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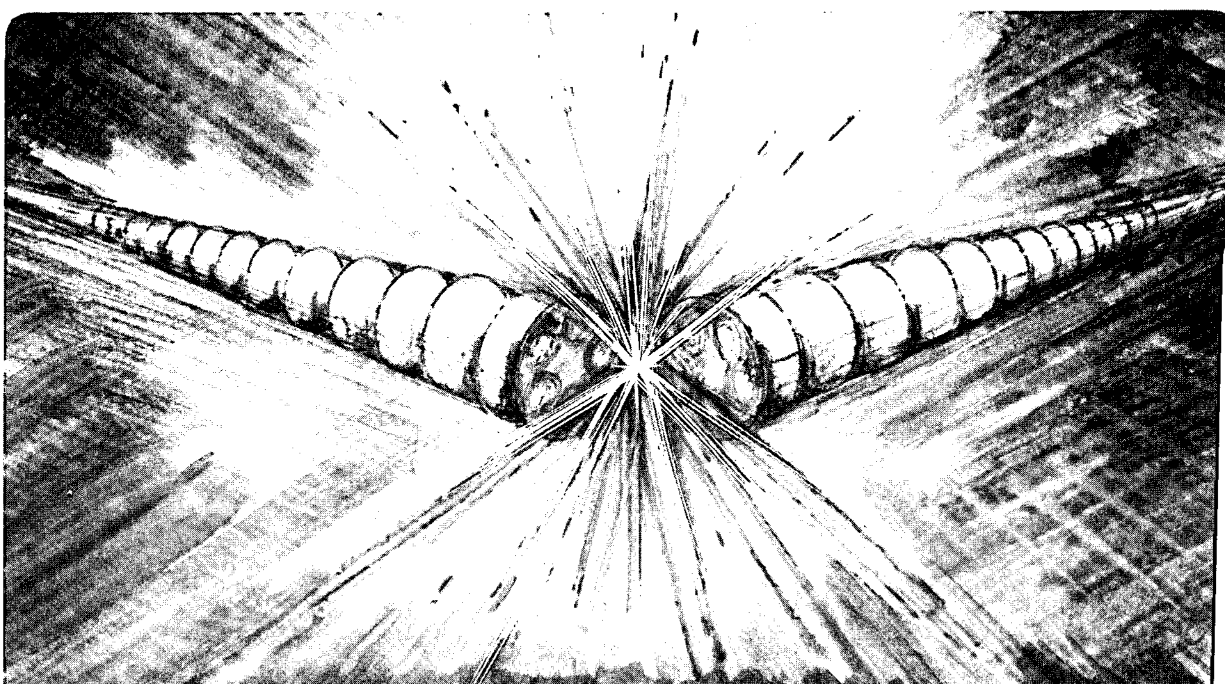

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COMPACT COMPLEX EXPRESSIONS FOR THE ELECTRIC FIELD OF 2-D ELLIPTICAL CHARGE DISTRIBUTIONS*

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ABSTRACT

We present a formula for the analytic calculation of the electric field in complex form for two-dimensional charge distributions with elliptical contours, in the absence of boundary conditions except at infinity. The formula yields compact and practical expressions for a significant class of distributions. The fact that the electric field vanishes inside an elliptical shell follows as a straightforward consequence of Cauchy's theorem. The known expressions for the field inside and outside a uniformly-charged ellipse are recovered in simple, concise form. The known expression for the field of a Gaussian distribution is recovered in a straightforward way as a special case of the more general formula. We present, as one new example, the field for a parabolic distribution.

1. Introduction.

It has long been recognized that a large class of complicated problems in two-dimensional electrostatics (and magnetostatics) in free space can be solved in a compact and elegant fashion by replacing the ordinary plane (x, y) by the complex plane $x+iy$. The reason for the great advantage of the complex plane can be succinctly stated as follows: if we confine our attention to a charge-free region of space, then the relevant Maxwell's equations to be solved are

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = 0 \tag{1}$$

subject to certain boundary conditions. In terms of the field components, these equations read

$$\frac{\partial E_x}{\partial x} = -\frac{\partial E_y}{\partial y}, \quad \frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x} \tag{2}$$

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which are nothing but the Cauchy-Riemann conditions for the complex conjugate $\overline{E}$ of the "complex electric field"

$$E \equiv E_x + iE_y \tag{3}$$

A fundamental theorem of complex analysis then implies that $\overline{E}$ is an analytic function of the complex variable $z \equiv x + iy$ or, equivalently, that the field itself E is an analytic function of $\bar{z}$,

$$E = E(\bar{z}) \tag{4}$$

There are two consequences from this analyticity property: (1), the complex electric field depends on x and y only through the combination $x - iy$ (the combination $x + iy$ is not allowed); and (2), the method of conformal mappings may then be used to transform a complicated boundary condition geometry into a new, simpler geometry in which it is easy to find an analytic function that satisfies the condition. Since conformal mappings preserve analyticity, the solution for the electric field satisfying the original boundary conditions is obtained by simply applying the inverse of the mapping.[1]

Now if the problem to be solved involves free charges, the divergence equation has a nonzero right-hand side,

$$\nabla \cdot \mathbf{E} = 4\pi\rho(\mathbf{x}) \tag{5}$$

and therefore the complex electric field does not satisfy the Cauchy-Riemann conditions, hence it is not, in general, an analytic function (at least not in the region where $\rho \neq 0$). As a result, the method of conformal mappings is useless, although the complex representation of the field can still be used profitably in many cases due to the compactness of the notation.[1]

For charge densities with elliptical contours, we show here that Cauchy's theorem allows the calculation of the field in a much simpler form than the conventional method,[1,2] and yields compact expressions that are convenient to use in algebraic or numerical computations. Although this class of densities is obviously a limited one, it is useful in many applications in beam physics, such as in the calculation of space-charge effects, or in beam-beam interaction problems. The method is advantageous both in free-space and also in regions where free charges are present. As shown below, this formalism requires working directly with the electric field rather than the electrostatic potential; in fact, working with the potential makes it much more complicated. The method seems applicable to other charge distributions provided its contours are simple closed curves, although it is clear that the basic case corresponds to elliptical contours.

In Sec. 2 we present the general setup of the calculation of the complex electric field in the presence of a charge density with elliptical contours. In Sec. 3 we carry out the most basic case, namely that in which the density is a delta-function over an elliptical shell. The well-known result that the field vanishes inside the shell follows as a simple consequence of Cauchy's theorem. The complex electric field for a general charge distribution with elliptical contours is the superposition of the field for the previous case weighted by the charge density. In Sec. 4 we carry out several examples of the calculation for specific charge densities. In particular, we recover in very simple way the known results for the cases of a uniformly-charged ellipse[3] and for a Gaussian[4] distribution. In addition we present, as a new example, the calculation of the field for a parabolic distribution, which might be used in space-charge problems in proton storage rings. In Sec. 5 we present some remarks, and in Sec. 6 our conclusions.

2. General setup of the calculation.

We start from the solution of Eq. (5) for the case of a single point charge Q located at the origin, which we label with a subscript 0,

$$\mathbf{E}_0(\mathbf{x}) = 2Q\frac{\mathbf{x}}{|\mathbf{x}|^2} \tag{6}$$

where $\mathbf{x}$ is the two-dimensional coordinate vector with components (x,y). The general solution for an arbitrary two-dimensional (surface) charge density $\rho(\mathbf{x})$ is obtained by the superposition principle,

$$\mathbf{E}(\mathbf{x}) = 2\int d^2\mathbf{x}'\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^2}\rho(\mathbf{x}') \tag{7}$$

In complex form these equations read

$$E_0 = \frac{2Q}{\bar{z}} \tag{8}$$

and

$$E(\mathbf{x}) = \int d^2\mathbf{x}'\frac{2}{\bar{z}-\bar{z}'}\rho(\mathbf{x}') \tag{9}$$

where the bar denotes complex conjugation, and $z' \equiv x' + iy'$. It should be noted that E_0 exhibits the analyticity property (4) for all $z \neq 0$, as it should.

So far, Eq. (9) is completely general and equivalent to Eq. (7). The trick to simplify (9) is to reduce the two-dimensional integral to a one-dimensional integral over a simple (i.e., nonintersecting) closed curve and then to take advantage of Cauchy's theorem. Obviously the success of this method depends on the properties of the charge density ρ.

In this note we are only concerned with a specific class of density functions, namely those with elliptical contours. That is to say, we assume that, with an appropriate choice of origin and orientation of the coordinate axes, ρ depends on x and y only through the combination

$$t \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} \tag{10}$$

rather than on x and y separately. We assume, without any loss of generality, that $a \geq b$; furthermore, if ρ is a rigorously localized distribution, we define a and b to be the semi-axes of the largest ellipse that encloses charge, namely that $\rho = 0$ for $x^2/a^2 + y^2/b^2 > 1$. If ρ extends over all space (such as in the case of the Gaussian distribution), the parameters a and b can be best interpreted as (or traded off for) the rms sizes of the distribution σ_x and σ_y respectively, by using Eq. (13) below. Eq. (10) implies that the charge density of an elliptical distribution is generally written

$$\rho\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = \int_0^\infty dt\, \rho(t) \delta\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - t\right) \tag{11}$$

With the change of variables described in Sec. 3 it is straightforward to show that the total charge Q of the distribution, which we assume finite, is given by

$$Q \equiv \int d^2\mathbf{x}\, \rho(\mathbf{x}) = \pi ab \int_0^\infty dt\, \rho(t) \tag{12}$$

and that the rms sizes are given by

$$\frac{\sigma_x^2}{a^2} = \frac{\sigma_y^2}{b^2} = \frac{1}{2} \int_0^\infty dt\, t \hat{\rho}(t) \tag{13}$$

where we have introduced the dimensionless density

$$\hat{\rho}(t) \equiv \frac{\pi ab}{Q} \rho(t) \tag{14}$$

which is normalized to unity,

$$\int_0^\infty dt\, \hat{\rho}(t) = 1 \tag{15}$$

Inserting Eq. (11) into Eq. (9) and interchanging the order of the integration,* the complex electric field for a general elliptical distribution becomes

* This step requires ρ to be integrable, namely $|Q| < \infty$. If ρ is not positive-definite or negative-definite, the requirement is that ρ must be absolutely integrable, namely $\int d^2\mathbf{x} |\rho(\mathbf{x})| < \infty$.

$$E(\mathbf{x}) = \int_0^\infty dt\,\hat{\rho}(t) \int d^2\mathbf{x}' \frac{2}{\bar{z} - \bar{z}'} \frac{Q}{\pi ab} \delta\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - t\right) \tag{16}$$

This analysis for density functions with elliptical contours was apparently first carried out by Smith[5] for the electrostatic potential. It is obviously applicable to the field as well, and is generalizable to three dimensions (although not within the complex-field formalism). The subexpression $\int d^2\mathbf{x}'(\cdots)$ in Eq. (16) is nothing but the electric field produced by a delta-function density over an elliptical shell of "radius" t, while the integral $\int dt\hat{\rho}(t)(\cdots)$ is the superposition of the fields produced by all shells of different radii with a weight given by $\hat{\rho}(t)$. In Sec. 3 we will evaluate the field produced by a single delta-function shell, and in Sec. 4 we will carry out the superposition integral for various densities $\hat{\rho}(t)$.

3. The basic case: delta-function elliptical shell.

The conventional calculation for the field (as well as that for the potential in two- and three-dimensional space) for this case uses elliptical coordinates and is not new.[1,5] We present here the corresponding calculation for the complex case. For simplicity we temporarily set $t = 1$ in the expression for the charge density, which then reads

$$\rho(\mathbf{x}) = \frac{Q}{\pi ab} \delta\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \tag{17}$$

and take the complex conjugate of the electric field, Eq. (9), so that

$$\bar{E}(\mathbf{x}) = \frac{2Q}{\pi ab} \int d^2\mathbf{x}' \frac{\delta\left(x'^2/a^2 + y'^2/b^2 - 1\right)}{z - z'} \tag{18}$$

(we will recover the result for $t \neq 1$ later in this section). The change of variables $x' = ar' \cos\varphi'$, $y' = br' \sin\varphi'$ implies

$$\int d^2\mathbf{x}'\,(\cdots) = ab \int_0^{2\pi} d\varphi' \int_0^\infty r'dr'\,(\cdots) \tag{19}$$

The integral over r' is straightforward, yielding

$$\bar{E}(\mathbf{x}) = \frac{Q}{\pi} \int_0^{2\pi} d\varphi' \frac{1}{z - z'} \tag{20}$$

where $z' \equiv a\cos\varphi' + ib\sin\varphi'$ represents the elliptical shell in complex parametric form. A further change of variable, $\zeta = \exp(i\varphi')$, transforms Eq. (20) into a Cauchy-type integral over the unit circle in the complex-ζ plane,

$$\overline{E}(\mathbf{x}) = \frac{2Q}{i\pi} \oint\limits_{|\zeta|=1} d\zeta \frac{1}{2\zeta z - (a+b)\zeta^2 - a + b} \tag{21}$$

which can be done by the method of residues. The poles $\zeta_\pm$ of the integrand are easily found to be

$$\zeta_\pm(z) = \frac{1}{a+b}\left[z \pm \sqrt{z^2 - a^2 + b^2}\right] \tag{22}$$

In order to provide a well-defined result, we need to make the square-root function in this expression unambiguous by means of an appropriate Riemann cut in the z-plane. There are two square-root type cuts that emanate out of the foci of the ellipse, which are located on the real axis at $x = \pm(a^2 - b^2)^{1/2}$. The cuts are, in principle, almost arbitrary: any pair of nonintersecting semi-infinite curves emanating out of the foci of the ellipse renders the square root in (22) unambiguous. It turns out, as discussed below, that the requirement that the electric field be an unambiguous, odd-parity function of z implies that there is a unique specification for the branch cuts. For the time being, however, we proceed with the evaluation of Cauchy's integral, keeping in mind that such a specification will be made explicit below.

We first note that the poles satisfy the relation

$$\zeta_+\zeta_- = \frac{a-b}{a+b} \tag{23}$$

which is a positive real number < 1. This implies that at least one of these poles (ζ_-) is always contained inside the unit circle, regardless of the value of z. The other pole, ζ_+, may or may not be inside the unit circle, depending on the value of z. More specifically, the functions $\zeta_\pm(z)$ are conformal mappings with the following properties: The mapping $\zeta_+(z)$ maps the ellipse $z = a\cos\varphi + ib\sin\varphi$ to the unit circle $|\zeta| = 1$, the exterior of the ellipse to the exterior of the unit circle and the interior of the ellipse to the annulus $[(a-b)/(a+b)]^{1/2} < |\zeta| < 1$. The mapping $\zeta_-(z)$ maps the ellipse $z = a\cos\varphi + ib\sin\varphi$ to the circle $|\zeta| = (a-b)/(a+b)$, the exterior of the ellipse to the disk $0 < |\zeta| < (a-b)/(a+b)$, and the interior of the ellipse to the annulus $(a-b)/(a+b) < |\zeta| < [(a-b)/(a+b)]^{1/2}$.

For the calculation of the field we first rewrite Eq. (21) in the form

$$\overline{E}(\mathbf{x}) = \frac{2iQ}{\pi(a+b)} \oint\limits_{|\zeta|=1} d\zeta \frac{1}{(\zeta - \zeta_+)(\zeta - \zeta_-)} \tag{24}$$

and consider two cases separately:

(1) If the observation point $\mathbf{x}$ is inside the elliptical charge shell, i.e., if z is inside the ellipse, then both poles ζ_+ and ζ_- are inside the unit circle $|\zeta| = 1$, and Cauchy's theorem yields

$$\overline{E}(\mathbf{x}) = 0 \qquad \text{(inside)} \tag{25}$$

(2) If the observation point $\mathbf{x}$ is outside the elliptical charge shell, then only ζ_- is inside the unit circle, and Cauchy's theorem yields

$$\begin{aligned} \overline{E}(\mathbf{x}) &= \frac{2iQ}{\pi(a+b)} \cdot \frac{2\pi i}{\zeta_- - \zeta_+} \\ &= \frac{2Q}{\sqrt{z^2 - a^2 + b^2}} \qquad \text{(outside)} \end{aligned} \tag{26}$$

This completes the calculation except that, as mentioned above, we need to specify the Riemann cuts in the z-plane in order to make $(z^2 - a^2 + b^2)^{1/2}$ well-defined. We first note that the original expression, Eq. (21), defines the electric field to be an unambiguous, odd-parity function of z (or $\mathbf{x}$), as it should be, on account of the even parity of the charge density (17). The expression (26), on the other hand, has the appearance of being of even parity, since it depends on z only through z^2. This is misleading: in general, determining the parity of a function that involves branch cuts is a topological question because it depends on *which path* one chooses to go from z to $-z$. Clearly, there are only two topologically distinct possibilities to define the Riemann cut structure for this square root: (a) the two cuts merge, joining together the two foci of the ellipse, as shown in Fig. 1a, and (b): the cuts emanate out of the foci of the ellipse and extend all the way to infinity, as in Fig. 1b. One can easily prove that the cut structure (a) makes $(z^2 - a^2 + b^2)^{1/2}$ an odd-parity function of z, while structure (b) makes it an even-parity function. Therefore the cut structure (a) provides the correct specification for the electric field.

Another way to establish that cut (a) is the correct specification is to note that cut (b) would lead to unphysical discontinuities of the electric field across the cut in the region outside the ellipse, thus violating the fact that Eq. (21) defines the field in a smooth and unambiguous fashion. In the region inside the ellipse the field vanishes identically, hence no discontinuity arises from a cut joining the two foci. Therefore cut (a) is the correct answer. The cut is almost arbitrary: any nonintersecting line joining the foci that is wholly contained within the ellipse will do. However, in practical applications, particularly for more complicated forms of $\hat{\rho}(t)$, it may be convenient to assume the cut to be a straight line, as shown in Fig. 1a.

In addition to being of odd parity, the real part of the square root $(z^2 - a^2 + b^2)^{1/2}$ is antisymmetric under the reflection $(x,y) \to (-x,y)$ and symmetric under $(x,y) \to (x,-y)$. The imaginary part has the opposite reflection properties. This is as it should be: if the charge density is fully even under reflections, the electric field must be fully odd. With a cut of type (a), the parity and reflection properties are made explicit by the following formula, valid for the region outside the ellipse:

$$\begin{aligned} \sqrt{z^2 - a^2 + b^2} &= \frac{\operatorname{sign}(x)}{\sqrt{2}} \sqrt{x^2 - y^2 - a^2 + b^2 + \sqrt{(x^2 - y^2 - a^2 + b^2)^2 + (2xy)^2}} \\ &\quad + i\frac{\operatorname{sign}(y)}{\sqrt{2}} \sqrt{y^2 - x^2 - b^2 + a^2 + \sqrt{(x^2 - y^2 - a^2 + b^2)^2 + (2xy)^2}} \end{aligned} \tag{27}$$

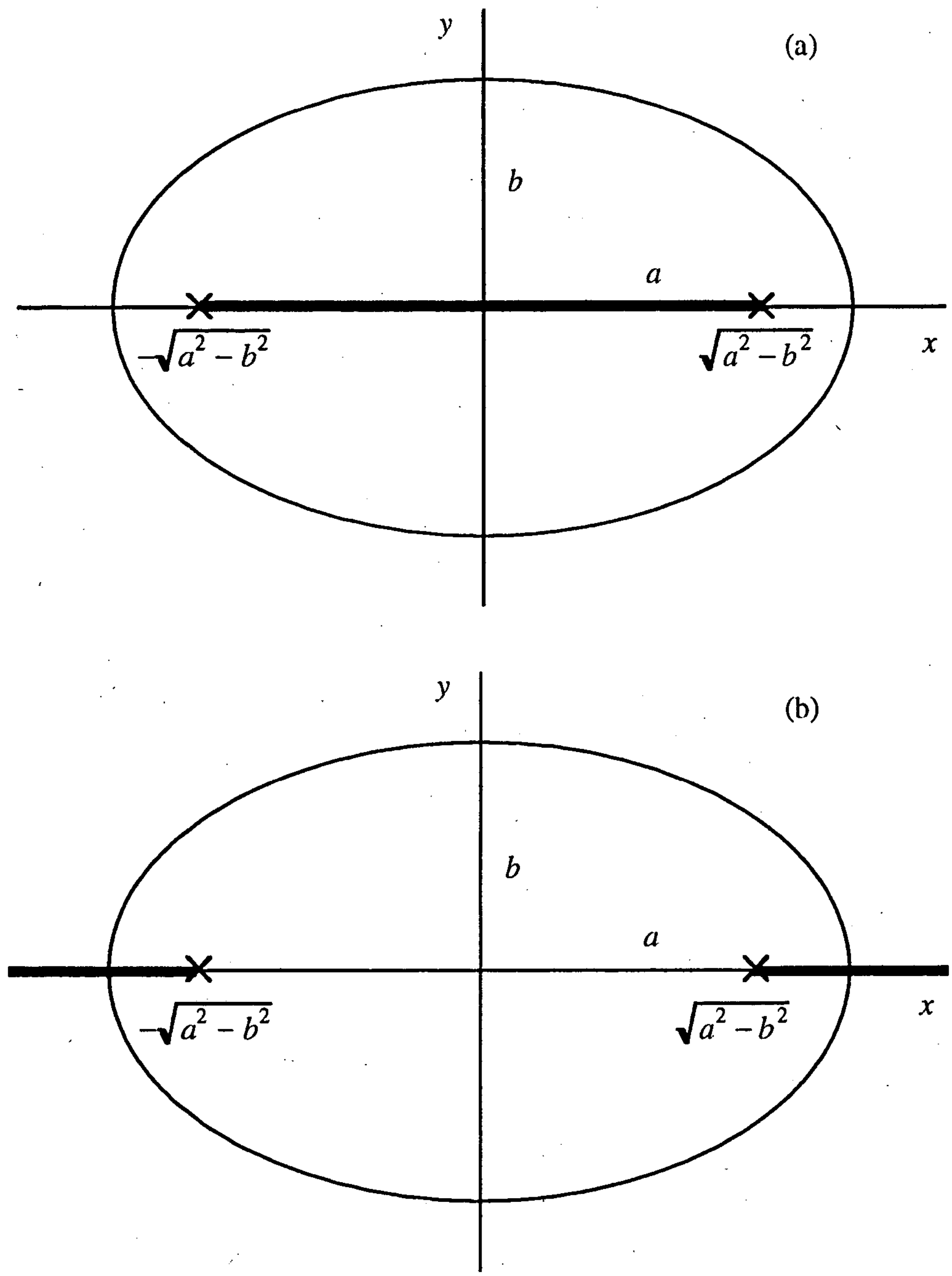

Fig. 1. (a): The square root $(z^2 - a^2 + b^2)^{1/2}$ is rendered an unambiguous odd-parity function of z for all z by joining into a single straight line the branch cuts that emanate out of the foci of the ellipse. (b): If the cuts extend out to infinity along the real axis, the square root $(z^2 - a^2 + b^2)^{1/2}$ is an even-parity function of z for all z, with unphysical discontinuities across the real axis. Thus case (a) gives the correct answer for the complex electric field. The cuts need not be straight lines, although this is the simplest choice.

Finally, we generalize our result for the field to the case in which $t \neq 1$, corresponding to

$$\rho(\mathbf{x}) = \frac{Q}{\pi ab}\,\delta\!\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-t\right) \tag{28}$$

which is obtained from the expression for $t = 1$, Eq. (26), by the replacements $(a,b) \to \sqrt{t}\cdot(a,b)$,

$$E(\mathbf{x}) = 2Q\frac{\theta(x^2/a^2 + y^2/b^2 - t)}{\sqrt{\bar{z}^2 - t(a^2-b^2)}} \tag{29}$$

where we have taken the complex conjugate and have used the step function $\theta(\cdots)$ to make explicit the fact that the field vanishes inside the elliptical shell. The square root in this last expression is made well-defined by a branch cut of the type shown in Fig. 1a, except that, for $t \neq 1$, the branch points are located at $\pm[t(a^2-b^2)]^{1/2}$.

4. Applications.

4.1 General remarks and limiting forms.

In general, as discussed in the introduction, the complex electric field is not a function of $\bar{z}$ alone because it is not, in general, an analytic function. It turns out, however, that in all but the simplest cases, the electric field can be expressed very compactly in terms of the auxiliary complex variables

$$\xi \equiv \frac{x}{a} + i\frac{y}{b} \qquad \text{and} \qquad \omega \equiv \frac{bx}{a} + i\frac{ay}{b} \tag{30}$$

and their complex conjugates, in addition to $z \equiv x + iy$. Obviously these variables are not independent; their relationship is most conveniently expressed by the easily-proven identities

$$z - \omega = (a-b)\bar{\xi} \tag{31a}$$
$$z + \omega = (a+b)\xi \tag{31b}$$

and their complex-conjugate counterparts. A relation that is particularly useful follows from multiplying these two together,

$$(a^2-b^2)|\xi|^2 = \bar{z}^2 - \bar{\omega}^2 = z^2 - \omega^2 \tag{32}$$

By combining Eqs. (16) and (29) we arrive at the general expression for the complex electric field for an elliptical distribution,

$$E(\mathbf{x}) = 2Q\int_0^{|\xi|^2} \frac{\hat{\rho}(t)\,dt}{\sqrt{\bar{z}^2 - t(a^2-b^2)}} \tag{33}$$

which constitutes the central result of this note. For each t, the cut is a nonintersecting line wholly contained within the ellipse $x^2/a^2 + y^2/b^2 = t$, joining the two foci at $\pm[t(a^2 - b^2)]^{1/2}$. Thus, overall, the cut extends out to $\pm|\xi|(a^2 - b^2)^{1/2}$. However, for a rigorously localized charged distribution, the cut extends only out to $\pm(a^2 - b^2)^{1/2}$ when the observation point is outside the region with charge, according to the discussion following Eq. (10). The integral can be carried out analytically for a significant class of weight functions $\hat{\rho}(t)$, of which we will provide a few examples below (the thin-shell case, Eq. (29), is recovered by setting $\hat{\rho}(t) = \delta(t-1)$).

If $\hat{\rho}(t)$ is regular at t=0, one can easily find from Eq. (33) the leading expression for the field at the origin,

$$E(\mathbf{x}) \to \frac{4Q\hat{\rho}(0)}{a+b}\xi \quad \text{as } |z| \to 0 \tag{34}$$

If $\hat{\rho}(t)$ has finite extent, or if it falls sufficiently rapidly as large distance, then Eq. (33) and the normalization property (15) imply that

$$E(\mathbf{x}) \to \frac{2Q}{\bar{z}} \qquad \text{as } |z| \to \infty \tag{35}$$

so that the field approaches that of a point charge, as it should be expected. Nonleading corrections to this limit, which constitute the multipole expansion of the field, can be easily found from Eq. (33) (the resulting multipole expansion is a convergent series only if the charge distribution is of finite extent; in all other cases the expansion is asymptotic).

4.2 Round distribution.

If the charge distribution is round rather than elliptical, Eq. (33) yields, for the field at distance r from the center,

$$E(\mathbf{x}) = \frac{2Q}{\bar{z}} \int_0^{r^2/a^2} dt\, \hat{\rho}(t) \tag{36}$$

which is the result one obtains in a straightforward manner from Gauss' theorem. In particular, if the charge distribution is a thin round shell, this yields the well-known result

$$E(\mathbf{x}) = \begin{cases} 0 & \text{inside} \\ E_0(\mathbf{x}) & \text{outside} \end{cases} \tag{37}$$

4.3 Uniformly charged ellipse.

This case was apparently first considered by Teng.[3] Our formalism allows to recover Teng's results in a much simpler and concise fashion. In this case the charge density is

$$\rho(\mathbf{x}) = \begin{cases} \dfrac{Q}{\pi ab} & \text{if } x^2/a^2 + y^2/b^2 \le 1 \\ 0 & \text{if } x^2/a^2 + y^2/b^2 > 1 \end{cases} \tag{38}$$

so that $\hat{\rho}(t) = \theta(1-t)$. Then the complex field is given by

$$E(\mathbf{x}) = 2Q \int_0^{\min(1,|\xi|^2)} \frac{dt}{\sqrt{\bar{z}^2 - t(a^2-b^2)}} \tag{39}$$

The integral is elementary. If the observation point is inside the ellipse the top limit in the integral is $t = |\xi|^2$ and we obtain

$$\begin{aligned} E(\mathbf{x}) &= \frac{4Q}{a^2-b^2}\left[\bar{z} - \sqrt{\bar{z}^2 - |\xi|^2(a^2-b^2)}\right] \\ &= \frac{4Q}{a^2-b^2}[\bar{z} - \bar{\omega}] \qquad \text{(inside)} \\ &= \frac{4Q}{a+b}\xi \end{aligned} \tag{40}$$

If the observation point is outside the ellipse, then the top limit is $t = 1$ and the result is

$$E(\mathbf{x}) = \frac{4Q}{a^2-b^2}\left[\bar{z} - \sqrt{\bar{z}^2 - a^2 + b^2}\right] \tag{41a}$$

$$= \frac{4Q}{\bar{z} + \sqrt{\bar{z}^2 - a^2 + b^2}} \qquad \text{(outside)} \tag{41b}$$

In arriving at the final result in Eq. (40) for the field inside the ellipse we have used the identity (32). Notice that this expression is manifestly devoid of possible discontinuities or ambiguities, as it should be according to the earlier discussion on the nature of the cut. Although the two expressions (41a) and (41b) for the field outside the ellipse are mathematically equivalent, (41b) has the advantage that it is manifestly regular in the round-beam limit, $a \to b$, and that it has a straightforward long-distance limit, $|z| \to \infty$. Therefore this second form is preferable in many numerical computations. Notice also that the field in the charge-free region, Eq. (41), is an analytic function of $\bar{z}$, as it should be according to Eq. (4).

At the edge of the ellipse, the two expressions (40) and (41) must coincide. This can be verified as follows: we first note that the edge of the ellipse is defined by $|\xi|^2 = 1$, so that the identity (32) reduces to $a^2 - b^2 = \bar{z}^2 - \bar{\omega}^2$. Substituting this into Eq. (41a) and using (31a) yields

$$E(\mathbf{x}) = \frac{4Q}{a^2 - b^2}(\bar{z} - \bar{\omega}) \qquad \text{(edge)}$$
$$= \frac{4Q}{a+b}\xi \tag{42}$$

which agrees with the expression for the field inside the ellipse, Eq. (40).

Inside the ellipse, Eq. (40) shows that the field is linear, with the x-component given by

$$E_x = \frac{4Q}{a(a+b)}x \qquad \text{(inside)} \tag{43}$$

This linearity property is well known, and is also true for a three-dimensional ellipsoid.[1,2] Outside the ellipse Eq. (41a) yields, for the real part,

$$E_x = \frac{4Q}{a^2 - b^2}\left[x - \frac{\operatorname{sign}(x)}{\sqrt{2}}\sqrt{x^2 - y^2 - a^2 + b^2 + \sqrt{(x^2 - y^2 - a^2 + b^2)^2 + (2xy)^2}}\right] \tag{44}$$

while the y-component can be obtained from this by exchanging $x \leftrightarrow y$ and $a \leftrightarrow b$. The real counterpart of Eq. (41b) is, of course, more complicated. The compactness and relative simplicity of the complex form, Eq. (41a), are obvious compared to this real form. If we specialize Eq. (41b) to the real axis, we obtain, for $x > 0$,

$$E_x = \frac{4Q}{x + \sqrt{x^2 - a^2 + b^2}} \tag{45}$$

which agrees with one of Teng's original expressions.[3]

4.3 Gaussian charge density.

An expression for the complex electric field was apparently first derived by Bassetti and Erskine.[4] We now show how to obtain their result from our general formula. The charge density is

$$\rho(\mathbf{x}) = \frac{Q}{2\pi\sigma_x\sigma_y}\exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right) \tag{46}$$

so that $\hat{\rho}(t) = \frac{1}{2}e^{-t/2}$ and the field is

$$E(\mathbf{x}) = Q\int_0^{|\xi|^2} \frac{e^{-t/2}\,dt}{\sqrt{\bar{z}^2 - t(\sigma_x^2 - \sigma_y^2)}} \tag{47}$$

Now making the change of integration variable $2(\sigma_x^2 - \sigma_y^2)s^2 = \bar{z}^2 - t(\sigma_x^2 - \sigma_y^2)$ and using the definition of the complex error function $w(z)$,[6]

$$\int_0^z ds\, e^{s^2} = \frac{\sqrt{\pi}}{2i}\left[e^{z^2} w(z) - 1\right] \tag{48}$$

gives

$$E(\mathbf{x}) = iQ\sqrt{\frac{2\pi}{\sigma_x^2 - \sigma_y^2}}\, e^{-s_1^2}\left[e^{s_2^2} w(s_2) - e^{s_1^2} w(s_1)\right] \tag{49}$$

where s_1 and s_2 are

$$\begin{aligned} s_1 &\equiv \frac{\bar{z}}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \\ s_2 &\equiv \frac{\sqrt{\bar{z}^2 - |\xi|^2(\sigma_x^2 - \sigma_y^2)}}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} = \frac{\bar{\omega}}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \end{aligned} \tag{50}$$

Substituting these and using the identity (32) the electric field becomes

$$E(\mathbf{x}) = iQ\sqrt{\frac{2\pi}{\sigma_x^2 - \sigma_y^2}}\left[e^{-|\xi|^2/2} w\left(\frac{\bar{\omega}}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}}\right) - w\left(\frac{\bar{z}}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}}\right)\right] \tag{51}$$

which is equivalent* to the function F in Ref. [4].

4.4 Parabolic charge density.

By using the elementary recursion relation

$$\int \frac{dt\, t^n}{\sqrt{1-kt}} = \frac{-2}{(2n+1)k}\left[t^n\sqrt{1-kt} - n\int \frac{dt\, t^{n-1}}{\sqrt{1-kt}}\right] \tag{52}$$

it is obviously possible to compute the complex electric field for a charge density that is an arbitrary polynomial function of t (many other forms for the density also yield expressions for the field in closed form). As an example, we consider the density

$$\rho(\mathbf{x}) = \begin{cases} \dfrac{2Q}{\pi ab}\left(1 - x^2/a^2 + y^2/b^2\right) & \text{if } x^2/a^2 + y^2/b^2 \le 1 \\ 0 & \text{if } x^2/a^2 + y^2/b^2 > 1 \end{cases} \tag{53}$$

* Our definition of E differs from Bassetti-Erskine's F by complex conjugation and a multiplicative factor of 2i.

which is perhaps the simplest case beyond the uniform-density case with possible physical meaning. This density might be useful, for example, in applications to space-charge problems for proton storage rings. The normalized density reads $\hat{\rho}(t) = 2(1-t)\theta(1-t)$. The field is computed following the same steps as in the uniform-charge-distribution case. The only challenge arises when one wants to express the final result in a form free from round-beam or large-distance apparent ambiguities (see Sec. 5 for a systematic solution to this challenge). The result is

$$E(\mathbf{x}) = \begin{cases} \dfrac{8Q\xi}{a+b}\left(1 - \dfrac{(2\bar{z}+\bar{\omega})\xi}{3(a+b)}\right) & \text{if } x^2/a^2 + y^2/b^2 \le 1 \\[2ex] \dfrac{2Q}{\bar{z}}\left(\dfrac{2\bar{z}}{\bar{z}+\sqrt{\bar{z}^2-a^2+b^2}}\right)^2\left(\dfrac{\bar{z}+2\sqrt{\bar{z}^2-a^2+b^2}}{3\bar{z}}\right) & \text{if } x^2/a^2 + y^2/b^2 > 1 \end{cases} \tag{54}$$

These expressions are in agreement with the short- and long-distance limits (34) and (35). It should also be noted that, in the region with charge, the field is manifestly unambiguous and devoid of discontinuities. In the charge-free region, the field obeys the analyticity property (4).

5. Remarks.

5.1 Form factor.

If a factor $\bar{z}$ is pulled outside the square root in Eq. (33), the field is written in the form

$$E(\mathbf{x}) = E_0(\mathbf{x}) \cdot F(\mathbf{x}) \tag{55}$$

where F is a dimensionless "form factor" that describes the finite-size effects,

$$F(\mathbf{x}) \equiv \int_0^{|\xi|^2} \frac{\hat{\rho}(t)\, dt}{\sqrt{1 - t(a^2-b^2)/\bar{z}^2}} \tag{56}$$

With the cut structure discussed earlier, it can be easily shown that the square root in this expression is an even-parity function, and hence so is $F(\mathbf{x})$ (the point-charge field $E_0(\mathbf{x})$, of course, is of odd parity, making the overall field $E(\mathbf{x})$ of odd parity). Since the field is well-defined for all z, so is the form factor. The leading behaviors shown in Eqs. (34) and (35) translate into

$$\begin{aligned} F(\mathbf{x}) &\to \frac{2\hat{\rho}(0)}{a+b}\xi\,\bar{z} \quad && \text{as } |z| \to 0 \\ F(\mathbf{x}) &\to 1 && \text{as } |z| \to \infty \end{aligned} \tag{57}$$

5.2 Coding in FORTRAN.

The ANSI-standard definition of the FORTRAN function CSQRT(Z) evaluates the square root $(\bar{z}^2 - a^2 + b^2)^{1/2}$ incorrectly for our purposes: CSQRT(Z) turns it into an even-parity function, corresponding to the cuts in Fig. 1b. By the same token, CSQRT(Z) also makes $[1-(a^2-b^2)/\bar{z}^2]^{1/2}$ an even-parity function. Therefore, if one wants to carry out computations in standard FORTRAN without conditioning the calculation to the quadrant to which z belongs, it is simplest to code the formulas with the replacement $(\bar{z}^2 - a^2 + b^2)^{1/2} \to \bar{z}[1-(a^2-b^2)/\bar{z}^2]^{1/2}$.

5.3 Computational speed.

Although the complex expressions for the field are simpler than those for the real and imaginary parts, they are slower to compute. Results from a benchmark with double-precision arithmetic on a VAX 6610 show that computing Eq. (44) (plus the y-component) is faster by a factor of 2 than computing Eq. (41a) and then taking the real and imaginary part. In single precision, the speed-up factor is 2.5. However, for more complicated cases, we expect that the complex expressions are more competitive from the perspective of computational speed due to their much simpler form.

5.4 Distributions that are polynomials in t.

As mentioned above, the recursion relation (52) allows the computation of the field when $\hat{\rho}(t)$ is an arbitrary polynomial. However, straightforward application of (52) yields an expression with $a^2 - b^2$ in the denominator. The general way to obtain a result with a manifestly regular round-beam limit is to use the definition of the hypergeometric function and to make a quadratic transformation,[7] thus

$$\begin{aligned}\int_0^T \frac{dt\, t^n}{\sqrt{1-kt}} &= T^{n+1}\int_0^1 \frac{ds\, s^n}{\sqrt{1-kTs}} \\ &= \frac{T^{n+1}}{n+1}\, {}_2F_1\!\left(\tfrac{1}{2}, n+1, n+2; kT\right) \\ &= \frac{T^{n+1}}{n+1}\left(\frac{2}{1+\sqrt{1-kT}}\right)^{n+1} {}_2F_1\!\left(-n, n+1, n+2; (1-\sqrt{1-kT})/2\right)\end{aligned} \tag{58}$$

where T is the appropriate top limit of the integral (for a finite-size distribution, $T \equiv \min(1, |\xi|^2)$ and $k \equiv (a^2 - b^2)/\bar{z}^2$). For integer values of n, the hypergeometric function in the last equation is a polynomial of degree n in the variable $(1-\sqrt{1-kT})/2$ and hence this form produces results in the form of Eq. (41b) or (54).

5.5 Distribution functions with contours other than elliptical.

The method described can be extended, in principle, to charge distributions that depend on x and y only through a positive-definite combination $c(x,y)$ such that

$$c(x,y) = t \tag{59}$$

represents a simple (i.e., nonintersecting) closed curve. However, it seems clear that the elliptical contours are the simplest.

6. Conclusions.

We have presented a formalism that yields simple, compact expressions for the electric field of two-dimensional charge distributions with elliptical contours. Cauchy's theorem plays a central role in the calculation. We have reproduced in closed form the known results for the cases of the uniformly charged ellipse and the Gaussian distribution. We have also presented a new result as an example of a large class of distribution functions that yield closed expressions for the electric field. Our formalism allows simplified coding in numerical simulations and easier analytical work in problems in beam physics such as those involving space-charge effects or the beam-beam interaction.

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8. References.

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