the matrix element

$$\langle s|\Phi^0(z)|\pi_n\rangle \tag{4.30}$$

must vanish for any state $|\pi_n\rangle$ in the orbit of the highest weight state in V_k^*, since

$$d_0'\Phi^0(z)|\pi_n\rangle = \Phi^1(z)d_0|\pi_n\rangle = 0. \tag{4.31}$$

This places a set of algebraic conditions on the weights h, k, k' for which vertex operators are allowed to exist. From the singular vector structure of the Verma module, we know that there are (at most) two singular vectors s, t, corresponding to the cosingular vectors v_1', w_1' in $V_{k'}^*$ with the property that the modules generated by s and t contain all other singular vectors in $V_{k'}$. Thus, the set of all algebraic conditions associated with singular vectors in $V_{k'}$ can be completely described by the condition that matrix elements containing s or t must vanish,

$$\begin{aligned} \langle s|\Phi(z)|\pi_n\rangle &= 0, \\ \langle t|\Phi(z)|\pi_n\rangle &= 0 \quad \forall |\pi_n\rangle \in V_k^*, n > 0. \end{aligned} \tag{4.32}$$

A further condition on the matrix elements of $\Phi(z)$ arises from the fact that at every level n at which there is a cosingular vector in V_k^*, there is a linear relation between the states $|\pi_n\rangle$. Each such linear relation gives a constraint to the matrix elements of Φ of the form

$$\sum_{\pi_n} c_{\pi_n}\langle \pi_m|\Phi(z)|\pi_n\rangle = 0. \tag{4.33}$$

Again, because of the cosingular vector structure, this infinite set of constraints can be completely described by the two families of constraints arising from the cosingular vectors v_1, w_1.

Thus, we have derived a set of algebraic constraints on the weights h, k, k' which must be satisfied for a vertex operator to exist as a map of dual Verma module resolutions. Although we have derived these constraints from the point of view of chain complexes, the constraints (4.32) and (4.33) also hold for the simpler construction of a vertex operator of weight h between irreducible representations with highest weights k, k'. In the case of

irreducible representations, the dual elements $\langle s|$ corresponding to singular vectors must vanish. Similarly, the linear conditions on vectors $|\pi_n\rangle$ associated with cosingular vectors in the dual Verma module still hold in the irreducible representation.

Just as the set of all matrix elements defines a vertex operator uniquely in the case where the resolutions are trivial, the determination of all matrix elements of the form

$$\langle \pi_n|\Phi(z)|\pi_m\rangle \tag{4.34}$$

determines the action of the vertex operator on the cohomology uniquely, and thus defines a vertex operator on the associated irreducible representation (the uniqueness of this representation essentially follows from the fact that the Shapovalov form is positive-definite on an irreducible unitary representation). Since the constraints (4.32) and (4.33) are the only extra conditions placed on a vertex operator between irreducible representations, we have essentially proven the existence of such vertex operators whenever these conditions are satisfied. As mentioned above, we do not have a proof that in all cases these vertex operators can be realized as chain maps, but we believe this to be the case.

We will now discuss some special cases of the constraints (4.32) and (4.33), and describe the resulting conditions on the fusion algebras of the related conformal field theories. For the purposes of this discussion, we shall find it useful to have explicit calculations of various matrix elements of a vertex operator $\Phi(z) = z^x \sum \phi_{-h-x+n} z^{-n}$ taking a representation with weight k to one with weight $k' = k + h + x$.

$$\langle\, |\phi_{-h-x}|\,\rangle = 1 \tag{4.35}$$

$$\langle\, |\phi_{-h-x+1}L_{-1}|\,\rangle = -x \tag{4.36}$$

$$\langle\, |\phi_{-h-x+2}L_{-2}|\,\rangle = h - x \tag{4.37}$$

$$\langle\, |\phi_{-h-x+2}L_{-1}^2|\,\rangle = x(x-1) \tag{4.38}$$

$$\langle\, |\phi_{-h-x+3}L_{-3}|\,\rangle = -2h - x \tag{4.39}$$

$$\langle\, |\phi_{-h-x+3}L_{-2}L_{-1}|\,\rangle = x(h+x-1) \tag{4.40}$$

$$\langle\, |\phi_{-h-x+3}L_{-1}^3|\,\rangle = -x(x-1)(x-2) \tag{4.41}$$

$$\langle\, |L_1\phi_{-h-x-1}|\,\rangle = 2h + x \tag{4.42}$$

$$\langle\, |L_2\phi_{-h-x-2}|\,\rangle = 3h + x \tag{4.43}$$

$$\langle\, |L_1^2\phi_{-h-x-2}|\,\rangle = (2h+x)(2h+x+1) \tag{4.44}$$

$$\langle\, |L_3\phi_{-h-x-3}|\,\rangle = 4h + x \tag{4.45}$$

$$\langle\, |L_1 L_2 \phi_{-h-x-3}|\, \rangle = (3h+x+1)(2h+x) \tag{4.46}$$

$$\langle\, |L_1^3 \phi_{-h-x-3}|\, \rangle = (2h+x)(2h+x+1)(2h+x+2) \tag{4.47}$$

First, we consider the case where $c > 1$. For any values of h, k and k', where $k, k' \neq 0$, all of the representations have trivial resolutions, and a vertex operator can be constructed through the above reasoning. It is interesting to note that although vertex operators between modules of this form are well-known in terms of multiple bosonic fields (we will discuss this further in the following section), we can use the above results to construct any vertex operator of this type in terms of a single bosonic field with modes $a_n, n \in \mathbb{Z}$. The representation of such a vertex operator is rather complicated algebraically, however, and does not appear to have as simple an expression as when multiple bosonic fields are used.

Now, let us consider the case where either $k = 0$ or $k' = 0$, indicating a vertex operator which either acts on or maps to the vacuum module. The dual Verma module with highest weight $h = 0$ has precisely one cosingular vector, at level 1. Thus, if $k' = 0$, the constraint that (4.42) is 0 indicates that $x = -2h$, so $h = k$. Similarly, if $k = 0$, setting (4.36) to 0 gives $k' = h$. Thus, we have the result that a vertex operator of weight h acting on the vacuum in either direction gives the module of weight h. This is precisely what we would expect from standard physical theories.

Finally, we consider a slightly more complicated example, that of a rational conformal field theory with $c = 1/2$. From the equations (2.75), (2.75) describing the discrete series of Virasoro representations, one finds that the only unitary representations corresponding to this choice of central charge are those with highest weights $h = 0, h = 1/16$, and $h = 1/2$. We know that there is a cosingular state in the module $V^*_{1/2}$ at level 2. The linear combination of states in this module which vanishes as a result can be calculated from (2.282) to be

$$(L_{-2} - \frac{3}{4}L_{-1}^2)|\,\rangle = 0. \tag{4.48}$$

If we take $k = 1/2$, this equation leads to the constraint on h and k',

$$3x^2 + x - 4h = 0. \tag{4.49}$$

The solutions for this where k' and h both correspond to unitary representations are $(h, k') = (0, 1/2)$, $(h, k') = (1/2, 0)$, and $(h, k') = (1/16, 1/16)$. We can perform a similar analysis

using the initial module with $k = 1/16$. The consequent vanishing condition in the dual Verma module is

$$(L_{-2} - \frac{4}{3}L_{-1}^2)|\,\rangle = 0. \tag{4.50}$$

The pairs h, k' which satisfy this vanishing condition are $(h, k') = (0, 1/16)$, $(h, k') = (1/16, 0)$, $(h, k') = (1/16, 1/2)$, and $(h, k') = (1/2, 1/16)$. We could go on and consider the constraints imposed by the singular vectors in the Verma module $V_{k'}$ at level 2. In order to ensure that all constraints are satisfied, it is also necessary to include the constraints arising from the second cosingular vector in the dual Verma module, which in this case appears at level 3, and also the constraints from the corresponding singular vectors at level 3 in the Verma module. However, none of these other constraints places any further restriction upon the vertex operators which are allowed in a $c = 1/2$ conformal field theory. We have thus calculated the restrictions on the fusion algebra of this CFT. Traditionally, a CFT fusion algebra is written in the form

$$[h][k] = \sum_{k'}[k'], \tag{4.51}$$

where k' is summed over all weights for which there exists a nontrivial vertex operator of weight h connecting the irreducible representations with weights k, k'. In this notation, our results for the $c = 1/2$ fusion rules are

$$\begin{aligned} [1/16]\,[1/16] &= [0] + [1/2] \\ [1/16]\,[1/2] = [1/2]\,[1/16] &= [1/16] \\ [1/16]\,[1/16] &= [0]\,. \end{aligned} \tag{4.52}$$

(We do not write the fusion rules for the vacuum module [0] explicitly, since these fusion rules are standard for all CFT's.) This is precisely the fusion algebra for the set of operators in the $c = 1/2$ CFT which corresponds to the Ising model [13].

We will now say a few words about the relationship between the analysis in this section and the differential equations approach to CFT. In their initial work on conformal field theory [13], BPZ defined vertex operators primarily as objects which appear inside correlation functions, and not in terms of an explicit action on a representation space. From this point of view, they analyzed the operator product expansion (OPE) of a product of two vertex operators, represented as a sum over other vertex operators and so-called "descendant fields" which are defined by applying Virasoro operators to the primary fields

(vertex operators). They showed that the resulting coefficient for every descendant field in a given conformal family (descending from a particular primary field) could be written in terms of the coefficient of the primary field, multiplied by a constant depending only upon the conformal weights of the three primary fields involved. This set of equations is precisely analogous to the determination of the matrix elements (4.23). This equivalence can be made precise by observing that these matrix elements are essentially components of a correlation function of 3 vertex operators of the form (4.1). In their work, BPZ also derive a set of constraint equations on the fusion algebra analogous to (4.32). They write these equations in terms of a set of differential operators which must be satisfied by certain correlation functions. Although these constraints are effectively equivalent to those imposed here, from the field theory formalism it is somewhat less clear that these constraints can all be consistently imposed, and what their effect is on the structure of the relevant Virasoro representations.

Finally, we conclude this section with a simple example of a vertex operator whose action can be computed purely in terms of the Virasoro generators. A standard result in conformal field theory is that any vertex operator $\Phi(z)$ satisfies the equation

$$\Phi(z) = \mathrm{e}^{zL_{-1}}\Phi(0)\mathrm{e}^{-zL_{-1}}. \tag{4.53}$$

This follows immediately from the differential equation

$$[L_{-1}, \Phi(z)] = \frac{\partial}{\partial z}\Phi(z). \tag{4.54}$$

A result of (4.53) is that the action of this vertex operator on the physical vacuum is described completely in terms of the Virasoro generator L_{-1} by

$$\Phi(z)|\Omega\rangle = \mathrm{e}^{zL_{-1}}|\,\rangle = \sum_{i=0}^{\infty}\frac{z^i L_{-1}^i}{i!}|\,\rangle, \tag{4.55}$$

where $|\,\rangle = \phi_{-h}|\Omega\rangle$ is the highest weight state in the module associated with the primary field Φ. This result follows because the physical vacuum $|\Omega\rangle$ is annihilated by L_{-1}. In general, acting on a highest weight state of an arbitrary module with the vertex operator (4.53) does not give a result which has such a simple description in terms of the Virasoro generators. We shall demonstrate in Section 4.4 that the explicit realization of the Virasoro generators $\hat{L}_n$ on coadjoint orbit dual Verma modules leads to a more general version of this result.

4.3 Free field vertex operators

In this section we briefly review the results of Felder [27] and others on the construction of screened vertex operators in terms of resolutions over Feigin-Fuchs modules. The fundamental object used in this construction is the standard bosonic vertex operator

$$V_\beta(z) = :\mathrm{e}^{\beta\phi(z)}: = z^{\beta a_0} \exp\left(\beta \sum_{n=1}^{\infty} \frac{a_{-n}}{n} z^n\right)\left(-\beta \sum_{n=1}^{\infty} \frac{a_n}{n} z^{-n}\right), \tag{4.56}$$

where the bosonic field

$$\phi(z) = a_0 \ln z + \sum_{n=1}^{\infty} \frac{a_n}{n} z^{-n} + \sum_{n=1}^{\infty} \frac{a_{-n}}{n} z^n \tag{4.57}$$

is expressed in terms of the bosonic modes (3.12). The operator $V_\beta(z)$ is a vertex operator of weight $\beta(\beta - 2\alpha_0)$ with respect to the realization of the Virasoro algebra (3.13) on Feigin-Fuchs type Virasoro modules. This operator changes the Fock space of the Feigin-Fuchs module by increasing the eigenvalue (2α) of a_0 by 2β.

Setting the background charge α_0 to 0, we can combine a set of D bosonic fields to give a realization of the Virasoro algebra on a Fock space with central charge $c = D$, with vertex operators carrying a momentum vector $\vec{\beta}$ and having a conformal weight of $\vec{\beta} \cdot \vec{\beta}$. These are the original vertex operators from string theory [37]. From these vertex operators, for $D > 1$ it is possible to construct a vertex operator with arbitrary weight h, acting on a module with highest weight k and resulting in a module with highest weight k' for any $k, k' > 0$. This can be accomplished by simply choosing an initial state of momentum $\vec{\beta}$ where $k = |\beta|^2$ and a vertex operator with momentum $\vec{\gamma}$ where $h = |\gamma|^2$ and $k' - h - k = 2\,\vec{\beta} \cdot \vec{\gamma}$. Note, however, that not only does this construction only work for integral c, but also that a larger Fock space, containing at least two sets of bosonic field modes, is necessary. The construction described in the previous section realizes the same vertex operators in turns of a single bosonic field. Whether this observation can be used to a useful end in conformal field theory is an open question; however, it is a useful example of the general applicability of the methods in this thesis.

Using the above description of a free field vertex operator, Dotsenko and Fateev were able to explicitly construct vertex operators on the discrete series of Virasoro modules in terms of Feigin-Fuchs representations [22]. This work was subsequently explained in the language of BRST cohomology and resolutions by Felder, who introduced explicitly the associated intertwining (BRST) operators [27]. In this approach, a value of $c = 1 - 24\alpha_0^2$

corresponding to a particular rational CFT is fixed, and a set of intertwining operators Q_m are defined in terms of the vertex operators (4.56) by

$$Q_m = \frac{1}{m}\int V_{\alpha_+}(z_0)\cdots V_{\alpha_+}(z_{m-1})\prod_{i=0}^{m-1}\mathrm{d}z_i\left[\frac{\mathrm{e}^{2\pi i\alpha_+^2 m}-1}{\mathrm{e}^{2\pi i\alpha_+^2}-1}\right]. \tag{4.58}$$

where

$$\alpha_\pm = \alpha_0 \pm \sqrt{1+\alpha_0^2}, \tag{4.59}$$

and the contours of the integration variables $z_0,\ldots,z_{m-1}$ are concentric circles about $z=0$. Similarly, the screened vertex operators are defined by

$$V_{n'n}^{r'r} = \int V_{1-n'\alpha_-/2-n\alpha_+/2}(z)V_{\alpha_-}(z_0)\cdots V_{\alpha_-}(z_{r'})V_{\alpha_+}(w_0)\cdots V_{\alpha_+}(w_r)\prod_{i=1}^{r'}\mathrm{d}z_i\prod_{i=1}^{r}\mathrm{d}w_i, \tag{4.60}$$

with similar integration contours. The integer indices n, n', r, r' in the screened vertex operators, and m in the intertwining operators, specify which representations in the discrete series these operators map to which other representations. We will not study these operators in any significant detail, but we will give a brief synopsis of the results achieved using this approach, and compare to the methods used here.

The vertex operators and intertwiners above do not exactly commute; heuristically, these operators obey a relation of the form

$$QV(z) = \mathrm{e}^{2\pi i\theta}V(z)Q, \tag{4.61}$$

with a generally nonzero phase θ. This relation is highly reminiscent of the q-commutation relations used to define quantum groups. In fact, in the related context of free field theories using Wakimoto realizations of affine algebras, a similar nontrivial commutation relation gives precisely the form of a particular class of quantum group relations, and leads into a rather subtle relationship between the representation theory of the affine algebras and quantum groups [18, 19]. In the case of the Virasoro algebra, these nontrivial phases can simply be removed by a redefinition of the phases of the vertex operators. It can then be shown [27] that the operators Q_m act on the Feigin-Fuchs modules associated with the discrete series Virasoro representations, such that $Q^2=0$, and in such a way that the resulting chain complex has a cohomology resolving the corresponding irreducible representation. The operators $V_{n'n}^{r'r}$ are then vertex operators describing a chain map between two resolutions of irreducible representations. These screened vertex operators can then be used to construct CFT correlation functions in the fashion discussed in Section 4.5

The elementary objects of the form

$$Q_1 = \int V_{\alpha_+}(z)\mathrm{d}z \tag{4.62}$$

play a role in this theory analogous to the screening operators defined in Section 4.1 (in fact, this operator has the effect of screening a single unit of charge, and was the initial motivation for the term "screening operator"). This object clearly commutes with the Virasoro generators, since the conformal weight of the bosonic vertex operator in the integral is 1. However, not only do the modes of this operator not obey the expected commutation relations among themselves, but the operator itself is not generally well-defined since in most cases the initial power of z (which is equal to $2\alpha_+\alpha$), is not an integer. Only for the proper combinations of m and α does the operator Q_m have a sensible definition as an operator on the bosonic Fock space. The connection between the field-theoretic point of view and the algebraic perspective in this case remains rather unclear.

To conclude this review of the free field approach to constructing vertex operators for Feigin-Fuchs representations, we now give an example of a specific computation of the action of an intertwining operator Q_m. This explicit calculation hopefully demonstrates clearly the connection between the language of fields and the more algebraic language of modules and resolutions, and illustrates the computational complexity of moving between these descriptions.

We consider again our favorite example, the discrete series Virasoro representation with $c = h = 1/2$. In terms of Feigin-Fuchs modules, this irreducible Virasoro representation has a two-sided resolution; denoting by F_h the Feigin-Fuchs representation with highest weight h, the central part of this resolution looks like

$$\cdots \to F_{7\frac{1}{2}} \xrightarrow{Q_1} F_{2\frac{1}{2}} \xrightarrow{Q_2} F_{\frac{1}{2}} \xrightarrow{Q_1} F_{3\frac{1}{2}} \xrightarrow{Q_2} F_{17\frac{1}{2}} \to \tag{4.63}$$

We will analyze the particular intertwining operator

$$Q_2 : F_{5/2} \to F_{1/2} \tag{4.64}$$

as it appears in this module (in the place of the usual operator d_{-1}). From the discussion earlier in this chapter, we expect that the image under Q_2 of the module $F_{5/2}$ will be a submodule of $F_{1/2}$ with the highest weight vector $|\,\rangle'$ from $F_{5/2}$ being mapped to a singular vector at level 2 in $F_{1/2}$. As we know, this singular vector must be the vector

$$(L_{-2} - \frac{3}{4}L_{-1}^2)|\,\rangle \tag{4.65}$$

where $|\,\rangle$ is the highest weight vector in $F_{1/2}$.

The appropriate values of α_0, α, and α_+ for the Feigin-Fuchs module $F_{5/2}$ are

$$\begin{aligned} \alpha_0 &= \frac{1}{4\sqrt{3}} \\ \alpha &= -10\alpha_0 \\ \alpha_+ &= 8\alpha_0 \end{aligned} \tag{4.66}$$

Writing the action of the operator Q_2 on the highest weight vector of this module explicitly, we have

$$\begin{aligned} Q_2|\,\rangle' &= \frac{1}{2}\int \mathrm{d}u\,\mathrm{d}v\; v^{-10/3}u^{-2/3}\exp(\alpha_+\sum_n \frac{a_{-n}}{n}u^n) \\ &\qquad \exp(-\alpha_+\sum_n \frac{a_n}{n}u^{-n})\exp(\alpha_+\sum_n \frac{a_{-n}}{n}v^n)|\,\rangle. \end{aligned} \tag{4.67}$$

It is a standard exercise to commute the annihilation operators from $V_{\alpha_+}(u)$ with the creation operators from $V_{\alpha_+}(u)$. This exchange gives an extra factor of $(1-v/u)^{2\alpha_+^2}$, so we have

$$Q_2|\,\rangle' = \frac{1}{2}\int \mathrm{d}u\,\mathrm{d}v\; v^{-10/3}u^{-2/3}(1-v/u)^{8/3}\exp(\alpha_+\sum_n \frac{a_{-n}}{n}(u^n+v^n))|\,\rangle. \tag{4.68}$$

Changing variables to $u = z, v = wz$ we have

$$Q_2|\,\rangle' = \frac{1}{2}\int \mathrm{d}z\,\mathrm{d}w\; z^{-3}w^{-10/3}(1-w)^{8/3}\exp\left(\alpha_+\sum_n \frac{a_{-n}}{n}(z^n(1+w^n))\right)|\,\rangle. \tag{4.69}$$

We can now explicitly integrate over z; the remaining integral over w is around a contour surrounding the branch points $w = 0$ and $w = 1$,

$$Q_2|\,\rangle' = \pi i\int \mathrm{d}w\; w^{-10/3}(1-w)^{8/3}\left[\alpha_+\frac{1+w^2}{2}a_{-2}|\,\rangle + \alpha_+^2\frac{(1+w)^2}{2}a_{-1}^2|\,\rangle\right]. \tag{4.70}$$

Deforming the contours to run directly from $w = 0$ to $w = 1$ (these are the original contours used by Dotsenko and Fateev in [22]), the integrals become standard β-function integrals. Writing the states $a_{-2}|\,\rangle$ and $a_{-1}^2|\,\rangle$ in the polynomial forms $|z_2\rangle$, $|z_1^2\rangle$ respectively, we have a result proportional to

$$\begin{aligned} \frac{\Gamma(-7/3)\Gamma(11/3)}{\Gamma(4/3)}\left(|z_2\rangle + 8\alpha_0|z_1^2\rangle\right) &+ \frac{\Gamma(-4/3)\Gamma(11/3)}{\Gamma(7/3)}\left(16\alpha_0|z_1^2\rangle\right) \\ &+ \frac{\Gamma(-1/3)\Gamma(11/3)}{\Gamma(10/3)}\left(|z_2\rangle + 8\alpha_0|z_1^2\rangle\right) \end{aligned} \tag{4.71}$$

Using the usual property $x\Gamma(x) = \Gamma(x+1)$, we see that this result is proportional to the vector

$$|z_2\rangle - 6\alpha_0|z_1^2\rangle. \tag{4.72}$$

From the explicit Feigin-Fuchs realization of the Virasoro algebra (3.13) with the value $\alpha = 6\alpha_0$, we can compute the singular vector

$$\begin{aligned}(L_{-2} - \frac{3}{4}L_{-1}^2)|\,\rangle &= (\alpha+\alpha_0)|z_2\rangle + \frac{1}{4}|z_1^2\rangle) - \frac{3}{4}(\alpha^2|z_1^2\rangle + \alpha|z_2\rangle) \\ &= \frac{5}{2}\alpha_0|z_2\rangle - \frac{15}{48}|z_1^2\rangle \\ &= \frac{5}{2}\alpha_0(|z_2\rangle - 6\alpha_0|z_1^2\rangle).\end{aligned} \tag{4.73}$$

Thus, the vector $Q_2|\,\rangle'$ is indeed proportional to the singular vector in $F_{1/2}$.

This example demonstrates that although the vertex operators and intertwining operators for the free field theory realizations of minimal models are expressed in a very different language from the algebraic concepts of modules and resolutions developed in the previous sections, the effective content of the two formalisms is equivalent. The field-theoretic approach used by authors such as Felder is more useful for computing certain properties of correlation functions and partition functions which are expressable in terms of that language. On the other hand, as this example calculation demonstrates, certain underlying mathematical properties of the theory are much more easily accessible using the algebraic formulation. Hopefully, in the future these disparate formalisms will merge somewhat, clarifying aspects both of conformal field theory and the underlying group theory. Recent work which will be discussed in the next chapter indicates that this convergence of perspectives is indeed occurring.

4.4 Coadjoint orbit vertex operators

We will now discuss the specific form of the vertex operators of weight h

$$\Phi(z) : V_k^* \to V_{k'}^* \tag{4.74}$$

which are defined on the coadjoint orbit realizations of the Virasoro dual Verma modules. As we showed in Section 4.2, these vertex operators exist for all values of the highest weights k, k' which correspond to Virasoro dual Verma modules without cosingular vectors. In addition, whenever such vertex operators can be defined on chain complexes of dual Verma

modules with cosingular vectors, their action on all physical states (states in the orbit of the highest weight vector of V_k^*) is completely determined by the matrix elements (4.23). Our first goal in this section is to translate these matrix elements into the coefficients of the modes of the vertex operator, when these modes are expressed as differential operators on the polynomial space R. We then show that when $x = 0$ the action of these operators on any highest weight state are closely related to the operator $\hat{L}_{-1}$ associated with the Virasoro generator L_{-1}, and that the action of these operators on any state can be defined completely in terms of Virasoro generators $\hat{L}_n$.

Explicitly, we can write the modes of a vertex operator (4.74) as individual differential operators

$$\phi_{-h-x-n} = \sum_{m,l>0:m-l=n}^{\infty} \sum_{\pi_m,\pi_l} p(\pi_m,\pi_l)(\prod_i z_{m_i} \prod_j \frac{\partial}{\partial z_{l_j}}) \tag{4.75}$$

which act directly on the space R. To calculate the coefficients $p(\pi_m, \pi_l)$ in these differential operators in terms of the matrix elements (4.23), we must relate the vectors $|\pi_n\rangle$ to the canonical basis for R of monomials. Similarly, we must relate the basis $\langle\pi_m|$ of the Verma module $V_{k'}$ to the set of differential operators

$$\partial_{\pi_n} = \left(\prod_i \frac{\partial}{\partial z_{n_i}}\right)|_{z_j=0} \tag{4.76}$$

where the evaluation at $z_j = 0$ for all $j > 0$ has the effect of extracting the constant term in a polynomial after the action of the product of partial derivatives.

The linear relationship between the vectors $|\pi_n\rangle$ and the monomials in R of degree n up to level 2 was given by (2.282); we expand this calculation explicitly to level 3,

$$\begin{aligned}
|\,\rangle &= 1 \\
\hat{L}_{-1}|\,\rangle &= -2h|z_1\rangle, \\
\hat{L}_{-1}^2|\,\rangle &= -6h|z_2\rangle + (4h^2+2h)|z_1^2\rangle \\
\hat{L}_{-2}|\,\rangle &= -(4h+\frac{c}{2})|z_2\rangle + 3h|z_1^2\rangle. \\
\hat{L}_{-1}^3|\,\rangle &= -24h|z_3\rangle + (24h+36h^2)|z_1z_2\rangle + (-8h^3-12h^2-4h)|z_1^3\rangle \\
\hat{L}_{-1}\hat{L}_{-2}|\,\rangle &= -(16h+2c)|z_3\rangle + (26h+c+8h^2+ch)|z_1z_2\rangle + (-6h^2+6h)|z_1^3\rangle \\
\hat{L}_{-3}|\,\rangle &= -(6h+2c)|z_3\rangle + (13h+c)|z_1z_2\rangle + 4h|z_1^3\rangle.
\end{aligned} \tag{4.77}$$

The relationship between the states $\langle\pi_n|$ and the derivative operators ∂_{π_m} is easy to calcu-

late; up to level 3, we have

$$\begin{aligned}
\langle\,| &= 1|_{z_j=0} \\
\langle\,|\hat{L}_1 &= -\frac{\partial}{\partial z_1}|_{z_j=0} \\
\langle\,|\hat{L}_1^2 &= \frac{\partial^2}{\partial z_1^2}|_{z_j=0} \\
\langle\,|\hat{L}_2 &= -\frac{\partial}{\partial z_2}|_{z_j=0} \\
\langle\,|\hat{L}_1^3 &= -\frac{\partial^3}{\partial z_1^3}|_{z_j=0} \\
\langle\,|\hat{L}_1\hat{L}_{-2} &= \frac{1}{2}\frac{\partial}{\partial z_3} + \frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}|_{z_j=0} \\
\langle\,|\hat{L}_3 &= -\frac{\partial}{\partial z_3}|_{z_j=0}
\end{aligned} \tag{4.78}$$

Clearly, at each level n, inverting these relationships involves calculating the inverse of a $\pi(n) \times \pi(n)$ matrix. The matrix relating the basis $\langle\pi_n|$ to the operators ∂_{π_m} is always invertible; the matrix relating $|\pi_n\rangle$ to monomials in R is only invertible when V_k^* contains no cosingular vectors. We do not have any kind of general prescription for calculating this inverse; for the relations (4.77) at level 2 the inverse relations are given by

$$\begin{aligned}
|z_2\rangle &= \frac{1}{\gamma}3h\hat{L}_{-1}^2|\,\rangle - 2h(2h+1)\hat{L}_{-2}|\,\rangle \\
|z_1^2\rangle &= \frac{1}{\gamma}(4h+\frac{c}{2})\hat{L}_{-1}^2|\,\rangle - 6h\hat{L}_{-2}|\,\rangle,
\end{aligned} \tag{4.79}$$

where

$$\gamma = 16h^3 + (2c-10)h^2 + ch \tag{4.80}$$

When $h = 0$ or the condition (2.283) is satisfied the determinant γ is 0 and the inverse transformation is undefined.

In Section 4.2, we calculated in (4.35-4.46) some of the matrix elements of $\Phi(z)$ between states of low level. The remaining matrix elements between states of level ≤ 2 are given by

$$\langle\,|L_1\phi_{-h-x}L_{-1}|\,\rangle = x + 2k - 2hx - x^2 \tag{4.81}$$

$$\langle\,|L_1\phi_{-h-x+1}L_{-1}^2|\,\rangle = (2h+x-2)x(x-1) - x(4k+2) \tag{4.82}$$

$$\langle\,|L_1\phi_{-h-x+1}L_{-2}|\,\rangle = (h-x-1)(2h+x) \tag{4.83}$$

$$\langle\,|L_1^2\phi_{-h-x-1}L_{-1}|\,\rangle = 8hk + 4kx - 4xh^2 - 4hx^2 - x^3 + 2hx + x^2 \tag{4.84}$$

$$\begin{aligned}
\langle\,|L_2\phi_{-h-x-1}L_{-1}|\,\rangle &= x-3hx-x^2 && (4.85)\\
\langle\,|L_1^2\phi_{-h-x}L_{-1}^2|\,\rangle &= 8k^2+4k+8kx-16hkx-8kx^2+2x-4xh^2-2hx+x^2\\
&\quad +4h^2x^2+4hx^3+x^4-10hx^2-4x^3 && (4.86)\\
\langle\,|L_1^2\phi_{-h-x}L_{-2}|\,\rangle &= 6k+4h^3-3hx^2-x^3-6h^2-9hx-3x^2+2h+4x && (4.87)\\
\langle\,|L_2\phi_{-h-x}L_{-1}^2|\,\rangle &= 6k+3hx^2+x^3+2x-3hx-3x^2 && (4.88)\\
\langle\,|L_2\phi_{-h-x}L_{-2}|\,\rangle &= c/2-2h+4k+2x+3h^2-2hx-x^2 && (4.89)
\end{aligned}$$

From (4.35-4.47), (4.77-4.78) and (4.81-4.89), we can write the explicit formulae for the low-order terms in ϕ_{-h-x-n} for small $|n|$. Using again the notation that $\mathcal{D}_n$ indicates terms containing derivatives $\frac{\partial}{\partial z_m}$ with $m \geq n$, we have

$$\begin{aligned}
\phi_{-h-x+2} &= \frac{1}{\gamma}\left[2h(2h+1)(x-h)+6hx(x-1)\right]\frac{\partial}{\partial z_2}\\
&\quad +\frac{1}{\gamma}\left[6h(x-h)+(c/2+4h)x(x-1)\right]\frac{\partial^2}{\partial z_1^2}+\mathcal{D}_3\\
\phi_{-h-x+1} &= \frac{x}{2h}\frac{\partial}{\partial z_1}+\mathcal{D}_2\\
\phi_{-h-x} &= 1+\frac{x+2k-2hx-x^2}{2h}z_1\frac{\partial}{\partial z_1}+\mathcal{D}_2 && (4.90)\\
\phi_{-h-x-1} &= -(2h+x)z_1+\mathcal{D}_1\\
\phi_{-h-x-2} &= -(3h+x)z_2+\frac{1}{2}(2h+x)(2h+x+1)z_1^2+\mathcal{D}_1\\
\phi_{-h-x-3} &= -(4h+x)z^3+(4h+3x/2+6h^2+5hx+x^2)z_1z_2\\
&\quad -\frac{1}{6}(2h+x)(2h+x+1)(2h+x+2)z_1^3+\mathcal{D}_1
\end{aligned}$$

Although this explicit computation of the initial terms in the vertex operators is in general rather unenlightening, we can nevertheless extract some interesting information from this calculation.

The first observation we can make is that the general form of the vertex operator has a nontrivial dependence on all three parameters k, h, and x (generally, this dependence appears for all terms after the leading order; here, we have only bothered to compute leading order terms except for ϕ_{-h-x}). Although one might naturally expect this dependence based on the method of construction, a cursory examination of the free field vertex operator (4.56) seems to indicate that $V_\beta(z)$ depends only upon the parameter β, and thus upon the conformal weight $h=\beta(\beta-2\alpha_0)$ of the vertex operator. Upon closer inspection, this problem becomes slightly less acute; the exponent βa_0 is equal to the quantity we denote

by x, and the vertex operator $V_\beta(z)$ always acts from a space of weight $k = \alpha(\alpha - 2\alpha_0)$ to a space of weight $k' = (\alpha+\beta)(\alpha+\beta-2\alpha_0)$, reducing the dimensionality of the space of allowed vertex operators to 2. Nonetheless, the action of this vertex operator on the Fock space certainly depends on one parameter fewer than the vertex operators we have constructed on the coadjoint orbit dual Verma modules. This may be an indication that finding a closed form expression for these coadjoint orbit vertex operators will be a particularly difficult task. Certainly, the construction of a field theory from such operators will be complicated by this fact; we would like to have a construction in which the vertex operators associated with a single primary field depend only upon the weight of that primary field.

It may be, however, that the extra dependencies which this operator exhibits are a result of two features of the explicit coadjoint orbit representations. The first feature is that the Virasoro operator realizations on these representations are somewhat *less* dependent upon the weight of the representation, in the sense that all the raising operators $L_n, n > 0$ take an identical form independently of the weight of the associated representation. The second feature is that the coadjoint orbit vertex operators can be defined for arbitrary triplets of weights h, k, k' when $c > 1$. All known free field representations on a single bosonic Fock space place strong restrictions on the weights and central charges for which vertex operators in this regime can be constructed. Thus, it may be that by increasing the generality of this type of vertex operator, we have lost some structure in the dependence of the vertex operator itself. In any case, further research is necessary to determine whether this formalism will actually lead to results which go beyond those accessible through the free field formalism.

The second result which we can see from the explicit construction of the initial vertex operator terms (4.90) is that these operators simplify considerably when we take $x = 0$ or $x = -2h$. When we set $x = 0$, for instance, the low-order terms in the vertex modes look like

$$\begin{aligned}
\phi_{-h-x+2} &= -\frac{1}{\gamma}2h^2(2h+1)\frac{\partial}{\partial z_2} + -\frac{1}{\gamma}6h^2\frac{\partial^2}{\partial z_1^2} + \mathcal{D}_3 \\
\phi_{-h-x+1} &= \mathcal{D}_2 \\
\phi_{-h-x} &= 1 + \frac{k}{h}z_1\frac{\partial}{\partial z_1} + \mathcal{D}_2 \\
\phi_{-h-x-1} &= -2hz_1 + 2kz_1^2\frac{\partial}{\partial z_1} + \mathcal{D}_2 \\
\phi_{-h-x-2} &= -3hz_2 + h(2h+1)z_1^2 + \mathcal{D}_1
\end{aligned} \tag{4.91}$$

$$\phi_{-h-x-3} = -4hz^3 + (4h + 6h^2)z_1 z_2 - \frac{1}{3}h(2h+1)(2h+2)z_1^3 + \mathcal{D}_1$$

Comparing to (4.77), we see that for $n < 4$ at least, the polynomial terms in ϕ_{-h-x-n} are precisely the result of acting on the highest weight vector of weight h with the operator $L_{-1}^n/n!$. In fact, this correspondence is true to all orders; when $k' = k + h$, the action of a vertex operator of weight h on the highest weight state $|\,\rangle$ of a module of weight k can be described by

$$\Phi(z)|\,\rangle = e^{z\hat{L}_{-1}^{(h)}}|\,\rangle' = \sum_{i=0}^{\infty} \frac{z^i \hat{L}_{-1}^{(h)i}}{i!}|\,\rangle', \tag{4.92}$$

where $|\,\rangle'$ is the highest weight state in $V_{k'}^*$ and we denote by $\hat{L}_{-1}^{(h)}$ the realization of the operator $\hat{L}_{-1}$ on the module of highest weight h. From the simplification (4.55) of the action of the vertex operator on the vacuum, we expect that this relation should hold for a vertex operator acting on the vacuum. However, when a vertex operator acts on a highest weight state with weight $k \neq 0$, there is no similar reason to expect a similar result. Using the vertex operator construction (4.56) on a single bosonic Fock space, no analogous situation can arise due to the relation $x = 2\alpha\beta$. We would not expect this type of result to be possible for the free field realizations in any case, however, since for the realizations (3.13),

$$[L_n^{(h+k)}, L_{-1}^{(h)}] \neq [L_n^{(h)}, L_{-1}^{(h)}] \tag{4.93}$$

when $k \neq 0$. Because the coadjoint orbit realizations have raising operators independent of the conformal weight, we do have the relation

$$[\hat{L}_n^{(h+k)}, \hat{L}_{-1}^{(h)}] = [\hat{L}_n^{(h)}, \hat{L}_{-1}^{(h)}] \quad \forall n > 0. \tag{4.94}$$

for these representations.

For the coadjoint orbit representations, (4.92) follows from (4.94) and the result in the special case $k = 0$. We can also demonstrate this property explicitly. Generally, we wish to show that

$$\hat{L}_n \frac{\hat{L}_{-1}^{(h)m}}{m!}|\,\rangle = (n(h-1) + h + m)\frac{\hat{L}_{-1}^{(h)m-n}}{(m-n)!}|\,\rangle, \tag{4.95}$$

where $|\,\rangle$ is a highest weight state of any weight k. From $[\hat{L}_1, \hat{L}_{-1}^{(h)}] = 2h + 2\xi_0$, it is easy to see that this equation holds for $n = 1$, since

$$\hat{L}_1 \frac{\hat{L}_{-1}^{(h)m}}{m!}|\,\rangle = [2h + (2h+2) + \cdots + (2h + 2m - 2)]\frac{\hat{L}_{-1}^{(h)m-1}}{(m-1)!}|\,\rangle, \tag{4.96}$$

We can now prove (4.95) by induction on n. Performing the commutator of $\hat{L}_n$ with each factor of $\hat{L}_{-1}$, and assuming the result for all values $n' < n$, we have

$$\begin{aligned}\hat{L}_n \frac{\hat{L}_{-1}^{(h)m}}{m!}|\,\rangle &= \sum_{i=n-1}^{m-1} (n+1)\hat{L}_{-1}^{(h)m-i-1}\hat{L}_{n-1}\frac{\hat{L}_{-1}^{(h)i}}{m!}|\,\rangle, \\ &= \sum_{i=n-1}^{m-1} (n+1)\,[nh-n+i+1]\,\frac{i!(m-n)!}{(i-n+1)!m!}\frac{\hat{L}_{-1}^{(h)m-n}}{(m-n)!}|\,\rangle,\end{aligned} \tag{4.97}$$

For the desired result to be proven, it suffices to demonstrate that

$$\sum_{i=n-1}^{m-1} (n+1)\,[nh-n+i+1]\,\frac{i!}{(i-n+1)!} = \frac{m!}{(m-n)!}[h(n+1)+m-n]. \tag{4.98}$$

But we can again use induction to prove this relation. The relation is trivial for $m = n$. Fixing n and assuming the relation for $m' < m$, we have

$$\begin{aligned}&\sum_{i=n-1}^{m-1} (n+1)\,[nh-n+i+1]\,\frac{i!}{(i-n+1)!} \\ &= \frac{(m-1)!}{(m-n-1)!}[h(n+1)+m-n-1] + (n+1)\,[nh-n+m]\,\frac{(m-1)!}{(m-n)!} \\ &= \frac{(m-1)![mh(n+1)+m^2-mn]}{(m-n)!}.\end{aligned} \tag{4.99}$$

Thus, we have proven (4.95). This gives us a formula for the action of a vertex operator on any highest weight state when $x = 0$ in terms of the Virasoro generator $\hat{L}_{-1}^{(h)}$. We can extend this result to give an expression for an arbitrary mode of the vertex operator acting on an arbitrary state in V_k^* of the form $|\pi_n\rangle$ by using the relations (4.16) to move the creation operators to the left, and then expressing the result in terms of modes of the vertex operator acting on the highest weight state $|\rangle$ of V_k^*. For example, we have (still assuming $k' = k+h$),

$$\begin{aligned}\phi_{-h-n}\hat{L}_{-m}|\,\rangle &= \hat{L}_{-m}\phi_{-h-n}|\,\rangle + (h(m-1)-n-m)\phi_{-h-n-m}|\,\rangle \\ &= \hat{L}_{-m}^{(h+k)}\frac{\hat{L}_{-1}^{(h)n}}{n!}|\,\rangle + (h(m-1)-n-m)\frac{\hat{L}_{-1}^{(h)n+m}}{m!}|\,\rangle.\end{aligned} \tag{4.100}$$

This gives us an alternative approach to calculating the action of the vertex operators in this special case, and indicates an interesting relationship between the vertex operators and the Virasoro generators. However, it is still rather difficult to see how this relationship might be used to construct any sort of closed form expression for the vertex operators. It is also unclear whether this relationship has any natural geometric interpretation when we describe the vertex operators as global differential operators acting on the holomorphic sections of various line bundles.

4.5 Correlation functions

We end this chapter with a short discussion of correlation functions in conformal field theories. For any conformal field theory, the correlation functions (4.1) are the primary quantities necessary to describe the physics of the theory. The free field approach and the use of differential equations give a fairly powerful set of tools for the computation of correlation functions in a wide variety of physical theories. The approach presented in this thesis gives a simple mathematical structure to the definition of such correlation functions, but so far has not given rise to any particularly useful approach to the explicit calculation of these objects. In this section, we review briefly some of the main methods and simplest results about correlation functions, check that in the simplest cases our formalism is in agreement with these results, and finally present some speculation about possible approaches to computing correlation functions using this approach.

The simplest correlation functions in any Virasoro conformal field theory are the 2-point and 3-point correlation functions. It is easy to see, using the scaling properties of vertex operators under the transformation $z \to cz$, that the 2-point functions are of the form

$$\langle\Omega|\Phi_1(z_1)\Phi_2(z_2)|\Omega\rangle \sim (z_1 - z_2)^{-2h}, \tag{4.101}$$

up to an undetermined proportionality constant, where the conformal weights h_1, h_2 of the vertex operators Φ_1, Φ_2 satisfy $h = h_1 = h_2$. Similarly, the 3-point functions are generally of the form

$$\langle\Omega|\Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)|\Omega\rangle \sim (z_1 - z_2)^{h_3-h_2-h_1}(z_1 - z_3)^{h_2-h_3-h_1}(z_2 - z_3)^{h_1-h_2-h_3}. \tag{4.102}$$

For 4 or more vertex operators, the form of the correlation functions can become more complex. For example, the 4-point function can depend upon an arbitrary function of the anharmonic quotient

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}. \tag{4.103}$$

There are several main approaches currently used to calculate these more complicated correlation functions. One approach, which was first developed in the original work of BPZ [13] on conformal field theories, is to use the differential equations analogous to the constraint equations (4.32) as conditions on the correlation functions. Applying this method to the 4-point functions of the minimal models, BPZ showed that these correlation functions

could be expressed in terms of hypergeometric functions. This method has subsequently been generalized and applied to a wide variety of conformal field theories. Perhaps the most famous example are the Knizhnik-Zamolodchikov differential equations which apply to the correlation functions for rational conformal field theories with affine algebra symmetries [46].

Another approach to the explicit calculation of conformal field theory correlation functions comes from the free field approach. The correlation function of an arbitrary product of (unscreened) vertex operators (4.56) is known to have the form

$$\langle\Omega|V_{\beta_1}(z_1)\cdots V_{\beta_n}(z_n)|\Omega\rangle = \prod_{i<j}(z_i - z_j)^{2\beta_i\beta_j}, \tag{4.104}$$

when $0 = \sum\beta_i$, and to vanish when this condition is not satisfied. Using this result and the definition of the screened vertex operators (4.60), the correlation functions of minimal models can be calculated in terms of a set of contour integrals [22, 26].

The correlation functions we have discussed so far are restricted to the case of conformal field theories on the sphere. In particular, the free field calculation described above does not need to explicitly deal with resolutions of the irreducible representations in terms of a chain complex of Feigin-Fuchs modules because the vacuum $|\Omega\rangle$ is a BRST invariant state, and therefore any matrix element of the form of (4.1) will contain no contribution from the "ghost" states in the module which are cohomologically trivial. One may also consider a general conformal field theory on a higher genus Riemann surface. For example, a correlation function on a torus could be calculated by taking a trace over the irreducible representation space of a product of vertex operators acting on that irreducible representation space. It was pointed out by Felder that the free field approach gives a simple way of computing a correlation function of this type; a theorem due to Lefschetz states that when a module I has a resolution of the form (M_i, d_i), and an operator $\mathcal{O} : I \to I$ has a realization as a chain map

$$\mathcal{O}^i : M_i \to M_i, \tag{4.105}$$

then the trace of $\mathcal{O}$ is given by

$$\mathrm{Tr}\,\mathcal{O} = \sum_{i=-\infty}^{\infty}(-1)^i \mathrm{Tr}\,\mathcal{O}^i. \tag{4.106}$$

A simple example of such a calculation is the formula for the character of an irreducible Virasoro representation I,

$$\chi_I(q) = \mathrm{Tr}_I\, q^{L_0}. \tag{4.107}$$

Although the character formula for a member of the discrete series of Virasoro representations is somewhat nontrivial, the character of a Verma, dual Verma, or Feigin-Fuchs module M of highest weight h is easily computed to be

$$\mathrm{Tr}_M\, q^{L_0} = q^h \prod_{i=1}^{\infty} \left(\frac{1}{1-q^i}\right) = q^{h+1/24}\eta(q)^{-1}. \tag{4.108}$$

Using this formula, along with (4.106) and a resolution of the discrete series irreducible representations in terms of Verma modules, the characters of the discrete series were first computed by Rocha-Caridi [55].

In the last few years, there has been a fairly large body of work in which the free field method for calculating correlation functions has been generalized to higher-genus Riemann surfaces [12, 28, 29]. The approach taken in these papers is essentially to sew together correlation functions on the sphere by using 3-point functions to define a "vertex" connecting spheres with punctures. Some care must the taken to ensure that the ghost states combine correctly with this sewing procedure such that their contribution to physical correlation functions vanishes.

In principle, correlation functions for any conformal field theory on an arbitrary Riemann surface can be constructed using the coadjoint orbit representations in much the same fashion that the free field representations were used in the work described above. Because we do not yet have a general closed form expression for a vertex operator between coadjoint orbit realizations on dual Verma modules, we cannot use this abstract construction to calculate any particularly sophisticated correlation functions using the methods of this thesis at this time. We can however, do some elementary computations to verify that for the simple correlation functions such as the 2- and 3-point functions, our formalism gives the expected results (4.101) and (4.102). In the following calculations, we denote by $\phi^{(h,k,k')}_{-h-x+n}$ the nth mode of the Virasoro vertex operator of weight h mapping $V^*_k \to V^*_{k'}$ (4.15). In the case of the 2-point function, the expression for a correlation function of two vertex operators of weight h in terms of modes is

$$\langle\Omega|\Phi^{(h)}(z)\Phi^{(h)}(w)|\Omega\rangle = \sum_{n=0}^{\infty} w^n z^{-2h-n} \phi^{(h,0,h)}_{-h-n}\phi^{(h,h,0)}_{h-n} \tag{4.109}$$

Expanding around $w = 0$ and using the result (4.90), we have the expression

$$\langle\Omega|\Phi^{(h)}(z)\Phi^{(h)}(w)|\Omega\rangle \;=\; z^{-2h}\left(1 + \frac{w}{z}\phi^{(h,0,h)}_{-h-1}\phi^{(h,h,0)}_{h+1} + \frac{w^2}{z^2}\phi^{(h,0,h)}_{-h-2}\phi^{(h,h,0)}_{h+2} + \mathcal{O}(\frac{w^3}{z^3})\right)$$

$$= \quad z^{-2h}\left(1+2h\frac{w}{z}+h(2h+1)\frac{w^2}{z^2}+\mathcal{O}(\frac{w^3}{z^3})\right), \tag{4.110}$$

in agreement with (4.101). A similar argument shows that the 3-point function computed in an expansion around $z_2 = z_3 = 0$ agrees with (4.102). In fact, from the general result describing the action of a vertex operator on the vacuum in terms of $\hat{L}_{-1}$ and $\hat{L}_1$ when acting on the left and right respectively, we can calculate (4.101) and (4.102) to all orders for the coadjoint orbit vertex operators. It is reassuring, however, to see that the explicit calculations in (4.90) are in agreement with the results of other methods.

The calculation of a correlation function of 4 or more vertex operators using the approach of this thesis would be a rather arduous task. In fact, without a more general or abstract description of the coadjoint orbit vertex operators, there is absolutely no reason to pursue such a computation. It is hoped, however, that eventually a better description of these operators will be possible, with which the calculation of such correlation functions will be simplified; since this approach gives a description of conformal field theories which is in some ways much simpler than the free field picture, it is possible that this viewpoint may even give insights into the theory which are inaccessible from the free field point of view.

For now, however, we can only speculate about what form a more concise description of the vertex operators described in this thesis might take. One possible approach is to attain a better geometrical picture of the nature of the vertex operator as a set of differential operators on line bundles over the coadjoint orbits. From this point of view, there is a very rich geometrical and group-theoretical structure to these operators, and it seems likely that a simple intrinsic definition of the vertex operators might be given in a geometrical language so that index theorems and other powerful geometrical tools might be of assistance in the explicit calculation of correlation functions. It is also possible that by purely algebraic manipulations, one might be able to construct a closed form expression for the vertex operators similar to the expressions derived in this thesis for the differential operators associated with the action of the algebra generators and the screening operators and intertwiners. If such a closed form expression were found, it might be possible to use this result to study more deeply certain geometrical, algebraic, and physical properties of conformal field theories.

Chapter 5

Conclusions

5.1 Results

In this thesis, we have presented a basic conceptual framework in which conformal field theory has a completely geometric interpretation in terms of mathematical objects with well-understood geometric, algebraic, and group-theoretic structures.

We began by considering a general Lie group G which could be either a finite-dimensional group, the Virasoro group, or a centrally extended loop group. We outlined the coadjoint orbit construction, which associates with each irreducible unitary representation I_b of G of highest weight b a complex line bundle $\mathcal{L}_b$ over a complex homogeneous space of the form G/T where T is a maximal subtorus of G. The group G has a natural action on the space $\mathcal{H}_b$ of holomorphic sections of $\mathcal{L}_b$. We described the associated action of the algebra $\mathfrak{g}$ explicitly in terms of a set of (first-order) differential operators acting on $\mathcal{H}_b$ in a particular choice of local gauge. For finite-dimensional groups and centrally extended loop groups, the space $\mathcal{H}_b$ carries precisely the irreducible representation I_b under the natural action of G. For the Virasoro group, we find that the resulting representation on the space of holomorphic sections is that of the dual Verma module associated with the highest weight b. In those cases where this representation is reducible, we can use an infinite sequence of line bundles associated with different highest weights b_i, b_i' to construct a chain complex of line and vector bundles

$$0 \to \mathcal{L}_b \xrightarrow{d_0} \mathcal{L}_{b_1} \oplus \mathcal{L}_{b_1'} \xrightarrow{d_1} \mathcal{L}_{b_2} \oplus \mathcal{L}_{b_2'} \xrightarrow{d_2} \cdots \tag{5.1}$$

where the operators d_i are combinations of intertwining operators which map holomorphic

sections of one line bundle to holomorphic sections of another line bundle in such a way that the intertwining operator commutes with the $\mathfrak{g}$-action on the spaces of sections. The cohomology of this complex is nontrivial only in the first bundle, and gives precisely the irreducible representation I_b of the Virasoro algebra.

Thus, we have a geometric construction of every irreducible unitary representation of the groups of interest as either a space of all holomorphic sections of a line bundle, or as the space of sections of a line bundle which are in the kernel of a particular differential operator. We then proceeded to define vertex operators, which map between irreducible representation spaces by acting on the spaces of holomorphic sections of the associated bundles. Given three weights b_1, b_2, b_3, a vertex operator of weight b_1 is defined by a set of maps

$$\Phi_u : \mathcal{L}_{b_2} \to \mathcal{L}_{b_3}, \tag{5.2}$$

where u takes a value in a representation of the algebra $\mathfrak{g}$ associated with the weight b_1. In the case of finite-dimensional groups, this space is merely the irreducible representation space with highest weight b_1; in the case of the Virasoro and affine algebras, the space is again an irreducible representation space of $\mathfrak{g}$, but is no longer highest weight. The operators Φ_u satisfy the vertex operator condition

$$[x, \Phi_u] = \Phi_{xu} \quad \forall x \in \mathfrak{g}. \tag{5.3}$$

For the Virasoro algebra, the vertex operator construction is slightly more complex when b_2 or b_3 are associated with degenerate representations. In this case, the vertex operator is actually a chain map between complexes of the form (5.1).

In order to make the connection to conformal field theory, a vacuum is chosen which corresponds to a particular line bundle $\mathcal{L}_0$ with extra symmetries under the group action. We denote the holomorphic section of $\mathcal{L}_0$ which is highest weight under the $\mathfrak{g}$ action by $|\Omega\rangle$. Generally, a conformal field theory correlation function is then written in the form

$$\langle\Omega|\Phi^{(1)}_{u_1}\Phi^{(2)}_{u_2}\cdots\Phi^{(k)}_{u_k}|\Omega\rangle \tag{5.4}$$

Although through most of this thesis, we dealt most directly with local expressions for the differential operators associated with vertex operators, intertwiners, and the $\mathfrak{g}$ action on $\mathcal{H}_b$, the fact that these operators take globally holomorphic sections of one line bundle to globally holomorphic sections of another indicates that these are actually globally defined complex-analytic differential operators on the bundles in question.

Thus, the picture of conformal field theory which emerges is that of a fairly simple geometrical structure which is closely related to the geometry of the Lie symmetry group of a particular conformal field theory. Although we have completed the basic skeleton of this geometrical picture of conformal field theory in this thesis, clearly much more work is necessary to arrive at a real understanding of the geometric significance of the vertex operators (5.2) and the associated correlation functions. In particular, the local formulae we have derived in this thesis are useful for analyzing and understanding the basic structures involved, but are not as powerful for specific computations as the related algebraic construction of free field theories. It is hoped that a better understanding of the underlying geometry of this construction will lead to more powerful tools for the analysis of conformal field theories in general. In the next section we discuss briefly some recent work which is related to the material, outlook, and/or methods of this thesis. In the final section of this chapter, we discuss some open questions and possible directions for future research.

5.2 Related work

In this section we discuss several issues involving a variety of recent work which is related to the subject matter of this thesis, but which was not mentioned or significantly referenced in the main discussion.

5.2.1 Virasoro coadjoint orbits

One of the main developments which motivated the part of this thesis in which coadjoint orbit representations of infinite dimensional groups were explicitly constructed was the paper by Witten [64], in which progress was made in understanding the structure of the coadjoint orbit representations of the Virasoro group. There has been a great deal of previous work in the mathematics literature on the structure of the Virasoro group, its representations, and the homogeneous space $\mathrm{Diff}S^1/S^1$, which we do not attempt to survey here; as starting points, the interested reader is referred to the works of Kirillov [44], Goodman and Wallach [36], Bowick and Rajeev [20], and Freed [30]. The coadjoint orbits of $\widehat{\mathrm{Diff}}S^1$ were first classified by Segal [58] and Lazutkin and Pankrotova [47]. These coadjoint orbits were further studied by Kirillov [44]. In [64], Witten presented some partial results relating the Virasoro coadjoint orbits to the irreducible unitary representations of $\widehat{\mathrm{Diff}}S^1$. By using perturbative techniques and the fixed point version of the Atiyah-Singer index

theorem, Witten was able to calculate the characters of the representations associated with the $\mathrm{Diff}S^1/S^1$ orbits, which he found to be the standard bosonic partition function (4.108) associated with a nondegenerate Virasoro representation. The perturbative methods used by Witten, however, were only valid in the semi-classical $c \gg 1$ domain. In particular, the structure of the $c \leq 1$ discrete series of unitary representations could not be understood in terms of coadjoint orbits using these techniques. Witten conjectured that these representations would be found in the $\mathrm{Diff}S^1/SL^{(n)}(2,\mathbb{R})$ orbits, but since these spaces do not admit Kähler structures, it has not yet been possible to perform geometric quantization in these cases, and the representations associated with these orbits are still not understood.

More recently, related investigations provided further clues to the structure of the Virasoro coadjoint orbit representations. By using a technique involving quantization on a group manifold, Aldaya and Navarro-Salas were able to construct representations of the Virasoro group on spaces of polarized functions on the group manifold $\widehat{\mathrm{Diff}S^1}$ itself [3]. For those values of c and h where the Kac determinant vanishes, they made the observation that the representation constructed through their method is reducible, yet contains only a single highest weight vector. By taking the orbit of the highest weight vector under the Virasoro action, they found a subspace of the original representation space which corresponds exactly to the appropriate irreducible unitary representation in the $c \leq 1$ discrete series. Furthermore, they introduced an auxiliary set of operators whose kernel gave the irreducible representation. By analogizing their techniques to the coadjoint orbit method, Aldaya and Navarro-Salas conjectured that a similar situation would arise in the $\mathrm{Diff}S^1/S^1$ coadjoint orbit representations with $c \leq 1$.

The results in this thesis are in agreement with Witten's results for all c, since the dual Verma module indeed has the partition function (4.108). Instead of finding the discrete series of representations in $\mathrm{Diff}S^1/SL^{(n)}(2,\mathbb{R})$ orbits as Witten suggested, however, we find that they can be constructed through resolutions of the dual Verma module representations arising from the $\mathrm{Diff}S^1/S^1$ orbits. In addition, the structure of the dual Verma module resolutions described in Chapter 3 corresponds precisely to the results of Aldaya and Navarro-Salas. The dual Verma modules indeed contain a single highest weight vector whose orbit is the irreducible representation for the discrete series of representations. Furthermore, the operators defined by Aldaya and Navarro-Salas are closely related to the screening operators and intertwiners introduced to construct the dual Verma resolution.

5.2.2 Coadjoint orbits and actions

Recently, the coadjoint orbits of infinite-dimensional groups have been studied in connection with conformal field theories for reasons which ostensibly are rather distinct from those underlying the discussion in this thesis. Specifically, coadjoint orbits have been used to construct the actions for Lagrangian descriptions of a wide variety of conformal field theories. The basic idea behind this construction was first developed in the context of infinite-dimensional groups by Alekseev and Shatashvili [4, 5], and essentially follows from the observation that the symplectic form ω on a coadjoint orbit space can be treated as the natural symplectic form associated with a classical mechanical system. Under this interpretation, the orbit space is interpreted as a classical phase space, and the process of geometric quantization is simply the usual quantization procedure on the associated classical system. By writing the symplectic form ω as the exterior derivative of a 1-form, $\omega = \mathrm{d}\alpha$, a candidate action $S = \int \alpha$ (corresponding to a vanishing Hamiltonian) can be given for any trajectory of a particle in the classical phase space. Because ω is closed, but not generally an exact form, this procedure can only be applied locally. However, in the quantum theory the fact that $2\pi\omega$ is constrained to be an integral homology class means that the action is always well-defined up to a phase of $\mathrm{e}^{2\pi i}$. Using this approach, it was shown by Alekseev and Shatashvili that the resulting action when this procedure is applied to a loop group coadjoint orbit is precisely the action of the WZW model. Similarly, the result of applying this procedure to the Virasoro $\mathrm{Diff}S^1/S^1$ orbits is precisely the action originally derived by Polyakov in [52] for two-dimensional quantum gravity. From this point of view, the hidden $SL(2,\mathbb{R})$ symmetry found by Polyakov has a natural geometrical significance. Similar results have been achieved by Aldaya, Navarro-Salas, and Navarro using their approach of quantization on the group manifold [2].

More recently, the coadjoint orbit method for constructing field theory actions has been applied to a wide variety of infinite-dimensional algebras, including the super Virasoro and $N = 2$ super Virasoro algebras [7, 8, 21], W_n and W_∞ algebras [10, 49, 50], and the algebras associated with groups of maps from 2-dimensional and 4-dimensional manifolds into a target space [23, 48]. For a review of this work, see [21].

Another interesting and related development originated with the work of Bershadsky and Ooguri [16]. In their paper, they considered the quantum version of Hamiltonian reduction on a theory of gauge fields coupled to the WZW model. They found that the

resulting constrained $SL(2,\mathbb{R})$ WZW model is closely related to the quantum field theory on Virasoro coadjoint orbits which is described by the Polyakov gravitational WZW model action. Similar constructions for $SL(n,\mathbb{R})$ current algebras give rise to W_n-algebra actions. In their construction, Bershadsky and Ooguri explicitly used the bosonic free field realizations of the affine and Virasoro algebras, along with the relevant BRST complexes. Since the original paper, there has been a large amount of literature extending this work; see for example [57] and references therein.

The Hilbert spaces associated with the quantum field theories defined by the natural coadjoint orbit actions should in fact be precisely the spaces of polarized wave-functions on the orbits, which are the states in the irreducible representations studied in this thesis. It is natural to speculate that there may be a deeper relationship between the infinite-dimensional symmetry groups of conformal field theories and the theories themselves than is so far understood using either the methods of this thesis or the methods used to construct field theory actions. Such a relationship was hinted at in [9], where the coadjoint orbit action construction was extended to give Ward identities for correlation functions. Clearly, what is needed is some synthesis of these viewpoints in which the action and the Hilbert space associated with a coadjoint orbit enter into a unified geometric picture of a conformal field theory. Possibly, this could be accomplished by having a better understanding of how a Hamiltonian picture of conformal field theories relates to the action which naturally arises from coadjoint orbits. In this case, it is possible that the formal structure defined in this thesis which gives an association between coadjoint orbits and conformal field theory correlation functions and vertex operators may be more than just a convenient way of expressing the physical theory in a consistent mathematical fashion. In fact, it is possible that the vertex operators constructed here may be in some way naturally connected with the intrinsic action associated with coadjoint orbits. This could be a very promising direction for future research.

5.3 Remaining questions

In this section, we discuss some of the open questions which remain regarding the work presented in this thesis.

- **Field theoretic formulation**

Although we have given explicit realizations of the affine and Virasoro algebras in terms of a bosonic Fock space, the associated description in terms of bosonic free fields is not completely clear. Unlike the Feigin-Fuchs and Wakimoto realizations, where the expressions for the generators of the algebra are quadratic in the field modes, the coadjoint orbit realizations have generators which contain an arbitrarily large number of modes in a single term. In the field theory picture, this implies that the current algebras of the corresponding physical theory must have an analogous representation, which seems to indicate that any associated field theory would not only have interactions, but might contain interaction terms of arbitrarily high order which are related in a rather nontrivial fashion. Although a simpler formulation of such a theory might be possible (such as the action formulation described in the previous section), the essential distinction between these representations and the free field representations remains. Because the structure of the Fock space is that of a dual Verma module for the coadjoint orbits, it is probably impossible to write the theory purely as a free field theory. A better understanding of the connection with field theory will probably be an essential component of any application of the methods in this thesis.

- **Vertex operators**

There are several questions remaining regarding the vertex operators constructed in this thesis. First, we have not proven that any resolution of an irreducible representation through dual Verma modules admits a set of vertex operators of the desired form, even when such a vertex operator exists for the irreducible representations. It would be nice to have a formal proof of this result. Second, we have given here a rather inconvenient characterization of the vertex operators by defining them through their matrix elements on the bosonic Fock space. For a better understanding of the structure of these operators, it is desirable to have a more abstract geometric or group-theoretic description. Such a description might well involve relating these vertex operators more directly to the associated field theory and the action arising from the coadjoint orbit.

- **Coset models**

Another interesting question is whether the class of coset conformal field theories has a natural description in the language of coadjoint orbit representations. Some work has been done using the free field representations of affine algebras to construct the representations associated with a coset theory (see for instance [18]). Due to the complications

involved with the associated two-sided resolutions, however, there are some difficulties with this construction. Hopefully, using the one-sided resolutions of dual Verma modules could simplify this construction.

- **Other groups**

A natural extension of this work would be to consider a larger class of groups and algebras, such as the super algebras and W-algebras. Just as free field W-algebra representations can be constructed through the process of Hamiltonian reduction from an affine algebra using free field realizations [16], and W-algebra actions can be constructed from coadjoint orbits, one might expect to find similar constructions using the coadjoint orbit dual Verma modules.

5.4 Conclusion

In conclusion, we have constructed an alternative approach to conformal field theory based on fundamental principles of geometry and group symmetry. This construction unifies the coadjoint orbit structure associated with a group and the description of vertex operators through their action on algebraic resolutions. The main problems remaining with this construction are the connection with a field theory and the challenge of finding a more convenient geometric description of the vertex operators. This construction may provide a link between the construction of field theory actions using coadjoint orbits and the physics of the associated field theory which is described in terms of correlation functions.

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