LBL-34833
UC-414

Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

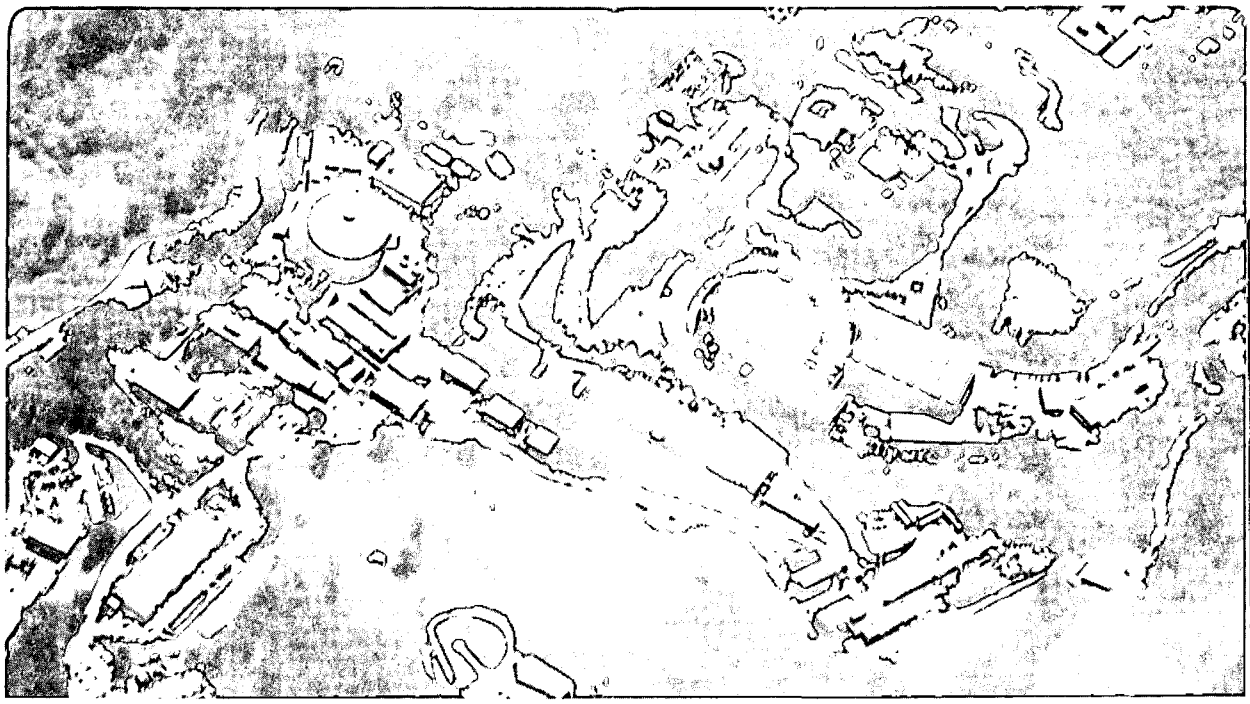
Physics Division

Presented at the XXIInd International Conference on Differential Geometric Methods in Theoretical Physics, Ixtapa, Mexico, September 20–25, 1993, and to be published in the Proceedings

Cartan Calculus on Quantum Lie Algebras

P. Schupp, P. Watts, and B. Zumino

December 1993



REFERENCE COPY
Does Not
Circulate
Bldg. 50 Library.
LBL-34833
Copy 1

Prepared for the U.S. Department of Energy under Contract Number DE-AC03-76SF00098

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor The Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or The Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof, or The Regents of the University of California.

Lawrence Berkeley Laboratory is an equal opportunity employer.

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

December 9, 1993

LBL-34833
UCB-PTH-93/32

Cartan Calculus on Quantum Lie Algebras[1]

Peter Schupp, Paul Watts[2] and Bruno Zumino

Department of Physics
University of California
and
Theoretical Physics Group
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

Abstract

A generalization of the differential geometry of forms and vector fields to the case of quantum Lie algebras is given. In an abstract formulation that incorporates many existing examples of differential geometry on quantum spaces we combine an exterior derivative, inner derivations, Lie derivatives, forms and functions all into one big algebra, the "Cartan Calculus".

(This is an extended version of a talk presented by P. Schupp at the XXIIth International Conference on Differential Geometric Methods in Theoretical Physics, Ixtapa, Mexico, September 1993)

[1]This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY-90-21139.

[2]watts@theorm.lbl.gov

recycled paper

1 Introduction

The central idea behind Connes' Universal Calculus [1] in the context of noncommutative geometry was to retain the classical differential geometric properties of $\mathbf{d}$, *i.e.* nilpotency and the undeformed Leibniz rule[4]: $\mathbf{d}\alpha = \mathbf{d}(\alpha) + (-1)^p \alpha \mathbf{d}$ for any p-form α. Here we want to base the construction of a differential calculus on quantum groups on two additional classical formulas: to extend the definition of a Lie derivative from functions and vector fields to forms we postulate

$$\pounds \circ \mathbf{d} = \mathbf{d} \circ \pounds; \tag{1}$$

this is essential for a geometrical interpretation of vector fields. The second formula that we can — somewhat surprisingly — keep undeformed in the quantum case is

$$\pounds_{\chi_i} = i_{\chi_i}\mathbf{d} + \mathbf{d}i_{\chi_i}, \qquad \textit{(Cartan identity)} \tag{2}$$

where $\{\chi_i\}$ are the generators of some quantum Lie algebra.

2 Quantum Lie Algebras

A quantum Lie algebra is a Hopf algebra $\mathcal{U}$ with a finite-dimensional biinvariant subvector space $\mathcal{T}_q$ spanned by generators $\{\chi_i\}$ with coproduct

$$\Delta\chi_i = \chi_i \otimes 1 + O_i{}^j \otimes \chi_j. \tag{3}$$

More precisely we will call this a quantum Lie algebra of **type II**. Let $\{\omega^j \in \mathcal{T}_q{}^*\}$ be a dual basis of 1-forms corresponding to a set of functions $b^j \in \mathcal{A}$ via $\omega^j \equiv Sb^j_{(1)}\mathbf{d}b^j_{(2)}$; *i.e.*

$$\begin{aligned}
{}_{\mathcal{A}}\Delta(\chi_i) &= 1 \otimes \chi_i, && (4)\\
\Delta_{\mathcal{A}}(\chi_i) &= \chi_j \otimes T^j{}_i, \quad T^j{}_i \in \mathrm{Fun}(G_q), && (5)\\
i_{\chi_i}(\omega^j) &= -\left\langle \chi_i, Sb^j \right\rangle = \delta^j_i, && (6)\\
{}_{\mathcal{A}}\Delta(\omega^i) &= 1 \otimes \omega^i, && (7)\\
\Delta_{\mathcal{A}}(\omega^i) &= \omega^j \otimes S^{-1}T^i{}_j. && (8)
\end{aligned}$$

[4]We use parentheses to delimit operations like $\mathbf{d}$, i_x and $\pounds_x$, *e.g.* $\mathbf{d}a = \mathbf{d}(a) + a\mathbf{d}$. However, if the limit of the operation is clear from the context, we will suppress the parentheses, *e.g.* $\mathbf{d}(i_x\mathbf{d}a) \equiv \mathbf{d}(i_x(\mathbf{d}(a)))$.

If the functions b^i also close under adjoint coaction $\Delta^{Ad}(b^i) = b^j \otimes S^{-1}T^i{}_j$, we will call the corresponding quantum Lie algebra one of **type I**.

We can derive two alternate expressions for the exterior derivative of a function from the Cartan identity (2) in terms of these bases:

$$\mathbf{d}(f) = \omega^j \pounds_{\chi_j}(f) = -\pounds_{S\chi_j}(f)\omega^j. \tag{9}$$

Combining the two expressions for $\mathbf{d}$ one easily derives the well-known $f-\omega$ commutation relations

$$f\omega^i = \omega^j \pounds_{O_j{}^i}(f). \tag{10}$$

The classical limit is given by $O_j{}^i \to 1\delta^i_j$, so that forms commute with functions.

3 Generators, Metrics and the Pure Braid Group

How does one go about finding the basis of generators $\{\chi_i\}$ and the set of functions $\{b^i\}$ that define the basis of 1-forms $\{\omega^i\}$? Here we would like to present a method that utilizes pure braid group elements as introduced in [2].

Let us recall that a pure braid element Υ is an element of $\mathcal{U}\hat{\otimes}\mathcal{U}$ that commutes with all coproducts of elements of $\mathcal{U}$, *i.e.*

$$\Upsilon\Delta(y) = \Delta(y)\Upsilon, \qquad \forall y \in \mathcal{U}. \tag{11}$$

Υ maps elements of $\mathcal{A}$ to elements of $\mathcal{U}$ with special transformation properties under the right coaction:

$$\begin{aligned} &\Upsilon : \mathcal{A} \to \mathcal{U} : \quad b \mapsto \Upsilon_b \equiv \langle \Upsilon, b \otimes \mathrm{id}\rangle\,; \\ &\Delta_{\mathcal{A}}(\Upsilon_b) = \Upsilon_{b_{(2)}} \otimes S(b_{(1)})b_{(3)} = \left\langle \Upsilon \otimes \mathrm{id}, \tau^{23}(\Delta^{Ad}(b) \otimes \mathrm{id})\right\rangle. \end{aligned} \tag{12}$$

An element Υ of the pure braid group defines furthermore a bilinear quadratic form on $\mathcal{A}$

$$(\ ,\ ) : \mathcal{A} \otimes \mathcal{A} \to k : \quad a \otimes b \mapsto (a,b) = -\langle \Upsilon, a \otimes S(b)\rangle \in k, \tag{13}$$

with respect to which we can construct orthonormal bases $\{b_i\}$ and $\{b^j\}$ of functions (*i.e.* $(b_i, b^j) = \delta^j_i$) that in turn will give generators $\chi_i := \Upsilon_{b_i}$ and

1-forms $\omega^j := S(b^j_{(1)})\mathrm{d}b^j_{(2)}$. Typically, one can choose $\mathrm{span}\{b_i\} = \mathrm{span}\{b^j\}$; then one starts by constructing one set, say $\{b_i\}$, of functions that close under adjoint coaction

$$\Delta^{Ad} b_i = b_j \otimes T^j{}_i. \tag{14}$$

If the numerical matrix

$$\eta_{ij} := -\langle \Upsilon, b_i \otimes S b_j \rangle \qquad \textit{(metric)} \tag{15}$$

is invertible, *i.e.* $\det(\eta) \neq 0$, then we can use its inverse $\eta^{ij} := (\eta^{-1})_{ij}$ to raise indices via

$$b^i = b_j \eta^{ji}. \tag{16}$$

This metric is invariant — or T is orthogonal — in the sense that

$$\eta_{ji} = \eta_{kl} T^k{}_j T^l{}_i. \tag{17}$$

Once we have obtained a metric η, we can truncate the pure braid element Υ and work instead with

$$\Upsilon \to \Upsilon_{trunc} = -S(\chi_i) \otimes \chi^i = -S(\chi_i) \otimes \chi_j \eta^{ji}, \tag{18}$$

which also commutes with all coproducts. Casimir operators can also be constructed from elements of the pure braid group. The truncated pure braid element gives, for instance, the quadratic casimir

$$[\cdot \circ \tau \circ (S^{-1} \otimes \mathrm{id})](\Upsilon_{trunc}) = \eta^{ji} \chi_j \chi_i. \qquad \textit{(casimir)} \tag{19}$$

Now we would like to show that we have actually obtained a quantum Lie algebra of type I[5]:

$$-\left\langle \chi_i, S b^j \right\rangle = -\langle \Upsilon, b_i \otimes S b_k \rangle \eta^{kj} = \eta_{ik} \eta^{kj} = \delta_i^j, \tag{20}$$

$$\Delta_{\mathcal{A}}(\chi_i) = \Upsilon_{b_{i(2)}} \otimes S(b_{i(1)}) b_{i(3)} = \Upsilon_{b_j} \otimes T^j{}_i = \chi_j \otimes T^j{}_i \tag{21}$$

and

$$\Delta^{Ad}(b^i) = b_k \otimes T^k{}_j \eta^{ji} = b_k \otimes \eta^{kl} \eta_{ln} T^n{}_j \eta^{ji} = b^k \otimes S^{-1} T^i{}_k. \tag{22}$$

[5]Note, that Υ has to be carefully chosen to insure the correct number of generators. Furthermore, we still have to check the coproduct of the generators. If they are not of the form $\Delta\chi_i = \chi_i \otimes 1 + O_i{}^j \otimes \chi_j$ then we might still consider a calculus with deformed Leibniz rule.

3.1 Examples

3.1.1 The R-matrix approach

Often one can take $b_i \in \text{span}\{t^n{}_m\}$, where $t^n{}_m$ is a quantum matrix in the defining representation of the quantum group under consideration. If we are dealing with a quasitriangular Hopf algebra with universal $\mathcal{R} \equiv \alpha_i \otimes \beta^i$, a natural choice for the pure braid element is

$$\Upsilon_R = \frac{1}{\lambda}\left(1 \otimes 1 - \mathcal{R}^{21}\mathcal{R}^{12}\right), \tag{23}$$

where the term $\mathcal{R}^{21}\mathcal{R}^{12}$ has been introduced and extensively studied in [3] and later in [4, 5, 2]. These choices of b_is and Υ lead to the R-matrix approach to differential geometry on quantum groups. The metric is

$$\eta = -\langle X_1, St_2\rangle = \frac{1}{\lambda}\left(\left[\left(R_{21}{}^{-1}\right)^{t_2}\left(R_{12}{}^{t_2}\right)^{-1}\right]^{t_2} - I\right), \tag{24}$$

where $X_1 = \langle \Upsilon_R, t_1 \otimes \mathrm{id}\rangle$, $R_{12} = \langle \mathcal{R}, t_1 \otimes t_2\rangle$, and I is the identity matrix. In the case of $\mathrm{GL}_q(2)$ we find

$$\eta_{\mathrm{GL}_q(2)} = \begin{pmatrix} -q^{-3} & 0 & 0 & 0 \\ 0 & 0 & -q^{-1} & 0 \\ 0 & -q^{-3} & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}. \tag{25}$$

Using this metric we recover — as expected — the well-known [6, 7] expression of the exterior derivative $\mathbf{d}$ on functions in terms of the quantum trace over X and the Cartan-Maurer form $\Omega = t^{-1}\mathrm{d}t$:

$$\mathbf{d} = \omega^i \chi_i = -\mathrm{tr}(D^{-1}\Omega X) \qquad \textit{(on functions)}. \tag{26}$$

(This follows essentially from $D_2^{-1}\eta_{12} = P_{12}$, where $D = \langle u, t\rangle$ with $u = S(\beta^i)\alpha_i$ and P is the permutation matrix.)

3.1.2 Trace formula for the metric

Again in the case where $\mathcal{U}$ is a quasitriangular Hopf algebra, there exists an alternate way of defining a Killing form; let $\rho : \mathcal{U} \to M_n(k)$ be an $n \times n$

matrix representation of $\mathcal{U}$ with entries in k. Define the map $\eta^{(\rho)} : \mathcal{U} \otimes \mathcal{U} \to k$ as[6]

$$\eta^{(\rho)}(x \otimes y) := \mathrm{tr}_\rho \left[S(uxy) \right], \tag{27}$$

where $x, y \in \mathcal{U}$, tr_ρ is the trace over the given representation, and u (see above) implements the square of the antipode [8]. The map $\eta^{(\rho)}$ has the following properties:

$$\eta^{(\rho)}(y \otimes x) = \eta^{(\rho)}(x \otimes S^2(y)), \tag{28}$$

$$\eta^{(\rho)}((z_{(1)} \overset{\mathrm{ad}}{\triangleright} x) \otimes (z_{(2)} \overset{\mathrm{ad}}{\triangleright} y)) = \eta^{(\rho)}(x \otimes y)\epsilon(z), \tag{29}$$

for all $x, y, z \in \mathcal{U}$. Respectively, these express the symmetry of $\eta^{(\rho)}$ and its invariance under the adjoint action. In the case when $\mathcal{U}$ is a quantum Lie algebra with generators $\{\chi_i\}$, we can define the Killing metric for the representation ρ as

$$\eta^{(\rho)}_{ij} := \eta^{(\rho)}(\chi_i \otimes \chi_j). \tag{30}$$

For the quantum group $\mathrm{GL}_q(2)$, with ρ being the fundamental representation, a calculation gives the Killing metric (expressed as a matrix in the basis $(\chi_1, \chi_+, \chi_-, \chi_2)$) as

$$\begin{aligned} \eta^{(\mathrm{fund\ GL}_q(2))} &= \begin{pmatrix} q^{-7} & 0 & 0 & 0 \\ 0 & 0 & q^{-5} & 0 \\ 0 & q^{-7} & 0 & 0 \\ 0 & 0 & 0 & q^{-5} \end{pmatrix} \\ &= -q^{-4}\eta_{\mathrm{GL}_q(2)}, \end{aligned} \tag{31}$$

so we see that the two differ only by a multiplicative constant (which is $-q^{-2n}$ for the $\mathrm{GL}_q(n)$ case). If we reduce the two matrices (25) and (31) into $1 \oplus 3$ matrices, corresponding to the basis $(\chi_1 + q^{-2}\chi_2, \chi_+, \chi_-, \chi_1 - \chi_2)$, we find

$$\eta_{\mathrm{GL}_q(2)} = -q^4 \eta^{(\mathrm{fund\ GL}_q(2))} = -\begin{pmatrix} q^{-3} + q^{-5} & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & q^{-3} & 0 & 0 \\ 0 & 0 & 0 & q^{-1} + q^{-3} \end{pmatrix}. \tag{32}$$

[6]The map $\eta^{(\rho)}$ as defined in the Proceedings of the XXIIth DGM Conference differs from the one appearing here by an antipode. This can be compensated by choosing the contragredient representation in the former case.

Here we see the decomposition which in the undeformed case is expressed as GL(2)$\simeq$U(1)$\times$SL(2).

Further properties of the Killing metric as defined in this subsection will be examined in a forthcoming paper [9].

3.1.3 The 2-dim quantum euclidean group

This is an example of a quantum Lie algebra that seems to have no universal $\mathcal{R}$ and where the set of functions $\{b_i\}$ does not arise from the matrix elements of some quantum matrix. In [2] we constructed such a set of functions b_0, b_+, b_-, b_1, and a pure braid element $\Upsilon_e = \frac{1}{\lambda}(\Delta c - c \otimes 1)$ from the casimir $c := P_+P_-$ of $\mathrm{e}_q(2)$. Now we can put the new machinery to work and calculate the (invertible) metric

$$\eta_{\mathrm{E}_q(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -q^{-2} & 0 \end{pmatrix}, \tag{33}$$

which immediately gives an expression for $\mathbf{d}$ on functions:

$$\mathbf{d} = \omega_0\chi_1 + \omega_1\chi_0 - q^2\omega_+\chi_- - \omega_-\chi_+. \tag{34}$$

3.1.4 Universal Calculus

Given (countably infinite) linear bases $\{e_i\}$ and $\{f^i\}$ of the Hopf algebras $\mathcal{U}$ and $\mathcal{A}$ respectively, we can always construct new counit-free elements $e_i - 1_{\mathcal{U}}\epsilon(e_i)$ and $f^i - 1_{\mathcal{A}}\epsilon(f^i)$ that each span (infinite) biinvariant spaces $\mathcal{T}_q$ and $\mathcal{C}_q$ respectively and have coproducts of the form (3); in fact $1_{\mathcal{U}} \oplus \mathcal{T}_q = \mathcal{U}$ and $1_{\mathcal{A}} \oplus \mathcal{C}_q = \mathcal{A}$ as vector spaces. Using some Gram-Schmitt orthogonalization procedure one can rearrange the infinite bases of $\mathcal{U}$ and $\mathcal{A}$ in such a way that $e_0 = 1_{\mathcal{U}}$, $f^0 = 1_{\mathcal{A}}$ and e_i, f^i with $\epsilon(e_i) = \epsilon(f^i) = 0$ for $i = 1, \ldots, \infty$ span $\mathcal{T}_q$ and $\mathcal{C}_q$ respectively. (In the rest of this section roman indices $i, j, k, \ldots$ will only take on values from 1 to ∞.) To avoid confusion with the finite-dimensional quantum Lie algebras, we will use the symbol $\boldsymbol{\delta}$ instead of $\mathbf{d}$ for the exterior derivative.

Given orthonormal linear bases $\{e_i\}$ and $\{f^i\}$ of $\mathcal{T}_q$ and $\mathcal{C}_q$ we can now express $\boldsymbol{\delta}$ on functions $a \in \mathcal{A}$ as

$$\boldsymbol{\delta}(a) = -\omega_{S^{-1}f^i}\,\mathcal{L}_{e_i - 1\epsilon(e_i)}(a); \tag{35}$$

note, however, that *all* of these $\omega_{S^{-1}f^i}$s are treated as linearly independent and remain so even in the classical limit, because (35) in conjunction with the Leibniz rule for δ only gives trivial commutation relations between forms and functions ($a\omega_b = \omega_{bS^{-1}a_{(2)}}a_{(1)} - \epsilon(b)\omega_{S^{-1}a_{(2)}}a_{(1)}$); therefore, it is not generally possible to reorganize the infinite set of $\omega_{S^{-1}f^i}$s into a finite basis of 1-forms. This is the case for Connes' noncommutative geometry [1] and is in contrast to the ordinary "textbook" treatment of differential calculi that has forms commuting with functions.

4 Calculus of Functions, Vector Fields and Forms

Here we will generalize the Cartan calculus of ordinary *commutative* differential geometry to the case of quantum Lie algebras.

As in the classical case, the Lie derivative of a function is given by the action of the corresponding vector field, *i.e.*

$$\begin{aligned} \pounds_{\chi_i}(a) &= \chi_i \triangleright a = a_{(1)}\left\langle \chi_i, a_{(2)}\right\rangle, \\ \pounds_{\chi_i} a &= a_{(1)}\left\langle \chi_{i(1)}, a_{(2)}\right\rangle \pounds_{\chi_{i(2)}}. \end{aligned} \tag{36}$$

The action on products is given through the coproduct of χ_i:

$$\chi_i \triangleright ab = (\chi_{i(1)} \triangleright a)(\chi_{i(2)} \triangleright b). \tag{37}$$

The Lie derivative along χ_i of an element $y \in \mathcal{U}$ is given by the adjoint action in $\mathcal{U}$:

$$\pounds_{\chi_i}(y) = \chi_i \overset{\mathrm{ad}}{\triangleright} y = \chi_{i(1)} y S(\chi_{i(2)}). \tag{38}$$

To find the action of $\boldsymbol{i}_{\chi_i}$ we can now use the Cartan identity (2):

$$\chi_i \triangleright a = \pounds_{\chi_i}(a) = \boldsymbol{i}_{\chi_i}(\mathrm{d}a) + \mathrm{d}(\boldsymbol{i}_{\chi_i} a). \tag{39}$$

As the inner derivation $\boldsymbol{i}_{\chi_i}$ contracts 1-forms and is zero on 0-forms like a, we find

$$\boldsymbol{i}_{\chi_i}(\mathrm{d}a) = \chi_i \triangleright a = a_{(1)}\left\langle \chi_i, a_{(2)}\right\rangle. \tag{40}$$

Next consider that for any form α,

$$\pounds_{\chi_i}(\mathrm{d}\alpha) = \mathrm{d}(\boldsymbol{i}_{\chi_i}\mathrm{d}\alpha) + \boldsymbol{i}_{\chi_i}(\mathrm{dd}\alpha) = \mathrm{d}(\pounds_{\chi_i}\alpha) + 0, \tag{41}$$

which shows that Lie derivatives commute with the exterior derivative; $\pounds_{\chi_i}\mathbf{d} = \mathbf{d}\pounds_{\chi_i}$ (we will also assume $\pounds_x$ commutes with $\mathbf{d}$ for *all* $x \in \mathcal{U}$ as well). From this and (36) we find

$$\pounds_{\chi_i}\mathbf{d}(a) = \mathbf{d}(a_{(1)})\left\langle \chi_{i(1)}, a_{(2)}\right\rangle \pounds_{\chi_{i(2)}}. \tag{42}$$

To find the complete commutation relations of $\boldsymbol{i}_{\chi_i}$ with functions and forms rather than just its action on them, we next compute the action of $\pounds_{\chi_i}$ on a product of functions a, $b \in \mathcal{A}$, *i.e.*

$$\pounds_{\chi_i}(ab) = \boldsymbol{i}_{\chi_i}\mathbf{d}(ab) = \boldsymbol{i}_{\chi_i}(\mathbf{d}(a)b + a\mathbf{d}(b)), \tag{43}$$

and compare with equation (37). Recalling that the χ_i have coproducts of the form $\Delta\chi_i = \chi_i \otimes 1 + O_i{}^j \otimes \chi_j$, $O_i{}^j \in \mathcal{U}$, we obtain

$$\boldsymbol{i}_{\chi_i}a = (O_i{}^j \triangleright a)\, \boldsymbol{i}_{\chi_j} = \pounds_{O_i{}^j}(a)\, \boldsymbol{i}_{\chi_j} \tag{44}$$

if we assume that the commutation relation of $\boldsymbol{i}_{\chi_i}$ with $\mathbf{d}(a)$ is of the general form

$$\boldsymbol{i}_{\chi_i}\mathbf{d}(a) = \underbrace{\boldsymbol{i}_{\chi_i}(\mathbf{d}a)}_{\in\mathcal{A}} + \text{“braiding term”} \cdot \boldsymbol{i}_{\chi_?}\,. \tag{45}$$

A calculation of $\pounds_{\chi_i}(\mathbf{d}(a)\mathbf{d}(b))$ along similar lines gives in fact

$$\begin{aligned} \boldsymbol{i}_{\chi_i}\mathbf{d}(a) &= (\chi_i \triangleright a) - \mathbf{d}(O_i{}^j \triangleright a)\, \boldsymbol{i}_{\chi_j} \\ &= \boldsymbol{i}_{\chi_i}(\mathbf{d}a) - \pounds_{O_i{}^j}(\mathbf{d}a)\, \boldsymbol{i}_{\chi_j}, \end{aligned} \tag{46}$$

and we propose for any p-form α:

$$\boldsymbol{i}_{\chi_i}\alpha = \boldsymbol{i}_{\chi_i}(\alpha) + (-1)^p \pounds_{O_i{}^j}(\alpha)\, \boldsymbol{i}_{\chi_j}. \tag{47}$$

Using the Cartan identity we can derive commutation relations for (Lie) derivatives and functions from equation (44), which can be written in Hopf algebra language as

$$\chi_i a = a_{(1)}\left\langle \chi_{i(1)}, a_{(2)}\right\rangle \chi_{i(2)}. \tag{48}$$

This actually defines the product in the cross-product algebra $\mathcal{A}\rtimes\mathcal{U}$ of general vector fields that one obtains by combining the Hopf algebras $\mathcal{A}$ and $\mathcal{U}$; see *e.g.* [2].

4.1 Maurer-Cartan Forms

The most general left-invariant 1-form can be written as [10]

$$\omega_b := S(b_{(1)})\mathrm{d}(b_{(2)}) = -\mathrm{d}(Sb_{(1)})b_{(2)} \tag{49}$$

$$(\textit{left-invariance:}\ \ {}_{\mathcal{A}}\Delta(\omega_b) = S(b_{(2)})b_{(3)} \otimes S(b_{(1)})\mathrm{d}(b_{(4)}) = 1 \otimes \omega_b), \tag{50}$$

corresponding to a function $b \in \mathcal{A}$. If this function happens to be $t^i{}_k$, where $t \in M_m(\mathcal{A})$ is an $m \times m$ matrix representation of $\mathcal{U}$ with $\Delta(t^i{}_k) = t^i{}_j \otimes t^j{}_k$ and $S(t) = t^{-1}$, we obtain the well-known Cartan-Maurer form $\omega_t = t^{-1}\mathrm{d}(t)$. Here is a nice formula for the exterior derivative of ω_b:

$$\mathrm{d}(\omega_b) = -\omega_{b_{(1)}}\omega_{b_{(2)}}. \tag{51}$$

The Lie derivative is

$$\pounds_{\chi_i}(\omega_b) = \omega_{b_{(2)}}\left\langle \chi_i, S(b_{(1)})b_{(3)}\right\rangle. \tag{52}$$

The contraction of left-invariant forms with $\boldsymbol{i}_\chi$ is

$$\boldsymbol{i}_{\chi_i}(\omega_b) = -\left\langle \chi_i, S(b)\right\rangle \in k. \tag{53}$$

4.2 Tensor Product Realization of the Wedge

From (52) and (53) we find commutation relations for $\boldsymbol{i}_{\chi_i}$ with ω^j,

$$\begin{aligned} \boldsymbol{i}_{\chi_i}\omega^j &= \delta_i^j - \pounds_{O_i{}^k}(\omega^j)\boldsymbol{i}_{\chi_k} \\ &= \delta_i^j - \omega^m\left\langle O_i{}^k, S^{-1}(T^j{}_m)\right\rangle \boldsymbol{i}_{\chi_k}, \end{aligned} \tag{54}$$

which can be used to define the wedge product $\wedge$ of forms as a kind of antisymmetrized tensor product[7]. As in the classical case we make an ansatz for the product of two forms in terms of tensor products

$$\omega^i \wedge \omega^j = \omega^i \otimes \omega^j - \hat{\sigma}^{ij}{}_{mn}\omega^m \otimes \omega^n, \tag{55}$$

with as yet unknown numerical constants $\hat{\sigma}^{ij}{}_{mn} \in k$, and define $\boldsymbol{i}_{\chi_i}$ to act on this product by contracting in the first tensor product space, *i.e.*

$$\boldsymbol{i}_{\chi_i}(\omega^j \wedge \omega^k) = \delta_i^j\omega^k - \hat{\sigma}^{jk}{}_{mn}\delta_i^m\omega^n. \tag{56}$$

[7]So far we have suppressed the $\wedge$-symbol; to avoid confusion we will reinsert it in this paragraph.

But from (54) we already know how to compute this, and we find

$$\hat{\sigma}^{ij}{}_{mn} = \left\langle O_m{}^j, S^{-1}(T^i{}_n) \right\rangle, \tag{57}$$

or

$$\begin{aligned} \omega^i \wedge \omega^j &= (I - \hat{\sigma})^{ij}{}_{mn} \omega^m \otimes \omega^n \\ &= \omega^i \otimes \omega^j - \omega^k \otimes \pounds_{O_k{}^j}(\omega^i). \end{aligned} \tag{58}$$

These equations give implicit (anti)commutation relations between the ω^is. Note that $(I - \hat{\sigma})$ has a sensible classical limit — it becomes $(I - P)$ where P is the permutation matrix. Using the same method as for ω we can also obtain a tensor product decomposition of products of inner derivations.

Example: Maurer-Cartan Equation

$$\begin{aligned} \mathbf{d}\omega^j &= \mathbf{d}\omega_{b^j} = -\omega_{b^j_{(1)}} \wedge \omega_{b^j_{(2)}} \\ &= -\omega_{S^{-1}(Sb^j_{(1)} b^j_{(3)})} \otimes \omega_{b^j_{(2)}} \\ &= -\omega^k \otimes \omega^l \left\langle -S\chi_k, S^{-1}(Sb^j_{(1)} b^j_{(3)}) \right\rangle \left\langle -S\chi_l, b^j_{(2)} \right\rangle \\ &= -\omega^k \otimes \omega^l \left\langle \underbrace{(S^{-1}\chi_k)_{(1)} \chi_l S(S^{-1}\chi_k)_{(2)}}_{S^{-1}\chi_k \overset{\mathrm{ad}}{\triangleright} \chi_l}, Sb^j \right\rangle \\ &= -f'^j_k{}_l \omega^k \otimes \omega^l. \end{aligned} \tag{59}$$

In the previous equation we have introduced the adjoint action of a left-invariant vector field on another vector field. A short calculation gives

$$S^{-1}\chi_k \overset{\mathrm{ad}}{\triangleright} \chi_l = \chi_b \chi_c (\delta^c_k \delta^b_l - \hat{\sigma}^{cb}{}_{kl}) = \chi_a \left\langle S^{-1}\chi_k, T^a{}_l \right\rangle = \chi_a f'{}_k{}^a{}_l \tag{60}$$

as compared to

$$\chi_k \overset{\mathrm{ad}}{\triangleright} \chi_l \equiv \pounds_{\chi_k}(\chi_l) = \chi_b \chi_c (\delta^c_k \delta^b_l - \hat{R}^{cb}{}_{kl}) = \chi_a f_k{}^a{}_l, \tag{61}$$

with $\hat{R}^{cb}{}_{kl} = \left\langle O_k{}^b, T^c{}_l \right\rangle$. The two sets of structure constants are related by $\langle \chi_k, T^a{}_l \rangle = f_k{}^a{}_l = -f'^a_i{}_l R^{ij}{}_{kl}$. See [11] for a detailed discussion of such structure constants.

4.2.1 The "Anti-Wedge" Operator

There is actually an operator W that recursively translates wedge products into the tensor product representation:

$$\begin{array}{l} W : \Lambda_q^p \to \mathcal{T}_q^* \otimes \Lambda_q^{p-1}, \quad p \geq 1, \\ W(\alpha) = \omega^n \otimes \boldsymbol{i}_{\chi_n}(\alpha), \end{array} \tag{62}$$

for any p-form α. Two examples:

$$\begin{aligned} \omega^j \wedge \omega^k &= \omega^n \otimes \boldsymbol{i}_{\chi_n}(\omega^j \wedge \omega^k) \\ &= \omega^n \otimes (\delta_n^j \omega^k - \pounds_{O_n{}^m}(\omega^j)\delta_m^k) \\ &= \omega^j \otimes \omega^k - \omega^n \otimes \pounds_{O_n{}^k}(\omega^j) \\ &= \omega^j \otimes \omega^k - \omega^n \otimes \omega^m \hat{\sigma}^{jk}{}_{nm} \end{aligned} \tag{63}$$

and, after a little longer computation that uses W twice,

$$\begin{aligned} \omega^a \wedge \omega^b \wedge \omega^c &= \omega^a \otimes (\omega^b \wedge \omega^c) - \omega^i \otimes (\omega^j \wedge \omega^c)\hat{\sigma}^{ab}{}_{ij} \\ &\quad + \omega^i \otimes (\omega^j \wedge \omega^k)\hat{\sigma}^{al}{}_{ij}\hat{\sigma}^{bc}{}_{lk} \\ &= \omega^a \otimes \omega^b \otimes \omega^c - \omega^a \otimes \omega^j \otimes \omega^k \hat{\sigma}^{bc}{}_{jk} \\ &\quad - \omega^i \otimes \omega^j \otimes \omega^c \hat{\sigma}^{ab}{}_{ij} + \omega^i \otimes \omega^j \otimes \omega^k \hat{\sigma}^{lc}{}_{jk}\hat{\sigma}^{ab}{}_{il} \\ &\quad + \omega^i \otimes \omega^j \otimes \omega^k \hat{\sigma}^{al}{}_{ij}\hat{\sigma}^{bc}{}_{lk} - \omega^i \otimes \omega^j \otimes \omega^k \hat{\sigma}^{an}{}_{il}\hat{\sigma}^{bc}{}_{nm}\hat{\sigma}^{lm}{}_{jk}. \end{aligned} \tag{64}$$

In some cases this expression can be further simplified with the help of the characteristic equation of $\hat{\sigma}$.

4.3 Summary of Relations in the Cartan Calculus

Commutation Relations For any p-form α:

$$\mathbf{d}\alpha = \mathbf{d}(\alpha) + (-1)^p \alpha \mathbf{d} \tag{65}$$

$$\boldsymbol{i}_{\chi_i}\alpha = \boldsymbol{i}_{\chi_i}(\alpha) + (-1)^p \pounds_{O_i{}^j}(\alpha)\boldsymbol{i}_{\chi_j} \tag{66}$$

$$\pounds_{\chi_i}\alpha = \pounds_{\chi_i}(\alpha) + \pounds_{O_i{}^j}(\alpha)\pounds_{\chi_j} \tag{67}$$

Actions For any function $f \in \mathcal{A}$, 1-form $\omega_f \equiv Sf_{(1)}\mathbf{d}f_{(2)}$ and vector field $\phi \in \mathcal{A}\rtimes\mathcal{U}$:

$$\boldsymbol{i}_{\chi_i}(f) = 0 \tag{68}$$

$$
\begin{aligned}
i_{\chi_i}(\mathbf{d}f) &= \mathbf{d}f_{(1)}\left\langle \chi_i, f_{(2)}\right\rangle && (69)\\
i_{\chi_i}(\omega_f) &= -\left\langle \chi_i, Sf\right\rangle && (70)\\
\pounds_{\chi_i}(f) &= \chi_i(f) = f_{(1)}\left\langle \chi_i, f_{(2)}\right\rangle && (71)\\
\pounds_{\chi_i}(\omega_f) &= \omega_{f_{(2)}}\left\langle \chi_i, S(f_{(1)})f_{(3)}\right\rangle && (72)\\
\pounds_{\chi_i}(\phi) &= \chi_{i(1)}\phi S(\chi_{i(2)}) && (73)
\end{aligned}
$$

Graded Quantum Lie Algebra of the Cartan Generators

$$
\begin{aligned}
\mathbf{dd} &= 0 && (74)\\
\mathbf{d}\pounds_x &= \pounds_x\mathbf{d} && (75)\\
\pounds_{\chi_i} &= \mathbf{d}i_{\chi_i} + i_{\chi_i}\mathbf{d} && (76)\\
\left[\pounds_{\chi_i}, \pounds_{\chi_k}\right]_q &= \pounds_{\chi_l} f_i{}^l{}_k && (77)\\
\left[\pounds_{\chi_i}, i_{\chi_k}\right]_q &= i_{\chi_l} f_i{}^l{}_k && (78)
\end{aligned}
$$

The quantum commutator $[\,,\,]_q$ is here defined as follows:

$$
\left[\pounds_{\chi_i}, \Box\right]_q := \pounds_{\chi_i}\Box - \pounds_{O_i{}^j}(\Box)\pounds_{\chi_j}. \qquad (79)
$$

This quantum Lie algebra becomes infinite-dimensional as soon as we introduce derivatives along general vector fields.

4.4 Universal Cartan Calculus

In the case of a Universal Calculus (see section 3.1.4) the relations of a Cartan Calculus can be expressed in a basis-free form in Hopf algebra language. Here is a summary of commutation relations valid on any form. All of these equations are identical to the corresponding quantum Lie algebra relations when written in terms of the bases $\{e_\beta\}$ and $\{f^\beta\}$, where $\beta = 0, 1, \ldots, \infty$. $x, y \in \mathcal{U}$, $a \in \mathcal{A}$, α is a p-form and $\phi \in \mathcal{A}\rtimes\mathcal{U}$ is a vector field.

$$
\begin{aligned}
\pounds_x a &= a_{(1)}\left\langle x_{(1)}, a_{(2)}\right\rangle \pounds_{x_{(2)}} && (80)\\
\pounds_x \delta(a) &= \delta(a_{(1)})\left\langle x_{(1)}, a_{(2)}\right\rangle \pounds_{x_{(2)}} && (81)\\
\pounds_x \alpha &= \pounds_{x_{(1)}}(\alpha)\, \pounds_{x_{(2)}} && (82)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{i}_x a &= a_{(1)} \left\langle x_{(1)}, a_{(2)} \right\rangle \boldsymbol{i}_{x_{(2)}} && (83)\\
\boldsymbol{i}_x \boldsymbol{\delta}(a) &= a_{(1)} \left\langle x - 1\epsilon(x), a_{(2)} \right\rangle - \boldsymbol{\delta}(a_{(1)}) \left\langle x_{(1)}, a_{(2)} \right\rangle \boldsymbol{i}_{x_{(2)}} && (84)\\
\boldsymbol{i}_x \alpha &= \boldsymbol{i}_x(\alpha) + (-1)^p \pounds_{x_{(1)}}(\alpha)\, \boldsymbol{i}_{x_{(2)}} && (85)\\
\boldsymbol{\delta}\alpha &= \boldsymbol{\delta}(\alpha) + (-1)^p \alpha \boldsymbol{\delta} && (86)\\
\boldsymbol{\delta}\boldsymbol{\delta}(\alpha) &= -(-1)^p \boldsymbol{\delta}(\alpha)\boldsymbol{\delta} && (87)
\end{aligned}
$$

$$
\pounds_x(\phi) = x_{(1)} \phi S(x_{(2)}) \qquad (88)
$$

$$
\begin{aligned}
\boldsymbol{\delta}^2 &= 0 && (89)\\
\boldsymbol{\delta}\pounds_x &= \pounds_x \boldsymbol{\delta} && (90)\\
\pounds_x &= \boldsymbol{\delta}\boldsymbol{i}_x + 1\epsilon(x) + \boldsymbol{i}_x\boldsymbol{\delta} \qquad \textit{(universal Cartan identity)} && (91)\\
\pounds_x \pounds_y &= \pounds_{y^{(1)}} \left\langle x_{(1)}, y^{(2)'} \right\rangle \pounds_{x_{(2)}} && (92)\\
\pounds_x \boldsymbol{i}_y &= \boldsymbol{i}_{y^{(1)}} \left\langle x_{(1)}, y^{(2)'} \right\rangle \pounds_{x_{(2)}} && (93)
\end{aligned}
$$

This type of Cartan calculus on an arbitrary Hopf algebra will be treated in detail in an upcoming paper [12].

4.5 Lie Derivatives Along General Vector Fields

So far we have focused on Lie derivatives and inner derivations along *left-invariant* vector fields, *i.e.* along elements of $\mathcal{T}_q$. The classical theory allows functional coefficients, so that a general vector field need not be left-invariant. Here we may introduce derivatives along elements in the $\mathcal{A}\rtimes\mathcal{T}_q$ plane by the following set of equations valid on forms (recall $\epsilon(\chi) = 0$ for $\chi \in \mathcal{T}_q$):

$$
\begin{aligned}
\boldsymbol{i}_{f\chi_i} &= f\boldsymbol{i}_{\chi_i}, && (94)\\
\pounds_{f\chi_i} &= \mathbf{d}\boldsymbol{i}_{f\chi_i} + \boldsymbol{i}_{f\chi_i}\mathbf{d}, && (95)\\
\pounds_{f\chi_i} &= f\pounds_{\chi_i} + \mathbf{d}(f)\boldsymbol{i}_{\chi_i}, && (96)\\
\pounds_{f\chi_i}\mathbf{d} &= \mathbf{d}\pounds_{f\chi_i}. && (97)
\end{aligned}
$$

Equation (96) can be used to define Lie derivatives recursively on any form.

Acknowledgements

We would like to thank P. Aschieri, S. Majid and N. Yu. Reshetikhin for helpful discussions.

This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grants PHY-90-21139 and PHY-89-04035.

References

[1] A. Connes, *Publ. Math. IHES* **62** 257 (1985)

[2] P. Schupp, P. Watts and B. Zumino, *Commun. Math. Phys.* **157** 305 (1993)

[3] N. Yu. Reshetikhin and M. A. Semenov-Tian-Shansky, *Lett. Math. Phys.* **19** (1990)

[4] B. Jurčo, *Lett. Math. Phys.* **22** 177 (1991)

[5] S. Majid, private communication (1993)

[6] B. Zumino, in X^{th} IAMP Conf. Leipzig (1991), ed. K. Schmüdgen, Springer-Verlag (1992)

[7] P. Schupp, P. Watts and B. Zumino, *Lett. Math. Phys.* **25** 139 (1992)

[8] V. G. Drinfel'd, *Leningrad Math. J.* **1** 321 (1989)

[9] P. Watts, "Killing Form on Quasitriangular Hopf Algebras and Quantum Lie Algebras", in preparation

[10] S. L. Woronowicz, *Commun. Math. Phys.* **122** 125 (1989)

[11] L. Castellani and A. R-Monteiro, preprint DFTT-18/93 (1993)

[12] P. Schupp and P. Watts, "Universal Cartan Calculus on Hopf Algebras", in preparation

LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
TECHNICAL INFORMATION DEPARTMENT
BERKELEY, CALIFORNIA 94720