LBL-34667
UC-414

Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

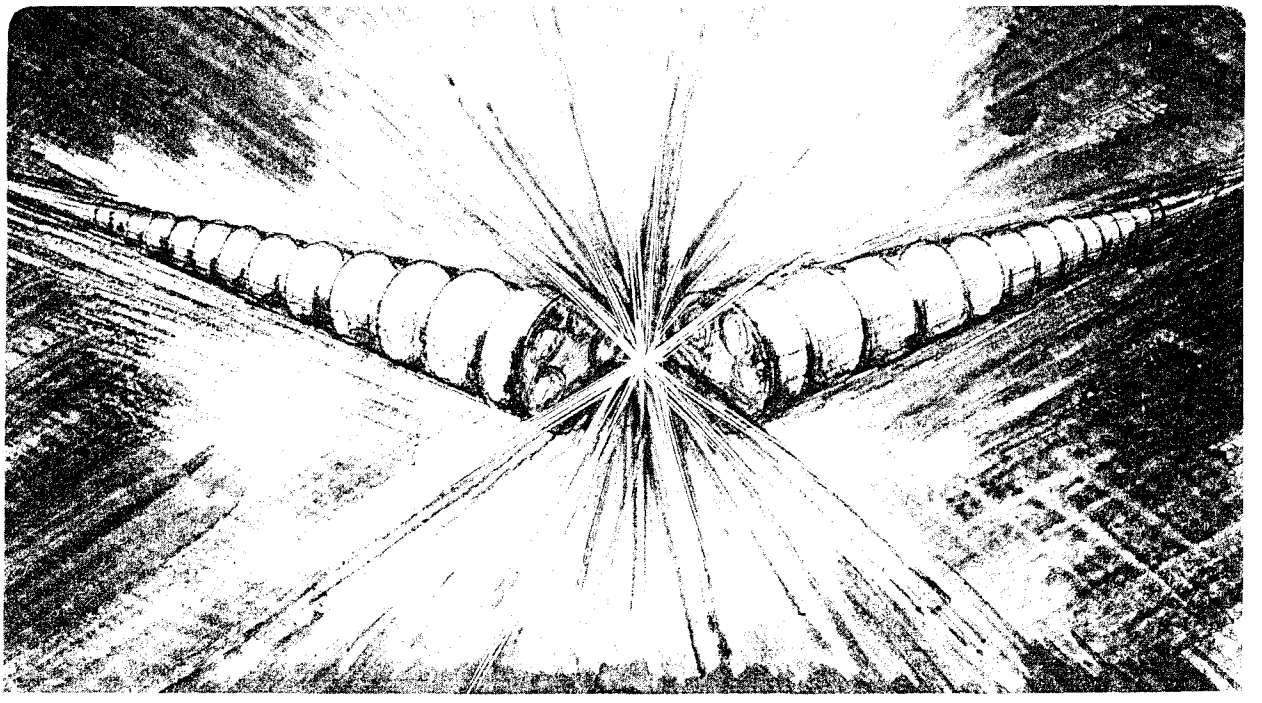
Accelerator & Fusion Research Division

Presented at the Workshop on Beam Cooling and Related Topics, Montreux, Switzerland, October 3–8, 1993, and to be published in the Proceedings

Liouville's Theorem and Phase-Space Cooling

R.L. Mills and A.M. Sessler

September 1993



Prepared for the U.S. Department of Energy under Contract Number DE-AC03-76SF00098

LBL-34667
CBP Note-039

LIOUVILLE'S THEOREM AND PHASE-SPACE COOLING*

Robert L. Mills
Ohio State University
Columbus, OH 43210
and
Andrew M. Sessler
Lawrence Berkeley Laboratory
University of California
Berkeley, CA 94720

September 28, 1993

ABSTRACT
A discussion is presented of Liouville's theorem and its consequences for conservative dynamical systems. A formal proof of Liouville's theorem is given. The Boltzmann equation is derived, and the collisionless Boltzmann equation is shown to be rigorously true for a continuous medium. The Fokker-Planck equation is derived. Discussion is given as to when the various equations are applicable and, in particular, under what circumstances phase space cooling may occur.

1. INTRODUCTION

This workshop is devoted to cooling, so why start with a paper on Liouville's theorem [1], a theorem well-known to be "against cooling"? For very good reason, really, for it is only with a deep understanding of when one can *not* cool, that one can design—and properly analyze—cooling systems.

Back in the "old days", starting in 1955, the MURA (Midwestern Universities Research Assoc.) physicists were well aware of the limits imposed by Liouville's theorem [2]. They tried, in fact, to produce damping with tapered foils (a scheme which works in principle, but is not practical) and slaunched cavities (wrong in principle). Most of this work—for good reason—was unpublished, although publication might have prevented others from wasting many hours [3].

Besides Liouville's theorem there are other invariants, enumerated systematically by Poincare, which any dynamical system must observe [4]. In this paper we shall only consider the first Poincare invariant, i.e., Liouville's theorem. This invariant concerns over-all preservation of phase space volume and, therefore, is of relevance to the subject of cooling.

Starting in 1958 attention at MURA turned to evaluating space charge, or collective, effects; first static effects and then dynamic effects [5,6]. It seemed proper to base this analysis upon the collisionless Boltzmann equation, or Vlasov equation, but it was rather unclear, in those old days, just when that equation was a valid approximation. The present authors made a study of that matter, but never published their work [7].

It seems appropriate to re-visit the subject and now—some 35 years later—to publish the work. At the same time, it seems useful to present—in one review paper—derivations of Liouville's theorem, the Vlasov equation, and the Fokker-Planck equation. It is hoped that this will provide useful background for a proper understanding of phase space damping, i.e., cooling.

*This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, of the U.S. Department of Energy under contract no. DE-AC03-76SF00098.

2. LIOUVILLE'S THEOREM

This dynamical theorem applies in a conservative Hamiltonian system such as a single particle in external magnetic and electric fields. In this case the phase density of many non-interacting systems (having slightly different initial conditions) is preserved as one follows the motion of the system.

Thus, this theorem is applicable to a beam of particles when the interaction between particles is negligible and can be ignored, which is often true—but not always true—for high energy beams.

Even within the restrictions of Liouville's theorem it is possible to arrange to interchange phase space (between, say, longitudinal and transverse degrees of freedom) or, equally interestingly, not even to exchange phase space, but to introduce correlations between the degrees of freedom, i.e., "push phase space around". A device that does just this, to distinct advantage for free-electron lasers, has recently been proposed [8].

A proof of Liouville's theorem can be found in many text books [9,10]. A simple way of looking at this theorem is to think of it as equivalent to the condition of incompressible flow in the phase space of a given system. Let the system be described by the N coordinates q_α and the N conjugate nomenta p_α, $\alpha=1,2,...,N$. The phase space is just the 2N-dimensional space with coordinates q_α and p_α, and the development in time of the state of the system is represented by the trajectory of a single point in phase space. Just as with fluid flow, there is a well-defined velocity field at each instant of time, which assigns to each point in phase space a definite velocity, with components $\dot{q}_\alpha$ and $\dot{p}_\alpha$ given as functions of the q's and p's by Hamilton's equations. For fluid flow in any number of dimensions the condition that volumes are preserved by the flow is equivalent to the vanishing of the divergence of the velocity field:

$$\nabla \cdot v(x) = 0 \ . \tag{1}$$

For phase space the components of the velocity field are

$$\dot{q}_\alpha = v_\alpha(q,p,t) = \partial H/\partial p_\alpha \ , \tag{2}$$

$$\dot{p}_\alpha = F_\alpha(q,p,t) = -\partial H/\partial q_\alpha \ , \tag{3}$$

The divergence condition (1) then becomes

$$\Sigma_\alpha(\partial v_\alpha/\partial q_\alpha + \partial F_\alpha/\partial p_\alpha) = \partial^2 H/\partial p_\alpha \partial q_\alpha - \partial^2 H/\partial q_\alpha \partial p_\alpha = 0 \ , \tag{4}$$

which is automatically satisfied as a consequence of Hamilton's equations, and thus demonstrates the validity of Liouville's theorem. It is important to note that the Hamiltonian may depend explicitly on the time t.

Often a dynamical system is represented by an ensemble of possible states, with a distribution function f giving the number $n(\Delta v)$ of particles systems in a small volume ΔV of phase space:

$$n(\Delta v) = f(q,p,t)\,\Delta V \ . \tag{5}$$

If the q_i and p_i are Hamiltonian variables, then ΔV is conserved, and since the number of particles is clearly invariant, we deduce that moving with the particles (or set of systems) the density function f is constant:

$$\frac{df}{dt} = 0 \ . \tag{6}$$

It is interesting to write this out:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{\alpha=1}^{N}\left(\frac{\partial f}{\partial q_\alpha}\frac{dq_\alpha}{dt} + \frac{\partial f}{\partial p_\alpha}\frac{dp_\alpha}{dt}\right) = 0 \quad . \tag{7}$$

From Hamilton's equations:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{\alpha=1}^{N}\left(\frac{\partial f}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha}\right) = 0 \quad . \tag{8}$$

From this we can see that a stationary distribution — one for which $\partial f/\partial t=0$ — must satisfy the relationship

$$\{f,H\} \equiv \Sigma_\alpha\left(\partial f/\partial q_\alpha\ \partial H/\partial p_\alpha - \partial f/\partial p_\alpha\ \partial H/\partial q_\alpha\right) = 0 \quad . \tag{9}$$

The center expression here is the Poisson bracket of f with the Hamiltonian, and the vanishing of the Poisson bracket, for a function that doesn't depend explicitly on the time, is just the condition that the expression f(q,p) be a constant of the motion and therefore, normally, a function of the standard conserved quantities. For many systems the only such constant of the motion if the energy itself, in which case f must be equal to a function of the Hamiltonian function H:

$$f = f[H(q,p)]. \tag{10}$$

This result, applied to interacting particles, for which it is not strictly true, but is a very good approximation, as we shall see in the next section, was first used in beam physics in 1958 [5]. It has been widely used since that time.

3. LIOUVILLE'S THEOREM FOR A CONTINUOUS MEDIUM WITH CONSERVATIVE INTERACTIONS

The MURA Report, which the authors wrote 35 years ago, still reads very well. This section simply consists of the old report [7], somewhat streamlined. The formal proof, of that report, is presented in Appendix A.

The study of the motion of particles in an accelerator becomes a many-body problem when the interactions between particles are taken into account. It is thus important to investigate the possibility of establishing the validity—or approximate validity—of general dynamical theorems applicable to the n-body problem. Such a powerful theorem is the one proved here to be an extremely good approximate theorem for particles in an accelerator.

Liouville's theorem asserts that in a 2fN dimensional space (f is the number of degrees of freedom of one particle), spanned by the coordinates and momenta of all particles (called γ space), the density in phase space is a constant as one moves along with any state point. It is thus a statement about the density of points, each point representing a dynamical system. The systems constitute an ensemble and of course do not interact.

The theorem proven here refers to a system of many interacting particles, and asserts that in the 2f-dimensional space spanned by the coordinates and momenta of a single system (called a μ space), the density in phase is a constant as one moves along with any phase point. It is thus a statement about the behavior of interacting particles, and thus really quite different from Liouville's theorem.

The validity of the theorem, as well as the limits of its validity, may readily be seen by the following intuitive argument:

Consider first a system of many particles, N. Suppose these particles are subject to external forces derivable from a Hamiltonian (which may even be time-dependent), but there are no interactions between the particles. Clearly density in phase in μ space is a constant of the motion as one follows the motion of a phase point. This follows then immediately from Liouville's Theorem in γ space, since with no interactions between particles μ space for N particles is simply γ space for a single particle.

Consider now a system of a great many particles N, with interactions between the particles. Imagine that the solution has been obtained so that we know the motion of all the particles as a function of time. Concentrate now on a "small" number of particles n, which initially are localized in μ space. We will define what "small" means shortly. Let all the other particles move along the trajectories appropriate to the solution of the N-body problem. If the interactions among the n particles can be neglected compared to the interactions between the N-n particles and one of the n particles, then these particles are subject to "external forces" and by the first case the density in μ space is a constant as one moves along with the sample group of n particles. This is clearly true for any sample, and hence the theorem is established.

That is, the sample size must be small enough compared to N that the influence of the n sample particles on one of their number is negligible compared with the influence of the N-n remaining particles. At the same time, n must be large enough that fluctuations within the sample can be neglected. Both of these conditions can be met if N is sufficiently large, in which case the theorem is valid to a good approximation. In the argument given below, the limit of a continuous medium is taken so that fluctuation phenomena do not exist. For applications to particle accelerators where we consider a number of particles $N \approx 10^{13}$ this approximation is very valid, corresponding to neglect of particle–particle collisions which throw a particle out of the accelerator, but *not* neglecting long range electromagnetic interactions which are responsible for space–charge limits, plasma oscillations, beam–beam interactions and possible two–stream amplification mechanisms.

Each particle of a system moves under the influence of the force fields due to all the other particles of the system, in addition to the externally imposed fields that act equally on all the particles. That particle, then, moves according to a perfectly good single-particle Hamiltonian, and you might think that Liouville's theorem would be exact for each particle. The reason this doesn't work is that the single-particle Hamiltonians are all different, and the velocity fields in μ space determined by those Hamiltonians are different also. That is, the direction that the point moves in μ space depends on which particle you are following, not on just the location in μ space, and this spoils the picture of incompressible flow that constitutes Liouville's theorem.

What makes Liouville's theorem valid to a very good approximation in μ space is that the individual single-particle Hamiltonians are so nearly similar, because the contribution of any one particle to the force fields is so small. In the continuum limit that contribution is truly negligible, and Liouville's theorem for μ space becomes exactly valid. The proof is trivially equivalent to the proof given above for an entire system; the only approximation that needs to be made is that each particle moves in the <u>same</u> force field and is therefore described by the <u>same</u> time-dependent single-particle Hamiltonian.

The density $\rho(\mathbf{r},\mathbf{p},t)$ of particles in μ space is logically equivalent to the distribution function $f(q,p,t)$ defined for an entire system by Eq. 5, and in the continuum limit it remains constant along particle trajectories in exactly the same way. In section 4 we shall see for the explicit case of relativistic particles interacting electromagnetically how this approximation yeilds the relativistic version of the Vlasov equation, or collisionless Boltzmann equation. In section 5 we shall discuss the additional collision term of the Fokker-Planck equation.

4. THE VLASOV EQUATION

For a single particle of mass m and charge e moving in an external field described by the potentials $(\mathbf{A},\varphi)$ we have the well-known Hamiltonian [11]

$$H = c\left[m^2c^2 + (\mathbf{p}-e\mathbf{A})^2\right]^{1/2} + e\varphi \quad , \tag{11}$$

with the momentum given in terms of the velocity $\mathbf{v}$ by:

$$\mathbf{p} = m\gamma\mathbf{v} + e\mathbf{A} \quad , \tag{12}$$

and

$$\gamma = \left[1 + (\mathbf{p} - e\mathbf{A})^2/m^2c^2\right]^{1/2} \quad . \tag{13}$$

The collisionless Boltzmann equation is equivalent to Eq. 6, the statement that the phase-space density, now equal to $\rho(\mathbf{r},\mathbf{v},t)$, the particle density in μ space, is constant as one follows a particle trajectory, with ρ expressed now in terms of velocity rather than momentum:

$$d\rho/dt = \partial\rho/\partial t + \Sigma v_i \partial\rho/\partial x_i + \Sigma \dot{v}_i \partial\rho/\partial v_i = 0 \ . \tag{14}$$

It is necessary only to find the acceleration $\dot{\mathbf{v}}$ in terms of the electromagnetic force $\mathbf{F}$:

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{15}$$

$$= d(m\gamma\mathbf{v})/dt \tag{16}$$

$$= m\gamma(\dot{\mathbf{v}} + \gamma^2 \mathbf{v}\mathbf{v}\cdot\dot{\mathbf{v}}/c^2) \ . \tag{17}$$

Solving for $\dot{\mathbf{v}}$ yields

$$\dot{\mathbf{v}} = (1/m\gamma)(\mathbf{F} - \mathbf{v}\mathbf{v}\cdot\mathbf{F}/c^2) \ , \tag{18}$$

and substituting this result into Eq. 14 then gives the desired result:

$$\partial\rho/\partial t + \sum_i v_i \partial\rho/\partial x_i + \sum_{i,k} (1/m\gamma) F_i (\delta_{ik} - v_i v_k/c^2) \partial\rho/\partial v_k = 0. \tag{19}$$

In the non-relativistic limit

$$\frac{\partial\rho}{\partial t} + \mathbf{v}\cdot\nabla_r\rho + \frac{\mathbf{F}}{m}\nabla_v\rho = 0 \ , \tag{20}$$

the well-known Vlasov, or collisionless Boltzmann, equation.

Now the discussion of the last section allows us to extend this to many particles. In that case we must include collisions between particles which means we have on the RHS, $\left.\frac{\partial\rho}{\partial t}\right)_{\text{collisions}}$, rather than zero.

If, however, the particles *only* interact through the fields E and B then there are no collisions and we recover the Vlasov equation; but with E and B the *total* field (external field plus self-field) and ρ now the distribution function of *interacting* particles. Obviously Liouville's theorem is applicable.

5. THE FOKKER-PLANCK EQUATION

We would like to turn, now, to evaluation of the collision term $\left.\frac{\partial\rho}{\partial t}\right)_{\text{collisions}}$ on the RHS of the Boltzmann equation. Clearly, evaluation depends upon the nature of the forces between particles. For hard scattering, such as molecules in a gas undergo, Boltzmann, himself, addressed the subject.

For Coulomb forces the scattering is primarily into very small angles (i.e., hardly any change in the direction of the particle). In this case we must consider the effect of multiple scattering, thus deriving the change in ρ from many, very small, collisions. The result is the Fokker-Planck equation, which has been employed widely in physics. In accelerator physics it has been employed, amongst other things, to study intra-beam scattering, the effect of noise on the applied radio frequency, and the turbulence in intense charge bunches [12].

Let $W(\mathbf{v},\Delta\mathbf{v})$ be the probability density for a particle changing its velocity from $\mathbf{v}$ to $\mathbf{v}+\Delta\mathbf{v}$ in time Δt, and suppose for this discussion that the only changes in ρ are those due to collisions. Then

$$\rho(\mathbf{r},\mathbf{v},t+\Delta t) = \int \rho(\mathbf{r},\mathbf{v}-\Delta\mathbf{v},t) W(\mathbf{v}-\Delta\mathbf{v},\Delta\mathbf{v}) d(\Delta\mathbf{v}) \ . \tag{21}$$

Since

$$\left.\frac{\partial\rho}{\partial t}\right)_{\text{collisions}} \Delta t = \rho(\mathbf{r},\mathbf{v},t+\Delta t) - \rho(\mathbf{r},\mathbf{v},t) \quad , \tag{22}$$

we can Taylor expand and get

$$\begin{aligned} \rho(\mathbf{r},\mathbf{v}-\Delta\mathbf{v},t)W(\mathbf{v}-\Delta\mathbf{v},\Delta\mathbf{v}) &= \rho(\mathbf{r},\mathbf{v},t)W(\mathbf{v},\Delta\mathbf{v}) \\ &-\sum_i \frac{\partial(\rho W)}{\partial v_i}\Delta v_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2(\rho W)}{\partial v_i \partial v_j}\Delta v_i \Delta v_j + \ldots \end{aligned} \tag{23}$$

By definition

$$\int W d(\Delta\mathbf{v}) = 1 \ . \tag{24}$$

Defining

$$\langle \Delta\mathbf{v}\rangle_t \, \Delta t = \int W \, \Delta\mathbf{v} \, d(\Delta\mathbf{v}) \ , \tag{25}$$

$$\langle \Delta v_i \Delta v_j\rangle_t \, \Delta t = \int W \, \Delta v_i \, \Delta v_j \, d(\Delta\mathbf{v}) \ , \tag{26}$$

we have the Fokker-Planck collision term:

$$\left.\frac{\partial\rho}{\partial t}\right)_{\text{collisions}} = -\sum_i \frac{\partial}{\partial v_i}\left(\langle \Delta v_i\rangle_t \rho\right) + \sum_{i,j}\frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}\left(\langle \Delta v_i \Delta v_j\rangle_t \rho\right) \ . \tag{27}$$

The coefficients have been studied extensively in the literature.

6. DISCUSSION

Let us, first of all, review the theorems discussed in this note. Liouville's theorem applies to any Hamiltonian dynamical system, that is, a system describable by a Hamiltonian, which may be time-dependent. Thus the theorem applies, for example, to a single particle subject to a conservative external force, and also to a collection of non-interacting particles, each subject to the same external conservative force.

Does Liouville's theorem apply to highly nonlinear, stochastic motion of non-interacting particles? Clearly it does, since each particle still constitutes a Hamiltonian system governed by the same single-particle Hamiltonian. How then can we reconcile the theorem with the very non-Liouvillean motion characterized by non-zero Liapunov exponents? From a fine-grained point of view, there is no change in phase space density; Liouville's theorem is quite valid. From a coarse-grained point of view, however, the density in phase space changes significantly. The filaments of constant density become ever finer and lots of "air" mixes in, so the coarse-grained density is ever decreasing. Notice that the coarse-grained density becomes more and more relevant because, given any degree of precision with which density is observed, one only has to wait sufficiently long and the coarse-grained density will become appropriate. Notice also that the coarse-grained density can only decrease, that is, that only "heating", but not "cooling", can occur through this mechanism.

The Vlasov equation, or collisionless Boltzmann equation as usually used, is only a nonrelativistic approximation. It is applicable to a collection of particles, each subject to the same

(possibly time-dependent) external forces described by a conservative Hamiltonian, where the strong collisional interactions between particles are negligible and the particles interact with each other only through their mutual electromagnetic interactions. This interaction may be very significant; the collective behavior of plasmas, for example, which is very complicated indeed, is accurately described by the Vlasov equation.

The Fokker-Planck equation is valid when the interacting particles undergo many collisions, eah one of which only deflects the particle slightly (multiple scattering). [Is this right? The derivation doesn't seem to be limited to small momentum-transfer collisions.] It is thus a natural extension of the Vlasov equation. The Fokker-Planck equation can be used to study, for example, the effect of intra-beam scattering.

Secondly, let us now discuss under what situations one can have "cooling", that is, damping of phase space. The simplest is to have a dissipative system, as, for example, a charged particle radiatively coupled to the electromagnetic field[13]. Actually, there are quantum fluctuations in this coupling, which are usually studied by means of the Fokker-Planck equation.

A second example of a dissipative system is a beam of particles interacting with a foil[14]. This is called "ionization damping", and is very similar to radiation damping. It has been proposed for cooling a mu meson beam to make a muon collider.

A third example is provided by interaction between the "hot" beam one wishes to cool and a "cold" beam of electrons[15]. In this case, "hot" and "cold" refer to transverse temperature. Electron cooling has been employed in high-energy physics, though its most important applications have been in nuclear physics.

A rather different approach to "circumventing Liouville" is to develop a system which probes the distribution of particles in mu space and to improve that distribution by decreasing the emittance of the beam. Stochastic cooling is of this nature and, as we all know, has been very effectively used in high-energy physics[16].

Finally, let us comment on the transition from the Vlasov equation, where there is no damping of phase space, to a damping situation, such as is provided by radiation damping. Clearly, in the latter case, if we include all the modes of the electromagnetic field (photons, if we think quantum mechanically) with the particles, then the whole system is described by a Hamiltonian and Liouville's theorem applies. Thus, from this "big point of view" we only have transfer of phase volume among the various degrees of freedom and no damping of the total phase space. Nevertheless, there is damping of the particle phase space, spanned by all the particle coordinates and momenta, and "undamping" of the region spanned by electromagnetic mode coordinates and momenta.

If we have a group of particles coupled without radiative damping to one or more modes of the electromagnetic field, then the motion of the particles is described by the Vlasov equation and there is no damping in the particle phase space. In this situation of particles coupled dynamically to the electromagnetic field, one might think - erroneously - that Liouville's theorem refers only to the total phase space spanned by particle coordinates and momenta and field coordinates and momenta. Thus, especially in the case where the electromagnetic field is self-generated and growing in time (as in a free-electron laser), one might expect damping in the particle phase space and undamping in the electromagnetic field phase space, as described in the previous paragraph. The Vlasov equation shows, however, that this thinking is quite wrong. Provided the number of particles is large, as discussed in sec.3, each particle moves in the same well-determined electromagnetic field, described by a good single-particle Hamiltonian, and particle phase space is conserved. Notice that this is true whether or not the electromagnetic field is removed from the particle's surroundings (and, for example, used for some purpose); i.e., phase space is conserved whether or not the system is dissipative of energy.

What, then, is needed to achieve a violation of Liouville's theorem? There are just two ways in which the mu-space Liouville's theorem is violated at the microscopic level: through radiation damping, the dissipative self-force on the charged particles, and through the failure of the condition discussed above, that the contribution of any one particle to the electromagnetic field distribution can be neglected. The effects of radiation damping are well understood, and are not under discussion here, while the number of particles in a typical accelerator beam is so great that the latter condition is satisfied to an excellent degree. For this discussion, then, Liouville's theorem holds for mu space.

How, then, do we achieve the apparent violations of Liouville's theorem in various methods of beam cooling? As always, Liouville's theorem breaks down as soon as you work with a coarse-grained average density in phase space. The usual effect of this, due to the normal filamenting of volumes in phase space, is a reduction of phase space density, but the game we are playing is to reverse this normal effect by playing Maxwell's demon. We probe the distribution in mu space, determine which parts are occupied and which are not, and then adjust the electromagnetic fields so that the occupied parts move closer together and the unoccupied parts go elsewhere. The time-dependent fields that we introduce do not alter the fact that the same single-particle Hamiltonian governs the motion of every particle, and thus do not destroy the validity of Liouville's theorem at the microscopic level. The coarse-grained average density, though, increases in the occupied regions of phase space and decreases, necessarily, for the unoccupied regions. For occupied and unoccupied, of course, you can substitute more densely and less densely occupied.

What, then, are the limitations on how far you can play this game? The question is simply how precisely you can probe mu space, and how precisely you can dissect and manipulate it; i.e., what is the spacial resolution of the probe. In some cases a measure of this precision is the wavelength of the electromagnetic signals involved in the process; in other cases, such as in stochastic cooling, it is more the "effective wavelength", since the dipole mode sensed by the pickup has a very long wavelength, but its amplitude is measured with very great precision. If the spacial resolution is greater than the inter-particle spacing, then no further cooling is possible. Others have argued that the spacvial resolution should be compared with the final beam emittance. We believe that this question is unsettled, and note that 155 years after Liouville's work there are still interesting and unresolved aspects of his remarkable theorem.

7. ACKNOWLEDGMENTS

We wish to thank Swapan Chattopadhyay, Kwang-Je Kim, Hiromi Okamoto, George Schmidt, and Simon Yu for illuminating discussions.

REFERENCES

[1] J. Liouville, Journ. de Math. 3, 349 (1838).
[2] K.R. Symon and A.M. Sessler, *Methods of radio frequency acceleration in fixed field accelerators,* Proc. of CERN Symp. on High Energy Accelerators, p. 44, CERN, Geneva (1956).
[3] Members of the MURA Group (private communication).
[4] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York (1978), p. 238.
[5] C.E. Nielsen and A.M. Sessler, Rev. Sci. Inst. 30, 80 (1959).
[6] C.E. Nielsen, A.M. Sessler and K.R. Symon, *Longitudinal instabilities in intense relativistic beams*, Proc. of CERN Symp. on High Energy Accelerators, p. 239, CERN, Geneva (1959); A.A. Kolomenskij and A.N. Lebedev, *Certain beam-stacking effects in fxed-feld magnet systems*, ibid p. 115.
[7] R.L. Mills and A.M. Sessler, *Liouville's theorem for a continuous medium with conservative interactions*, Midwestern Universities Research Association, MURA-433, Oct. 8, 1958 (unpublished).
[8] A.M. Sessler, D.H. Whittum and L.-H. Yu, Phys. Rev. Lett. 64, 309 (1992)..
[9] H. Goldstein, *Classical Mechanics*, Addison-Wesley, Reading, MA (1980), p. 426; A.J. Lichtenberg, *Phase Space Dynamics of Particles*, Wiley, New York (1969).
[10] K. Miyamoto, *Plasma Physics for Nuclear Fusion*, The MIT Press, Cambridge, p. 102 ff (1976).
[11] J.D. Jackson, *Classical Electrodynamics*, Wiley, New York (1975), p. 574.

[12] P.J. Channell and A.M. Sessler, Nucl. Instr. and Methods 136, 473 (1976); D. Boussard, G. Dome and C. Grazini, *The influence of rf noise on the lifetime of bunched proton beams*, 11th Inter. Conf. on High-Energy Accelerators, CERN, Birkhäuser, Basel (1980), p. 620; A. Piwinski, *Intra-beam scattering*, 14th Inter. Conf. on High Energy Accelerators, SLAC, Stanford (1974), p. 405.
[13] K. Robinson, Phys. Rev. 111, 373 (1958); M. Sands, Phys. Rev. 97, 470 (1955).
[14] Members of the MURA Group (private communication); D. Neuffer, Particle Accelerators 14, 75 (1983).
[15] G.I. Budker, *Status report of works on storage rings at Novosibirsk*, Proc. Int. Symp. on Electron and Positron Storage Rings, Saclay (1966), p. II-1-1; W. Kells, *Electron cooling*, Physics of High Energy Particle Accelerators, AIP Conf. Proc. 87, AIP, New York (1982), p. 656.
[16] S. van der Meer, *Stochastic damping of betatron oscillations in the ISR*, CERN Internal Report CERN/ISR-PO/72-31 (1972); S. van der Meer, *An introduction to stochastic cooling*, Physics of Particle Accelerators, AIP Conf. Proc. 153, AIP, New York (1987), p. 1628.

APPENDIX A : Formal Proof of Liouville's Theorem for a Continuous Media

Let λ_i (i = 1, ---, 2f,) be parameters labelling the particles of the medium (2f dimensional phase space; this is the μ space); $dn = \sigma d\lambda_1 ... d\lambda_{2f}$ = number of particles in 'volume' element $d\lambda$; σ = constant 'density' with respect to λ.

Let $\pi_\alpha(\lambda)$ = momentum density = $\sigma\rho_x$, $q_\alpha(\lambda)$ = position of particle λ.

The Hamiltonian is

$$H = \sigma\int h[q(\lambda),\pi(\lambda)]d\lambda + \sigma^2\iint v[q(\lambda),\pi(\lambda),q(\lambda')\pi(\lambda')]d\lambda d\lambda', \tag{A1}$$

where h and v do not depend explicitly on λ, and the equations of motion are

$$\dot{q}_\alpha(\lambda) = \frac{\delta H}{\delta\pi_\alpha(\lambda)} = \sigma\frac{\partial h[\lambda]}{\partial\pi_\alpha(\lambda)} + \sigma^2\int\frac{\partial}{\partial\pi_\alpha(\lambda)}v[\lambda,\lambda']d\lambda' \tag{A2}$$

$$\dot{\pi}_\alpha(\lambda) = -\frac{\delta H}{\delta q_\alpha(\lambda)} = -\sigma\frac{\partial h[\lambda]}{\partial q_\alpha[\lambda]} - \sigma^2\int\frac{\partial}{\partial q_\alpha(\lambda)}v[\lambda,\lambda']d\lambda' . \tag{A3}$$

Let us introduce a condensed notation:

$$\begin{aligned}\xi_s &= q_s\ ,\ \ s = 1,\ldots,f\\ &= -\pi_s\ ,\ \ s = f+1,\ldots,2f\ .\end{aligned} \tag{A4}$$

The equations of motion may be written

$$\dot{\xi}_s(\lambda) = \sum_t \varepsilon_{st}\frac{\delta H}{\delta\xi_t(\lambda)} = \sum_t \varepsilon_{st}\left[\sigma\frac{\partial h}{\partial\xi_t} + \sigma^2\int\frac{\partial v}{\partial\xi_t}d\lambda'\right], \tag{A5}$$

where

$$\begin{aligned}\varepsilon_{s,f+s} &= 1,\ \ s = 1,\ldots,f\\ \varepsilon_{f+s,s} &= -1,\ \ s = 1,\ldots,f\\ \varepsilon_{st} &= 0,\ \ \text{otherwise}\ .\end{aligned} \tag{A6}$$

Clearly ε_{st} is antisymmetric. Now the density $\rho(q,\pi)$ in phase space is given by

$$\rho dq d\pi = \sigma d\lambda \tag{A7}$$

or

$$\rho d\xi = \sigma d\lambda \ . \tag{A8}$$

$$\begin{aligned} \frac{1}{\rho} &= \frac{1}{\sigma} \times \text{Jacobian} \\ &= \frac{1}{\sigma} \times \det\left|\frac{\partial \xi_s}{\partial \lambda_i}\right| \\ &= \frac{1}{\sigma}\Delta, \text{ say}. \end{aligned} \tag{A9}$$

The inverse matrix to $\partial\xi_s/\partial l_i$ is of course $\partial\lambda_i/\partial\xi_s$, and the adjoint to $\partial\xi_s/\partial\lambda_i$ is $\Delta\frac{\partial\lambda_i}{\partial\xi_s}$ .Thus we have for the rate of change of $\frac{1}{\rho}$ along a trajectory (λ = constant):

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{\rho}\right) &= \frac{1}{\sigma}\frac{\partial\Delta}{\partial t}\Big|_{\lambda=\text{constant}} \\ &= \frac{1}{\sigma}\sum_{i,s}\left(\Delta\frac{\partial\lambda_i}{\partial\xi_s}\right)\frac{\partial^2\xi_s}{\partial t\partial\lambda_i}, \end{aligned} \tag{A10}$$

but

$$\begin{aligned} \frac{\partial^2\xi_s}{\partial t\partial\lambda_i} &= \sum_t \frac{\partial}{\partial\lambda_i}\varepsilon_{st}\frac{\delta H}{\delta\xi_t(\lambda)} \\ &= \sum_t \varepsilon_{st}\frac{\partial}{\partial\lambda_1}\left[\sigma\frac{\partial h[\lambda]}{\partial\xi_t(\lambda)} + \sigma^2\int\frac{\partial v[\lambda,\lambda']}{\partial\xi_t(\lambda)}d\lambda'\right] \\ &= \sum_{r,t}\varepsilon_{st}\frac{\partial\xi_r}{\partial\lambda_i}\left[\sigma\frac{\partial^2 h[\lambda]}{\partial\xi_t\partial\xi_r} + \sigma^2\int\frac{\partial^2 v[\lambda,\lambda']}{\partial\xi_t(\lambda)\partial\xi_r(\lambda)}d\lambda'\right] \\ &= \sum_{r,t}\varepsilon_{st}\frac{\partial\xi_r}{\partial\lambda_i}\left[\sigma\frac{\partial^2 h[\lambda]}{\partial\xi_t\partial\xi_r} + \sigma^2\int\frac{\partial^2 v[\lambda,\lambda']}{\partial\xi_t(\lambda)\partial\xi_r(\lambda)}d\lambda'\right] \\ &= \sum_{r,t}\varepsilon_{st}\frac{\partial\xi_r}{\partial\lambda_i}M_{tr} \quad \text{and } M_{tr} \text{ is symmetric,} \end{aligned} \tag{A11}$$

so

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{1}{\rho}\right) &= \frac{1}{\sigma}\Delta \cdot \sum_{i,s,t,r} \frac{\partial \lambda_i}{\partial \xi_s} \varepsilon_{st} \frac{\partial \xi_r}{\partial \lambda_i} M_{tr} \\
&= \frac{\Delta}{\sigma} \sum_{s,t,r} \delta_{r,s} \varepsilon_{st} M_{tr} \\
&= \frac{\Delta}{\sigma} \sum_{s,t} \varepsilon_{st} M_{ts} \\
&= 0
\end{aligned}
\tag{A12}
$$

since ε is antisymmetric and M is symmetric. So the density in μ space does indeed remain constant along any trajectory.